Chapter 2
Algebraic Moments, Elementary Exponential Inequalities

2.1 Introduction

In this chapter, we start by giving upper bounds for algebraic moments of partial sums from a strongly mixing sequence. These inequalities are similar to Rosenthal's inequalities (1970) concerning moments of sums of independent random variables. They may be applied to provide estimates of deviation probabilities of partial sums from their mean value, which are more efficient than the results derived from the Marcinkiewicz–Zygmund type moment inequalities given in Ibragimov (1962) or Billingsley (1968) for uniformly mixing sequences, or in Yokoyama (1980) for strongly mixing sequences, in particular for partial sums with a small variance. For example, Rosenthal type inequalities may be used to obtain precise upper bounds for integrated L^p-risks of kernel density estimators. They provide the exact rates of convergence, in contrast to Marcinkiewicz–Zygmund type moment inequalities, as shown first by Bretagnolle and Huber (1979) in the independent case.

In Sects. 2.2 and 2.3, we follow the approach of Doukhan and Portal (1983), for algebraic moments in the strong mixing case. In Sect. 2.4 we give a second method, which provides explicit constants in inequalities for the algebraic moments of order $2p$. Applying then the Markov inequality to S_n^{2p}, and minimizing the so obtained deviation bound with respect to p, we then get exponential Hoeffding type inequalities in the uniform mixing case. We also apply this method to obtain upper bounds for non-algebraic moments in Sect. 2.5.

2.2 An Upper Bound for the Fourth Moment of Sums

In this section, we adapt the method introduced in Billingsley (1968, Sect. 22) to bound the moment of order 4 of a sum of random variables satisfying a uniform mixing condition in the context of strongly mixing sequences. We start by introducing some notation that we shall use throughout the sequel.

E. Rio, *Asymptotic Theory of Weakly Dependent Random Processes*,
Probability Theory and Stochastic Modelling 80,
DOI 10.1007/978-3-662-54323-8_2

Notation 2.1 Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued random variables. Set $\mathcal{F}_k = \sigma(X_i : i \le k)$ and $\mathcal{G}_l = \sigma(X_i : i \ge l)$. By convention, if the sequence $(X_i)_{i\in T}$ is defined on a subset T of $\mathbb{Z}$, we set $X_i = 0$ for i in $\mathbb{Z} \setminus T$.

Throughout Sects. 2.2 and 2.3, the strong mixing coefficients $(\alpha_n)_{n\ge 0}$ of $(X_i)_{i\in\mathbb{Z}}$ are defined, as in Rosenblatt (1956), by

$$\alpha_0 = 1/2 \text{ and } \alpha_n = \sup_{k\in\mathbb{Z}} \alpha(\mathcal{F}_k, \mathcal{G}_{k+n}) \text{ for any } n > 0. \tag{2.1}$$

Starting from Theorem 1.1(a), we now give an upper bound for the fourth moment of the partial sums for nonstationary sequences.

Theorem 2.1 *Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of centered real-valued random variables with finite fourth moments. Let $Q_k = Q_{|X_k|}$ and set*

$$M_{4,\alpha,n}(Q_k) = \sum_{k=1}^{n} \int_0^1 [\alpha^{-1}(u) \wedge n]^3 Q_k^4(u)du.$$

Then

$$\mathbb{E}(S_n^4) \le 3\Big(\sum_{i=1}^{n}\sum_{j=1}^{n} |\mathbb{E}(X_i X_j)|\Big)^2 + 48 \sum_{k=1}^{n} M_{4,\alpha,n}(Q_k).$$

Proof For $i \notin [1, n]$, let us replace the initial random variables X_i by the null random variable. With this convention

$$S_n^4 = 24 \sum_{i<j<k<l} X_i X_j X_k X_l + 12 \sum_{\substack{j<k \\ i\notin\{j,k\}}} X_i^2 X_j X_k + 6 \sum_{i<j} X_i^2 X_j^2 + 4 \sum_{i\ne j} X_i^3 X_j + \sum_i X_i^4. \tag{2.2}$$

It follows that

$$\mathbb{E}(S_n^4) \le 3 \sum_{i\le j\le k\le l} |\mathbb{E}(X_i X_j X_k X_l)|(1 + \mathbb{1}_{i<j})(1 + \mathbb{1}_{j<k})(1 + \mathbb{1}_{k<l}). \tag{2.3}$$

We now apply Theorem 1.1(a) to the product $X_i X_j X_k X_l$ at the maximal spacing. So, let $m = \sup(j - i, k - j, l - k)$. If $m = k - j > 0$, then Theorem 1.1(a) applied to $X = X_i X_j$ and $Y = X_k X_l$ yields

$$|\mathbb{E}(X_i X_j X_k X_l)| \le |\mathbb{E}(X_i X_j)\mathbb{E}(X_k X_l)| + 2\int_0^{\alpha_m} Q_{X_i X_j}(u) Q_{X_k X_l}(u)du. \tag{2.4}$$

If $m = j - i$ and $k - j < m$, Theorem 1.1(a) applied to $X = X_i$ and $Y = X_j X_k X_l$ yields

$$|\mathbb{E}(X_i X_j X_k X_l)| \le 2\int_0^{\alpha_m} Q_{X_i}(u) Q_{X_j X_k X_l}(u)du. \tag{2.5}$$

The case $m = l - k$ and $\sup(k - j, j - i) < m$ can be treated in the same way and gives the same inequality. To complete the proof, we will need the technical lemma below, due to Bass (1955) in the case $p = 2$.

Lemma 2.1 *Let $Z_1, \ldots Z_p$ be nonnegative random variables. Then*

$$\mathbb{E}(Z_1 \ldots Z_p) \leq \int_0^1 Q_{Z_1}(u) \ldots Q_{Z_p}(u) du. \qquad (a)$$

Furthermore,

$$\int_0^1 Q_{Z_1 Z_2}(u) Q_{Z_3}(u) \ldots Q_{Z_p}(u) du \leq \int_0^1 Q_{Z_1}(u) Q_{Z_2}(u) \ldots Q_{Z_p}(u) du \qquad (b)$$

and

$$\int_0^1 Q_{Z_1+Z_2}(u) Q_{Z_3}(u) \ldots Q_{Z_p}(u) du \leq \int_0^1 (Q_{Z_1}(u) + Q_{Z_2}(u)) Q_{Z_3}(u) \ldots Q_{Z_p}(u) du. \qquad (c)$$

Proof of Lemma 2.1 We first prove (a). By Fubini's Theorem,

$$\begin{aligned} \mathbb{E}(Z_1 \ldots Z_p) &= \int_{\mathbb{R}^p} \mathbb{P}(Z_1 > z_1, \ldots, Z_p > z_p) dz_1 \ldots dz_p \\ &\leq \int_{\mathbb{R}^p} \inf_{i \in [1,p]} \mathbb{P}(Z_i > z_i) dz_1 \ldots dz_p. \end{aligned} \qquad (2.6)$$

Now

$$\inf_{i \in [1,p]} \mathbb{P}(Z_i > z_i) = \int_0^1 \mathbb{1}_{z_1 < Q_{Z_1}(u)} \ldots \mathbb{1}_{z_p < Q_{Z_p}(u)} du. \qquad (2.7)$$

Plugging (2.7) into (2.6) and again applying Fubini's theorem, we then get (a).

Let us now prove (b). Let U be a random variable with the uniform distribution over [0, 1]. For any nonnegative random variable Z, $Q_Z(U)$ has the distribution of Z. Now (cf. Exercise 1, Chap. 1), if $H(t) = \mathbb{P}(Z_1 Z_2 > t)$, then, for any random variable δ with uniform distribution over [0, 1] independent of (Z_1, Z_2),

$$W = 1 - V = H(Z_1 Z_2 - 0) + \delta(H(Z_1 Z_2) - H(Z_1 Z_2 - 0))$$

has the uniform law. Let $(T_1, T_2, \cdots, T_p) = (Z_1, Z_2, Q_{Z_3}(W), \ldots, Q_{Z_p}(W))$. Then the random variable $(T_1 T_2, T_3, \ldots, T_p)$ has the same law as $(Q_{Z_1 Z_2}(U), Q_{Z_3}(U), \ldots, Q_{Z_p}(U))$. Hence, by Lemma 2.1(a),

$$\int_0^1 Q_{Z_1 Z_2}(u) Q_{Z_3}(u) \ldots Q_{Z_p}(u) du \leq \int_0^1 Q_{Z_1}(u) Q_{Z_2}(u) \ldots Q_{Z_p}(u) du,$$

which completes the proof of (b). The proof of (c), being similar, is omitted. ■

We now complete the proof of Theorem 2.1. Both inequalities (2.4) and (2.5) together with Lemma 2.1(b) applied repeatedly yield

$$|\mathbb{E}(X_i X_j X_k X_l)| \leq 2\int_0^{\alpha_m} Q_i(u)Q_j(u)Q_k(u)Q_l(u)du \\ + |\mathbb{E}(X_i X_j)\mathbb{E}(X_k X_l)|\,\mathbb{I}_{k-j>\max(j-i,l-k)}, \tag{2.8}$$

where $m = \max(j-i, k-j, l-k) > 0$ is the maximal spacing. In the case $m = 0$, (2.8) still holds since

$$E(X_i^4) = \int_0^1 Q_i^4(u)du \leq 2\int_0^{1/2} Q_i^4(u)du.$$

Now

$$\sum_{i\leq j<k\leq l} |\mathbb{E}(X_i X_j)\mathbb{E}(X_k X_l)|(1+\mathbb{I}_{i<j})(1+\mathbb{I}_{k<l}) \leq \Big(\sum_{(i,j)\in[1,n]^2} |\mathbb{E}(X_i X_j)|\Big)^2.$$

Hence, by (2.3) and (2.8),

$$\mathbb{E}(S_n^4) - 3\Big(\sum_{i=1}^n\sum_{j=1}^n |\mathbb{E}(X_i X_j)|\Big)^2 \leq 12\sum_{i\leq j\leq k\leq l}\int_0^{\alpha_m}(Q_i^4(u)+Q_j^4(u)+Q_k^4(u)+Q_l^4(u))du \\ \leq 48\sum_{m=0}^{n-1}\sum_{k=1}^{n}\int_{\alpha_{m+1}}^{\alpha_m}(m+1)^3 Q_k^4(u)du, \tag{2.9}$$

with the convention that $\alpha_n = 0$ in (2.9). Hence Theorem 2.1 holds ■

Application of Theorem 2.1 to bounded random variables. Suppose that $\|X_i\|_\infty \leq 1$ for any $i > 0$. Then by Theorem 2.1 and Corollary 1.2,

$$\mathbb{E}(S_n^4) \leq 3\Big(\sum_{i=1}^n\sum_{j=1}^n |\mathbb{E}(X_i X_j)|\Big)^2 + 144n\sum_{m=0}^{n-1}(m+1)^2\alpha_m \\ \leq 48n^2\Big(\sum_{m=0}^{n-1}\alpha_m\Big)^2 + 144n\sum_{m=0}^{n-1}(m+1)^2\alpha_m. \tag{2.10}$$

Let us compare this result with Lemma 4, Sect. 20, in Billingsley (1968). This lemma gives, in our setting (note that Billingsley's proof can be adapted to strongly mixing sequences),

$$\mathbb{E}(S_n^4) \leq 768n^2\Big(\sum_{m=0}^{n-1}\sqrt{\alpha_m}\Big)^2. \tag{2.11}$$

For any $p > 0$, set

$$\Lambda_p(\alpha^{-1}) = \sup_{0 \le m < n} (m+1)(\alpha_m)^{1/p}. \tag{2.12}$$

Applying (2.10), we get

$$\mathbb{E}(S_n^4) \le (8\pi^2 + 144)(n\Lambda_2(\alpha^{-1}))^2 \le 223n^2(\Lambda_2(\alpha^{-1}))^2. \tag{2.13}$$

Since $(\alpha_m)_{m\ge 0}$ is nonincreasing,

$$\Lambda_2(\alpha^{-1}) \le \sum_{m=0}^{n-1} \sqrt{\alpha_m}, \tag{2.14}$$

which shows that (2.13) implies (2.11). Now, if the strong mixing coefficients α_m satisfy $\alpha_m = O(m^{-2})$, then (2.13) ensures that $\mathbb{E}(S_n^4) = O(n^2)$. In that case (2.11) leads to a logarithmic loss. ■

2.3 Even Algebraic Moments

In this section, we extend Theorem 2.1 to moments of order $2p$ with $p > 2$ an integer. Our main result is the following.

Theorem 2.2 *Let $p > 0$ be an integer and $(X_i)_{i\in\mathbb{N}}$ be a sequence of centered real-valued random variables with finite moments of order $2p$. Set $Q_k = Q_{X_k}$. Then there exist positive constants a_p and b_p such that*

$$\mathbb{E}\big(S_n^{2p}\big) \le a_p\Big(\int_0^1 \sum_{k=1}^n [\alpha^{-1}(u) \wedge n] Q_k^2(u)du\Big)^p + b_p \sum_{k=1}^n \int_0^1 [\alpha^{-1}(u) \wedge n]^{2p-1} Q_k^{2p}(u)du.$$

Remark 2.1 Recall that $Q_k(U)$ and $|X_k|$ have the same law. The weighted moments on the right-hand side of the above inequality play the same role as the usual moments in the independent case. We refer to Annex C for more comparisons between these quantities and the usual moments.

Doukhan and Portal (1983) give recursive relations which allow us to bound a_p and b_p by induction on p. These upper bounds can be used to derive exponential inequalities for geometrically strongly mixing sequences or random fields (cf. Doukhan et al. (1984) or Doukhan 1994). For nonalgebraic moments, one can derive moment inequalities from the algebraic case via interpolation inequalities (see Utev (1985) or Doukhan 1994). Nevertheless, interpolation inequalities lead to suboptimal mixing conditions. In Chap. 6, we will give another way to prove moment inequalities, which leads to unimprovable mixing conditions.

Proof of Theorem 2.2 We follow the line of proof of Doukhan and Portal (1983); cf. also Doukhan (1994). For any positive integer q, let

$$A_q(n) = \sum_{1 \le i_1 \le \cdots \le i_q \le n} |\mathbb{E}(X_{i_1} \dots X_{i_q})|. \tag{2.15}$$

It is easy to check that

$$\mathbb{E}(S_n^{2p}) \le (2p)! A_{2p}(n). \tag{2.16}$$

Theorem 2.2 then follows from similar upper bounds on $A_q(n)$. We will bound these quantities by induction on q via Lemma 2.2 below.

Lemma 2.2 *Suppose that the random variables $X_1, \dots X_n$ are centered and with finite absolute moments of order q. Then*

$$A_q(n) \le \sum_{r=1}^{q-1} A_r(n) A_{q-r}(n) + 2 \sum_{k=1}^{n} \int_0^1 [\alpha^{-1}(u) \wedge n]^{q-1} Q_k^q(u) du.$$

Proof As in the proof of Theorem 2.1, we may assume that $\alpha_n = 0$. Let

$$m(i_1, \dots, i_q) = \sup_{k \in [1,q[} (i_{k+1} - i_k)$$

and

$$j = \inf\{k \in [1, q[\, : i_{k+1} - i_k = m(i_1, \dots, i_q)\}. \tag{2.17}$$

Theorem 1.1(a) applied to $X = X_{i_1} \dots X_{i_j}$ and $Y = X_{i_{j+1}} \dots X_{i_q}$ together with Lemma 2.1(b) ensures that

$$|\mathbb{E}(X_{i_1} \dots X_{i_q})| \le |\mathbb{E}(X_{i_1} \dots X_{i_j}) \mathbb{E}(X_{i_{j+1}} \dots X_{i_q})| + 2 \int_0^{\alpha_{m(i_1,\dots,i_q)}} Q_{i_1}(u) \dots Q_{i_q}(u) du. \tag{2.18}$$

Summing (2.18) over $(i_1, \dots, i_q)$ we infer that

$$A_q(n) \le \sum_{r=1}^{q-1} A_r(n) A_{q-r}(n) + 2 \sum_{i_1 \le \cdots \le i_q} \int_0^{\alpha_{m(i_1,\dots,i_q)}} Q_{i_1}(u) \dots Q_{i_q}(u) du. \tag{2.19}$$

Now, starting from the elementary inequality

$$Q_{i_1}(u) \dots Q_{i_q}(u) \le q^{-1} (Q_{i_1}^q(u) + \cdots + Q_{i_q}(u)),$$

and interchanging the sum and the integral, we get that

$$\sum_{i_1 \le \cdots \le i_q} \int_0^{\alpha_{m(i_1,\ldots,i_q)}} Q_{i_1}(u) \ldots Q_{i_q}(u)du \le \frac{1}{q} \sum_{l=1}^{q} \sum_{i_l=1}^{n} \sum_{m=0}^{n-1} \int_{\alpha_{m+1}}^{\alpha_m} \chi(i_l, m) Q_{i_l}^q(u)du,$$

where $\chi(i_l, m)$ is the cardinality of the set of $(q-1)$-tuples $(i_1, .., i_{l-1}, i_{l+1}, .., i_q)$ such that

$$i_1 \le \cdots \le i_{l-1} \le i_l \le i_{l+1} \le \cdots \le i_q \quad \text{and} \quad \sup_{k \in [1,q[} (i_{k+1} - i_k) \le m.$$

Noting that $\chi(i_l, m) \le (m+1)^{q-1}$, we then get Lemma 2.2. ■

End of the proof of Theorem 2.2. Let

$$M_{q,\alpha,n} = \sum_{k=1}^{n} \int_0^1 [\alpha^{-1}(u) \wedge n]^{q-1} Q_k^q(u)du. \tag{2.20}$$

We will prove by induction on q that

$$A_q(n) \le a_q M_{2,\alpha,n}^{q/2} + b_q M_{q,\alpha,n}. \tag{$\mathcal{H}(q)$}$$

By Corollary 1.2, $\mathcal{H}(2)$ holds true with $a_2 = 2$ and $b_2 = 0$. Suppose now that $\mathcal{H}(r)$ holds for any $r \le q-1$ Then, from Lemma 2.2 we get that

$$A_q(n) \le \sum_{r=2}^{q-2} (a_r M_{2,\alpha,n}^{r/2} + b_r M_{r,\alpha,n})(a_{q-r} M_{2,\alpha,n}^{(q-r)/2} + b_{q-r} M_{q-r,\alpha,n}) + 2M_{q,\alpha,n}.$$

Hence $\mathcal{H}(q)$ will hold true if we prove that, for any r in $[2, q-2]$,

$$(a_r M_{2,\alpha,n}^{r/2} + b_r M_{r,\alpha,n})(a_{q-r} M_{2,\alpha,n}^{(q-r)/2} + b_{q-r} M_{q-r,\alpha,n}) \le a_{q,r} M_{2,\alpha,n}^{q/2} + b_{q,r} M_{q,\alpha,n}. \tag{2.21}$$

To prove (2.21) we apply Young's inequality $qxy \le rx^{q/r} + (q-r)y^{q/(q-r)}$ to the left-hand side of (2.21). Noting that $(v+w)^s \le 2^{s-1}(v^s + w^s)$ for any $s \ge 1$, we get that (2.21) will hold true if

$$M_{r,\alpha,n}^{q/r} \le c_{q,r}(M_{2,\alpha,n}^{q/2} + M_{q,\alpha,n}). \tag{2.22}$$

Now, let

$$M_{p,\alpha,n}(Q_k) = \int_0^1 [\alpha^{-1}(u) \wedge n]^{p-1} Q_k^p(u)du.$$

By Hölder's inequality,

$$M_{r,\alpha,n}(Q_k) \le (M_{q,\alpha,n}(Q_k))^{(r-2)/(q-2)} (M_{2,\alpha,n}(Q_k))^{(q-r)/(q-2)}.$$

Therefore

$$M_{r,\alpha,n} = \sum_{k=1}^{n} M_{r,\alpha,n}(Q_k) \le \sum_{k=1}^{n} (M_{q,\alpha,n}(Q_k))^{(r-2)/(q-2)} (M_{2,\alpha,n}(Q_k))^{(q-r)/(q-2)}.$$

Hence, by Hölder's inequality applied with exponents $(q-2)/(r-2)$ and $(q-2)/(q-r)$ together with the appropriate Young's inequality,

$$M_{r,\alpha,n} \le M_{q,\alpha,n}^{(r-2)/(q-2)} M_{2,\alpha,n}^{(q-r)/(q-2)} \le c'_{r,q}(M_{q,\alpha,n}^{r/q} + M_{2,\alpha,n}^{r/2}),$$

which implies (2.22). Whence (2.21) holds, and Lemma 2.2 follows by induction on q. Both (2.16) and Lemma 2.2 then imply Theorem 2.2. ■

Application to bounded random variables. Suppose that $\|X_i\|_\infty \le 1$ for any $i > 0$. Then

$$\mathbb{E}(S_n^{2p}) \le (2a_p + b_p)n^p(\Lambda_p(\alpha^{-1}))^p. \tag{2.23}$$

Consequently, if the strong mixing coefficients $(\alpha_m)_{m\ge 0}$ satisfy $\alpha_m = O(m^{-p})$, then (2.23) implies the Marcinkiewicz–Zygmund type inequality $\mathbb{E}(S_n^{2p}) = O(n^p)$. In that case Yokoyama's inequalities (1980) are not efficient (cf. Annex C for more details). ■

2.4 Exponential Inequalities

The constants a_p and b_p appearing in Theorem 2.2 can be bounded by explicit constants. Nevertheless, in the case of geometrically mixing sequences, it seems that it is difficult to obtain the exact dependence in p of the constants (recall that one can derive exponential inequalities from moment inequalities with explicit constants). In this section, we give a different way to obtain moment inequalities, which is more suitable for deriving exponential inequalities. Next we will derive exponential inequalities for geometrically strongly mixing inequalities from these new inequalities. We will also obtain the so-called Collomb inequalities (1984) for uniformly mixing sequences via this method. We refer to Delyon (2015) and Wintenberger (2010) for additional results.

Notation 2.2 Let $\mathcal{F}_i = \sigma(X_j : j \le i)$. We set $\mathbb{E}_i(X_k) = \mathbb{E}(X_k \mid \mathcal{F}_i)$.

The fundamental tool of this section is the equality below.

Theorem 2.3 *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued random variables and ψ be a convex differentiable map from $\mathbb{R}$ into $\mathbb{R}^+$, with $\psi(0) = 0$, and such that the second derivative of ψ in the sense of distributions is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Let ψ'' denote the density of the second derivative of ψ. Suppose that for any i in $[1, n]$ and any k in $[i, n]$,*

$$\mathbb{E}(|(\psi'(S_i) - \psi'(S_{i-1}))X_k|) < \infty. \tag{a}$$

Then

$$\mathbb{E}(\psi(S_n)) = \sum_{i=1}^{n} \int_0^1 \mathbb{E}\Big(\psi''(S_{i-1} + tX_i)\Big(-tX_i^2 + X_i \sum_{k=i}^{n} \mathbb{E}_i(X_k)\Big)\Big)dt.$$

Proof By the Taylor integral formula of order 2

$$\begin{aligned}\psi(S_n) &= \sum_{i=1}^{n}(\psi(S_i) - \psi(S_{i-1})) \\ &= \sum_{k=1}^{n} \psi'(S_{k-1})X_k + \sum_{i=1}^{n} \int_0^1 (1-t)\psi''(S_{i-1} + tX_i)X_i^2 dt.\end{aligned}$$

Now

$$\psi'(S_{k-1}) = \sum_{i=1}^{k-1}(\psi'(S_i) - \psi'(S_{i-1})) = \sum_{i=1}^{k-1} \int_0^1 \psi''(S_{i-1} + tX_i)X_i dt.$$

Plugging this equality into the Taylor formula, we get that

$$\psi(S_n) = \sum_{i=1}^{n} \int_0^1 \psi''(S_{i-1} + tX_i)\Big(-tX_i^2 + X_i \sum_{k=i}^{n} X_k\Big)dt. \tag{2.24}$$

Now, taking the mean of the above equality, noticing that, under assumption (a), the random variables $(1-t)\psi''(S_{i-1} + tX_i)X_i^2$ and $\psi''(S_{i-1} + tX_i)X_i X_k$ are integrable with respect to the product measure $\lambda \otimes \mathbb{P}$ and applying Fubini's theorem, we get that

$$\mathbb{E}(\psi(S_n)) = \sum_{i=1}^{n} \int_0^1 \mathbb{E}\Big(\psi''(S_{i-1} + tX_i)\Big(-tX_i^2 + X_i \sum_{k=i}^{n} X_k)\Big)\Big)dt.$$

Theorem 2.3 then follows from this equality and the fact that

$$\mathbb{E}\big(\psi''(S_{i-1} + tX_i)X_i X_k\big) = \mathbb{E}\big(\psi''(S_{i-1} + tX_i)X_i \mathbb{E}_i(X_k)\big). \quad \blacksquare$$

We now derive a Hoeffding type inequality from Theorem 2.3 (cf. Theorem B.4, Annex B, for Hoeffding's inequality for bounded and independent random variables). This inequality is an extension of the Azuma inequality (1967) for martingales to dependent sequences.

Theorem 2.4 *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of real-valued bounded random variables and let $(m_1, m_2, \ldots, m_n)$ be an n-tuple of positive reals such that*

$$\sup_{j\in[i,n]}\Big(\|X_i^2\|_\infty + 2\,\|X_i\sum_{k=i+1}^{j}\mathbb{E}_i(X_k)\|_\infty\Big) \le m_i \ \textit{ for any } i \in [1,n], \tag{a}$$

with the convention $\sum_{k=i+1}^{i}\mathbb{E}_i(X_k)=0$. *Then, for any nonnegative integer* p*,*

$$\mathbb{E}(S_n^{2p}) \le \frac{(2p)!}{2^p\,p!}\Big(\sum_{i=1}^{n} m_i\Big)^p. \tag{b}$$

Consequently, for any positive x*,*

$$\mathbb{P}(|S_n|\ge x) \le \sqrt{e}\exp\big(-x^2/(2m_1+\cdots+2m_n)\big). \tag{c}$$

Proof Define the functions ψ_p by $\psi_0(x)=1$ and $\psi_p(x)=x^{2p}/(2p)!$ for $p>0$. Set $M_i=\|X_i\|_\infty^2$. We prove (b) by induction on p. At range 0, (b) holds true for any sequence $(X_i)_{i\in\mathbb{Z}}$, since $S_n^0=1$. If (b) holds at range p for any sequence $(X_i)_{i\in\mathbb{Z}}$, then, applying Theorem 2.3 to $\psi=\psi_{p+1}$ and noting that $\psi''_{p+1}=\psi_p$, we get that

$$2\mathbb{E}(\psi_{p+1}(S_n)) \le \sum_{i=1}^{n}\int_0^1 \mathbb{E}(\psi_p(S_{i-1}+tX_i))(m_i+(1-2t)M_i)dt. \tag{2.25}$$

We now apply the induction hypothesis to the sequence $(X'_l)_{l\in\mathbb{Z}}$ defined by $X'_l=X_l$ for any $1\le l<i$, $X'_i=tX_i$ and $X'_l=0$ for $l\notin[1,i]$. For $l<i$ and $j<i$,

$$X'_l\sum_{m=l+1}^{j}\mathbb{E}_l(X'_m)=X_l\sum_{m=l+1}^{j}\mathbb{E}_l(X_m).$$

For $l<i$ and $j\ge i$,

$$X'_l\sum_{m=l+1}^{j}\mathbb{E}_l(X'_m)=tX_l\sum_{m=l+1}^{i}\mathbb{E}_l(X_m)+(1-t)X_l\sum_{m=l+1}^{i-1}\mathbb{E}_l(X_m).$$

Hence the sequence $(X'_l)_{l\in\mathbb{Z}}$ satisfies assumption (a) with the new sequence $(m'_i)_i$ defined by $m'_l=m_l$ for $l<i$ and $m'_i=t^2M_i$. Consequently, applying (b) to $S'_i=X'_1+\cdots+X'_i$, we get that

$$2^p p!\ \mathbb{E}(\psi_p(S_{i-1}+tX_i)) \le (m_1+\cdots+m_{i-1}+t^2M_i)^p.$$

Now $m_i+(1-2t)M_i\ge m_i-M_i\ge 0$. Hence

$$
\begin{aligned}
2^{p+1} p! \int_0^1 & \mathbb{E}(\psi_p(S_{i-1} + tX_i))(m_i + (1-2t)M_i)dt \\
& \le \int_0^1 (m_1 + \cdots + m_{i-1} + t^2 M_i)^p (m_i + (1-2t)M_i)dt \\
& \le \int_0^1 (m_1 + \cdots + m_{i-1} + tm_i + t(1-t)M_i)^p (m_i + (1-2t)M_i)dt,
\end{aligned}
\tag{2.26}
$$

since $tm_i + t(1-t)M_i \ge t^2 M_i$. Now

$$
\begin{aligned}
(p+1) \int_0^1 (m_1 + \cdots + m_{i-1} + tm_i + t(1-t)M_i)^p (m_i + (1-2t)M_i)dt = \\
(m_1 + \cdots + m_i)^{p+1} - (m_1 + \cdots + m_{i-1})^{p+1},
\end{aligned}
\tag{2.27}
$$

whence

$$
\begin{aligned}
2^{p+1}(p+1)! \int_0^1 \mathbb{E}(\psi_p(S_{i-1} + tX_i))(m_i + (1-2t)M_i)dt \le \\
(m_1 + \cdots + m_i)^{p+1} - (m_1 + \cdots + m_{i-1})^{p+1}.
\end{aligned}
\tag{2.28}
$$

Finally, both (2.25) and (2.28) ensure that the induction hypothesis holds at range $p+1$ for the sequence $(X_i)_{i \in \mathbb{Z}}$. Hence (b) holds true by induction on p.

In order to prove (c), we will apply the Markov inequality to S_n^{2p} for some appropriate p. Set

$$A = x^2/(2m_1 + \cdots + 2m_n) \text{ and } p = [A + (1/2)],$$

the square brackets designating the integer part. (c) holds trivially for $A \le 1/2$. Hence we may assume that $A \ge 1/2$. Then $p > 0$, and applying the Markov inequality to S_n^{2p}, we get that

$$\mathbb{P}(|S_n| \ge x) \le (4A)^{-p}(2p)!/p! \tag{2.29}$$

If A belongs to $[1/2, 3/2]$, (2.29) yields

$$\mathbb{P}(|S_n| \ge x) \le (2A)^{-1} \le \sqrt{e}\exp(-A),$$

since $2A \ge \exp(A - 1/2)$ for A in $[1/2, 3/2]$. Next, if $A \ge 3/2$, using the fact that the sequence $(2\pi n)^{-1/2}(e/n)^n n!$ is nonincreasing, we get that $(2p)! \le \sqrt{2}(4p/e)^p p!$, whence

$$\mathbb{P}(|S_n| \ge x) \le \sqrt{2}\,(eA)^{-p} p^p.$$

Now, taking the logarithm of this inequality, we obtain

$$A + \log \mathbb{P}(|S_n| \ge x) \le \log\sqrt{2} + f_p(A),$$

with $f_p(A) = (A - p) - p\log(A/p)$. Here $p \geq 2$ and A belongs to $[p - 1/2, p + 1/2[$. Since $f_p'(A) = (A - p)/A$ and $f_p''(A) = p/A^2$, the function f_p is convex. Consequently the maximum of f_p is attained at $A = p - 1/2$ or $A = p + 1/2$. Since f_p reaches its minimum at point p and f_p'' is decreasing, the maximum of f_p is attained for $A = p - 1/2$. Hence

$$A + \log \mathbb{P}(|S_n| \geq x) \leq \frac{\log 2 - 1}{2} + p\log\Big(\frac{2p}{2p-1}\Big) \leq \frac{\log 2 - 1}{2} + 2\log(4/3),$$

since $p \geq 2$. Thus we get that

$$\mathbb{P}(|S_n| \geq x) \leq \frac{16\sqrt{2}}{9\sqrt{e}}\exp(-A) \leq \sqrt{e}\exp(-A),$$

which completes the proof of Theorem 2.4(c). ■

We now apply Theorem 2.4 to uniformly mixing sequences, as defined below.

Definition 2.1 The uniform mixing coefficients of $(X_i)_{i\in\mathbb{Z}}$ are defined by

$$\varphi_0 = 1 \text{ and } \varphi_n = \sup_{k\in\mathbb{Z}} \varphi(\mathcal{F}_k, \sigma(X_{k+n})) \text{ for any } n > 0.$$

The sequence $(X_i)_{i\in\mathbb{Z}}$ is said to be uniformly mixing if φ_n converges to 0 as n tends to ∞.

Corollary 2.1 below provides a Hoeffding type inequality for uniformly mixing sequences of bounded random variables.

Corollary 2.1 *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of centered and real-valued bounded random variables. Set $\theta_n = 1 + 4(\varphi_1 + \cdots + \varphi_{n-1})$ and $M_i = \|X_i\|_\infty^2$. Then, for any positive integer p,*

$$\mathbb{E}(S_n^{2p}) \leq \frac{(2p)!}{p!}\Big(\frac{\theta_n}{2}\Big)^p (M_1 + \cdots + M_n)^p. \tag{a}$$

Next, for any positive x,

$$\mathbb{P}(|S_n| \geq x) \leq \sqrt{e}\exp\Big(-x^2/(2\theta_n M_1 + \cdots + 2\theta_n M_n)\Big). \tag{b}$$

Proof Let us apply Theorem 2.4 to the sequence $(X_i)_{i\in\mathbb{Z}}$. Since the random variables X_k are centered at expectation, by Theorem 1.4(b) and the Riesz–Fisher theorem,

$$\|\mathbb{E}_i(X_k)\|_\infty \leq 2\varphi_{k-i}\|X_k\|_\infty.$$

Hence we may apply Theorem 2.4 with

$$m_i = M_i + 4 \sum_{k=i+1}^{n} \sqrt{M_i M_k}\, \varphi_{k-i}.$$

Summing on i, we have:

$$\begin{aligned} m_1 + \cdots + m_n &\le \sum_{i=1}^{n} M_i + 4 \sum_{1 \le i < k \le n} \sqrt{M_i M_k}\, \varphi_{k-i} \\ &\le \sum_{i=1}^{n} M_i + 2 \sum_{1 \le i < k \le n} (M_i + M_k)\varphi_{k-i} \le \theta_n \sum_{i=1}^{n} M_i. \end{aligned}$$

Corollary 2.1 then follows from both Theorem 2.4 and the above upper bound.

2.5 New Moment Inequalities

In this section, we derive from Theorem 2.3 new moment inequalities for strongly mixing sequences. These inequalities are similar to the Marcinkiewicz–Zygmund type inequalities for independent random variables. Throughout the section, the strong mixing coefficients are defined in the following way:

$$\alpha_0 = 1/2 \text{ and } \alpha_n = \sup_{k \in \mathbb{Z}} \alpha(\mathcal{F}_k, X_{k+n}) \text{ for any } n > 0. \tag{2.30}$$

Our main result is as follows.

Theorem 2.5 *Let p be any real in $]1, \infty[$. Let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence of real-valued random variables with mean 0 and finite moment of order $2p$. Set $Q = Q_{X_0}$. Then, with the notations of Sect. 2.4, for any positive n,*

$$\mathbb{E}(|S_n|^{2p}) \le (4np)^p \sup_{l \in [1,n]} \mathbb{E}\Big(\Big|X_0 \sum_{i=0}^{l-1} \mathbb{E}_0(X_i)\Big|^p\Big). \tag{a}$$

Consequently,

$$\mathbb{E}(|S_n|^{2p}) \le (8np)^p \int_0^1 [\alpha^{-1}(u) \wedge n]^p Q^{2p}(u)du. \tag{b}$$

Remark 2.2 Inequality (a) may be applied to some dynamical systems with hyperbolicity, as shown by Melbourne and Nicol (2008). Inequality (b) can be improved if the strong mixing coefficients are defined by (2.1). We shall obtain Marcinkiewicz–Zygmund type inequalities Marcinkiewicz–Zygmund type inequalities under a weaker mixing condition in Chap. 6 (see Sect. 6.4 and (C.15) in Annex C).

Proof We prove Theorem 2.5 by induction on n. Our induction hypothesis is the following. For any integer $k \le n$ and any t in [0, 1],

$$\mathbb{E}(|S_{k-1} + tX_k|^{2p}) \le (4p)^p (k-1+t)^p \sup_{l \in [1,k]} \mathbb{E}\Big(\Big|X_0 \sum_{i=0}^{l-1} \mathbb{E}_0(X_i)\Big|^p\Big).$$

First, for any integer $k \le 4p$,

$$\|S_{k-1} + tX_k\|_{2p} \le (k-1+t)\|X_0\|_{2p} \le \sqrt{4p(k-1+t)}\ \|X_0\|_{2p}.$$

Hence the induction hypothesis holds for $k \le [4p]$.

Now let $n > 4p$. If the induction hypothesis holds at range $n-1$, then, applying Theorem 2.3 with $\psi(x) = |x|^{2p}$, and setting

$$h_n(t) = \mathbb{E}(|S_{n-1} + tX_n|^{2p}) \ \text{ and } \ \Gamma_n = \sup_{l \in [1,n]} \|X_0 \sum_{i=0}^{l-1} \mathbb{E}_0(X_i)\|_p,$$

we obtain that

$$\frac{h_n(t)}{4p^2} \le \sum_{i=1}^{n-1} \int_0^1 \mathbb{E}\Big(|S_{i-1} + sX_i|^{2p-2} X_i \sum_{k=i}^{n} \mathbb{E}_i(X_k)\Big) ds + \int_0^t \mathbb{E}(|S_{n-1} + sX_n|^{2p-2} X_n^2) ds.$$

We now apply Hölder's inequality with exponents $p/(p-1)$ and p:

$$\mathbb{E}\Big(|S_{i-1} + sX_i|^{2p-2} X_i \sum_{k=i}^{n} \mathbb{E}_i(X_k)\Big) \le (h_i(s))^{(p-1)/p} \|X_i \sum_{k=i}^{n} \mathbb{E}_i(X_k)\|_p.$$

From the stationarity of $(X_i)_{i \in \mathbb{Z}}$,

$$h_n(t) \le 4p^2 \Gamma_n \Big(\sum_{i=1}^{n-1} \int_0^1 (h_i(s))^{(p-1)/p} ds + \int_0^t (h_n(s))^{(p-1)/p} ds\Big).$$

Now if the induction hypothesis holds at range $n-1$, then

$$\begin{aligned}\int_0^1 (h_i(s))^{(p-1)/p} ds &\le (4p\Gamma_n)^{p-1} \int_0^1 (i-1+s)^{p-1} ds \\ &\le (4\Gamma_n)^{p-1} p^{p-2} (i^p - (i-1)^p).\end{aligned}$$

Set $g_n(s) = (4p(n-1+s)\Gamma_n)^p$. The above inequalities ensure that

$$h_n(t) \le g_n(0) + 4p^2\Gamma_n \int_0^t (h_n(s))^{(p-1)/p} ds.$$

Now, let

$$H_n(t) = \int_0^t (h_n(s))^{(p-1)/p} ds.$$

The above differential inequality may be written as

$$H_n'(s)(g_n(0) + 4p^2\Gamma_n H_n(s))^{-1+1/p} \le 1.$$

Integrating this differential inequality between 0 and t yields

$$(h_n(t))^{1/p} - (g_n(0))^{1/p} \le 4pt\Gamma_n,$$

which implies that $h_n \le g_n$. Hence (a) holds true.

To prove (b), it is enough to prove that

$$\Gamma_n \le \|(\alpha^{-1} \wedge n)Q^2\|_p.$$

Let $q = p/(p-1)$. Clearly

$$\Gamma_n \le \|\sum_{i=0}^{n-1} |\mathbb{E}_0(X_i)|X_0\|_p.$$

Hence, by the Riesz–Fisher theorem, there exists a random variable Y in $L^q(\mathcal{F}_0)$ such that $\|Y\|_q = 1$ and

$$\Gamma_n \le \mathbb{E}(Y \sum_{i=0}^{n-1} |X_0 \mathbb{E}_0(X_i)|) \le \sum_{i=0}^{n-1} \|Y X_0 \mathbb{E}_0(X_i)\|_1.$$

Hence, by (1.11c),

$$\Gamma_n \le 2 \sum_{i=0}^{n-1} \int_0^{\alpha_i} Q_{YX_0}(u) Q_{X_i}(u) du.$$

Finally, by Lemma 2.1(b)

$$\Gamma_n \le 2 \int_0^1 Q_Y(u)[\alpha^{-1}(u) \wedge n] Q^2(u) du,$$

which implies (b) via Hölder's inequality on [0, 1] applied to the functions Q_Y and $[\alpha^{-1} \wedge n]Q^2$. ■

To conclude this section, we give a pseudo exponential inequality for geometrically strongly mixing sequences. Our result is similar to the results of Theorem 6 in Doukhan et al. (1984).

Corollary 2.2 *Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of centered real-valued random variables each bounded a.s. by* 1, *and $(\alpha_n)_{n\geq 0}$ be defined by (2.30). Suppose that, for some $a < 1$, $\limsup_n \alpha_n^{1/n} < a$. Then there exists some positive x_0 such that, for any $x \geq x_0$ and any positive integer n,*

$$\mathbb{P}\Big(|S_n| \geq x\sqrt{n\log(1/a)}\Big) \leq a^{x/2}.$$

Proof It is easy to check that

$$\limsup_{p\to\infty} p^{-1}\|\alpha^{-1}Q^2\|_p < (-e\log a)^{-1}.$$

Hence there exists some $p_0 > 1$ such that, for any $p \geq p_0$,

$$\|S_n\|_{2p}^2 \leq 4np^2(-e\log a)^{-1}.$$

By the Markov inequality applied to S_n^{2p}, we infer that

$$\mathbb{P}\Big(|S_n| \geq x\sqrt{n\log(1/a)}\Big) \leq e^{-p}\Big(\frac{-2p}{x\log a}\Big)^{2p}.$$

Set $p = -(x/2)\log a$. Then the above inequality yields Corollary 2.2 if $p \geq p_0$, which holds true as soon as $x \geq -(2p_0/\log a)$. ■

Exercises

(1) Let $(X_i)_{i\in\mathbb{Z}}$ be a sequence of centered real-valued random variables with finite fourth moments, and let $(\alpha_n)_{n\geq 0}$ be defined by (2.1).

(a) Let $i \leq j \leq k \leq l$ be natural integers. Prove that

$$|\mathbb{E}(X_iX_jX_kX_l)| \leq 2\int_0^1 \mathbb{1}_{u<\alpha_{j-i}}\mathbb{1}_{u<\alpha_{l-k}}Q_i(u)Q_j(u)Q_k(u)Q_l(u)du. \tag{1}$$

(b) Prove that

$$\mathbb{E}(S_n^4) \leq 12\sum_{1\leq i\leq j\leq k\leq l\leq n}|\mathbb{E}(X_iX_jX_kX_l)|(1+\mathbb{1}_{j<k}).$$

(c) Prove that

$$\mathbb{E}(S_n^4) \le 24 \sum_{j=1}^{n} \sum_{k=1}^{n} \int_0^1 [\alpha^{-1}(u) \wedge n]^2 Q_j^2(u) Q_k^2(u) du. \tag{2}$$

(d) Suppose now that $\|X_k\|_\infty \le 1$ for any k in $[1, n]$. Derive from the above inequalities that

$$\mathbb{E}(S_n^4) \le 24 n^2 \sum_{m=0}^{n-1} (2m+1)\alpha_m. \tag{3}$$

Compare (3) with (2.13) and (2.11).
(2) Let $(S_n)_{n \ge 0}$ be a martingale sequence in L^p for some $p > 2$ and $X_n = S_n - S_{n-1}$. Either use Inequality (2.3) in Pinelis (1994) or adapt the proof of Theorem 2.5 to prove the inequality (4) below, given in Rio (2009):

$$\|S_n\|_p^2 \le \|S_0\|_p^2 + (p-1) \sum_{k=1}^{n} \|X_k\|_p^2. \tag{4}$$

Springer
http://www.springer.com/978-3-662-54322-1

Asymptotic Theory of Weakly Dependent Random Processes
Rio, E.
2017, XVIII, 204 p., Hardcover
ISBN: 978-3-662-54322-1