Chapter 2
Linear Wave Equations

2.1 Expression of Solutions

In this section we consider the following Cauchy problem of linear wave equations:

$$\Box u = F(t, x), \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.1}$$

$$t = 0 : u = f(x), u_t = g(x), \quad \in I\!R^n, \tag{2.1.2}$$

where $x = (x_1, \ldots, x_n)$,

$$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \tag{2.1.3}$$

is the $n-$dimensional wave operator, and F, f and g are given functions with suitable regularities.

According to the superposition principle and the Duhamel's principle based on this, to solve the Cauchy problem (2.1.1)–(2.1.2), it suffices to solve the Cauchy problem of the following homogeneous wave equation:

$$\Box u = 0, \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.4}$$

$$t = 0 : u = 0, u_t = g(x), \quad x \in I\!R^n. \tag{2.1.5}$$

Denote the solution of this problem by

$$u = S(t)g. \tag{2.1.6}$$

Here

$$S(t) : g \to u(t, \cdot), \tag{2.1.7}$$

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being a linear operator whose specific properties reflect the nature of wave equations, is the key object of study of this chapter.

If the solution of the Cauchy problem (2.1.4)–(2.1.5) is known, then it is easy to know that the solution to the Cauchy problem

$$\Box u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.1.8}$$

$$t = 0 : u = f(x), u_t = 0, \quad x \in \mathbb{R}^n \tag{2.1.9}$$

can be expressed by

$$u = \frac{\partial}{\partial t}(S(t)f); \tag{2.1.10}$$

while, the solution to the Cauchy problem of the inhomogeneous wave equation:

$$\Box u = F(t, x), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.1.11}$$

$$t = 0 : u = 0, u_t = 0, \quad x \in \mathbb{R}^n \tag{2.1.12}$$

can be expressed, according to the Duhamel's principle, by

$$u = \int_0^t S(t - \tau)F(\tau, \cdot)d\tau. \tag{2.1.13}$$

Therefore, in general the solution to the Cauchy problem (2.1.1)–(2.1.2) of wave equations can be represented uniformly by

$$u = \frac{\partial}{\partial t}(S(t)f) + S(t)g + \int_0^t S(t - \tau)F(\tau, \cdot)d\tau. \tag{2.1.14}$$

On the other hand, the solution to the Cauchy problem (2.1.4)–(2.1.5) can also be obtained by solving the Cauchy problem of the forms (2.1.8)–(2.1.9) or (2.1.11)–(2.1.12). In fact, if the solution v to the Cauchy problem

$$\Box v = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{2.1.15}$$

$$t = 0 : v = g, v_t = 0, \quad x \in \mathbb{R}^n \tag{2.1.16}$$

is already known, then

$$u = \int_0^t v(\tau, \cdot)d\tau \tag{2.1.17}$$

is exactly the solution to the Cauchy problem (2.1.4)–(2.1.5). Moreover, it is easy to show from (2.1.14) that the solution to the Cauchy problem

$$\Box u = g(x)\delta(t), \quad (t, x) \in I\!R \times I\!R^n, \tag{2.1.18}$$

$$t = -1 : u = 0, u_t = 0, \quad x \in I\!R^n \tag{2.1.19}$$

is exactly the solution to the Cauchy problem (2.1.4)–(2.1.5), where δ is the Dirac function.

2.1.1 Expression of Solutions When n ≤ 3

When $n = 1$, as $t \geq 0$, the solution to the Cauchy problem (2.1.4)–(2.1.5) of the one-dimensional wave equation is given by the well-known d'Alembert formula:

$$u(t, x) = \frac{1}{2} \int_{x-t}^{x+t} g(y) dy. \tag{2.1.20}$$

When $n = 2$, as $t \geq 0$, the solution to the Cauchy problem (2.1.4)–(2.1.5) of the two-dimensional wave equation is given by the two-dimensional Poisson formula:

$$u(t, x) = \frac{1}{2\pi} \int_{|y-x| \leq t} \frac{g(y)}{\sqrt{t^2 - |y - x|^2}} dy, \tag{2.1.21}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, and

$$|y - x| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}.$$

When $n = 3$, as $t \geq 0$, the solution to the Cauchy problem (2.1.4)–(2.1.5) of the three-dimensional wave equation is given by the three-dimensional Poisson formula:

$$u(t, x) = \frac{1}{4\pi t} \int_{|y-x|=t} g(y) dS_y, \tag{2.1.22}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$,

$$|y - x| = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2},$$

and dS_y stands for the area element of the sphere $|y - x| = t$.

The derivation of formulas (2.1.20)–(2.1.22) can be found, say, in Gu Chaohao, Li Tatsien et al. 1987.

We can find out from (2.1.20)–(2.1.22) that, when the space dimension $n \leq 3$, the expressions of the solution $u = u(t, x)$ to the Cauchy problem (2.1.4)–(2.1.5) involve only $g(x)$ itself but not its derivatives. Besides, when

$$g(x) \geq 0, \quad \forall x \in I\!R^n, \tag{2.1.23}$$

we always have

$$u(t, x) \geq 0, \quad \forall (t, x) \in I\!R_+ \times I\!R^n, \tag{2.1.24}$$

where $n = 1$, 2 and 3. This property is called the **positivity of the fundamental solution** (See Remark 2.2).

When $n \geq 4$, the fundamental solution does not have the positivity any longer. This can be shown by the expression of solutions, which will be derived later soon.

2.1.2 Method of Spherical Means

Here and throughout this section, we always assume that $n > 1$.

For any given function $\psi(x) = \psi(x_1, \ldots, x_n)$, denote by

$$h(x, r) = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \psi(y) dS_y \tag{2.1.25}$$

the integral mean of ψ on the sphere centered at $x = (x_1, \ldots, x_n)$ with radius r, where ω_n stands for the area of the unit sphere S^{n-1} in $I\!R^n$, dS_y is the area element of the sphere $|y - x| = r$, and $\omega_n r^{n-1}$ is the area of this sphere. The above formula can be easily rewritten as

$$h(x, r) = \frac{1}{\omega_n} \int_{|\xi|=1} \psi(x + r\xi) d\omega_\xi, \tag{2.1.26}$$

where $d\omega_\xi$ is the area element of the unit sphere S^{n-1}, and $\xi = (\xi_1, \ldots, \xi_n)$.

From the above formula, it turns out that the function $h(x, r)$ is well-defined not only for $r \geq 0$ but also for $r < 0$, and is an even function of r.

If $\psi \in C^2$, then it is obvious that $h \in C^2$, and

$$h(x, 0) = \psi(x), \tag{2.1.27}$$

and since h is an even function of r, we have

$$\frac{\partial h}{\partial r}(x, 0) = 0. \tag{2.1.28}$$

In addition, from (2.1.26) we have

$$\frac{\partial h(x, r)}{\partial r} = \frac{1}{\omega_n} \int_{|\xi|=1} \sum_{i=1}^{n} \psi_{x_i}(x + r\xi) \xi_i d\omega_\xi$$

$$= \frac{1}{\omega_n r^{n-1}} \int_{|\tilde{\xi}|=r} \sum_{i=1}^{n} \psi_{x_i}(x+\tilde{\xi})\xi_i dS,$$

where $\tilde{\xi} = r\xi$, and dS stands for the area element of the sphere $|\tilde{\xi}| = r$. Then, from the Green's formula we get

$$\frac{\partial h(x,r)}{\partial r} = \frac{1}{\omega_n r^{n-1}} \int_{|y-x|\le r} \triangle\psi(y)dy, \tag{2.1.29}$$

where

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{2.1.30}$$

is the n-D Laplacian operator.

Differentiating (2.1.29) once with respect to r, and using (2.1.29) again, we obtain

$$\begin{aligned}\frac{\partial^2 h(x,r)}{\partial r^2} &= -\frac{n-1}{\omega_n r^n} \int_{|y-x|\le r} \triangle\psi(y)dy \\ &\quad + \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \triangle\psi(y)dS_y \\ &= -\frac{n-1}{r} \frac{\partial h(x,r)}{\partial r} \\ &\quad + \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \triangle\psi(y)dS_y.\end{aligned} \tag{2.1.31}$$

On the other hand, from (2.1.26) we have

$$\begin{aligned}\triangle_x h(x,r) &= \frac{1}{\omega_n} \int_{|\xi|=1} \triangle_x\psi(x+r\xi)d\omega_\xi \\ &= \frac{1}{\omega_n r^{n-1}} \int_{|y-x|=r} \triangle\psi(y)dS_y,\end{aligned} \tag{2.1.32}$$

where $\triangle_x$ stands for the Laplacian operator with respect to x (see (2.1.30)).

Combining (2.1.31)–(2.1.32) and noting (2.1.27)–(2.1.28), we obtain the following

Lemma 2.1 *Assume that $\psi(x) \in C^2$, then its spherical mean function $h(x,r) \in C^2$, and satisfies the following Darboux equation*

$$\frac{\partial^2 h(x,r)}{\partial r^2} + \frac{n-1}{r} \frac{\partial h(x,r)}{\partial r} = \triangle_x h(x,r) \tag{2.1.33}$$

and the initial condition

$$r = 0 : h = \psi(x), \frac{\partial h}{\partial r} = 0. \tag{2.1.34}$$

In particular, taking

$$\psi(x_1, \ldots, x_n) = \phi(x_1) \tag{2.1.35}$$

as a function depending only on x_1 *but not on* $x_2, \ldots, x_n$*, we can prove that its spherical mean function has the expression*

$$h(x, r) = \frac{\omega_{n-1}}{\omega_n} \int_{-1}^{1} \phi(x_1 + r\mu)(1 - \mu^2)^{\frac{n-3}{2}} d\mu, \tag{2.1.36}$$

where, ω_{n-1} *is taken to be* 2 *when* $n = 2$*, i.e., it is defined artificially that* $\omega_1 = 2$*, and the same below. This coincides with the value of* ω_1 *deduced by using (2.4.7) in this chapter when* $n = 2$*.*

In fact, from (2.1.26) we easily get

$$\begin{aligned} h(x, r) &= \frac{1}{\omega_n r^{n-1}} \int_{|y|=r} \psi(x + y) dS \\ &= \frac{1}{\omega_n r^{n-1}} \frac{\partial}{\partial r} \int_{|y|\le r} \psi(x + y) dy. \end{aligned} \tag{2.1.37}$$

Noticing (2.1.35), we have

$$\int_{|y|\le r} \psi(x + y) dy = \int_{\lambda^2 + |\tilde{y}|^2 \le r^2} \phi(x_1 + \lambda) d\lambda d\tilde{y},$$

where $\tilde{y} = (y_2, \ldots, y_n)$. Adopting polar coordinates to the variable $\tilde{y}$ and denoting $\rho = |\tilde{y}|$, the above formula can be rewritten as

$$\begin{aligned} &\int_{|y|\le r} \psi(x + y) dy \\ &= \omega_{n-1} \int_{\lambda^2 + \rho^2 \le r^2} \phi(x_1 + \lambda) \rho^{n-2} d\lambda d\rho \\ &= \omega_{n-1} \int_{-r}^{r} d\lambda \int_0^{\sqrt{r^2 - \lambda^2}} \phi(x_1 + \lambda) \rho^{n-2} d\rho, \end{aligned}$$

then it is easy to show that

$$\begin{aligned}
&\frac{\partial}{\partial r}\int_{|y|\le r}\psi(x+y)dy \\
&=\omega_{n-1}r\int_{-r}^{r}\phi(x_1+\lambda)(r^2-\lambda^2)^{\frac{n-3}{2}}d\lambda \\
&=\omega_{n-1}r^{n-1}\int_{-1}^{1}\phi(x_1+r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu.
\end{aligned}$$

Thus, (2.1.36) follows from (2.1.37).

The spherical mean function $h(x, r)$ given by (2.1.36) depends only on x_1 and r, then the corresponding Darboux equation (2.1.33) is reduced to

$$\frac{\partial^2 h}{\partial r^2}+\frac{n-1}{r}\frac{\partial h}{\partial r}=\frac{\partial^2 h}{\partial x_1^2}, \tag{2.1.38}$$

moreover,

$$\frac{\partial^2 h}{\partial x_1^2}=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\phi''(x_1+r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu. \tag{2.1.39}$$

Taking $x_1 = 0$ in (2.1.38)–(2.1.39), we obtain

Lemma 2.2 *Suppose that*

$$h(r)=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\phi(r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu, \tag{2.1.40}$$

then we have

$$h''(r)+\frac{n-1}{r}h'(r)=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}\phi''(r\mu)(1-\mu^2)^{\frac{n-3}{2}}d\mu. \tag{2.1.41}$$

Now we apply the above results to solving the Cauchy problem of wave equations.

Suppose that $v = v(t, x)$ is the solution to the Cauchy problem (2.1.15)–(2.1.16). It is clear that v is an even function of t. Let

$$w(x,r)=\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1}v(r\mu,x)(1-\mu^2)^{\frac{n-3}{2}}d\mu. \tag{2.1.42}$$

Regarding x as a parameter, from Lemma 2.2 and using equation (2.1.15), it yields

$$\begin{aligned}
&\frac{\partial^2 w(x,r)}{\partial r^2} + \frac{n-1}{r}\frac{\partial w(x,r)}{\partial r} \\
&= \frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1} v_{tt}(r\mu, x)(1-\mu^2)^{\frac{n-3}{2}} d\mu \\
&= \frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1} \triangle_x v(r\mu, x)(1-\mu^2)^{\frac{n-3}{2}} d\mu \\
&= \triangle_x w(x,r),
\end{aligned}$$

i.e., $w = w(x,r)$ satisfies the Darboux equation (2.1.33). Meanwhile, from (2.1.16), and taking particularly $\phi \equiv 1$ (thus its spherical mean is $h \equiv 1$) in (2.1.36), we have

$$\frac{\omega_{n-1}}{\omega_n}\int_{-1}^{1} (1-\mu^2)^{\frac{n-3}{2}} d\mu = 1, \tag{2.1.43}$$

then it is clear that

$$r = 0 : w = g(x), \ \frac{\partial w}{\partial r} = 0. \tag{2.1.44}$$

Hence, it follows from Lemma 2.1 that

$$w(x,r) = \frac{1}{\omega_n}\int_{|\xi|=1} g(x + r\xi) d\omega_\xi. \tag{2.1.45}$$

Combining (2.1.42) and (2.1.45) and noting that v is an even function of t, we obtain

$$\frac{2\omega_{n-1}}{\omega_n}\int_{0}^{1} v(r\mu, x)(1-\mu^2)^{\frac{n-3}{2}} d\mu = \frac{1}{\omega_n}\int_{|\xi|=1} g(x + r\xi) d\omega_\xi. \tag{2.1.46}$$

Equation (2.1.46) is an integral equation satisfied by the solution $v = v(t,x)$ to the Cauchy (2.1.15)–(2.1.16). Therefore, the Cauchy problem (2.1.15)–(2.1.16) can be solved through inversion of (2.1.46).

Applying in (2.1.46) the variable substitution

$$r = \sqrt{s}, \ r\mu = \sqrt{\sigma}, \tag{2.1.47}$$

and denoting

$$Q(r,x) = \frac{1}{\omega_n}\int_{|\xi|=1} g(x + r\xi) d\omega_\xi, \tag{2.1.48}$$

we obtain

$$\frac{\omega_{n-1}}{\omega_n}\int_0^s \frac{v(\sqrt{\sigma},x)}{\sqrt{\sigma}}(s-\sigma)^{\frac{n-3}{2}}d\sigma = s^{\frac{n-2}{2}}Q(\sqrt{s},x). \tag{2.1.49}$$

Ignoring for the time being the dependence with respect to x, and denoting

$$w(s) = s^{\frac{n-2}{2}}Q(\sqrt{s},x),\ \ \chi(\sigma) = \frac{v(\sqrt{\sigma},x)}{\sqrt{\sigma}}, \tag{2.1.50}$$

Equation (2.1.49) can be rewritten as

$$\frac{\omega_{n-1}}{\omega_n}\int_0^s \chi(\sigma)(s-\sigma)^{\frac{n-3}{2}}d\sigma = w(s). \tag{2.1.51}$$

Next we will solve the integral equation (2.1.51) so as to derive the expression of solutions to the Cauchy problem of wave equations as $n > 1$.

2.1.3 Expression of Solutions When $n(> 1)$ Is Odd

When $n(> 1)$ is odd, $\frac{n-3}{2}$ is a nonnegative integer, by taking derivatives of order $\frac{n-1}{2}$ on both sides of (2.1.51), we can solve that

$$\chi(s) = \frac{\omega_n}{\omega_{n-1}\cdot(\frac{n-3}{2})!}\left(\frac{d}{ds}\right)^{\frac{n-1}{2}} w(s), \tag{2.1.52}$$

thus, noting (2.1.50), we have

$$\frac{v(\sqrt{s},x)}{\sqrt{s}} = \frac{\omega_n}{\omega_{n-1}\cdot(\frac{n-3}{2})!}\left(\frac{d}{ds}\right)^{\frac{n-1}{2}}(s^{\frac{n-2}{2}}Q(\sqrt{s},x)). \tag{2.1.53}$$

Taking $s = t^2$ in the above formula, we get that the solution to the Cauchy problem (2.1.15)–(2.1.16) is

$$v(t,x) = \frac{\omega_n}{\omega_{n-1}\cdot(\frac{n-3}{2})!}t\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}}(t^{n-2}Q(t,x)),$$
$$(t,x)\in I\!\!R_+\times I\!\!R^n. \tag{2.1.54}$$

Using Theorem 2.5 in the appendix (Sect. 2.4) of this chapter, namely,

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}, \tag{2.1.55}$$

the above formula can also be written as

$$v(t,x) = \frac{\sqrt{\pi}}{\Gamma(\frac{n}{2})} t \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-1}{2}} (t^{n-2}Q(t,x)),$$
$$(t,x) \in I\!R \times I\!R^n. \tag{2.1.56}$$

Finally, using (2.1.17), we obtain the following

Theorem 2.1 *When $n(>1)$ is odd, the solution to the Cauchy problem (2.1.4)–(2.1.5) is*

$$u(t,x) = \frac{\sqrt{\pi}}{2\Gamma(\frac{n}{2})} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} (t^{n-2}Q(t,x)), \tag{2.1.57}$$

where

$$Q(t,x) = \frac{1}{\omega_n}\int_{|\xi|=1} g(x+t\xi)d\omega_\xi. \tag{2.1.58}$$

Taking particularly $n=3$ in Theorem 2.1, and noting that $\omega_3 = 4\pi$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, the three-dimensional Poisson formula (2.1.22) follows immediately.

2.1.4 Expression of Solutions When $n(\geq 2)$ Is Even

When $n(\geq 2)$ is even, to obtain the solution $u = u(t,x)$ to the Cauchy problem (2.1.4)–(2.1.5), we can add an argument x_{n+1} artificially, and regard u as the solution to the following Cauchy problem

$$\Box_{n+1} u = 0, \tag{2.1.59}$$
$$t = 0 : u = 0, u_t = g(x), \tag{2.1.60}$$

where $x = (x_1, \ldots, x_n)$, and

$$\Box_{n+1} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_{n+1}^2} \tag{2.1.61}$$

is the $(n+1)$-dimensional wave operator.

Applying Theorem 2.1 to the Cauchy (2.1.59)–(2.1.60), we get

$$u(t,x) = \frac{\sqrt{\pi}}{2\Gamma(\frac{n+1}{2})} \left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} (t^{n-1}\overline{Q}(t,x)), \tag{2.1.62}$$

where

$$\overline{Q}(t,x) = \frac{1}{\omega_{n+1}} \int_{|\xi'|=1} g(x_1 + t\xi_1, \ldots, x_n + t\xi_n) d\omega_{\xi'}, \tag{2.1.63}$$

and $\xi' = (\xi, \xi_{n+1}) = (\xi_1, \ldots, \xi_n, \xi_{n+1})$.

Denote $y' = (y, y_{n+1})$. It is clear that

$$\begin{aligned}
\overline{Q}(t,x) &= \frac{1}{\omega_{n+1}t^n} \int_{|y'|=t} g(x+y) dS_{y'} \\
&= \frac{1}{\omega_{n+1}t^n} \frac{\partial}{\partial t} \int_{|y'|\le t} g(x+y) dy' \\
&= \frac{1}{\omega_{n+1}t^n} \frac{\partial}{\partial t} \int_{|y|\le t} \int_{-\sqrt{t^2-|y|^2}}^{\sqrt{t^2-|y|^2}} g(x+y) dy_{n+1} dy \\
&= \frac{2}{\omega_{n+1}t^n} \frac{\partial}{\partial t} \int_{|y|\le t} \sqrt{t^2-|y|^2} g(x+y) dy \\
&= \frac{2}{\omega_{n+1}t^{n-1}} \int_{|y|\le t} \frac{g(x+y)}{\sqrt{t^2-|y|^2}} dy \\
&= \frac{2}{\omega_{n+1}t^{n-1}} \int_{|y-x|\le t} \frac{g(y)}{\sqrt{t^2-|y-x|^2}} dy.
\end{aligned} \tag{2.1.64}$$

Thus, using (2.4.7) in the appendix (Sect. 2.4) of this chapter, namely,

$$\frac{\omega_{n+1}}{\omega_n} = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})} \sqrt{\pi}, \tag{2.1.65}$$

we obtain

Theorem 2.2 *When $n (\geq 2)$ is even, the solution to the Cauchy problem (2.1.4)–(2.1.5) is*

$$u(t,x) = \frac{1}{\omega_n \Gamma(\frac{n}{2})} \left(\frac{1}{2t} \frac{\partial}{\partial t} \right)^{\frac{n-2}{2}} R(t,x), \tag{2.1.66}$$

where

$$R(t,x) = \int_{|y-x|\le t} \frac{g(y)}{\sqrt{t^2-|y-x|^2}} dy. \tag{2.1.67}$$

Taking particularly $n = 2$ in Theorem 2.2, and noting that $\omega_2 = 2\pi$, the two-dimensional Poisson formula (2.1.21) follows immediately.

Some of the results in Sects. 2.1.2–2.1.4 can be found in Courant and Hilbert (1989).

2.2 Expression of Fundamental Solutions

The solution $E = E(t,x)$ of the following Cauchy problem of wave equation

$$\Box E = 0, \quad (t,x) \in I\!R_+ \times I\!R^n, \tag{2.2.1}$$

$$t = 0 : E = 0,\ E_t = \delta(x), \quad x \in I\!R^n \tag{2.2.2}$$

in the sense of distributions, is called the **fundamental solution** of the wave operator. In (2.2.2), $\delta(x)$ is the Dirac function.

Obviously, when we find the fundamental solution $E = E(t,x)$, the solution to the Cauchy problem (2.1.4–2.1.5) can be expressed by

$$S(t)g = E(t,\cdot) * g, \quad \forall t \geq 0, \tag{2.2.3}$$

where $*$ stands for the convolution of distributions.

Conversely, if there exists a distribution E such that (2.2.3) holds for any given function g, then E must be the fundamental solution of the wave operator.

Now we derive the expression of the fundamental solution of the wave operator.

For any given $a > 0$, define the function

$$\chi_+^a(y) = \frac{(\max(y,0))^a}{\Gamma(a+1)} = \begin{cases} \frac{y^a}{\Gamma(a+1)}, & y \geq 0, \\ 0, & y < 0. \end{cases} \tag{2.2.4}$$

$\chi_+^a(y)$ is a continuous function of y, whose support is $\{y \geq 0\}$. It is easy to show that, as $a > 0$ we have

$$\frac{d}{dy}\chi_+^{a+1}(y) = \chi_+^a(y). \tag{2.2.5}$$

Since one can keep differentiating a continuous function in the sense of distributions, $\chi_+^a(y)$ can be defined inductively for $a \leq 0$ in the category of distributions by using the above formula. Hence, for any given real number a, the function $\chi_+^a(y)$ with support $\subseteq \{y \geq 0\}$ can be defined. It is easy to know that $\chi_+^a(y)$ is a homogeneous function of degree a with respect to y, and

$$\text{sing supp}\chi_+^a \subseteq \{y = 0\}, \tag{2.2.6}$$

where sing supp stands for the singular support of distributions.

In particular, we have

$$\chi_+^0(y) = \frac{d}{dy}\chi_+^1(y) = H(y), \tag{2.2.7}$$

where

$$H(y) = \begin{cases} 1, & y > 0, \\ 0, & y < 0 \end{cases} \tag{2.2.8}$$

is the Heaviside function. Then

$$\chi_+^{-1}(y) = \frac{d}{dy}\chi_+^0(y) = \delta(y). \tag{2.2.9}$$

In addition, noticing that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, it is easy to show that

$$\chi_+^{-\frac{1}{2}}(y) = \frac{d}{dy}\chi_+^{\frac{1}{2}}(y) = \begin{cases} \frac{1}{\sqrt{\pi y}}, & y > 0, \\ 0, & y < 0. \end{cases} \tag{2.2.10}$$

Theorem 2.3 *The fundamental solution of the $n(\geq 1)$-dimensional wave operator is*

$$E(t,x) = \frac{1}{2\pi^{\frac{n-1}{2}}}\chi_+^{-\frac{n-1}{2}}(t^2 - |x|^2). \tag{2.2.11}$$

Proof It suffices to verify (2.2.3).

When $n = 1$, from (2.2.11) and noting (2.2.7) we have

$$\begin{aligned} E(t,\cdot) * g &= \frac{1}{2}\int H(t^2 - |x-y|^2)g(y)dy \\ &= \frac{1}{2}\int H(t - |x-y|)g(y)dy \\ &= \frac{1}{2}\int_{|y-x|\leq t} g(y)dy \\ &= \frac{1}{2}\int_{x-t}^{x+t} g(y)dy. \end{aligned}$$

From D'Alembert formula (2.1.20), it yields (2.2.3) as $n = 1$.

When $n(\geq 2)$ is even, noting that due to (2.2.10) we have

$$\chi_+^{-\frac{1}{2}}(t^2 - |\cdot|^2) * g = \frac{1}{\sqrt{\pi}}\int_{|y-x|\leq t}\frac{g(y)}{\sqrt{t^2 - |x-y|^2}}dy,$$

then from Theorem 2.2 and noting (2.1.55) we have

$$S(t)g = \frac{\sqrt{\pi}}{\omega_n\Gamma(\frac{n}{2})}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}(\chi_+^{-\frac{1}{2}}(t^2 - |\cdot|^2) * g)$$

$$
\begin{aligned}
&= \frac{\sqrt{\pi}}{\omega_n \Gamma(\frac{n}{2})} \chi_+^{-\frac{n-1}{2}}(t^2 - |\cdot|^2) * g \\
&= E(t, \cdot) * g,
\end{aligned}
$$

i.e., (2.2.3) is satisfied when $n(\geq 2)$ is even.

When $n(\geq 3)$ is odd, noting that due to (2.2.9) we have

$$
\begin{aligned}
&\chi_+^{-1}(t^2 - |\cdot|^2) * g \\
&= \int \delta(t^2 - |x-y|^2) g(y) dy \\
&= \int \delta((t + |x-y|)(t - |x-y|)) g(y) dy \\
&= \int \delta(2t(t - |x-y|)) g(y) dy \\
&= \frac{1}{2t} \int \delta(t - |x-y|) g(y) dy \\
&= \frac{1}{2t} \int_{|y-x|=t} g(y) dS_y \\
&= \frac{t^{n-2}}{2} \int_{|\xi|=1} g(x + t\xi) d\omega_\xi,
\end{aligned}
$$

then from Theorem 2.1 and noting (2.1.55) we have

$$
\begin{aligned}
S(t)g &= \frac{\sqrt{\pi}}{\omega_n \Gamma(\frac{n}{2})} \left(\frac{1}{2t} \frac{\partial}{\partial t} \right)^{\frac{n-3}{2}} (\chi_+^{-1}(t^2 - |\cdot|^2) * g) \\
&= \frac{\sqrt{\pi}}{\omega_n \Gamma(\frac{n}{2})} \chi_+^{-\frac{n-1}{2}}(t^2 - |\cdot|^2) * g \\
&= E(t, \cdot) * g,
\end{aligned}
$$

i.e., (2.2.3) is satisfied when $n(\geq 3)$ is odd.

The proof of Theorem 2.3 is finished. □

Remark 2.1 Noting (2.2.9) and (2.2.5), from Theorem 2.3 we easily know that: when $n(> 1)$ is odd, the support of the fundamental solution $E(t, x)$ is the characteristic cone $|x| = t$.

Remark 2.2 From Theorem 2.3, it is easy to show that the fundamental solution of the wave operator as $n = 1$ is

$$
E(t, x) = \begin{cases} \frac{1}{2}, & |x| \leq t, \\ 0, & |x| > t; \end{cases} \tag{2.2.12}
$$

as $n = 2$ it is

$$E(t, x) = \begin{cases} \frac{1}{2\pi\sqrt{t^2-|x|^2}}, & |x| \le t, \\ 0, & |x| > t, \end{cases} \tag{2.2.13}$$

where $x = (x_1, x_2)$; while, as $n = 3$ it is

$$E(t, x) = \frac{\delta(|x| - t)}{4\pi|x|}, \tag{2.2.14}$$

where $x = (x_1, x_2, x_3)$. These coincide with the results shown by (2.1.20)–(2.1.22), and indicate directly the positivity of fundamental solutions as $n = 1, 2$ and 3 shown in Sect. 2.1.1.

2.3 Fourier Transform

The solution of the Cauchy problem to linear wave equations can also be obtained by the Fourier transform.

Taking the Fourier transform in the Cauchy problem (2.1.4)–(2.1.5) with respect to the argument x, we have

$$\hat{u}_{tt}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) = 0, \tag{2.3.1}$$

$$t = 0 : \hat{u} = 0, \hat{u}_t = \hat{g}(\xi), \tag{2.3.2}$$

where $\hat{u}$ and $\hat{g}$ stand for the Fourier transforms of u and g, respectively. Regarding ξ as a parameter and solving the above Cauchy problem of ordinary differential equation, we immediately get

$$\hat{u}(t, \xi) = \frac{\sin(|\xi|t)}{|\xi|} \hat{g}(\xi). \tag{2.3.3}$$

Using (2.1.14), we obtain the following

Theorem 2.4 *Suppose that $u = u(t, x)$ is the solution of the Cauchy problem (2.1.1)–(2.1.2), then the Fourier transform of u with respect to x is*

$$\begin{aligned} \hat{u}(t, \xi) = {} & \cos(|\xi|t) \hat{f}(\xi) + \frac{\sin(|\xi|t)}{|\xi|} \hat{g}(\xi) \\ & + \int_0^t \frac{\sin(|\xi|(t - \tau))}{|\xi|} \hat{F}(\tau, \xi) d\tau. \end{aligned} \tag{2.3.4}$$

Hereafter, we will utilize Theorem 2.4 to establish some estimates on solutions to the Cauchy problem of wave equations.

2.4 Appendix—The Area of Unit Sphere

It is known that Γ function is defined by (see Chen and Yu 2010):

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \forall z > 0. \tag{2.4.1}$$

We have

$$\Gamma(z+1) = z\Gamma(z), \quad \forall z > 0, \tag{2.4.2}$$

and when z is a positive integer,

$$\Gamma(z+1) = z!. \tag{2.4.3}$$

Moreover

$$\Gamma(1) = 1 \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{2.4.4}$$

B function is defined by (see Chen and Yu 2010)

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad \forall p, q > 0, \tag{2.4.5}$$

and we have

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.4.6}$$

Taking $x = \mu^2$ in the following operations, and noting (2.4.6) and (2.4.4), when $n > 1$ we have

$$\begin{aligned}
\int_{-1}^1 (1-\mu^2)^{\frac{n-3}{2}} d\mu &= 2\int_0^1 (1-\mu^2)^{\frac{n-3}{2}} d\mu \\
&= \int_0^1 x^{-\frac{1}{2}}(1-x)^{\frac{n-3}{2}} dx \\
&= B\left(\frac{1}{2}, \frac{n-1}{2}\right) \\
&= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \\
&= \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})},
\end{aligned}$$

then from (2.1.43) we obtain: when $n > 1$ we have

$$\frac{\omega_n}{\omega_{n-1}} = \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}\sqrt{\pi}. \tag{2.4.7}$$

This shows that $\{\Gamma(\frac{n}{2})\omega_n\}$ forms a geometric sequence with common ratio $\sqrt{\pi}$. Hence, noticing that $\omega_2 = 2\pi$, we have

$$\Gamma\left(\frac{n}{2}\right)\omega_n = \pi^{\frac{n-2}{2}}(\Gamma(1)\omega_2) = 2\pi^{\frac{n}{2}},$$

then we obtain the following

Theorem 2.5 *The area of the unit sphere S^{n-1} in $n(> 1)$-dimensional space $\mathbb{R}^n$ is*

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}. \tag{2.4.8}$$

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