Introduction

But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate new ways, new and fruitful problems, and appears then itself as the real questioner.

David Hilbert, *Mathematical Problems*

The study of locally nilpotent derivations and $\mathbb{G}_a$-actions has recently emerged from the long shadows of other branches of mathematics, branches whose provenance is older and more distinguished. The subject grew out of the rich environment of Lie theory, invariant theory, and differential equations and continues to draw inspiration from these and other fields.

At the heart of the present exposition lie 16 principles for locally nilpotent derivations, laid out in *Chap. 1*. These provide the foundation on which the subsequent theory is built. We would like to distinguish which properties of a locally nilpotent derivation are due to its being a derivation and which are special to the locally nilpotent condition. Thus, we first consider general properties of derivations. The 16 First Principles which follow can then be seen as belonging especially to the locally nilpotent derivations.

Of course, one must choose one's category. While $\mathbb{G}_a$-actions can be investigated in a characteristic-free environment, locally nilpotent derivations are, by nature, objects belonging to rings of characteristic zero. Most of the basic results about derivations in *Chaps. 1* and *2* are stated for a commutative k-domain B, where k is a field of characteristic zero.

In discussing geometric aspects of the subject, it is also generally assumed that B is affine and that the underlying field k is algebraically closed. The associated geometry falls under the rubric of affine algebraic geometry. Miyanishi writes: "There is no clear definition of affine algebraic geometry. It is one branch of algebraic geometry which deals with the affine spaces and the polynomial rings, hence affine algebraic varieties as subvarieties of the affine spaces and finitely generated algebras as the residue rings of the polynomial rings" [306]. Due to their obvious importance, special attention is given throughout the book to polynomial rings and affine spaces $\mathbb{A}^n_k$.

Chapter 3 explores the case of polynomial rings over k. Here, the Jacobian derivations are of central importance. *Chapter 4* considers polynomial rings in two variables, first over a field k and second over other base rings. Two proofs of Rentschler's Theorem are given, and this result is applied to give proofs for Jung's Theorem and the Structure Theorem for the planar automorphism group. This effectively classifies all locally nilpotent derivations of $k[x, y]$ and likewise all algebraic $\mathbb{G}_a$-actions on the plane $\mathbb{A}^2$. *Chapter 5* documents the progress in our understanding of the three-dimensional case which has been made over the past three decades, beginning with the Bass-Nagata example and Miyanishi's Theorem. We now have a large catalogue of interesting and instructive examples, in addition to the impressive Daigle-Russell classification in the homogeneous case and Kaliman's classification of the free $\mathbb{G}_a$-actions. These feats notwithstanding, a classification of the locally nilpotent derivations of $k[x, y, z]$ remains elusive.

Chapter 6 examines the case of linear actions of $\mathbb{G}_a$ on affine spaces, and it is here that the oldest literature on the subject of $\mathbb{G}_a$-actions can be found. One of the main results of the chapter is the Maurer-Weitzenböck Theorem, a classical result showing that a linear action of $\mathbb{G}_a$ on $\mathbb{A}^n$ has a finitely generated ring of invariants.[1]

Nagata's famous counterexamples to the Fourteenth Problem of Hilbert showed that the Maurer-Weitzenböck Theorem does not generalize to higher-dimensional groups, i.e., it can happen that a linear $\mathbb{G}_a^m$-action on affine space has a nonfinitely generated ring of invariants when $m > 1$. It can also happen that a nonlinear $\mathbb{G}_a$-action has nonfinitely generated invariant ring, and these actions form the main topic of *Chap. 7*.

Chapter 8 discusses various algorithms associated with locally nilpotent derivations, including the van den Essen algorithm for calculating kernels of finite type. *Chapter 9* focuses on the Makar-Limanov and Derksen invariants of a ring and illustrates how they can be applied. *Chapter 10* shows how locally nilpotent derivations can be found and used in a range of important problems, such as the cancellation problem and embedding problem. The concluding chapter, *Chap. 11*, is devoted to questions and open problems.

In addition to the articles found in the *References*, there are four larger works used in preparing this manuscript. These are the books of Nowicki (1994) and van den Essen (2000) and the extensive lecture notes written by Makar-Limanov (1998) and Daigle (2003). In addition, I found in the books of Kraft (1985), Popov (1992), Grosshans (1997), Borel (2001), Derksen and Kemper (2002), and Dolgachev (2003) a wealth of pertinent references and historical background regarding invariant theory.

[1]This result is commonly attributed only to R. Weitzenböck, but after reading Borel's *Essays in the History of Lie Groups and Algebraic Groups*, it becomes clear that L. Maurer should receive at least equal credit.

The reader will find that this book focuses on the algebraic aspects of locally nilpotent derivations, as the book's title indicates. The subject is simply too large and diverse to include a complete geometric treatment in a volume of this size. The outstanding survey articles of Kaliman [228] and Miyanishi [306] will serve to fill this void.

Historical Overview

The study of locally nilpotent derivations in its present form appears to have emerged in the 1960s, and was first made explicit in the work of several mathematicians working in France, including Dixmier, Gabriel and Nouazé, and Rentschler. Their motivation came from the areas of Lie algebras and Lie groups, where the connections between derivations, vector fields, and group actions were well-explored.

Linear $\mathbb{G}_a$-actions were one of the main objects of interest for invariant theory in the nineteenth century. Gordan (1868) gave an algorithm to calculate the invariants of the basic $\mathbb{G}_a$-actions and found their invariant rings up to dimension 7. Stroh (1888) gave a transcendence basis for the invariants of the basic $\mathbb{G}_a$-actions, and Hilbert calculated the invariants of the basic actions up to integral closure (see [188], §10, Note 1). In 1899, Maurer outlined his proof showing the finite generation of invariant rings for one-dimensional group actions. In 1932, Weitzenböck gave a more complete version of Maurer's proof, which used ideas of Gordan and M. Roberts dating to 1868 and 1871, respectively, in addition to the theory developed by Hilbert. In their paper dating to 1876, Gordan and M. Nöther studied certain systems of differential operators and were led to investigate special kinds of non-linear $\mathbb{G}_a$-actions on $\mathbb{C}^n$, though they did not use this language; see *Chap. 3*.

It seems that the appearance of Nagata's counterexamples to Hilbert's Fourteenth Problem in 1958 spurred a renewed interest in $\mathbb{G}_a$-actions and more general unipotent actions, since the theorem of Maurer and Weitzenböck could then be seen in sharp contrast to the case of higher-dimensional vector group actions. It was shortly thereafter, in 1962, that Seshadri published his well-known proof of the Maurer-Weitzenböck result. Nagata's 1962 paper [322] contains significant results about connected unipotent groups acting on affine varieties, and his classic Tata lecture notes [323] appeared in 1965. The case of algebraic $\mathbb{G}_a$-actions on affine varieties was considered by Białynicki-Birula in the mid-1960s [30–32]. In 1966, Hadziev published his famous theorem [198], which is a finiteness result for the maximal unipotent subgroups of reductive groups. In 1967, Shioda gave the first complete system of generators for the basic $\mathbb{G}_a$-action in dimension 9. The 1969 article of Horrocks [212] considered connectedness and fundamental groups for certain kinds of unipotent actions, and the 1973 paper of Hochschild and Mostow [208] remains a standard reference for unipotent actions. Grosshans began his work on unipotent actions in the early 1970s; his 1997 book [188] provides an excellent overview of the subject. Another notable body of research from the 1970s is due to Fauntleroy [148–150] and Fauntleroy and Magid [151, 152]. The papers of Pommerening also began to appear in the late 1970s (see [188, 340]), and Tan's algorithm for computing invariants of basic $\mathbb{G}_a$-actions appeared in 1989. In his 2002 thesis, Cröni gave a complete set of generators for the basic $\mathbb{G}_a$-action in dimension 8.

In a famous paper published in 1968, Rentschler classified the locally nilpotent derivations of the polynomial ring in two variables over a field of characteristic zero and pointed out how this gives the equivalent classification of all the algebraic $\mathbb{G}_a$-actions on the plane $\mathbb{A}^2$. This article is highly significant, in that it was the first publication devoted to the study of certain locally nilpotent derivations (even though its title mentions only $\mathbb{G}_a$-actions). Indeed, Rentschler's landmark paper crystallized the definitions and concepts for locally nilpotent derivations in their modern form, and further provided a compelling illustration of their importance, namely, a simple proof of Jung's Theorem using locally nilpotent derivations.

It should be noted that the classification of planar $\mathbb{G}_a$-actions in characteristic zero was first given by Ebey in 1962 [133]. Ebey's paper clearly deserves more recognition than it receives. Of the more than 400 works listed in the *References* of this book, only the papers of Shafarevich (1966) and Koshevoi (1967) cite it [252, 380]. Ebey's paper was an outgrowth of his thesis, written under the direction of Max Rosenlicht. Rather than using the standard theorems of Jung (1942) or van der Kulk (1953) on planar automorphisms, the author used an equivalent result of Engel, dating to 1958.

The crucial Slice Theorem appeared in the 1967 paper of Gabriel and Nouazé [178], which is cited in Rentschler's paper. Other proofs of the Slice Theorem were given by Dixmier in 1974 ([116], 4.7.5), Miyanishi in 1978 ([297], 1.4), and Wright in 1981 ([426], 2.1). In Dixmier's proof we find the implicit definition and use of what is herein referred to as the Dixmier map. Wright's proof also uses such a construction. The first explicit definition and use of this map is found in van den Essen [141], 1993, and in Deveney and Finston [101], 1994. Arguably, the Dixmier map is to unipotent actions what the Reynolds operator is to reductive group actions (see [142], 9.2).

Certainly, one impetus for the study of locally nilpotent derivations is the Jacobian Conjecture. This famous problem and its connection to derivations is briefly described in *Chap. 3* and is thoroughly investigated in the book of van den Essen [142]. It seems likely that the conjecture provided, at least partly, the motivation behind Vasconcelos's Theorem on locally nilpotent derivations, which appeared in 1969. In the paper of Wright mentioned above, locally nilpotent derivations play a central role in his discussion of the conjecture.

There are not too many papers about locally nilpotent derivations or $\mathbb{G}_a$-actions from the decade of the 1970s. A notable exception is found in the work of Miyanishi, who was perhaps the first researcher to systematically investigate $\mathbb{G}_a$-actions throughout his career. Already in 1968, his paper [293] dealt with locally finite higher iterative derivations. These objects were first defined by Hasse and Schmidt [204] in 1937 and serve to generalize the definition of locally nilpotent derivations in order to give a correspondence with $\mathbb{G}_a$-actions in arbitrary characteristic.

Miyanishi's 1971 paper [294] is about planar $\mathbb{G}_a$-actions in positive characteristic, giving the analogue of Rentschler's Theorem in this case. His 1973 paper [295] uses $\mathbb{G}_a$-actions to give a proof of the cancellation theorem of Abhyankar, Eakin, and Heinzer. In his 1978 book [297], Miyanishi entitled the first section "Locally Nilpotent Derivations" (Sect. 1.1). In these few pages, Miyanishi organized and proved many of the fundamental properties of locally nilpotent derivations: the correspondence of locally nilpotent derivations and exponential automorphisms (Lemma 1.2) the fact that the kernel is factorially closed (Lemma 1.3.1) the Slice Theorem (Lemma 1.4), and its local version (Lemma 1.5). While these results already existed elsewhere in the literature, this publication constituted an important new resource for the study of locally nilpotent derivations. A later section of the book, called "Locally Nilpotent Derivations in Connection with the Cancellation Problem" (Sect. 1.6), proved some new cases in which the cancellation problem has a positive solution, based on locally nilpotent derivations. Miyanishi's 1980 paper [298] and 1981 book [299] include some of the earliest results about $\mathbb{G}_a$-actions on $\mathbb{A}^3$. Ultimately, his 1985 paper [301] outlined the proof of his well-known theorem about invariant rings of $\mathbb{G}_a$-actions on $\mathbb{A}^3$. In many other papers, Miyanishi used $\mathbb{G}_a$-actions extensively in the classification of surfaces, characterization of affine spaces, and the like.

In 1984, Bass produced a non-triangularizable $\mathbb{G}_a$-action on $\mathbb{A}^3$, based on the automorphism published by Nagata in 1972. This example, together with the 1985 theorem of Miyanishi, marked the beginning of the next generation of research on $\mathbb{G}_a$-actions and locally nilpotent derivations. The subject gathered momentum in the late 1980s, with significant new results of Popov, Snow, M. Smith, Winkelmann, and Zurkowski [344, 386–388, 421, 431, 432].[2] This trend continued in the early 1990s, especially in several papers due to van den Essen, and Deveney and Finston, which began a more systematic approach to the study of locally nilpotent derivations. Paul Roberts' counterexample to the Fourteenth Problem of Hilbert appeared in 1990, and it was soon realized that his example was the invariant ring of a $\mathbb{G}_a$-action on $\mathbb{A}^7$. The 1994 book of Nowicki [333] includes a chapter about locally nilpotent derivations. The book of van den Essen, published in 2000, is about polynomial automorphisms and the Jacobian Conjecture and takes locally nilpotent derivations as one of its central themes.

By the mid-1990s, Daigle, Kaliman, Makar-Limanov, Russell, Bhatwadekar, and Dutta began making significant contributions to our understanding of the subject. The introduction by Makar-Limanov in 1996 of the ring of absolute constants (now called the Makar-Limanov invariant) brought widespread recognition to locally nilpotent derivations as a tool in understanding affine geometry and commutative ring theory. Extensive (unpublished) lecture notes on the subject of locally nilpotent derivations were written by Makar-Limanov (1998) and by Daigle (2003). Papers of Kaliman which appeared in 2004 contain important results about $\mathbb{C}^+$-actions

[2]My own work in this area began in 1993, and I "went to school" on these papers.

on threefolds, bringing to bear a wide range of tools from topology and algebraic geometry.

The Makar-Limanov invariant is currently one of the central themes in the classification of algebraic surfaces. In particular, families of surfaces having a trivial Makar-Limanov invariant have been classified by Bandman and Makar-Limanov, Daigle and Russell, Dubouloz, and Gurjar and Miyanishi [9, 88, 126, 193]. Already in 1983, Bertin [24] had studied surfaces which admit a $\mathbb{C}^+$-action.

By the late 1990s, locally nilpotent derivations began to appear in some thesis work, especially from the Nijmegen School, i.e., students of van den Essen at the University of Nijmegen. It appears that Z. Wang's 1999 PhD thesis, written under the direction of Daigle at the University of Ottawa, holds the distinction of being the first thesis devoted to the subject of locally nilpotent derivations.

The foregoing overview is by no means a complete account of the subject's development. Significant work in this area from many other researchers can be found in the *References*, much of which is discussed in the following chapters. In a conversation with the author about locally nilpotent derivations and $\mathbb{G}_a$-actions, Białynicki-Birula remarked: "I believe that we are just at the beginning of our understanding of this wonderful subject."

Notes on the Second Edition

New material presented in the second edition includes an overview of results about linear $\mathbb{G}_a$-actions from the nineteenth century, with the disclaimer that, given the vast body of literature on classical invariant theory, this is done in only the most cursory fashion. In this volume, I have also endeavored to better represent the work of certain researchers, including that of Bhatwadekar and Dutta and of Deveney and Finston.

There remain 16 First Principles, but a new principle (the Generating Principle) is introduced, taking the place of the original Principle 14, which can be seen as a consequence of Principle 15. *Chapter 1* discusses degree functions, gradings, and associated graded rings in a more general setting and devotes a new section to degree modules for locally nilpotent derivations and the canonical factorization of the quotient morphism for the induced $\mathbb{G}_a$-action; *Chap. 8* then gives a new algorithm for calculating degree modules. *Chapter 2* has been expanded to become a gathering place for a large number of fundamental results used in later chapters. New topics found there include cable algebras, transvectants, G-critical elements, and the AB and ABC theorems. In particular, the cable algebra structure on a ring induced by a locally nilpotent derivation can be viewed as a generalization of Jordan block form for a nilpotent linear operator. Other new topics include the down operator in *Chap. 7* and the Pham-Brieskorn surfaces in *Chap. 9*.

This edition features a new proof for the Abhyankar-Eakin-Heinzer Theorem for algebraically closed fields of characteristic 0, based on a proof given by Makar-Limanov. It also includes a new proof of nonfinite generation for the triangular derivation in dimension 5 due to Daigle and the author, showing that this kernel is a cable algebra of an especially simple type. *Chapter 10* gives a new proof that the Danielewski surfaces are not cancelative. In addition, readers are introduced to the theory of affine modifications relative to $\mathbb{G}_a$-actions, which was developed by Kaliman and Zaidenberg in the late 1990s and which underlies the discussion of canonical factorizations found in *Chap. 1*.

Above all, this second edition is intended to be an improved reference for locally nilpotent derivations and $\mathbb{G}_a$-actions and their applications.

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