Chapter 2
Allocation of Scarce Resources

2.1 Introduction

Radio frequency spectrum is a valuable natural resource, perhaps nowhere more so than India, where wireless telecommunication is an especially important sector of the economy. Similarly, coal is also an extremely valuable natural resource. A natural question that arises is the following. How should the government in an emerging economy allocate such a scarce resource? The allocation mechanism has two goals: (i) to achieve allocative efficiency, that is, to ensure the most efficient use of the spectrum, and (ii) to raise revenue for the government. Recently, in India there seems to be a broad consensus amongst judges and other constitutional bodies (like the 'Comptroller and Auditor General' of India) that auctions are probably superior to administrative mechanisms, where the bureaucrat or the politician decides who gets the right to use the scarce natural resources, in achieving these objectives.

In this chapter we would like to explore some aspects of this important issue. We do not intend to provide a complete solution. Instead, we would analyze what are the possible options in allocating scarce resources in an emerging economy like India and which option is more likely to serve the purpose of maximizing common good (total welfare). Our analysis suggests that auctions may not always serve the desired purpose.

We motivate our exercise by providing two recent examples from India.

2.1.1 The 2G Spectrum Scam in India

The term 2G stands for second-generation wireless telephone technology. 2G cellular telecom networks were first commercially launched on the GSM standard in Finland by Radiolinja in 1991.

The 2G spectrum scandal was an Indian telecommunications scam which involved politicians and government officials illegally undercharging mobile telephony

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companies for frequency allocation licenses, which they would then use to create 2G subscriptions for cell phones.

The 2G scam allegedly started in early 2008, with the farcically arbitrary allocation of 122 licences (and associated spectrum) at substantially below-market rates. A sequence of bizarre events led to the subsequent confusion and controversy. First, the Department of Telecom in India opened a window for spectrum allotment without prior notice. Apparently, the plan was that licenses would be distributed on a first come first serve basis. Suddenly at the last moment, the Department of Telecom mandated that companies must submit some additional documents and that too within 45 min. Otherwise, they would not be considered eligible for the spectrum allocation. This gave rise to the idea that it was designed to only benefit companies that had prior knowledge of the change in requirements, clearly implying corruption and nepotism. In 2010, following media reports that involved tapped phone conversations between corporate lobbyists, journalists and important politicians, the 2G scam came out in the open. Later, in a report on this by the 'Comptroller and Auditor General' of India listed numerous irregularities in the licensing procedure. This episode culminated in the arrest of then telecom minister in early 2011.[1]

The shortfall between the money collected and the money which could have been collected had the entire spectrum been auctioned off was estimated to be approximately Rs 1,76,6 billion. This figure was arrived at by the 'Comptroller and Auditor General' of India.[2]

On February 20, 2012, the Supreme Court of India delivered a judgement on a 'Public Interest Litigation' directly related to the 2G spectrum scam. Declaring the allotment of spectrum as "unconstitutional and arbitrary", the court quashed all licences issued in 2008 by the then minister for communications and information technology in India. According to the court, the minister "wanted to favour some companies at the cost of the public exchequer" and "virtually gifted away important national assets". The Supreme Court judgment ordering the cancellation of telecom licences noted,

> " Natural resources belong to the people but the State legally owns them on behalf of its people and from that point of view natural resources are considered as national assets. However, as they constitute public property/national asset, while distributing natural resources, the State is bound to act in consonance with the principles of equality and public trust and ensure that no action is taken which may be detrimental to public interest."[3]

The Supreme Court first observed that auction was the best way to allocate natural resources, be it oil and gas, spectrum, minerals and coal. However, later in September, 2012, the Supreme Court clarified that it does not believe that all national resources

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[1] We closely follow the report in 'Indian Express'. For details see ≪http://indianexpress.com/article/opinion/columns/because-the-price-was-right/≫.

[2] For a summary of the CAG report see ≪http://www.ndtv.com/india-news/2g-spectrum-scam-some-highlights-of-cag-report-439416≫.

[3] For this part we have closely followed the report in the 'Financial Express'. See ≪http://www.financialexpress.com/opinion/auctions-best-for-natural-resource-allocation/59266/≫.

must be auctioned. In fact, it opined auction cannot be the sole criteria for alienation of natural resources. The Supreme Court also made the following observations.

1. Maximization of revenue in the distribution of natural resources cannot be the sole criteria in all situations and circumstances.
2. Which policy is best is the wisdom of executive since judiciary does not have the expertise to decide which method is better for the disposal of a particular natural resource. Economic policy of the executive can be struck down only if it is found to be arbitrary.
3. The policy of allocation of natural resources for public good falls in the domain of the legislature and the executive. *However, when such a policy decision is not backed by a social or welfare purpose, and precious and scarce natural resources are alienated for commercial pursuits of profit maximizing private entrepreneurs, adoption of means other than those that are competitive and maximize revenue may be arbitrary.* In fact, one of the judges said,

> "I would, therefore, conclude by stating that no part of the natural resource can be dissipated as a matter of largess, charity, donation or endowment, for private exploitation. Each bit of natural resource expended must bring back a reciprocal consideration. The consideration may be in the nature of earning revenue or may be to 'best subserve the common good'. It may well be the amalgam of the two. There cannot be a dissipation of material resources free of cost or at a consideration lower than their actual worth. One set of citizens cannot prosper at the cost of another set of citizens, for that would not be fair or reasonable."[4]

We now proceed to our next example.

2.1.2 The Coal-Block Scam

The coal block scam happened in India during the allocation of coal blocks to certain corporate bodies. Between 1993 and 2011, the government of India gave away 206 coal blocks for free to government and private companies. The purported reason for giving the coal blocks for free was to increase the total coal production in the country. The government-owned Coal India Ltd, which accounted for 80% of the total coal production in the country, had not been able to produce enough to meet the growing energy needs of the country. As a result, the government argued that such free allocations of coal blocks were necessary as this would make production of coal economically viable since the companies that would get the coal blocks do not have to incur any fixed cost to purchase the blocks. Estimates made by Nomura Equity Research suggest that between 2006 and 2009, the coal blocks given away for free had geological reserves of around 40 billion tonne. India has around 286 billion tonne of geological reserves of coal. This means around 14% of total geological reserves of coal was given away free during the period.

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[4] I rely on following NDTV report for all the details on the observation of the Supreme Court. See ≪http://www.ndtv.com/india-news/highlights-supreme-courts-opinion-on-auction-of-natural-resources-500371≫.

There was a huge uproar in India when the report by the Comptroller and Auditor General (CAG) of India in 2012 highlighted that the failure to auction these coal blocks amounted to huge losses to the Indian government in terms of revenue foregone. When the CAG final report was tabled in parliament the losses were shown to be Rs 1860 billion. According to the CAG report, about 25 big industrial company names were involved. All these companies allegedly made huge windfall gains and many economists and political commentators cited this as an example of crony capitalism. The audit report strongly suggested that the Government of India should finalize the regulations of competitive bidding. This CAG report also observed that "auctioning of blocks was considered as one of the widely practiced and acceptable selection process which was transparent and objective". The report revealed that the delays in introducing the process of competitive bidding have largely helped the private companies.

In July 2014, the Supreme Court set up a special CBI (Central Bureau of Investigation) court to undertake the trials of all coal allocation cases. On August 25, 2014, the Supreme court passed the judgement, terming all coal allocations between 1993 and 2010 illegal. It observed that all the allocations suffer from arbitrariness and legal flaws. The apex court also pointed out that the scam resulted in the heavy suffering of *common good and public interest*.[5]

Remark It would appear from the observations of the Supreme Court and also from the report of the 'Comptroller and Auditor General' of India in case of both the 2G scam and the coal block scam in India that auctions should be preferred over other options as a means of allocating the right to scarce natural resources. The "common good" which was mentioned by the Supreme Court may be interpreted to be total welfare (consumer surplus plus producer surplus plus government revenue).

2.1.3 Appropriate Policy for Allocating Scarce Resource

As noted before, in this chapter we discuss possible options in allocating a scarce resource in an emerging economy like India. We will do so in the context of a multimarket oligopoly. While our set-up will roughly replicate the Indian scenario we will keep our analysis entirely theoretical.

Imagine that there are two firms in a market. They compete in a horizontally differentiated product market. An example of this would be where two mobile companies (like Airtel and Vodafone) operate in a city like Delhi or Mumbai. Now suppose that a new spectrum (say with 2G or 3G technology) is available. The government wants to allocate the right to use this spectrum in a new market (say a tier 2 city in India, which is a smaller town that has till date not been covered by this new technology). Only one of the two firms will be granted the right to use that spectrum in this new

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[5]For details see the following: ≪http://www.dnaindia.com/india/report-11-things-you-need-to-know-about-the-coal-block-allocation-scam-2013511≫ ≪http://www.dnaindia.com/money/report-all-you-wanted-to-know-about-the-coal-scam-1735936≫.

market. The government has the following two options. (i) It can allocate the right to operate in the new market by conducting a first-price auction. Here the winner pays the bid that it has quoted and the loser does not pay anything. (ii) It can also allocate this right randomly (by tossing a coin) so that each firm wins with probability half. In this case neither the winner nor the loser pays anything. After the decision on the allocation of right to access the new market has been taken, the winner gets to operate in both markets and the loser operates in only the first market. We then analyze the equilibrium outcome for both cases (allocation by auction or by tossing a coin).

In our model the two options available to the government approximately replicates the Indian scenario. Allocating the right to use scarce natural use through an auction has become common practice in India after the Supreme Court judgement in 2012. Note that earlier, before the Supreme Court judgement, rights to use spectrum were often distributed arbitrarily (say a first-come-first-serve basis as in the case of 2G). Often, such rights were given almost for free (as companies were charged a very nominal amount). While there was a lot of corruption involved in such a process (the Supreme Court adjudication proves this), we do not focus on this corruption angle. We model this arbitrary allocation by assuming that the right was allocated randomly (tossing a coin or rolling a dice). The government of the day claimed that such was indeed the case. A first come- first serve basis allocation without corruption would be equivalent to random allocation.[6]

We will try to check whether auctions are better than random allocations. We do so by studying the effects of these two policy regimes on the expected total welfare (consumer surplus plus producer surplus plus government revenue, if any). The Supreme Court judgement and the report by the Comptroller and Auditor General of India, as discussed before, seem to suggest that the expected equilibrium welfare is higher when the government allocates the rights through an auction. We will show that this need not always be true.

Market Quality

Market quality is essentially a multidimensional concept: it encompasses efficiency, fairness and non-discrimination. Since we assume that there is no corruption in allocation, we essentially assume implicitly that “fairness in pricing” is automatically guaranteed in this context. Since the allocation mechanisms analyzed here, auctions and lottery, are both non-discriminatory; in our set-up the principle of non-discrimination is automatically satisfied. Consequently, in this specific context, total welfare is a very good proxy for market quality.[7]

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[6]To make our model tractable we deal only with first-price auctions. In India, like in many other countries (for example, USA), simultaneous ascending auctions are often used to allocate the right to use new spectrum. See Milgrom (2004) for the details on simultaneous ascending auctions.

[7]In Dastidar and Yano (2017) and in Chap. 3 of this book market quality will be defined to be a convex combination of total welfare and total ‘fairness’. The maximum possible ‘fairness’ is zero (see Chap. 3). This will be true when there is no corruption in the system (as in economies like Denmark or New Zealand). In the present chapter, since we assume away corruption, total fairness is at its maximum (i.e. zero). Also, there is no discrimination and hence market quality is equivalent to total welfare.

2.1.4 Summary of Our Findings

We consider a scenario where there are two markets and two firms. In market 1 both firms, who produce differentiated products, have free access. In market 2 there is no free access. The government has to decide on the process by which one of the two firms gets the right to operate in market 2. As noted in the previous section, the government has two options. (i) It can allocate the right to operate in the new market by conducting a first-price auction, where the highest bidder wins. (ii) It can allocate the right randomly (by tossing a coin) so that each firm wins with probability half.

We consider a three stage game. In the first stage the government decides on how to allocate the right of access to the new market: use a first-price auction or a lottery. If the government decides to allocate through an auction then in the second-stage both firms bid simultaneously for right of access to market 2. The highest bidder wins the right and pays the bid. The loser does not pay anything. If the government decides to allocate through a lottery then in the second stage it tosses a coin and one of the firms is declared a winner. In this case none of the firms pays anything. In the third stage the winner of the 'right' gets to operate in both markets and the loser operates in only market 1. Both firms play a Cournot duopoly game in market 1 and the winner chooses a monopoly output in market 2. The governments payoff is total expected welfare and the firms' payoffs are profits.

In our model, on the demand side we consider a differentiated product market. The parameter γ measures the degree of product differentiation and $\gamma \in [-1, 1]$. When $\gamma < 0$ the goods are complements and when $\gamma > 0$ the goods are substitutes. We also show that $\gamma < 0$ implies that goods (or services delivered by the two companies) are *strategic complements* and $\gamma > 0$ implies that they are *strategic substitutes*. These terms and their meanings were introduced in the literature by Bulow et al. (1985).

On the cost side we consider three cases. (i) Costs are not interrelated across markets. This would arise when firms have constant marginal costs (say c) and no capacity constraints. In this case, the marginal payoff to a firm with respect to output choice in any one market is independent of the output choice in the other market. (ii) Costs are interrelated across markets. We specifically take the case of quadratic costs. In this case, the marginal payoff with respect to output in any one market is strictly decreasing in the output of the other market. Following Bulow et al. (1985) we say that for this specific case, there are 'diseconomies of scope'. (iii) Constant marginal costs with strict capacity constraints. In this case both firms can produce upto k with a constant marginal cost, c, and cannot produce beyond that.

We analyze the equilibrium outcomes and compute expected total welfare for each of the above cases. A summary of our main conclusions is as follows.

1. When costs are not interrelated across markets (firms have constant marginal costs and no capacity constraints), then regardless of the differentiation parameter, the expected equilibrium welfare is higher when the rights of market 2 is allocated through an auction. Hence, in equilibrium, in the first stage the government chooses to allocate the right through an auction. In this case, the observations

of both the Supreme Court and of the Comptroller and Auditor General of India seem to be economically sound.

2. When costs are interrelated across markets (i.e. costs are quadratic) then we identify a critical value of the differentiation parameter (denoted by $\underline{\gamma}$ where $\underline{\gamma} > 0$). We show that if $\gamma > \underline{\gamma}$ (the goods are close enough substitutes) then the expected equilibrium welfare is higher when the government allocates the rights of market 2 through an auction. Hence, if $\gamma > \underline{\gamma}$, in equilibrium, in the first stage the government chooses to allocate the right through an auction. However, when $\gamma \leq \underline{\gamma}$ (goods are either complements or not close enough substitutes) then the expected equilibrium welfare may be higher when the right to operate in market 2 is allocated through a lottery rather than through an auction. We demonstrate this with two illustrative examples. In such cases, in equilibrium, the government chooses to allocate the right through a lottery.
3. When there are strict capacity constraints, if the size of market 2 is very high relative to the size of market 1, then the expected equilibrium welfare is higher when the government allocates the rights of market 2 through an auction. In such cases, in equilibrium, the government chooses to allocate the right through an auction. However, when the size of market 2 is relatively low then the expected equilibrium welfare may be higher when the government allocates the rights of market 2 through a lottery rather than through an auction. Again, we demonstrate this possibility with two illustrative examples. Consequently, in such cases, in equilibrium, the government chooses to allocate the right through a lottery.
4. When the right is allocated through an auction a firm with the highest type wins just as in a symmetric independent private value model. When the right is allocated randomly, the firm with the highest type wins with probability half. It would appear that allocative efficiency is not ensured when the right is allocated randomly. That is, total welfare would be higher if the right is allocated through an auction. *It may however be noted here this would be indeed true if there was a single market. Since we have multimarket oligopoly, when costs are quadratic or when there are strict capacity constraints, the choice of output in one market affects the strategic decisions of the players in the other market. Consequently, total welfare (which indicates efficiency) need not be higher when the right is allocated through an auction.*
5. Note that in this chapter we take total welfare to be the key indicator of market quality. Since we assume that there is no corruption in allocation " fairness in pricing" is automatically guaranteed in this context. Since both auctions and lottery, are non-discriminatory, in our set-up the principle of non-discrimination is automatically satisfied. Consequently, in this specific context, total welfare is a very good proxy for market quality. The main takeaway from our examples is that when there are diseconomies of scope (costs are interrelated across markets) or there are capacity constraints, the products of the two firms are either complements or not close enough substitutes and the potential market size of the new market (i.e. market 2) is not high enough, then allocating the right of access to the new market through auctions need not always fetch a better outcome as compared to the case when such a right is allocated through a lottery. *In such cases, the observations*

of both the Supreme Court and of the Comptroller and Auditor General of India appear to be economically invalid. This has serious policy implications, especially in an emerging economy like India.

2.2 The Model

There are two markets and two firms who produce horizontally differentiated products. In market 1 both firms have free access and in market 2 entry is restricted. The government can give the right of access to market 2 to any firm either by conducting an auction or allocate the right randomly by tossing a coin.

On the demand side of market 1, the representative consumer's utility function of two differentiated products, q_1 and q_2, and a numeraire good, q_0, is given by the following:

$$U = a\left(q_1 + q_2\right) - \frac{1}{2}\left(q_1^2 + q_2^2 + 2\gamma q_1 q_2\right) + q_0.$$

The parameter γ measures the degree of product differentiation and $\gamma \in [-1, 1]$. When $\gamma < 0$ the goods are complements and when $\gamma > 0$ the goods are substitutes. Note that when γ is unity then the products are homogeneous (perfect substitutes) and when γ is zero the products are independent. We will consider cases where $\gamma \neq 0$.[8]

The utility function generates the following system of inverse demand functions:

$$p_1 = a - q_1 - \gamma q_2$$
$$p_2 = a - \gamma q_1 - q_2$$

In market 2, the inverse demand for firm i is given by $P_i = A_i - Q_i$. The parameter A_i is private information to firm i. That is, A_i is the type of firm i. We assume that A_1 and A_2 are identically and independently distributed over $\left[\underline{A}, \bar{A}\right]$ with distribution function $F(.)$ and density function $f(.)$. Note that $\bar{A} > \underline{A} > 0$. The parameter, A_i, is a proxy for market size. More efficient a firm is, higher will be its market size. This is because a more efficient firm is able to exploit business opportunities better and consequently able to sell its product or services to a larger number of customers.

We consider a three-stage game. The players are the government and two firms. The governments payoff is total expected welfare and the firms' payoffs are profits.

1. Stage 1: In the first stage, the government chooses either an auction or a lottery for allocation of rights to operate in market 2.

[8]This specific utility function is based on Dixit (1979). Scores of papers in the literature have used this. A small sample of such papers is as follows: Singh and Vives (1984), Hackner (2000), Bester and Petrakis (1993), Zanchettin (2006), Pal (2010), Alipranti et al. (2014) and Dastidar (2015a).

2. Stage 2: Only one firm is allocated the right to operate in market 2. If in the first stage the government had chosen to conduct an auction to allocate such a right, then in the second stage there is a first-price auction where both firms bid simultaneously. This is modelled as a simultaneous move game of incomplete information (types are not known at this stage). The highest bidder wins the right to operate in market 2 and pays its bid. The loser does not pay anything. It may be noted here that each firm has the option of not participating in the auction. If in the first stage the government had chosen to allocate the right randomly, then in the second stage it tosses a coin. In this case, each firm wins the right to operate in market 2 with probability half and no firm makes any payment. That is, the winner gets the right to operate in market 2 free of cost.
3. Stage 3: If an auction is conducted in the second stage, the bids are first revealed to both players and the winner is declared in the second stage. Since bids are revealed, the firms get to know about each other's type. Note that for the winner, the bid amount paid is like a sunk cost in the third stage. If the government allocates the right randomly then only the winner is declared in the second stage. We assume that in this case the firms get to know each other's type *before* choosing their output levels in the third stage. In either case the winner gets to operate in both markets and the loser operates in only market 1. That is, the winner chooses outputs for both markets (1 and 2) while the loser chooses the output only for market 1. We assume that the winner and the loser play a simultaneous move Cournot duopoly game in market 1. In market 2 the winner chooses the monopoly quantity. In short, since types are revealed, the third stage duopoly game is a complete information game.

Let the winner choose x in market 1 and q in market 2. The loser chooses y in market 1. We assume that both firms have symmetric costs given by the following:

$$\text{Winner's cost} : C_W(x, q) = \frac{c}{\alpha}(x + q)^{\alpha}$$
$$\text{Loser's cost} : C_L(y) = \frac{c}{\alpha} y^{\alpha}, \text{ where } \alpha \geq 1.$$

If $\alpha = 1$, and there are no capacity constraints then costs are not interrelated across markets. In this case, the marginal cost for the winner is independent of its output level in any market. If $\alpha > 1$ then costs are interrelated across markets. When costs are interrelated across markets, then, for the winner the marginal cost of operating in any market depends on the output in both markets. A change in the output in any one market will affect the marginal cost in both markets. Since the demand in the two markets are unrelated, we can say that when $\alpha > 1$, there are 'diseconomies of scope'. The marginal profit with respect to output in any one market is strictly decreasing in the output of the other market (see Bulow et al. 1985).

For simplicity, we will consider only the following cases.

1. Case (i) $\alpha = 1$: constant marginal cost and no capacity constraint.
2. Case (ii) $c = 1$ and $\alpha = 2$: quadratic costs.

3. Case (iii) $\alpha = 1$ and $k =$ capacity: constant marginal cost with strict capacity constraint.

This simplification will make our model tractable and the computations easier. However, our basic insights will go through with general cost functions as well.

2.3 Equilibrium Outcome When Costs Are Not Interrelated

We now proceed to analyze the case where costs are not interrelated across markets. Note that we have $\alpha = 1$. Firms have constant marginal cost and no capacity constraints. Essentially, this means there are neither diseconomies nor economies of scope. The marginal profit with respect to output in any one market is unaffected by any change in the output of the other market. We assume the following.

Assumption 1 $\underline{A} > c$.

Assumption 2 $a > c$.

The two assumptions, that are very standard, ensure that the firms produce strictly positive outputs in equilibrium (i.e. we have an interior solution) and each firm participates in the auction in the first stage.

2.3.1 Third Stage Equilibrium

The winner chooses x in market 1 and q in market 2. The loser chooses y in market 1. Since types are revealed in this stage, let the revealed type of the winner be A. The winner's gross payoff is

$$\pi_W = (A - q)\, q + (a - x - \gamma y)\, x - c\,(x + q)\,. \tag{2.1}$$

Note that if there if the government decides to hold an auction in the first stage, the winner's net payoff is [$\pi_W -$ (winner's bid)]. If there is no auction and the winner is decided by the toss of coin then the winner's net payoff is simply π_W. The loser's payoff is

$$\pi_L = (a - \gamma x - y)\, y - cy \tag{2.2}$$

Remark Note that $\frac{\partial^2 \pi_W}{\partial x \partial y} = \frac{\partial^2 \pi_L}{\partial y \partial x} = -\gamma < 0$ iff $\gamma > 0$. Following Bulow et al. (1985) we conclude that the products of the two firms are 'strategic substitutes' if $\gamma > 0$ and 'strategic complements' if $\gamma < 0$. The same holds true for the case where costs are interrelated across markets (see next section).

At an interior equilibrium, we must have the following.

(i) the first order conditions:

$$\frac{\partial}{\partial q}\pi_W = \frac{\partial}{\partial x}\pi_W = 0 \text{ and } \frac{\partial}{\partial y}\pi_L = 0$$

(ii) the second order conditions:

$$\det \begin{vmatrix} \frac{\partial^2 \pi_W}{\partial q^2} & \frac{\partial^2 \pi_W}{\partial q \partial x} \\ \frac{\partial^2 \pi_W}{\partial x \partial q} & \frac{\partial^2 \pi_W}{\partial x^2} \end{vmatrix} > 0 \text{ and } \frac{\partial^2 \pi_L}{\partial y^2} < 0.$$

From the first order conditions using the expressions in (2.1) and (2.2) we have the following.

$$A - 2q - c = 0 \tag{2.3a}$$

$$a - 2x - \gamma y - c = 0 \tag{2.3b}$$

$$a - \gamma x - 2y - c = 0 \tag{2.3c}$$

Note that the second order conditions are also satisfied. Solving the above equations we get the equilibrium values of q, x and y. Let such equilibrium values be q^*, x^* and y^*. Routine computations show that the equilibrium values of x, q and y are as follows:

$$x^* = \frac{a-c}{\gamma+2},\ q^* = \frac{(A-c)}{2},\ y^* = \frac{a-c}{\gamma+2} \tag{2.4}$$

Our assumptions ensure that q^*, x^* and y^* are strictly positive. Using the values of q^*, x^* and y^* we get the equilibrium gross payoffs for the winner and loser, π_W^* and π_L^*, and also their difference, $\pi_W^* - \pi_L^*$.

$$\pi_W^*(A,\gamma) = \frac{(A-c)^2}{4} + \frac{(a-c)^2}{(\gamma+2)^2} \tag{2.5a}$$

$$\pi_L^*(A,\gamma) = \frac{(a-c)^2}{(\gamma+2)^2} \tag{2.5b}$$

$$\pi_W^*(A,\gamma) - \pi_L^*(A,\gamma) = \frac{(A-c)^2}{4} \tag{2.5c}$$

Note that since $A > c$ (Assumption 1), $\frac{\partial}{\partial A}\left(\pi_W^* - \pi_L^*\right) = \frac{1}{2}(A-c) > 0$. Also, $\frac{\partial^2}{\partial A \partial \gamma}\left(\pi_W^* - \pi_L^*\right) = 0$. We summarize our results in terms of a proposition. The proof is straightforward.

Proposition 1 *If costs are not interrelated across markets, then in the third stage* $\forall A \in \left[\underline{A}, \bar{A}\right]$ *and* $\forall \gamma \in [-1, 1]$ *we get the following. (i)* $\pi_W^*(A,\gamma) - \pi_L^*(A,\gamma) > 0$ *(ii)* $\frac{\partial}{\partial A}\left(\pi_W^* - \pi_L^*\right) > 0$. *(iii)* $\frac{\partial^2}{\partial A \partial \gamma}\left(\pi_W^* - \pi_L^*\right) = 0$.

2.3.2 Second Stage Equilibrium

Note that if the government decides to allocate the right randomly (by tossing a coin) each firm wins the right to operate in market 2 with probability half and no firm makes any payment. If the government decides to allocate the right through an auction then in the first stage the two firms bid to win the rights to sell in market 2. We consider a first-price sealed bid auction. As noted before, the parameter A_i is private information to firm i. That is, A_i is the type of firm i. We assume that A_1 and A_2 are identically and independently distributed over $\left[\underline{A},\ \bar{A}\right]$ with distribution function $F\left(.\right)$ and density function $f\left(.\right)$. Each firm's strategy is the following:

$$b_i\left(A_i\right) : \left[\underline{A},\ \bar{A}\right] \rightarrow [0, \infty) \cup \{No\}$$

Note that $\{No\}$ means not to participate in the auction. We now show that there exists a symmetric strictly increasing Bayesian–Nash equilibrium, $b\left(.\right)$, where both firms participate in the auction regardless of type.

We use standard auction theoretic techniques to derive the bidding equilibrium. Let firm 2 follow the strategy $b\left(.\right)$. Let firm 1 choose a bid $b_1 = b\left(z\right)$. Firm 1 wins iff $b\left(A_2\right) < b\left(z\right) \Longleftrightarrow A_2 < z$. Hence, the probability that firm 1 wins the right to operate in market 2 is $F\left(z\right)$. The expected payoff to firm 1 by choosing a bid $b\left(z\right)$ is the following.

$$\begin{aligned} E_1\left(z, A_1, \gamma\right) &= F\left(z\right)\left[\pi_W^*\left(A_1, \gamma\right) - b\left(z\right)\right] + \left(1 - F\left(z\right)\right)\pi_L^*\left(A_1, \gamma\right) \\ &= F\left(z\right)\left[\pi_W^*\left(A_1, \gamma\right) - \pi_L^*\left(A_1, \gamma\right)\right] - F\left(z\right) b\left(z\right) + \pi_L^*\left(A_1, \gamma\right) \quad (2.6) \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial z} E_1\left(z, A_1, \gamma\right) &= f\left(z\right)\left[\pi_W^*\left(A_1, \gamma\right) - \pi_L^*\left(A_1, \gamma\right)\right] - f\left(z\right) b\left(z\right) - F\left(z\right) b'\left(z\right) \\ &= 0 \text{ at } z = A_1. \quad (2.7) \end{aligned}$$

From (2.7) above we get

$$f\left(A_1\right)\left[\pi_W^*\left(A_1, \gamma\right) - \pi_L^*\left(A_1, \gamma\right)\right] - f\left(A_1\right) b\left(A_1\right) - F\left(A_1\right) b'\left(A_1\right) = 0$$

$$\Longleftrightarrow \left[F\left(A_1\right) b\left(A_1\right)\right]' = f\left(A_1\right)\left[\pi_W^*\left(A_1, \gamma\right) - \pi_L^*\left(A_1, \gamma\right)\right]$$

Since $F\left(\underline{A}\right) = 0$ we get the equilibrium bidding strategy to be as follows:

$$b\left(A_1\right) = \frac{1}{F\left(A_1\right)} \int_{\underline{A}}^{A_1} \left[\pi_W^*\left(t, \gamma\right) - \pi_L^*\left(t, \gamma\right)\right] f\left(t\right) dt \quad (2.8)$$

Note that

$$b'(A_1) = \frac{\begin{bmatrix} F(A_1)\left[\left\{\pi_W^*(A_1,\gamma) - \pi_L^*(A_1,\gamma)\right\} f(A_1)\right] \\ -f(A_1)\int_{\underline{A}}^{A_1}\left[\pi_W^*(t,\gamma) - \pi_L^*(t,\gamma)\right] f(t)\,dt \end{bmatrix}}{[F(A_1)]^2}$$
$$= \frac{f(A_1)\begin{bmatrix} F(A_1)\left\{\pi_W^*(A_1,\gamma) - \pi_L^*(A_1,\gamma)\right\} \\ -\int_{\underline{A}}^{A_1}\left[\pi_W^*(t,\gamma) - \pi_L^*(t,\gamma)\right] f(t)\,dt \end{bmatrix}}{[F(A_1)]^2} \tag{2.9}$$

From Proposition 1 we know that $\left[\pi_W^*(t,\gamma) - \pi_L^*(t,\gamma)\right]$ is strictly increasing in t. This means

$$\int_{\underline{A}}^{A_1}\left[\pi_W^*(t,\gamma) - \pi_L^*(t,\gamma)\right] f(t)\,dt$$
$$< \left[\pi_W^*(A_1,\gamma) - \pi_L^*(A_1,\gamma)\right]\int_{\underline{A}}^{A_1} f(t)\,dt$$
$$= \left[\pi_W^*(A_1,\gamma) - \pi_L^*(A_1,\gamma)\right] F(A_1)$$

Using the above in the expression for $b'(A_1)$ we get that

$$\forall A_1 \in \left[\underline{A},\ \bar{A}\right],\ b'(A_1) > 0. \tag{2.10}$$

Using standard techniques we can now show that if one firm chooses a bidding strategy $b(.)$ as stated above, the best that the other firm can choose is the same bidding strategy (see Krishna 2010, Chap. 2).

We now proceed to demonstrate that both firms will choose to participate in the auction regardless of their types. First note that *in equilibrium*, the expected payoff to firm 1 by choosing a strategy $b(.)$ is as follows:

$$E_1^*(A_1,A_1,\gamma) = F(A_1)\left[\pi_W^*(A_1,\gamma) - \pi_L^*(A_1,\gamma)\right] - F(A_1)\,b(A_1) + \pi_L^*(A_1,\gamma) \tag{2.11}$$

Since $b(A_1) = \frac{1}{F(A_1)}\int_{\underline{A}}^{A_1}\left[\pi_W^*(t,\gamma) - \pi_L^*(t,\gamma)\right] f(t)\,dt$ from (2.11) we get that

$$E_1^*(A_1,A_1,\gamma) = F(A_1)\left[\pi_W^*(A_1,\gamma) - \pi_L^*(A_1,\gamma)\right]$$
$$- \int_{\underline{A}}^{A_1}\left[\pi_W^*(t,\gamma) - \pi_L^*(t,\gamma)\right] f(t)\,dt + \pi_L^*(A_1,\gamma) \tag{2.12}$$

Note that since $F\left(\underline{A}\right) = 0$

$$\int_{\underline{A}}^{A_1}\left[\pi_W^*(t,\gamma) - \pi_L^*(t,\gamma)\right] f(t)\,dt$$

$$
\begin{aligned}
&= \int_{\underline{A}}^{A_1} \left[\pi_W^* (t, \gamma) - \pi_L^* (t, \gamma)\right] dF(t) \\
&= \left[F(t) \left\{\pi_W^* (t, \gamma) - \pi_L^* (t, \gamma)\right\}\right]_{\underline{A}}^{A_1} - \int_{\underline{A}}^{A_1} F(t) \frac{d}{dt} \left[\pi_W^* (t, \gamma) - \pi_L^* (t, \gamma)\right] dt \\
&= F(A_1) \left[\pi_W^* (A_1, \gamma) - \pi_L^* (A_1, \gamma)\right] - \int_{\underline{A}}^{A_1} F(t) \frac{d}{dt} \left[\pi_W^* (t, \gamma) - \pi_L^* (t, \gamma)\right] dt
\end{aligned} \tag{2.13}
$$

Hence, using (2.13) in (2.12) we have

$$
E_1^* (A_1, A_1, \gamma) = \pi_L^* (A_1, \gamma) + \int_{\underline{A}}^{A_1} F(t) \frac{d}{dt} \left[\pi_W^* (t, \gamma) - \pi_L^* (t, \gamma)\right] dt
$$

Since $\frac{d}{dt} \left[\pi_W^* (t, \gamma) - \pi_L^* (t, \gamma)\right] > 0$ (Proposition 1) we get that $E_1^* (A_1, A_1, \gamma) > \pi_L^* (A_1, \gamma)$. Note that $\pi_L^* (A_1, \gamma)$ is the payoff that would accrue to firm 1 if it chooses not to participate in the auction. We just demonstrated that the expected net payoff to firm 1 by participating in the auction is strictly greater than the payoff by not participating. This means firm 1 will always participate in the auction. Since the firms have symmetric costs, the same holds true for firm 2 as well. We summarize these results in terms of a proposition.

Proposition 2 *When costs are not interrelated across markets, if the government decides to allocate the right through an auction in the first stage, then both firms will choose to participate in the auction in the second-stage. There is a symmetric, strictly increasing Bayesian–Nash equilibrium where a firm with type A bids the following:*

$$
b(A) = \frac{1}{F(A)} \int_{\underline{A}}^{A} \left[\frac{1}{4} (t - c)^2\right] f(t)\, dt.
$$

The most efficient firm, i.e. the firm with the highest type, wins the auction.

2.3.3 First-Stage Equilibrium

In the first-stage the government chooses either an auction or a lottery as a means of allocating the right to operate in market 2. The government's payoff is total expected welfare. We now proceed to the analysis of total welfare.

Total Welfare

Note that total welfare is the sum of the following: consumer surplus in both markets, profits of both firms and the revenue accruing to the government (bid of the winning firm).

We know that the winner chooses x in market 1 and q in market 2 and the loser chooses y in market 1. Let p_x and p_y be the prices of x and y in market 1. Given the representative consumer's utility function in market 1, the consumer surplus here is

$$CS_1 = a(x+y) - \frac{1}{2}\left(x^2 + y^2 + 2\gamma xy\right) - p_x x - p_y y$$

In market 2, since inverse demand is $P_i = A_i - Q_i$, the consumer surplus in market 2 when the winner chooses q is

$$CS_2 = \frac{1}{2}q^2.$$

The price of q is $(A-q)$ where A is the type of the winner. When the government decides to allocate the right through an auction the winner's *net profit* is

$$\pi_W(A,\gamma) - b(A) = p_x x + (A-q)q - c(x+q) - b(A).$$

When the government decides to allocate the right randomly the winner's *net profit* is just

$$\pi_W(A,\gamma) = p_x x + (A-q)q - c(x+q)$$

The losers net profit is

$$\pi_L(A,\gamma) = p_y y - cy$$

The government's revenue is zero when the right is allocated randomly and the revenue is the winner's bid when the right is allocated through an auction.

When the government decides to allocate the right through an auction the total welfare (W) is therefore

$$\begin{aligned}
&\text{Total consumer surplus} + \text{total profits} + \text{government revenue} \\
&= [CS_1 + CS_2] + [\pi_W(A,\gamma) - b(A) + \pi_L(A,\gamma)] + b(A) \\
&= [CS_1 + CS_2] + [\pi_W(A,\gamma) + \pi_L(A,\gamma)] \\
&= a(x+y) - x^2 - y^2 - q^2 - \gamma xy + Aq - xq
\end{aligned} \tag{2.14}$$

Comment Note that when the government decides to allocate the right randomly the total welfare is exactly the same as above ($b(A) = 0$ in the case of a lottery).

In equilibrium the winner chooses x^* and q^* and the loser chooses y^*. We had earlier derived these equilibrium values.

$$x^* = \frac{a-c}{\gamma+2},\ q^* = \frac{(A-c)}{2},\ y^* = \frac{a-c}{\gamma+2}.$$

Therefore, by using the above values in (2.14) we get the expression for equilibrium total welfare.

$$W^*(A,\gamma)=\frac{1}{8(\gamma+2)^2}\begin{pmatrix}5A^2\gamma^2+20A^2\gamma+20A^2-10Ac\gamma^2\\-40Ac\gamma-40Ac+8a^2\gamma+24a^2\\-16ac\gamma-48ac+5c^2\gamma^2+28c^2\gamma+44c^2\end{pmatrix}\tag{2.15}$$

We now state our next result.

Proposition 3 *When costs are not interrelated across markets the following is true. (i)* $\frac{\partial}{\partial a}W^*(A,\gamma)>0$ *and (ii)* $\frac{\partial}{\partial\gamma}W^*(A,\gamma)<0$.

Proof Since $a>c$ and $\gamma\in[0,1]$ using (2.15) and routine computations we get that

$$\frac{\partial}{\partial a}W^*(A,\gamma)=\frac{16(\gamma+3)(a-c)}{8(\gamma+2)^2}>0\text{ and}$$
$$\frac{\partial}{\partial\gamma}W^*(A,\gamma)=-\frac{(\gamma+4)(a-c)^2}{(\gamma+2)^3}<0.$$ ■

Comment Total welfare is strictly increasing in a and strictly decreasing in the differentiation parameter, γ. Note that the term 'a' may be interpreted as the market size in the market 1. It is intuitively obvious that an increase in market size will lead to an increase in the profits for both firms (note that both firms are there in market 1) and also an increase in the consumer surplus. Consequently, total welfare must rise. Also, note that profits, both for the winner and for the loser, is strictly decreasing in γ. Higher γ implies that the products are closer substitutes ($\gamma=1$ means that the products are perfect substitutes) and this in turn means that the intensity of competition increases. Consequently, there is a reduction in profits for both firms and this reduction in profits outweighs any possible increase in consumer surplus that may arise due to a increase in γ. As a result, total welfare decreases with an increase in γ. This clearly means that total welfare will be at its lowest if goods are perfect substitutes ($\gamma=1$).

Comparison of Two Policy Regimes

As noted in our introduction, the government can allocate the rights to operate in market 2 in either of the two following ways: (i) through a first-price auction or (ii) through a lottery where the government tosses a coin and decides on the winner.

Note that the equilibrium expost welfare is $W^*(A,\gamma)$, where A is the type of the winner. When the government decides on the policy in the first-stage it is unaware of the types of the two firms. Hence, it will compare the expected equilibrium welfare for the two policy regimes and take a decision. The Supreme Court judgement, as discussed in the introduction, seems to assume that expected equilibrium welfare is higher when the government allocates the rights through a first-price auction. We will show that this will always be the case when costs are not interrelated across markets.

Expected welfare when the right is allocated through an auction Note that A_1 and A_2 are identically and independently distributed over $[\underline{A},\ \bar{A}]$ with distribution function $F(.)$ and density function $f(.)$. The firm with the highest type wins the auction. Let $A_{(1)} = \max\{A_1, A_2\}$. From basic statistical techniques we know that the distribution function and density function of $A_{(1)}$ are $F_{(1)}(t) = [F(t)]^2$ and $f_{(1)}(t) = 2F(t) f(t)$ respectively. Hence, the expected equilibrium welfare when the right is allocated through an auction is as follows:

$$Exp\left[W^*_{auction}(A,\gamma)\right] = \int_{\underline{A}}^{\bar{A}} W^*(A,\gamma) f_{(1)}(A)\, dA = \int_{\underline{A}}^{\bar{A}} W^*(A,\gamma)\, 2F(A) f(A)\, dA \tag{2.16}$$

Expected welfare when the right is allocated by the toss of a coin Note that here each firm wins with probability half. In this case the expected equilibrium welfare is as follows:

$$Exp\left[W^*_{lottery}(A,\gamma)\right] = \int_{\underline{A}}^{\bar{A}} W^*(A,\gamma) f(A)\, dA \tag{2.17}$$

We proceed to state the final proposition of this section.

Proposition 4 *When costs are not interrelated across markets, for all* $\gamma \in [-1, 1]$*, we get*

$$Exp\left[W^*_{auction}(A,\gamma)\right] \geq Exp\left[W^*_{lottery}(A,\gamma)\right].$$

Hence, in the first-stage, for all $\gamma \in [-1, 1]$*, in equilibrium the government allocates the right to operate in market 2 through an auction.*

Proof Note that $\forall t \in \left[\underline{A}, \bar{A}\right]$, $F_{(1)}(t) \leq F(t)$. That is, $F_{(1)}(.)$ stochastically dominates $F(.)$. This means that if $\frac{\partial}{\partial A} W^*(A,\gamma) > 0$ then $Exp\left[W^*_{auction}(A,\gamma)\right] \geq Exp\left[W^*_{lottery}(A,\gamma)\right]$.[9]

In (2.15) we have already derived the expression for $W^*(A,\gamma)$. It is straightforward to compute that

$$\frac{\partial}{\partial A} W^*(A,\gamma) = \frac{5}{4}(A - c) \text{ and } \frac{\partial^2}{\partial A^2} W^*(A,\gamma) = 0 \tag{2.18}$$

Note that since $\forall A \in \left[\underline{A}, \bar{A}\right]$ we have $A > c$ (Assumption 1), $\frac{\partial}{\partial A} W^*(A,\gamma) > 0$. Hence,

$$\int_{\underline{A}}^{\bar{A}} W^*(A,\gamma) f_{(1)}(A)\, dA \geq \int_{\underline{A}}^{\bar{A}} W^*(A,\gamma) f(A)\, dA.$$

That is, $Exp\left[W^*_{auction}(A,\gamma)\right] \geq Exp\left[W^*_{lottery}(A,\gamma)\right]$. ∎

Comment When costs are not interrelated across markets, then regardless of the differentiation parameter, the expected equilibrium welfare is *higher* when the government allocates the rights of market 2 through an auction. The intuition behind this

[9] See the appendix in Krishna (2010) for a discussion on stochastic dominance.

is straightforward. Allocating through auction implies that the firm with the highest type (highest A) gets the right to operate in market 2. If the right was allocated through a lottery, the highest type would win with probability half. This means the expected market size in market 2 would be higher when the right is allocated through an auction rather than by a lottery. This in turn implies that both the consumer surplus and profits would be higher. Consumer surplus would be higher as the amount sold in market 2 would be higher. Profit would be higher as the market size is greater and there are no diseconomies of scope (marginal cost is constant when costs are not interrelated across markets). Hence, in this case, the observations of both the Supreme Court and of the Comptroller and Auditor General of India are economically sound.

2.4 Equilibrium Outcome with Interrelated Costs

We now analyze the more interesting case where costs are interrelated across markets. For this we need $\alpha > 1$. As noted before, this means there are 'diseconomies of scope'. The marginal profit with respect to output in any one market is strictly decreasing in the output of the other market (see Bulow et al. 1985). For simplicity we will consider the case where $c = 1$ and $\alpha = 2$. That is, we consider a case where costs are quadratic. This will make our model tractable and the computations easier.

We assume the following.

Assumption 1 $\underline{A} > \frac{a}{3-|\gamma|}$.

Assumption 2 $a > \frac{\bar{A}}{3-\gamma}$.

The two assumptions ensure that when costs are interrelated, the firms produce strictly positive outputs in equilibrium (i.e. we have an interior solution) and each firm, regardless of its type, participates in the auction in the first stage (in case the government holds an auction). Note that a firm will choose to participate in the first stage auction iff it's expected payoff from doing so is more than the expected payoff from operating only in market 1 (in which both firms have free access). It will be clear from our results that our main insights will go through for any $\alpha > 1$ and any $c > 0$ (with the assumptions suitably modified to ensure interior equilibria).

2.4.1 *Third Stage Equilibrium*

As before, let the winner choose x in market 1 and q in market 2. The loser chooses y in market 1. Since types are revealed in this stage, let the type of the winner be A. The winner's gross payoff is

$$\pi_W = (A - q)\, q + (a - x - \gamma y)\, x - \frac{1}{2}\,(x + q)^2\,. \tag{2.19}$$

As discussed before, if the government chooses to allocate the right through an auction in the first stage, the winner's net payoff is $\pi_W -$ (winner's bid). If there is no auction and the winner is decided by the toss of coin then the winner's net payoff is simply π_W. The loser's payoff is

$$\pi_L = (a - \gamma x - y)\, y - \frac{1}{2} y^2 \tag{2.20}$$

At an interior equilibrium, the first order conditions are the following.

$$A - 2q - (x + q) = 0 \tag{2.21a}$$
$$a - 2x - \gamma y - (x + q) = 0 \tag{2.21b}$$
$$a - \gamma x - 2y - y = 0 \tag{2.21c}$$

Note that the second order conditions are also satisfied. Solving the above equations we get the equilibrium values of q, x and y.

$$q^* = \frac{(3-\gamma)\,[A\,(3+\gamma) - a]}{3\left(8-\gamma^2\right)},\ x^* = \frac{a\,(3-\gamma) - A}{8-\gamma^2} \text{ and } y^* = \frac{a\,(8-3\gamma) + A\gamma}{3\left(8-\gamma^2\right)} \tag{2.22}$$

We claim that since $A \in \left[\underline{A},\ \bar{A}\right]$ and $\gamma \in [-1, 1]$, the two assumptions ensure that $q^*, x^*, y^* > 0$. First note that $8 - \gamma^2 > 0$ and $3 - \gamma > 0$. Also, $A\,(3+\gamma) - a \geq A\,(3 - |\gamma|) - a > 0$ (Assumption 1). Therefore, $q^* > 0$. Using Assumption 2 we get that $x^* > 0$. Note that since $a\,(3-\gamma) > A$ (Assumption 2) and since $8 - 3\gamma > 0$ we have

$$a\,(8-3\gamma) + A\gamma > A\left(\frac{(8-3\gamma)}{3-\gamma} + \gamma\right) = \frac{A\left(8-\gamma^2\right)}{3-\gamma} > 0.$$

The above shows that $y^* > 0$.

Using the values of q^*, x^* and y^* we get the equilibrium gross payoffs for the winner and loser, π_W^* and π_L^*, and also their difference, $\pi_W^* - \pi_L^*$.

$$\pi_W^*\,(A,\gamma) = \frac{\begin{pmatrix} A^2\gamma^4 - 16A^2\gamma^2 + 72A^2 + 16Aa\gamma \\ -48Aa + 8a^2\gamma^2 - 48a^2\gamma + 72a^2 \end{pmatrix}}{6\left(8-\gamma^2\right)^2} \tag{2.23a}$$

$$\pi_L^*\,(A,\gamma) = \frac{(8a + A\gamma - 3a\gamma)^2}{6\left(8-\gamma^2\right)^2} \tag{2.23b}$$

$$\pi_W^*\,(A,\gamma) - \pi_L^*\,(A,\gamma) = \frac{[A\,(3+\gamma) - a]\,[A\,(3-\gamma) - a]}{6\left(8-\gamma^2\right)} \tag{2.23c}$$

We now show that, given our assumptions, $\pi_W^* (A, \gamma) - \pi_L^* (A, \gamma) > 0$ regardless of whether goods are substitutes or complements. First, consider the case of substitutes ($\gamma > 0$). From Assumption 1 we get that $A(3 - |\gamma|) > a$. If $\gamma > 0$ then $3 - |\gamma| = 3 - \gamma$. Hence, Assumption 1 implies that $A(3 - \gamma) - a > 0$. Since $\gamma > 0$ we have $A(3 + \gamma) - a > A(3 - \gamma) - a > 0$. Using (2.23c) we get this means $\pi_W^* - \pi_L^* > 0$. Now, consider the case of complements ($\gamma < 0$). If $\gamma < 0$ then $3 - |\gamma| = 3 + \gamma$. From Assumption 1 we get $A(3 + \gamma) - a > 0$. Now note that since $\gamma < 0$ we have $A(3 - \gamma) - a > A(3 + \gamma) - a > 0$. From (2.23c) we again get $\pi_W^* - \pi_L^* > 0$.

Using routine computations and (2.23c) we have

$$\frac{\partial}{\partial A}\left(\pi_W^* - \pi_L^*\right) = \frac{2\left[A\left(9 - \gamma^2\right) - a\right]}{6\left(8 - \gamma^2\right)} \tag{2.24}$$

Also, note that from Assumption 1 we have $A > \frac{a}{3-|\gamma|}$. Now since $\gamma \in [-1, 1]$

$$\begin{aligned} A > \frac{a}{3 - |\gamma|} &\Longrightarrow A > \frac{a}{3 - |\gamma|}\left(\frac{3}{3 + |\gamma|}\right) = \frac{3a}{9 - \gamma^2} \\ &\Longrightarrow A\left(9 - \gamma^2\right) - a > 0 \end{aligned} \tag{2.25}$$

Using (2.25) and (2.24) we get

$$\frac{\partial}{\partial A}\left(\pi_W^* - \pi_L^*\right) = \frac{2\left[A\left(9 - \gamma^2\right) - a\right]}{6\left(8 - \gamma^2\right)} > 0 \tag{2.26}$$

Also note that

$$\frac{\partial^2}{\partial A \partial \gamma}\left(\pi_W^* - \pi_L^*\right) = \frac{2}{3}\frac{\gamma(A - a)}{\left(8 - \gamma^2\right)^2} \geq 0 \iff \gamma(A - a) \geq 0. \tag{2.27}$$

We summarize our preceding discussion in terms of a proposition.

Proposition 5 *When costs are interrelated across markets, then in the third stage equilibrium* $\forall A \in \left[\underline{A}, \bar{A}\right]$ *and* $\forall \gamma \in [-1, 1]$ *we get the following. (i)* $\pi_W^* (A, \gamma) - \pi_L^* (A, \gamma) > 0$ *(ii)* $\frac{\partial}{\partial A}\left[\pi_W^* (A, \gamma) - \pi_L^* (A, \gamma)\right] > 0$ *and (iii)* $\frac{\partial^2}{\partial A \partial \gamma}\left[\pi_W^* (A, \gamma) - \pi_L^* (A, \gamma)\right] \geq 0 \iff \gamma(A - a) \geq 0$.

Comment Since the winner gets a strictly higher gross payoff than the loser, in the second-stage there is an incentive to participate in the auction. The difference between the winner's and loser's payoff is increasing in the market size, A. This is also intuitively obvious.

We now proceed to compute the second stage equilibrium.

2.4.2 Second Stage Equilibrium

Note that if the government decides to allocate the right randomly (by tossing a coin) each firm wins the right to operate in market 2 with probability half and no firm makes any payment. If the government decides to allocate the right through a first-price auction, then in the first stage the two firms bid to win the rights to sell in market 2.

As before, we can show that the equilibrium bidding strategy is as follows:

$$\begin{aligned} b\left(A_1\right) &= \frac{1}{F\left(A_1\right)} \int_{\underline{A}}^{A_1} \left[\pi_W^*\left(t, \gamma\right) - \pi_L^*\left(t, \gamma\right)\right] f\left(t\right) dt \\ &= \frac{1}{F\left(A\right)} \int_{\underline{A}}^{A} \left[\frac{\left[t\left(3+\gamma\right)-a\right]\left[t\left(3-\gamma\right)-a\right]}{6\left(8-\gamma^2\right)}\right] f\left(t\right) dt \end{aligned} \tag{2.28}$$

We now proceed to demonstrate that both firms will choose to participate in the auction regardless of their types. First note that in equilibrium, the expected payoff to firm 1 by choosing a strategy $b\left(.\right)$ is as follows (see the discussion in the previous section):

$$E_1^*\left(A_1, A_1, \gamma\right) = \pi_L^*\left(A_1, \gamma\right) + \int_{\underline{A}}^{A_1} F\left(t\right) \frac{d}{dt} \left[\pi_W^*\left(t, \gamma\right) - \pi_L^*\left(t, \gamma\right)\right] dt \tag{2.29}$$

Since $\frac{d}{dt}\left[\pi_W^*\left(t, \gamma\right) - \pi_L^*\left(t, \gamma\right)\right] > 0$ (Proposition 5) we get that $E_1^*\left(A_1, A_1, \gamma\right) > \pi_L^*\left(A_1, \gamma\right)$. This means firm 1 will always participate in the second-stage auction. We summarize these results in terms of a proposition.

Proposition 6 *When costs are interrelated across markets, if the government decides to allocate the right through an auction in the first stage, then both firms will choose to participate in the auction in the second-stage. There is a symmetric, strictly increasing Bayesian–Nash equilibrium where a firm with type A bids the following:*

$$b\left(A\right) = \frac{1}{F\left(A\right)} \int_{\underline{A}}^{A} \left[\frac{\left[t\left(3+\gamma\right)-a\right]\left[t\left(3-\gamma\right)-a\right]}{6\left(8-\gamma^2\right)}\right] f\left(t\right) dt$$

The most efficient firm, i.e. the firm with the highest type, wins the auction.

2.4.3 First-Stage Equilibrium

As noted before, in the first-stage the government chooses either an auction or a lottery as a means of allocating the right to operate in market 2. The government's payoff is total expected welfare. We now proceed to the analysis of total welfare when costs are quadratic.

Total Welfare

Routine computations show (see the previous section) that the total welfare (W) for either case (auction or toss of a coin) is as follows:

$$\begin{aligned}
&\text{Total consumer surplus } + \text{ total profits } + \text{ government revenue}\\
&= [CS_1 + CS_2] + [\pi_W (A, \gamma) - b (A) + \pi_L (A, \gamma)] + b (A)\\
&= [CS_1 + CS_2] + [\pi_W (A, \gamma) + \pi_L (A, \gamma)]\\
&= a (x + y) - x^2 - y^2 - q^2 - \gamma xy + Aq - xq
\end{aligned} \tag{2.30}$$

In equilibrium the winner chooses x^* and q^* and the loser chooses y^*. In (2.22) we had earlier derived these equilibrium values. Therefore, by using the equilibrium values in (2.30) we get the expression for equilibrium total welfare.

$$W^* (A, \gamma) = \frac{1}{9 \left(8 - \gamma^2\right)^2} \begin{pmatrix} 2A^2\gamma^4 - 34A^2\gamma^2 + 153A^2 - 4Aa\gamma^3 \\ +9Aa\gamma^2 + 50Aa\gamma - 126Aa + 9a^2\gamma^3 \\ -16a^2\gamma^2 - 126a^2\gamma + 281a^2 \end{pmatrix} \tag{2.31}$$

We now state our next result.

Proposition 7 *When costs are interrelated across markets, the following is true. (i)* $\frac{\partial}{\partial a} W^* (A, \gamma) > 0$ *(ii) There exists* $\gamma^* \approx -0.100836133705$ *such that* $\gamma < \gamma^* \Rightarrow \frac{\partial}{\partial \gamma} W^* (A, \gamma) < 0$. *If* $\gamma > \gamma^*$ *then the sign of* $\frac{\partial}{\partial \gamma} W^* (A, \gamma)$ *is ambiguous.*

Proof (i) It may be noted that when costs are interrelated across markets

$$\frac{\partial}{\partial a} W^* (A, \gamma) = \frac{\begin{bmatrix} 562a - 126A + 50A\gamma - 252a\gamma \\ +9A\gamma^2 - 4A\gamma^3 - 32a\gamma^2 + 18a\gamma^3 \end{bmatrix}}{9 \left(8 - \gamma^2\right)^2} \tag{2.32}$$

Since $\gamma \in [-1, 1]$ the denominator of (2.32) is strictly positive. Hence $\frac{\partial}{\partial a} W^* (A, \gamma) > 0 \Leftrightarrow$ (Numerator) > 0. Now note that

$$Numerator = a \left(562 - 252\gamma - 32\gamma^2 + 18\gamma^3\right) + A \left(-126 + 50\gamma + 9\gamma^2 - 4\gamma^3\right) \tag{2.33}$$

Note that $562 - 252\gamma - 32\gamma^2 + 18\gamma^3 > 0$ for all $\gamma \in [-1, 1]$. Since $a > \frac{\bar{A}}{3-\gamma}$ (by Assumption 2) we have $a > \frac{A}{3-\gamma}$ for all $A \in \left[\underline{A}, \bar{A}\right]$. Therefore,

$$\begin{aligned}
Numerator &= a \left(562 - 252\gamma - 32\gamma^2 + 18\gamma^3\right) + A \left(-126 + 50\gamma + 9\gamma^2 - 4\gamma^3\right)\\
&> \frac{A}{3 - \gamma} \left(562 - 252\gamma - 32\gamma^2 + 18\gamma^3\right) + A \left(-126 + 50\gamma + 9\gamma^2 - 4\gamma^3\right)\\
&= \frac{A}{3 - \gamma} \left(4\gamma^4 - 3\gamma^3 - 55\gamma^2 + 24\gamma + 184\right)
\end{aligned} \tag{2.34}$$

Since $\gamma \in [-1, 1]$, from (2.34) we get that (Numerator) > 0. This means $\frac{\partial}{\partial a} W^* (A, \gamma) > 0$.

(ii) Using (2.31) and routine computations we get that

$$\frac{\partial}{\partial \gamma} W^* (A, \gamma) = \frac{\begin{pmatrix} 4A^2\gamma^3 - 68A^2\gamma + 4Aa\gamma^4 - 18Aa\gamma^3 \\ -54Aa\gamma^2 + 360Aa\gamma - 400Aa - 9a^2\gamma^4 \\ +32a^2\gamma^3 + 162a^2\gamma^2 - 868a^2\gamma + 1008a^2 \end{pmatrix}}{9\left(\gamma^2 - 8\right)^3}. \tag{2.35}$$

Since $\gamma \in [-1, 1]$ the denominator of (2.35) is strictly negative. Consequently, $\frac{\partial}{\partial \gamma} W^* (A, \gamma) < 0 \Leftrightarrow$ (Numerator) > 0. Now note that

$$\begin{aligned} & \mathit{Numerator\ of}\ (2.35) \\ &= 4A^2\gamma\left(\gamma^2 - 68\right) + 2Aa\begin{pmatrix} 2\gamma^4 - 9\gamma^3 \\ -27\gamma^2 + 180\gamma - 200 \end{pmatrix} + a^2\begin{pmatrix} -9\gamma^4 + 32\gamma^3 \\ +162\gamma^2 - 868\gamma + 1008 \end{pmatrix} \\ &= 4A^2\gamma\left(\gamma^2 - 68\right) + a\begin{pmatrix} -9\gamma^4 + 32\gamma^3 \\ +162\gamma^2 - 868\gamma + 1008 \end{pmatrix}\left(a + \frac{2A\begin{pmatrix} 2\gamma^4 - 9\gamma^3 \\ -27\gamma^2 + 180\gamma - 200 \end{pmatrix}}{\begin{pmatrix} -9\gamma^4 + 32\gamma^3 \\ +162\gamma^2 - 868\gamma + 1008 \end{pmatrix}}\right) \end{aligned}$$

Computations show that $\frac{2\left(2\gamma^4 - 9\gamma^3 - 27\gamma^2 + 180\gamma - 200\right)}{\left(-9\gamma^4 + 32\gamma^3 + 162\gamma^2 - 868\gamma + 1008\right)} \geq -\frac{2}{5} = -0.40$. To demonstrate this we plot this expression below for $\gamma \in [-1, 1]$ (Fig. 2.1).

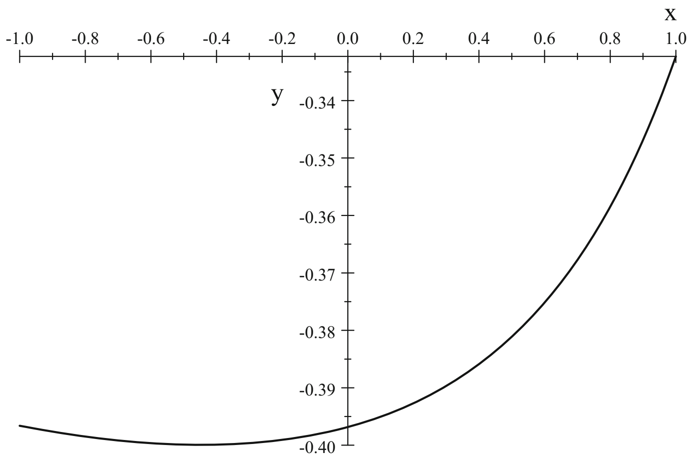

Fig. 2.1 $\frac{2\left(2\gamma^4 - 9\gamma^3 - 27\gamma^2 + 180\gamma - 200\right)}{\left(-9\gamma^4 + 32\gamma^3 + 162\gamma^2 - 868\gamma + 1008\right)}$

Now note that since $\gamma \in [-1, 1]$ we have $\gamma^2 - 68 < 0$. Since $\frac{2(2\gamma^4 - 9\gamma^3 - 27\gamma^2 + 180\gamma - 200)}{(-9\gamma^4 + 32\gamma^3 + 162\gamma^2 - 868\gamma + 1008)} \geq -\frac{2}{5}$ and $A < a(3 - \gamma)$ (by assumption) we get that

$$
\begin{aligned}
& \textit{Numerator of}\,(2.35) \\
&= 4A^2\gamma\left(\gamma^2 - 68\right) + a\begin{pmatrix} -9\gamma^4 + 32\gamma^3 \\ +162\gamma^2 - 868\gamma + 1008 \end{pmatrix}\left(a + \frac{2A\begin{pmatrix} 2\gamma^4 - 9\gamma^3 \\ -27\gamma^2 + 180\gamma - 200 \end{pmatrix}}{\begin{pmatrix} -9\gamma^4 + 32\gamma^3 \\ +162\gamma^2 - 868\gamma + 1008 \end{pmatrix}}\right) \\
&> 4A^2\gamma\left(\gamma^2 - 68\right) + a\left(-9\gamma^4 + 32\gamma^3 + 162\gamma^2 - 868\gamma + 1008\right)\left(a - \frac{2A}{5}\right) \\
&> 4a^2(3 - \gamma)^2\gamma\left(\gamma^2 - 68\right) + a\left(-9\gamma^4 + 32\gamma^3 + 162\gamma^2 - 868\gamma + 1008\right)\left(a - \frac{2a(3 - \gamma)}{5}\right) \\
&= a^2\left(4(3 - \gamma)^2\gamma\left(\gamma^2 - 68\right) + \left(-9\gamma^4 + 32\gamma^3 + 162\gamma^2 - 868\gamma + 1008\right)\left(1 - \frac{2(3 - \gamma)}{5}\right)\right) \\
&= a\left(\frac{2}{5}\gamma^5 - \frac{47}{5}\gamma^4 - \frac{888}{5}\gamma^3 + \frac{6262}{5}\gamma^2 - \frac{9356}{5}\gamma - \frac{1008}{5}\right) \qquad (2.36)
\end{aligned}
$$

We now plot $\left(\frac{2}{5}\gamma^5 - \frac{47}{5}\gamma^4 - \frac{888}{5}\gamma^3 + \frac{6262}{5}\gamma^2 - \frac{9356}{5}\gamma - \frac{1008}{5}\right)$ below over $\gamma \in [-1, 1]$ (Fig. 2.2).

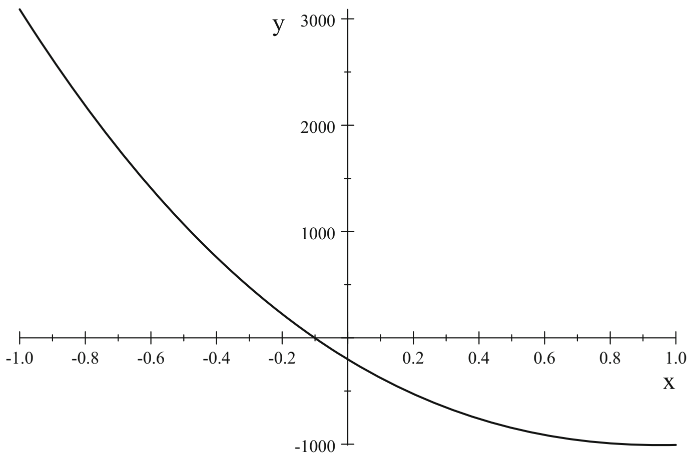

Fig. 2.2 $\frac{2}{5}\gamma^5 - \frac{47}{5}\gamma^4 - \frac{888}{5}\gamma^3 + \frac{6262}{5}\gamma^2 - \frac{9356}{5}\gamma - \frac{1008}{5}$

Using the plot above and routine computations we get that

$$
\begin{pmatrix} \frac{2}{5}\gamma^5 - \frac{47}{5}\gamma^4 - \frac{888}{5}\gamma^3 \\ +\frac{6262}{5}\gamma^2 - \frac{9356}{5}\gamma - \frac{1008}{5} \end{pmatrix} > 0 \iff \gamma < \gamma^* \approx -0.100836133705 \qquad (2.37)
$$

Using (2.36) and (2.37) we get $\gamma < \gamma^* \approx -0.100836133705 \Longrightarrow$ (Numerator) > 0. Consequently,

$$\gamma < \gamma^* \approx -0.100836133705 \Longrightarrow \frac{\partial}{\partial\gamma} W^*(A, \gamma) < 0 \tag{2.38}$$

Now we show that when $\gamma > \gamma^* \approx -0.100836133705$ the sign of $\frac{\partial}{\partial\gamma} W^*(A, \gamma)$ is ambiguous. Note that the numerator of $\frac{\partial}{\partial\gamma} W^*(A, \gamma)$ can be written as the following.

$$\begin{aligned} &\gamma^4 a (4A - 9a) + 2\gamma^3 \left(2A^2 - 9Aa + 16a^2\right) + 54\gamma^2 a (3a - A) \\ &+4\gamma \left(90Aa - 17A^2 - 217a^2\right) - 16a (25A - 63a) \end{aligned} \tag{2.39}$$

Let $a = 1$. This means any $A \in \left(\frac{1}{2}, 2\right)$ will satisfy both our assumptions. For illustrative purpose let $A = \frac{19}{10}$.

Then using these values in (2.39) we get

$$\begin{aligned} &\gamma^4 a (4A - 9a) + 2\gamma^3 \left(2A^2 - 9Aa + 16a^2\right) + 54\gamma^2 a (3a - A) \\ &+4\gamma \left(90Aa - 17A^2 - 217a^2\right) - 16a (25A - 63a) \\ = &-\frac{7}{5}\gamma^4 + \frac{306}{25}\gamma^3 + \frac{297}{5}\gamma^2 - \frac{10737}{25}\gamma + 248 \end{aligned} \tag{2.40}$$

Computations show that

$$-\frac{7}{5}\gamma^4 + \frac{306}{25}\gamma^3 + \frac{297}{5}\gamma^2 - \frac{10\,737}{25}\gamma + 248 < 0 \Leftrightarrow \gamma > 0.6412856 \tag{2.41}$$

To demonstrate this we plot $-\frac{7}{5}\gamma^4 + \frac{306}{25}\gamma^3 + \frac{297}{5}\gamma^2 - \frac{10\,737}{25}\gamma + 248$ below over $\gamma \in [-1, 1]$ (Fig. 2.3).

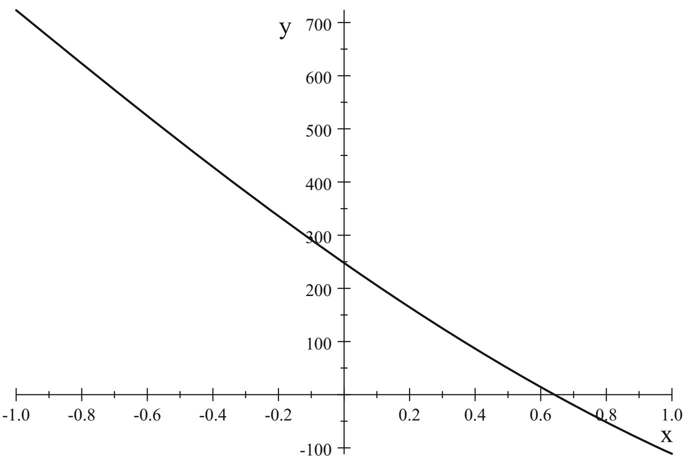

Fig. 2.3 $-\frac{7}{5}\gamma^4 + \frac{306}{25}\gamma^3 + \frac{297}{5}\gamma^2 - \frac{10\,737}{25}\gamma + 248$

The above means that when $a = 1$ and $A = \frac{19}{10}$, the numerator of $\frac{\partial}{\partial \gamma} W^* (A, \gamma) < 0$ $\Leftrightarrow \gamma > 0.6412856$. Consequently, we claim that the sign of $\frac{\partial}{\partial \gamma} W^* (A, \gamma)$ is ambiguous as it depends on the value of γ. ∎

Comment It may be noted that 'a' is proxy for market size of market 1. Given our assumptions, it is routine to show that both $\pi^*_W (.)$ and $\pi^*_L (.)$ is strictly increasing in 'a'. Total output in market 1, $(x^* + y^*)$, is also increasing in a and output in market 2, q^*, is decreasing in a. This means, as 'a' increases, although consumer surplus contracts in market 2, this will be outweighed by higher total profits and higher consumer surplus in market 1. Consequently, total welfare will increase as 'a' increases. This is somewhat similar to the case where costs are not interrelated across markets.

Proposition 7 implies that when goods are substitutes, then the sign of $\frac{\partial}{\partial \gamma} W^* (A, \gamma)$ is ambiguous. This stands in contrast to the case where costs are not interrelated across markets.

Comparison of Two Policy Regimes

As noted before, the government can allocate the rights to operate in market 2 in either of the two following ways: (i) through a first-price auction or (ii) through a lottery where the government flips a coin and decides on the winner.

The Supreme Court judgement and the report by the Comptroller and Auditor General of India, as discussed in the introduction, seem to suggest that the expected equilibrium welfare is always higher when the government allocates the rights through a first-price auction. *We will show that this need not be the case when costs are interrelated across markets.*

Earlier we showed that if $\frac{\partial}{\partial A} W^* (A, \gamma) > 0$ then $Exp \left[W^*_{auction} (A, \gamma) \right] \geq Exp \left[W^*_{lottery} (A, \gamma) \right]$. In (2.31) we have already derived the expression for $W^* (A, \gamma)$. It is straightforward to compute that

$$\frac{\partial}{\partial A} W^* (A, \gamma) = \frac{\left(306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3\right)}{9 \left(8 - \gamma^2\right)^2} \tag{2.42a}$$

$$\frac{\partial^2}{\partial A^2} W^* (A, \gamma) = \frac{\left(4\gamma^4 - 68\gamma^2 + 306\right)}{9 \left(8 - \gamma^2\right)^2} \tag{2.42b}$$

Since $\gamma \in [-1, 1]$ we must have $\frac{\partial^2}{\partial A^2} W^* (A, \gamma) > 0$. Note that the sign of $\frac{\partial}{\partial A} W^* (A, \gamma)$ is the same as that of $\left(306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3\right)$. Now

$$\begin{aligned} &306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3 \\ &= 2A \left(-34\gamma^2 + 2\gamma^4 + 153\right) + a \left(50\gamma + 9\gamma^2 - 4\gamma^3 - 126\right) \end{aligned} \tag{2.43}$$

Since $\gamma \in [-1, 1]$ it is clear that

$$-34\gamma^2 + 2\gamma^4 + 153 > 0 \text{ and } 50\gamma + 9\gamma^2 - 4\gamma^3 - 126 < 0 \tag{2.44}$$

From Assumption 1 we get that $A(3 - |\gamma|) > a$. We had earlier argued that this implies $A(3 - \gamma) > a$ regardless of whether the goods are substitutes ($\gamma > 0$) or complements ($\gamma < 0$). Since $-34\gamma^2 + 2\gamma^4 + 153 > 0$ and $\gamma \in [-1, 1]$ this in turn means

$$\begin{aligned}
&306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3 \\
&= 2A\left(-34\gamma^2 + 2\gamma^4 + 153\right) + a\left(50\gamma + 9\gamma^2 - 4\gamma^3 - 126\right) \\
&> \frac{a}{3-\gamma}\left(-34\gamma^2 + 2\gamma^4 + 153\right) + a\left(50\gamma + 9\gamma^2 - 4\gamma^3 - 126\right) \\
&= \frac{a}{3-\gamma}\left(8\gamma^4 - 21\gamma^3 - 91\gamma^2 + 276\gamma - 72\right) \\
&\geq 0 \iff 8\gamma^4 - 21\gamma^3 - 91\gamma^2 + 276\gamma - 72 > 0
\end{aligned} \tag{2.45}$$

Note that

$$\begin{aligned}
&\frac{\partial}{\partial\gamma}\left(8\gamma^4 - 21\gamma^3 - 91\gamma^2 + 276\gamma - 72\right) \\
&= 32\gamma^3 - 63\gamma^2 - 182\gamma + 276 > 0 \text{ for all } \gamma \in [-1, 1]
\end{aligned} \tag{2.46}$$

We now plot $8\gamma^4 - 21\gamma^3 - 91\gamma^2 + 276\gamma - 72$ over the range $[-1, 1]$ below (Fig. 2.4).

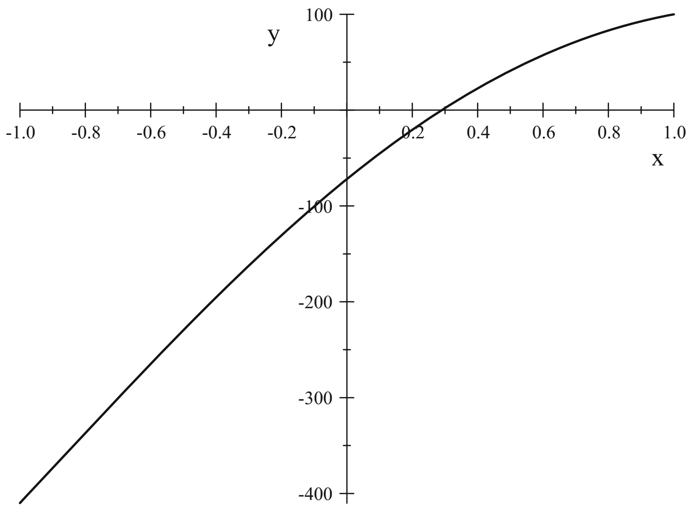

Fig. 2.4 $8\gamma^4 - 21\gamma^3 - 91\gamma^2 + 276\gamma - 72$

Using the plot above and routine computations we can show that for $\gamma \in [-1, 1]$

$$8\gamma^4 - 21\gamma^3 - 91\gamma^2 + 276\gamma - 72 > 0 \Longleftrightarrow \gamma > 0.2903141117 \tag{2.47}$$

This means $\gamma > 0.2903141117 \Longrightarrow 306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3 > 0$. Using this fact in our previous discussion we get that $\gamma > 0.2903141117 \Longrightarrow \frac{\partial}{\partial A} W^* (A, \gamma) > 0$. We use this result to state our next main result.

Proposition 8 *When costs are interrelated across markets, there exists* $\underline{\gamma} \approx 0.2903141117$ *such that*

$$\gamma > \underline{\gamma} \Longrightarrow Exp\left[W^*_{auction}(A, \gamma)\right] \geq Exp\left[W^*_{lottery}(A, \gamma)\right].$$

Hence, in the first-stage, if $\gamma > \underline{\gamma}$ *then in equilibrium the government allocates the right to operate in market 2 through an auction.*

Comment The proof of the above proposition follows from the earlier discussion. The intuition behind this is as follows. It may be noted that from Proposition 8 we get if the products are sufficiently strong substitutes (γ is positive and high enough). Then $Exp\left[W^*_{auction}(A, \gamma)\right] \geq Exp\left[W^*_{lottery}(A, \gamma)\right]$. Note that when costs are interrelated across markets (see (2.42a)),

$$\frac{\partial}{\partial A} W^* (A, \gamma) = \frac{\left(306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3\right)}{9\left(8 - \gamma^2\right)^2} \tag{2.48}$$

Using (2.48) and routine computations we get that

$$\frac{\partial^2}{\partial A \partial \gamma} W^* (A, \gamma) = \frac{2\left(200a + 68A\gamma - 180a\gamma - 4A\gamma^3 + 27a\gamma^2 + 9a\gamma^3 - 2a\gamma^4\right)}{9\left(8 - \gamma^2\right)^3} \tag{2.49}$$

Now note that

$$\begin{aligned} &200a + 68A\gamma - 180a\gamma - 4A\gamma^3 + 27a\gamma^2 + 9a\gamma^3 - 2a\gamma^4 \\ &= a\left(200 - 180\gamma + 27\gamma^2 + 9\gamma^3 - 2\gamma^4\right) + 4A\gamma\left(17 - \gamma^2\right) \end{aligned} \tag{2.50}$$

Since $\gamma \in [-1, 1]$, $a > 0$ and $A > 0$ we get that $\left(200 - 180\gamma + 27\gamma^2 + 9\gamma^3 - 2\gamma^4\right) > 0$ and $17 - \gamma^2 > 0$. Using these information in (2.50) we get

$$\begin{aligned} \gamma > 0 &\Longrightarrow 200a + 68A\gamma - 180a\gamma - 4A\gamma^3 + 27a\gamma^2 + 9a\gamma^3 - 2a\gamma^4 > 0 \\ &\Longrightarrow \frac{\partial^2}{\partial A \partial \gamma} W^* (A, \gamma) > 0 \end{aligned} \tag{2.51}$$

We noted earlier that allocating through auction implies that the firm with the highest type (highest A) gets the right to operate in market 2. If the right was allocated through a lottery, the highest type would win with probability half. This means the expected market size in market 2 would be higher when the right is allocated through an auction rather than by a lottery. When $\gamma > \underline{\gamma}$ both $\frac{\partial}{\partial A} W^* (A, \gamma)$ and $\frac{\partial^2}{\partial A \partial \gamma} W^* (A, \gamma)$ are strictly positive. Consequently, expected total welfare would be higher when the expected market size in market 2 (expected value of A) is higher.

Note that $\gamma > \underline{\gamma}$ is a *sufficient* condition for the expected equilibrium welfare to be *higher* when the government allocates the rights of market 2 through an auction. This means the supreme court judgment is economically sound for the case of interrelated costs where the goods are substitutes and the degree of differentiation, γ, is not too low.

Clearly, a *necessary* condition for the expected equilibrium welfare to be lower when the government allocates the rights of market 2 through an auction is $\gamma \leq \underline{\gamma}$. This means the products have to be either complements ($\gamma < 0$) or not very strong substitutes (γ positive but low enough). We now provide two examples, one with substitutes and the other with complements, to show that it is possible to have a scenario where the expected equilibrium welfare is strictly lower when the government allocates the rights of access to market 2 through an auction. In such cases, in equilibrium the government will allocate the right to operate in market 2 through lottery.

Example 1 (*substitutes*) Let $\gamma = \frac{1}{10}$, $a = 1$, $\underline{A} = 0.35$ and $\bar{A} = 0.36$. The types A_1 and A_2 are identically and independently distributed over $[0.35, 0.36]$ with uniform density. Here $f(t) = 100$ and $F(t) = 100t - 35$. Consequently, $F_{(1)}(t) = (100t - 35)^2$ and $f_{(1)}(t) = 200(100t - 35)$. It may be noted that both our assumptions are satisfied in this example. In (2.31) we had earlier computed the equilibrium total welfare to be as follows.

$$W^* (A, \gamma) = \frac{\begin{pmatrix} 2A^2\gamma^4 - 34A^2\gamma^2 + 153A^2 - 4Aa\gamma^3 + 9Aa\gamma^2 + 50Aa\gamma \\ -126Aa + 9a^2\gamma^3 - 16a^2\gamma^2 - 126a^2\gamma + 281a^2 \end{pmatrix}}{9\left(8 - \gamma^2\right)^2} \tag{2.52}$$

Using the values of γ and a we get

$$W^* \left(A, \frac{1}{10}\right) = \frac{1526\,602}{5745\,609} A^2 - \frac{1209\,140}{5745\,609} A + \frac{2682\,490}{5745\,609} \tag{2.53}$$

Expected welfare when the right is allocated through an auction:

$$\begin{aligned}
&\int_{\underline{A}}^{\bar{A}} W^*\left(A, \frac{1}{10}\right) f_{(1)}(A)\, dA \\
&= \int_{0.35}^{0.36} 200\left(\frac{1526602}{5745\,609}A^2 - \frac{1209140}{5745\,609}A + \frac{2682490}{5745\,609}\right)(100A - 35)\, dA \\
&= 0.42562
\end{aligned} \tag{2.54}$$

Expected welfare when the right is allocated randomly:

$$\begin{aligned}
&\int_{\underline{A}}^{\bar{A}} W^*\left(A, \frac{1}{10}\right) f(A)\, dA \\
&= \int_{0.35}^{0.36} 100\left(\frac{1526602}{5745\,609}A^2 - \frac{1209140}{5745\,609}A + \frac{2682490}{5745\,609}\right) dA \\
&= 0.42566
\end{aligned} \tag{2.55}$$

Using (2.54) and (2.55) we get that in this example

$$Exp\left[W^*_{auction}\left(A, \frac{1}{10}\right)\right] < Exp\left[W^*_{lottery}\left(A, \frac{1}{10}\right)\right].$$

Here in equilibrium, in the first stage the government will allocate the right to operate in market 2 through lottery.

Example 2 (*complements*) Let $\gamma = -\frac{9}{10}$, $a = 1$, $\underline{A} = \frac{1}{2}$ and $\bar{A} = \frac{3}{5}$. The types A_1 and A_2 are identically and independently distributed over $\left[\frac{1}{2}, \frac{3}{5}\right]$ with uniform density. Here $f(t) = 10$ and $F(t) = 10t - 5$. Consequently, $F_{(1)}(t) = (10t - 5)^2$ and $f_{(1)}(t) = 20\,(10t - 5)$. Again, both our assumptions are satisfied in this example.

Here we have

$$W^*\left(A, -\frac{9}{10}\right) = \frac{140858}{516\,961}A^2 - \frac{178660}{516\,961}A + \frac{3748790}{4652\,649} \tag{2.56}$$

Expected welfare when the right is allocated through an auction:

$$\begin{aligned}
&\int_{\underline{A}}^{\bar{A}} W^*\left(A, -\frac{9}{10}\right) f_{(1)}(A)\, dA \\
&= \int_{\frac{1}{2}}^{\frac{3}{5}} 20\left(\frac{140858}{516\,961}A^2 - \frac{178\,660}{516\,961}A + \frac{3748790}{4652\,649}\right)(10A - 5)\, dA \\
&= 0.69754
\end{aligned} \tag{2.57}$$

Expected welfare when the right is allocated randomly:

$$\begin{aligned}&\int_{\underline{A}}^{\bar{A}} W^*\left(A, -\frac{9}{10}\right) f\,(A)\, dA\\ &= \int_{\frac{1}{2}}^{\frac{3}{5}} 10\left(\frac{140\,858}{516\,961}A^2 - \frac{178\,660}{516\,961}A + \frac{3748\,790}{4652\,649}\right) dA\\ &= 0.6983\end{aligned} \tag{2.58}$$

Using (2.57) and (2.58) in this example too we have

$$Exp\left[W^*_{auction}\left(A, -\frac{9}{10}\right)\right] < Exp\left[W^*_{lottery}\left(A, -\frac{9}{10}\right)\right].$$

Like the previous example, here too in equilibrium, the government will allocate the right to operate in market 2 through lottery.

Remark The two examples above clearly suggest that when costs are interrelated across markets (there are diseconomies of scope), then allocation the right though auction may not be a good strategy. Lottery may be a better option. We now try to provide a simple intuition behind this. Note that when costs are interrelated the equilibrium output levels we have already shown that the sign of $\frac{\partial}{\partial A} W^*\,(A, \gamma)$ is the same as that of $306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3$. Now

$$\begin{aligned}&306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3\\ &= 2A\left(-34\gamma^2 + 2\gamma^4 + 153\right) + a\left(50\gamma + 9\gamma^2 - 4\gamma^3 - 126\right)\end{aligned} \tag{2.59}$$

Since $\gamma \in [-1, 1]$ it is clear that

$$-34\gamma^2 + 2\gamma^4 + 153 > 0 \text{ and } 50\gamma + 9\gamma^2 - 4\gamma^3 - 126 < 0 \tag{2.60}$$

From (2.59) and (2.60) we get that if $\bar{A}$ (the highest possible type) is relatively lower as compared to a (market size of market 1) then $306A - 126a + 50a\gamma - 68A\gamma^2 + 4A\gamma^4 + 9a\gamma^2 - 4a\gamma^3$ would be negative. This would imply that $\frac{\partial}{\partial A} W^*\,(A, \gamma)$ is negative. While allocating the right through an auction leads to a higher expected market size in market 2 as compared to the case where the right is allocated through a lottery; this also means total expected welfare is like to be lower, especially when $\bar{A}$ is small, as a higher expected A decreases total expected welfare (as $\frac{\partial}{\partial A} W^*\,(A, \gamma) < 0$). Our two examples illustrate this simple point.

The key takeaway is that when there are diseconomies of scope (costs are interrelated across markets) and the potential market size of the new market (i.e. market 2) is not high, then allocating the right of access to this new market through auctions need not always fetch a better outcome as compared to the case when such a right is allocated randomly. Hence, in such cases, the observations of both the Supreme Court and of the Comptroller and Auditor General of India are economically not very sound.

2.5 Strict Capacity Constraints

A natural question that arises is the following: what happens when both firms have strict capacity constraints? Suppose both firms can produce upto k with a constant marginal cost, c, and cannot produce beyond that. We now proceed to investigate this case. We first provide our assumptions below.

Assumption 1 $\underline{A} > c$.

Assumption 2 $a > c$.

Assumption 3 Both firms have the same capacity, k where for all $A \in \left[\underline{A}, \bar{A}\right]$

$$\max\left\{\frac{(a-c)}{(\gamma+2)}, \frac{(A-c)}{2}\right\} < k < \frac{(A-c)}{2} + \frac{(a-c)}{(\gamma+2)}$$

Assumption 4 We assume that for all $A \in \left[\underline{A}, \bar{A}\right]$ the following two conditions hold.

$$(A-a)^2 - 2(a-c)^2 + k\left(4-\gamma^2\right)(A-k) + k\left(4a - c\left(8-\gamma^2\right)\right) > 0$$
$$k\left(4-\gamma^2\right) + 2(A-a) > 0$$

Remark It may be noted that Assumption 3 is required to make our analysis non-trivial. If $k \geq \frac{(A-c)}{2} + \frac{(a-c)}{(\gamma+2)}$, then the equilibrium outcome would be exactly the same as demonstrated for the case where costs are not interrelated across markets (constant marginal costs). If $k \leq \max\left\{\frac{(a-c)}{(\gamma+2)}, \frac{(A-c)}{2}\right\}$ then the winner would not be able to operate in both the markets. Hence, Assumption 3 ensures that the capacity, k, is sufficiently high enough to enable the winner to operate in both markets. Assumption 4 ensures that the winner gets a strictly higher gross payoff than the loser. Also, it ensures that the difference between the winner's and loser's payoff is increasing in the market size, A. All these will be clear when we provide the details.

2.5.1 Third Stage Equilibrium

As before, let the winner choose x in market 1 and q in market 2. The loser chooses y in market 1. Since types are revealed in this stage, let the type of the winner be A. The winner's gross payoff is

$$\pi_W = (A - q)\, q + (a - x - \gamma y)\, x - \frac{1}{2}\,(x + q)^2 \tag{2.61}$$

As discussed before, if the government chooses to allocate the right through an auction in the first stage, the winner's net payoff is π_W − (winner's bid). If there is no auction and the winner is decided by the toss of coin then the winner's net payoff is simply π_W. The loser's payoff is

$$\pi_L = (a - \gamma x - y)\, y - \frac{1}{2} y^2 \tag{2.62}$$

From routine computations we get the equilibrium values of q, x and y.

$$q^* = \frac{2A - 2a + 4k + a\gamma - c\gamma - k\gamma^2}{8 - \gamma^2} \tag{2.63a}$$

$$x^* = \frac{2a - 2A + 4k - a\gamma + c\gamma}{8 - \gamma^2} \tag{2.63b}$$

$$y^* = \frac{4a - 4c + A\gamma - a\gamma - 2k\gamma}{8 - \gamma^2} \tag{2.63c}$$

$$k = q^* + x^* \tag{2.63d}$$

We claim that since $A \in \left[\underline{A},\ \bar{A}\right]$ and $\gamma \in [-1, 1]$, our assumptions ensure that $q^*, x^*, y^* > 0$. First note that $8 - \gamma^2 > 0$. Note that

$$q^* > 0 \Longleftrightarrow 2A - 2a + 4k + a\gamma - c\gamma - k\gamma^2 > 0 \tag{2.64}$$

Now

$$\begin{aligned}
&2A - 2a + 4k + a\gamma - c\gamma - k\gamma^2 \\
&= 2A - 2a + \gamma\,(a - c) + k\left(4 - \gamma^2\right) \\
&> 2A - 2a + \gamma\,(a - c) + \frac{(a - c)}{2 + \gamma}\left(4 - \gamma^2\right) \text{ (using Assumption 3)} \\
&= 2A - 2a + \gamma\,(a - c) + (2 - \gamma)\,(a - c) \text{ (since } 4 - \gamma^2 = (2 - \gamma)\,(2 + \gamma)\text{)} \\
&= 2A - 2c > 0 \text{ (using Assumption 1)}
\end{aligned} \tag{2.65}$$

Equations (2.64) and (2.65) together imply $q^* > 0$. Now note that

$$x^* > 0 \Longleftrightarrow 2a - 2A + 4k - a\gamma + c\gamma \tag{2.66}$$

Now

$$\begin{aligned} & 2a - 2A + 4k - a\gamma + c\gamma \\ > & \ 2a - 2A + 2\,(A - c) - a\gamma + c\gamma \text{ (using Assumption 3)} \\ = & \ (a - c)\,(2 - \gamma) > 0 \text{ (using Assumption 2)} \end{aligned} \tag{2.67}$$

Equations (2.66) and (2.67) together imply $x^* > 0$. Lastly, note that

$$y^* > 0 \Longleftrightarrow 4a - 4c + A\gamma - a\gamma - 2k\gamma > 0 \tag{2.68}$$

Now

$$\begin{aligned} & 4a - 4c + A\gamma - a\gamma - 2k\gamma \\ = & \ 4a - 4c + A\gamma - a\gamma - 2k\gamma - c\gamma + c\gamma \\ = & \ (a - c)\,(4 - \gamma) + \gamma\,[(A - c) - 2k] \\ \geq & \ 3\,(a - c) + \gamma\,[(A - c) - 2k] \text{ (since } \gamma \leq 1 \text{ and } a > c) \end{aligned} \tag{2.69}$$

From Assumption 3 we get $(A - c) - 2k < 0$. If $\gamma \leq 0$, then $3\,(a - c) + \gamma\,[(A - c) - 2k] > 0$. That is,

$$\gamma \leq 0 \Longrightarrow 4a - 4c + A\gamma - a\gamma - 2k\gamma > 0 \tag{2.70}$$

Now

$$\begin{aligned} \gamma > 0 \Longrightarrow & \ 3\,(a - c) + \gamma\,[(A - c) - 2k] \\ \geq & \quad 3\,(a - c) + (A - c) - 2k (\text{since } \gamma \leq 1 \text{ and } (A - c) - 2k < 0) \\ = & \quad 2\left[\frac{3\,(a - c)}{2} + \frac{(A - c)}{2} - k\right] \\ > & \quad 2\left[\frac{(a - c)}{2 + \gamma} + \frac{(A - c)}{2} - k\right]\left(\text{since} \frac{3}{2} > \frac{1}{2 + \gamma}\right) \\ > & \quad 0 \text{ (using Assumption 3)} \end{aligned} \tag{2.71}$$

Equations (2.68)–(2.71) imply $y^* > 0$.

Using the values of q^*, x^* and y^* we get the equilibrium gross payoffs for the winner and loser, π_W^* and π_L^*, and also their difference, $\pi_W^* - \pi_L^*$.

$$\begin{aligned}
\pi_W &= \left(A - q^*\right) q^* + \left(a - x^* - \gamma y^*\right) x^* - c\left(x^* + q^*\right) \\
&= \left(A - k + x^*\right)\left(k - x^*\right) + \left(a - x^* - \gamma y^*\right) x^* - ck(\text{since } k = x^* + q^*) \\
&= Ak - Ax^* - ck + ax^* + 2kx^* - k^2 - 2x^{*2} - x^* y^* \gamma \\
&= \frac{\begin{pmatrix} 8A^2 + 8Aa\gamma - 16Aa - 8Ac\gamma + Ak\gamma^4 - 16Ak\gamma^2 \\ +32Ak + 2a^2\gamma^2 - 8a^2\gamma + 8a^2 - 4ac\gamma^2 + 8ac\gamma - 16ak\gamma \\ +32ak + 2c^2\gamma^2 - ck\gamma^4 + 16ck\gamma^2 + 16ck\gamma - 64ck \\ -k^2\gamma^4 + 16k^2\gamma^2 - 32k^2 \end{pmatrix}}{\left(8 - \gamma^2\right)^2}
\end{aligned} \tag{2.72}$$

$$\begin{aligned}
\pi_L &= \left(a - \gamma x^* - y^*\right) y^* - cy^* \\
&= \frac{(4c - 4a - A\gamma + a\gamma + 2k\gamma)^2}{\left(8 - \gamma^2\right)^2}
\end{aligned} \tag{2.73}$$

$$\begin{aligned}
\pi_W^*(A, \gamma) - \pi_L^*(A, \gamma) &= \frac{\begin{pmatrix} A^2 - 2Aa - Ak\gamma^2 + 4Ak - a^2 + 4ac \\ +4ak - 2c^2 + ck\gamma^2 - 8ck + k^2\gamma^2 - 4k^2 \end{pmatrix}}{8 - \gamma^2} \\
&= \frac{\begin{bmatrix} (A - a)^2 - 2(a - c)^2 \\ +k\left(4 - \gamma^2\right)(A - k) + k\left[4a - c\left(8 - \gamma^2\right)\right] \end{bmatrix}}{8 - \gamma^2}
\end{aligned} \tag{2.74}$$

Using Assumption 4 in (2.74) we get $\pi_W^*(A, \gamma) - \pi_L^*(A, \gamma) > 0$. Now note that from Assumption 4 we also get

$$\frac{\partial}{\partial A}\left(\pi_W^* - \pi_L^*\right) = \frac{k\left(4 - \gamma^2\right) + 2(A - a)}{8 - \gamma^2} > 0 \tag{2.75}$$

Also note that

$$\frac{\partial^2}{\partial A \partial \gamma}\left(\pi_W^* - \pi_L^*\right) = \frac{4\gamma(A - a - 2k)}{\left(8 - \gamma^2\right)^2} \geq 0 \Longleftrightarrow \gamma(A - a - 2k) \geq 0\partial \tag{2.76}$$

We summarize the preceding discussion in terms of a proposition.

Proposition 9 *When each firm has strict capacity constraint, k, and constant marginal cost, c, then in the third stage equilibrium* $\forall A \in \left[\underline{A}, \bar{A}\right]$ *and* $\forall \gamma \in [-1, 1]$ *we get the following. (i)* $\pi_W^* (A, \gamma) - \pi_L^* (A, \gamma) > 0$ *(ii)* $\frac{\partial}{\partial A} \left(\pi_W^* - \pi_L^*\right) > 0$ *and (iii)* $\frac{\partial^2}{\partial A \partial \gamma} \left(\pi_W^* - \pi_L^*\right) \geq 0 \Longleftrightarrow \gamma (A - a - 2k) \geq 0.$

Comment Since the winner gets a strictly higher gross payoff than the loser, in the second-stage there is an incentive to participate in the auction. Like in the previous two models the difference between the winner's and loser's payoff is increasing in the market size, A.

We now proceed to compute the second stage equilibrium.

2.5.2 Second Stage Equilibrium

Note that if the government decides to allocate the right randomly (by tossing a coin) each firm wins the right to operate in market 2 with probability half and no firm makes any payment. If the government decides to allocate the right through a first-price auction, then in the second stage the two firms bid to win the rights to sell in market 2.

As before, we can show that the equilibrium bidding strategy is as follows:

$$b (A_1) = \frac{1}{F (A_1)} \int_{\underline{A}}^{A_1} \left[\pi_W^* (t, \gamma) - \pi_L^* (t, \gamma)\right] f (t)\, dt \tag{2.77}$$

As before, it is easy to demonstrate that both firms will choose to participate in the auction regardless of their types. We summarize these results in terms of a proposition.

Proposition 10 *When each firm has strict capacity constraint, k, and constant marginal cost, c, if the government decides to allocate the right through an auction in the first stage, then both firms will choose to participate in the auction in the second-stage. There is a symmetric, strictly increasing Bayesian–Nash equilibrium where a firm with type A bids the following:*

$$b (A) = \frac{1}{F (A)} \int_{\underline{A}}^{A} \left[\frac{(t - a)^2 - 2 (a - c)^2 + k \left(4 - \gamma^2\right) (t - k) + k \left[4a - c \left(8 - \gamma^2\right)\right]}{8 - \gamma^2}\right] f (t)\, dt.$$

The most efficient firm, i.e. the firm with the highest type, wins the auction.

2.5.3 First-Stage Equilibrium

As noted before, in the first-stage the government chooses either an auction or a lottery as a means of allocating the right to operate in market 2. The government's payoff is total expected welfare. We now proceed to the analysis of total welfare when there are strict capacity constraints and firms can produce upto capacity with constant marginal cost, c.

Total Welfare

We now proceed to the analysis of total welfare. Routine computations show (see the previous section) that the total welfare (W) for either case (auction or toss of a coin) is as follows:

$$\begin{aligned}
&\text{Total consumer surplus} \;+\; \text{total profits} \;+\; \text{government revenue} \\
&= [CS_1 + CS_2] + [\pi_W(A,\gamma) - b(A) + \pi_L(A,\gamma)] + b(A) \\
&= [CS_1 + CS_2] + [\pi_W(A,\gamma) + \pi_L(A,\gamma)] \\
&= a(x+y) - x^2 - y^2 - q^2 - \gamma xy + Aq - xq
\end{aligned}$$

In equilibrium the winner chooses x^* and q^* and the loser chooses y^*. We had earlier derived these equilibrium values. Therefore, by using the equilibrium values we get the expression for equilibrium total welfare.

$$W^*(A,\gamma) = \frac{\begin{pmatrix} 12A^2 - A^2\gamma^2 - Aa\gamma^3 + 2Aa\gamma^2 + 12Aa\gamma - 24Aa - 4Ac\gamma \\ +Ak\gamma^4 - 14Ak\gamma^2 + 32Ak + a^2\gamma^3 - 2a^2\gamma^2 - 12a^2\gamma + 28a^2 \\ -2ac\gamma^2 + 4ac\gamma + ak\gamma^3 - 2ak\gamma^2 - 16ak\gamma + 32ak + 3c^2\gamma^2 \\ -16c^2 + ck\gamma^3 - k^2\gamma^4 + 16k^2\gamma^2 - 48k^2 \end{pmatrix}}{\left(8-\gamma^2\right)^2} \tag{2.78}$$

$$\frac{\partial}{\partial A}\left(W^*(A,\gamma)\right) = \frac{\begin{bmatrix} 24A - 24a + 32k + 12a\gamma - 4c\gamma \\ -2A\gamma^2 + 2a\gamma^2 - a\gamma^3 - 14k\gamma^2 + k\gamma^4 \end{bmatrix}}{\left(8-\gamma^2\right)^2} \tag{2.79a}$$

$$\frac{\partial}{\partial a}\left(W^*(A,\gamma)\right) = \frac{\begin{bmatrix} -24A + 56a + 32k + 12A\gamma - 24a\gamma \\ +4c\gamma - 16k\gamma + 2A\gamma^2 - A\gamma^3 - 4a\gamma^2 \\ +2a\gamma^3 - 2c\gamma^2 - 2k\gamma^2 + k\gamma^3 \end{bmatrix}}{\left(8-\gamma^2\right)^2} \tag{2.79b}$$

We now state our next result which follows from (2.79a) and (2.79b).

Proposition 11 *When each firm has strict capacity constraint, k, and constant marginal cost, c, then we get the following. (i)* $\underline{A} \geq 2a \Longrightarrow \frac{\partial}{\partial A} W^*(A, \gamma) > 0$ *and (ii)* $32a - 33\bar{A} \geq 0 \Longrightarrow \frac{\partial}{\partial a} W^*(A, \gamma) > 0$.

Proof (i) Since $\left(8 - \gamma^2\right)^2 > 0$ the sign of $\frac{\partial W^*}{\partial A}$ is the same as the sign of the numerator in (2.79a). Note that

$$\text{Numerator in (79a)} = \begin{bmatrix} -24a + 24A + 12a\gamma - 4c\gamma - 2A\gamma^2 \\ +2a\gamma^2 - a\gamma^3 + k\left(32 - 14\gamma^2 + \gamma^4\right) \end{bmatrix} \tag{2.80}$$

Since $k > \frac{a-c}{\gamma+2}$ (Assumption 3) and since $32 - 14\gamma^2 + \gamma^4 > 0$ (as $-1 \leq \gamma \leq 1$) we get

$$\begin{aligned} \text{Numerator in (79a)} &> \begin{bmatrix} -24a + 24A + 12a\gamma - 4c\gamma - 2A\gamma^2 \\ +2a\gamma^2 - a\gamma^3 + \left(\frac{(a-c)}{(\gamma+2)}\right)\left(32 - 14\gamma^2 + \gamma^4\right) \end{bmatrix} \\ &= \frac{\begin{pmatrix} -16a + 48A - 32c + 24A\gamma - 8c\gamma \\ -4A\gamma^2 - 2A\gamma^3 + 2a\gamma^2 + 10c\gamma^2 - c\gamma^4 \end{pmatrix}}{\gamma + 2} \end{aligned} \tag{2.81}$$

Note that since $A > c$ and $32 + 8\gamma - 10\gamma^2 + \gamma^4 > 0$ (as $-1 \leq \gamma \leq 1$) we get

$$\begin{aligned} & -16a + 48A - 32c + 24A\gamma - 8c\gamma - 4A\gamma^2 - 2A\gamma^3 + 2a\gamma^2 + 10c\gamma^2 - c\gamma^4 \\ &= -2a\left(8 - \gamma^2\right) + 2A\left(12 - \gamma^2\right)(2 + \gamma) - c\left(32 + 8\gamma - 10\gamma^2 + \gamma^4\right) \\ &> -2a\left(8 - \gamma^2\right) + 2A\left(12 - \gamma^2\right)(2 + \gamma) - A\left(32 + 8\gamma - 10\gamma^2 + \gamma^4\right) \\ &= \left(8 - \gamma^2\right)(A\gamma(\gamma + 2) + 2A - 2a) \end{aligned} \tag{2.82}$$

Note that the minimum value of $\gamma(\gamma + 2)$ over the range $[-1, 1]$ is obtained at $\gamma = -1$ and the minimum value is -1. This means

$$\left(8 - \gamma^2\right)(A\gamma(\gamma + 2) + 2A - 2a) > \left(8 - \gamma^2\right)(A - 2a) \tag{2.83}$$

Now $\underline{A} \geq 2a \Longrightarrow \forall A \in \left[\underline{A}, \bar{A}\right]$, we have $A \geq 2a$. From (2.83) this implies $\left(8 - \gamma^2\right)$ $(A\gamma(\gamma + 2) + 2A - 2a) > 0$. From (2.80) to (2.83) we get that the Numerator in (2.79a) is strictly positive. This means if $\underline{A} \geq 2a$ then $\frac{\partial W^*}{\partial A} > 0$.

(ii) Since $\left(8 - \gamma^2\right)^2 > 0$ the sign of $\frac{\partial W^*}{\partial a}$ is the same as the sign of the numerator in (2.79b). Note that since $k > \frac{(a-c)}{(\gamma+2)}$ (Assumption 3) and since $32 - 16\gamma - 2\gamma^2 + \gamma^3 > 0$ (as $-1 \leq \gamma \leq 1$) we have

$$\text{Numerator in (2.79b)} = \begin{bmatrix} -24A + 56a + 12A\gamma - 24a\gamma + 4c\gamma + 2A\gamma^2 - A\gamma^3 \\ -4a\gamma^2 + 2a\gamma^3 - 2c\gamma^2 + k\left(32 - 16\gamma - 2\gamma^2 + \gamma^3\right) \end{bmatrix}$$
$$> \begin{bmatrix} -24A + 56a + 12A\gamma - 24a\gamma + 4c\gamma + 2A\gamma^2 - A\gamma^3 \\ -4a\gamma^2 + 2a\gamma^3 - 2c\gamma^2 + \left(\frac{(a-c)}{(\gamma+2)}\right)\left(32 - 16\gamma - 2\gamma^2 + \gamma^3\right) \end{bmatrix}$$
$$= \frac{\begin{bmatrix} -48A + 144a - 32c - 8a\gamma + 24c\gamma + 16A\gamma^2 \\ -A\gamma^4 - 34a\gamma^2 + a\gamma^3 + 2a\gamma^4 + 2c\gamma^2 - 3c\gamma^3 \end{bmatrix}}{\gamma + 2} \tag{2.84}$$

Note that since $a > c$ and $(2-\gamma)\left(16 - 4\gamma - 3\gamma^2\right) > 0$ (as $-1 \le \gamma \le 1$) we have

$$\begin{aligned} &\begin{bmatrix} -48A + 144a - 32c - 8a\gamma + 24c\gamma + 16A\gamma^2 \\ -A\gamma^4 - 34a\gamma^2 + a\gamma^3 + 2a\gamma^4 + 2c\gamma^2 - 3c\gamma^3 \end{bmatrix} \\ &= \begin{bmatrix} a\left(8-\gamma^2\right)\left(18-\gamma-2\gamma^2\right) - A(2-\gamma)(2+\gamma)\left(12-\gamma^2\right) \\ -c(2-\gamma)\left(16-4\gamma-3\gamma^2\right) \end{bmatrix} \\ &> \begin{bmatrix} a\left(8-\gamma^2\right)\left(18-\gamma-2\gamma^2\right) - A(2-\gamma)(2+\gamma)\left(12-\gamma^2\right) \\ -a(2-\gamma)\left(16-4\gamma-3\gamma^2\right) \end{bmatrix} \\ &= (\gamma+2)\left(-24A + 56a + 12A\gamma - 20a\gamma + 2A\gamma^2 - A\gamma^3 - 6a\gamma^2 + 2a\gamma^3\right) \\ &= (\gamma+2)\left[2a\left(28 - 10\gamma - 3\gamma^2 + \gamma^3\right) - A(2-\gamma)\left(12-\gamma^2\right)\right] \end{aligned} \tag{2.85}$$

It may be noted that the functions $28 - 10\gamma - 3\gamma^2 + \gamma^3$ and $(2-\gamma)\left(12-\gamma^2\right)$ are strictly decreasing in γ over the range $[-1, 1]$. To demonstrate this we plot the two functions in Fig. 2.5 over the range $[-1, 1]$. $\left(28 - 10\gamma - 3\gamma^2 + \gamma^3\right)$ is plotted with solid line and $(2-\gamma)\left(12-\gamma^2\right)$ is plotted with dashed lines.

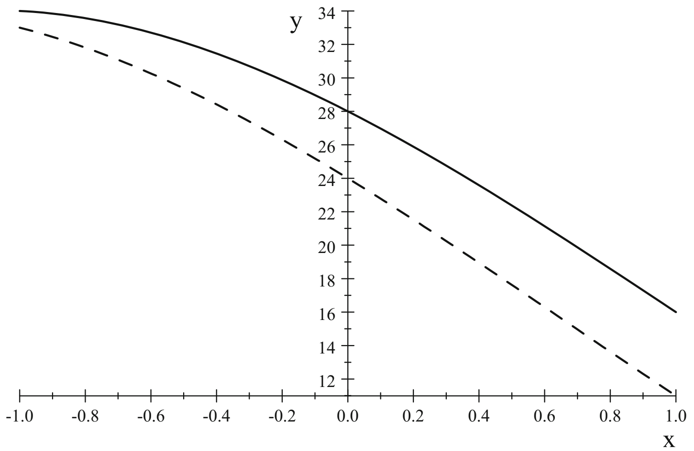

Fig. 2.5 $28 - 10\gamma - 3\gamma^2 + \gamma^3$ and $(2-\gamma)\left(12-\gamma^2\right)$

This means the minimum value of $28 - 10\gamma - 3\gamma^2 + \gamma^3$ is obtained at $\gamma = 1$ and the minimum value is 16. Similarly the maximum value of $(2 - \gamma)\left(12 - \gamma^2\right)$ is obtained at $\gamma = -1$ and the maximum value is 33. Therefore, from above we get

$$2a\left(28 - 10\gamma - 3\gamma^2 + \gamma^3\right) - A(2 - \gamma)\left(12 - \gamma^2\right) > 32a - 33A \tag{2.86}$$

From (2.84) to (2.86) it is clear that if $32a \geq 33\bar{A}$ then the Numerator in (2.79b) is strictly positive. This means $32a \geq 33A \Longrightarrow \frac{\partial W^*}{\partial a} > 0$. ■

Comment It may be noted that 'a' and A are proxy for market sizes of market 1 and 2 respectively. Proposition 11 shows that if the size of market 2 is very high relative to the size of market 1 then $\frac{\partial}{\partial A} W^*(A, \gamma) > 0$. In both the previous models, where either costs are not interrelated across markets (constant MC with no capacity constraint) or when costs are quadratic, we got that total welfare will increase as 'a' increases. When each firm has strict capacity constraint, k,we get that if the size of market 1 is high relative to the size of market 2 then $\frac{\partial}{\partial a} W^*(A, \gamma) > 0$. This result is different from the previous two cases.

Comparison of Two Policy Regimes

As noted before, the government can allocate the rights to operate in market 2 in either of the two following ways: (i) through a first-price auction or (ii) through a lottery where the government flips a coin and decides on the winner.

Earlier we showed that if $\frac{\partial}{\partial A} W^*(A, \gamma) > 0$ then $Exp\left[W^*_{auction}(A, \gamma)\right] \geq Exp\left[W^*_{lottery}(A, \gamma)\right]$. From Proposition 11 we know that $\underline{A} \geq 2a \Longrightarrow \frac{\partial}{\partial A} W^*(A, \gamma) > 0$. We now state our next main result.

Proposition 12 *When each firm has strict capacity constraint, k, and constant marginal cost, c,*

$$\underline{A} \geq 2a \Longrightarrow Exp\left[W^*_{auction}(A, \gamma)\right] \geq Exp\left[W^*_{lottery}(A, \gamma)\right].$$

Hence, in the first-stage, if $\underline{A} \geq 2a$ then in equilibrium the government allocates the right to operate in market 2 through an auction.

Comment Note that A is a proxy for market size in market 2. Proposition 12 shows that if the size of market 2 is high enough relative to market 2 then it is better to allocate the right to operate in market 2 through an auction. However, when the size of market 2 is relatively low then this need not be the case. We now provide two examples similar to the ones used before; one with substitutes and the other with complements, to show that it is possible to have a scenario where the expected equilibrium welfare is strictly lower when the government allocates the rights of access to market 2 through an auction.

Example 1 (*substitutes*) Let $\gamma = \frac{1}{10}$, $a = 1$, $c = 0.1$, $\underline{A} = 0.35$ and $\bar{A} = 0.36$. Also, let $k \in (0.42857, 0.55357)$. The types A_1 and A_2 are identically and independently distributed over $[0.35, 0.36]$ with uniform density. Here $f(t) = 100$ and $F(t) =$

$100t - 35$. Consequently, $F_{(1)}(t) = (100t - 35)^2$ and $f_{(1)}(t) = 200\,(100t - 35)$. We now show that all our assumptions are satisfied in the example. First note that

$$\max\left\{\frac{a-c}{\gamma+2}, \frac{\bar{A}-c}{2}\right\} = \frac{a-c}{\gamma+2} = \frac{1-0.1}{0.1+2} = 0.42857$$
$$\frac{a-c}{\gamma+2} + \frac{\underline{A}-c}{2} = 0.42857 + \frac{0.35-0.1}{2} = 0.55357$$

Since $0.42857 < k < 0.55357$, Assumption 3 is satisfied. Assumption 4 states that

$$(A-a)^2 - 2\,(a-c)^2 + k\left(4-\gamma^2\right)(A-k) + k\left(4a - c\left(8-\gamma^2\right)\right) > 0$$
$$k\left(4-\gamma^2\right) + 2\,(A-a) > 0$$

Now here since $A \in [0.35, 0.36]$ and $\bar{A} = 0.36 < a = 1$ we have

$$\begin{aligned}
&(A-a)^2 - 2\,(a-c)^2 + k\left(4-\gamma^2\right)(A-k) + k\left(4a - c\left(8-\gamma^2\right)\right)\\
&> \left(\bar{A}-a\right)^2 - 2\,(a-c)^2 + k\left(4-\gamma^2\right)\left(\underline{A}-k\right) + k\left(4a - c\left(8-\gamma^2\right)\right)\\
&= (0.36-1)^2 - 2\,(1-0.1)^2 + k\,(4-0.01)\,(0.35-k) + k\,(4-0.1\,(8-0.01))\\
&= -3.99k^2 + 4.597\,5k - 1.2104
\end{aligned} \tag{2.87}$$

In Fig. 2.6 we plot $-3.99k^2 + 4.5975k - 1.2104$ over the range [0.40, 0.55]. It is clear that $-3.99k^2 + 4.5975k - 1.2104 > 0$ over the range [0.428, 0.553].

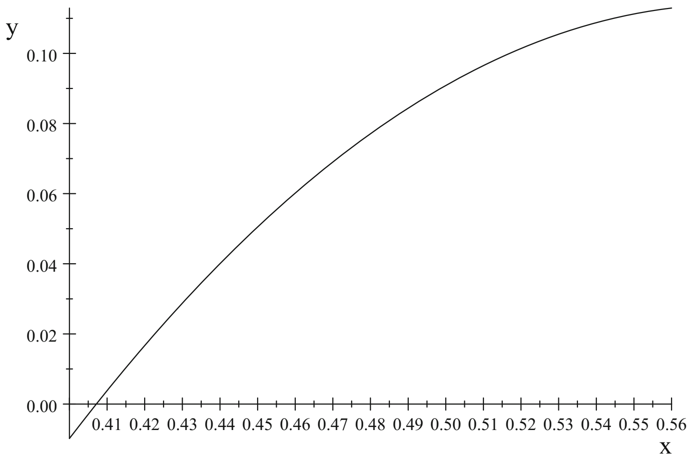

Fig. 2.6 $-3.99k^2 + 4.5975k - 1.2104$

Also,

$$\begin{aligned} k\left(4-\gamma^2\right)+2\left(A-a\right) &> k\left(4-\gamma^2\right)+2\left(\underline{A}-a\right) \\ &= k\left(4-0.01\right)+2\left(0.35-1\right) \\ &= 3.99k-1.3>0 \text{ for all } k>0.428 \end{aligned} \tag{2.88}$$

Therefore, Assumption 4 is satisfied in our example. Now routine computations yield that

$$\begin{aligned} W^*\left(A,\frac{1}{10}\right) &= 0.187\,81A^2+0.499\,06Ak-0.357\,47A-0.749\,37k^2+0.475\,89k \\ &\quad +0.417\,59 \end{aligned} \tag{2.89}$$

Hence, expected welfare when the right is allocated through an auction:

$$\begin{aligned} &\int_{\underline{A}}^{\bar{A}} W^*\left(A,\frac{1}{10}\right) f_{(1)}\left(A\right)dA \\ &= \int_{0.35}^{0.36} 200\begin{pmatrix} 0.187\,81A^2+0.499\,06Ak-0.357\,47A \\ -0.749\,37k^2+0.475\,89k+0.417\,59 \end{pmatrix}\left(100A-35\right)dA \\ &= -0.74937k^2+0.65389k+0.31398 \end{aligned} \tag{2.90}$$

Expected welfare when the right is allocated randomly:

$$\begin{aligned} &\int_{\underline{A}}^{\bar{A}} W^*\left(A,\frac{1}{10}\right) f\left(A\right)dA \\ &= \int_{0.35}^{0.36} 100\begin{pmatrix} 0.187\,81A^2+0.499\,06Ak-0.357\,47A \\ -0.749\,37k^2+0.475\,89k+0.417\,59 \end{pmatrix}dA \\ &= -0.74937k^2+0.65306k+0.31436 \end{aligned} \tag{2.91}$$

Using (2.90) and (2.91) it is now easy to show that

$$\int_{\underline{A}}^{\bar{A}} W^*\left(A,\frac{1}{10}\right) f\left(A\right)dA > \int_{\underline{A}}^{\bar{A}} W^*\left(A,\frac{1}{10}\right) f_{(1)}\left(A\right)dA \Longleftrightarrow k<0.45783$$

Note that in our example, $k \in (0.42857, 0.55357)$. This implies

$$k \in (0.42857, 0.45783) \Longrightarrow Exp\left[W^*_{auction}\left(A,\frac{1}{10}\right)\right] < Exp\left[W^*_{lottery}\left(A,\frac{1}{10}\right)\right].$$

Here in equilibrium, in the first stage the government will allocate the right to operate in market 2 through lottery.

Example 2 (*complements*) Let $\gamma=-\frac{9}{10}$, $a=1$, $c=0.25$, $\underline{A}=\frac{1}{2}$ and $\bar{A}=\frac{3}{5}=0.6$. Let $k \in (0.68182, 0.688)$. The types A_1 and A_2 are identically and independently distributed over $\left[\frac{1}{2}, \frac{3}{5}\right]$ with uniform density. Here $f(t) = 10$ and $F(t) = 10t - 5$. Consequently, $F_{(1)}(t) = (10t - 5)^2$ and $f_{(1)}(t) = 20(10t - 5)$.

We now show that all our assumptions are satisfied in the example.

$$\max\left\{\frac{a-c}{\gamma+2}, \frac{\bar{A}-c}{2}\right\} = \frac{a-c}{\gamma+2} = \frac{1-0.25}{-0.9+2} = 0.68182$$
$$\frac{a-c}{\gamma+2} + \frac{\underline{A}-c}{2} = 0.681\,82 + \frac{0.5-0.25}{2} = 0.80682$$

Since $k \in (0.68182, 0.688)$ clearly Assumption 3 is satisfied. Assumption 4 states that

$$(A-a)^2 - 2(a-c)^2 + k\left(4-\gamma^2\right)(A-k) + k\left(4a - c\left(8-\gamma^2\right)\right) > 0$$
$$k\left(4-\gamma^2\right) + 2(A-a) > 0$$

Now here since $A \in [0.5, 0.6]$ and $\bar{A} = 0.6 < a = 1$ we have

$$\begin{aligned}
&(A-a)^2 - 2(a-c)^2 + k\left(4-\gamma^2\right)(A-k) + k\left(4a - c\left(8-\gamma^2\right)\right)\\
&> \left(\bar{A}-a\right)^2 - 2(a-c)^2 + k\left(4-\gamma^2\right)\left(\underline{A}-k\right) + k\left(4a - c\left(8-\gamma^2\right)\right)\\
&= (0.6-1)^2 - 2(1-0.25)^2 + k(4-0.81)(0.5-k) + k(4 - 0.25(8-0.01))\\
&= -3.19k^2 + 3.597\,5k - 0.965
\end{aligned} \tag{2.92}$$

In Fig. 2.7 we plot $-3.19k^2 + 3.597\,5k - 0.965$ over the interval [0.680, 0.689].

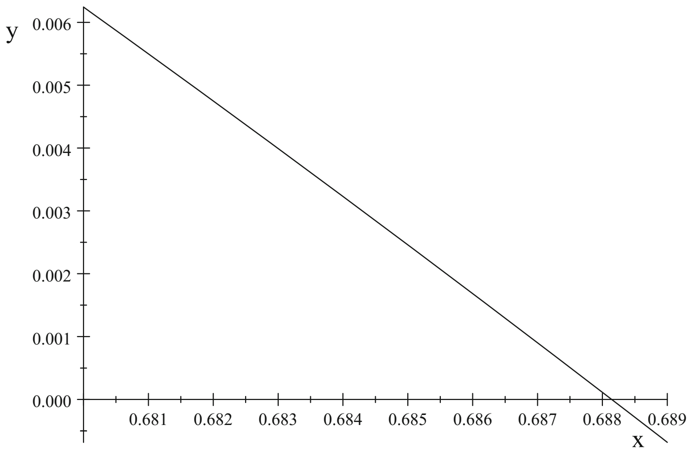

Fig. 2.7 $-3.19k^2 + 3.597\,5k - 0.965$

It is clear that $-3.19k^2 + 3.597\,5k - 0.965 > 0$ over the range $(0.68182, 0.688)$. Routine computations show that $-3.19k^2 + 3.597\,5k - 0.965 < 0$ for $k > 0.68814$.

Also,

$$\begin{aligned} k\left(4-\gamma^2\right) + 2\left(A-a\right) &> k\left(4-\gamma^2\right) + 2\left(\underline{A}-a\right) \\ &= k\left(4-0.81\right) + 2\left(0.5-1\right) \\ &= 3.\,19k - 1.0 > 0 \text{ for all } k > 0.68 \end{aligned}$$

Therefore, Assumption 4 is satisfied in our example.

Now routine computations yield that

$$W^*\left(A, -\frac{9}{10}\right) = 0.21646A^2 + 0.41233Ak - 0.61032A - 0.69050k^2 + 0.84859k + 0.66345 \tag{2.93}$$

Expected welfare when the right is allocated through an auction:

$$\begin{aligned} &\int_{\underline{A}}^{\bar{A}} W^*\left(A, -\frac{9}{10}\right) f_{(1)}\left(A\right) dA \\ &= \int_{\frac{1}{2}}^{\frac{3}{5}} 20 \begin{pmatrix} 0.216\,46A^2 + 0.412\,33Ak - 0.610\,32A \\ -0.690\,50k^2 + 0.848\,59k + 0.663\,45 \end{pmatrix} \left(10A - 5\right) dA \\ &= -0.6905k^2 + 1.0822k + 0.38723 \end{aligned} \tag{2.94}$$

Expected welfare when the right is allocated randomly:

$$\begin{aligned} &\int_{\underline{A}}^{\bar{A}} W^*\left(A, -\frac{9}{10}\right) f\left(A\right) dA \\ &= \int_{\frac{1}{2}}^{\frac{3}{5}} 10 \begin{pmatrix} 0.216\,46A^2 + 0.412\,33Ak - 0.610\,32A \\ -0.690\,50k^2 + 0.848\,59k + 0.663\,45 \end{pmatrix} dA \\ &= -0.6905k^2 + 1.0754k + 0.39343 \end{aligned} \tag{2.95}$$

Using (2.94) and (2.95) it is now easy to show that

$$\int_{\underline{A}}^{\bar{A}} W^*\left(A, \frac{1}{10}\right) f\left(A\right) dA > \int_{\underline{A}}^{\bar{A}} W^*\left(A, \frac{1}{10}\right) f_{(1)}\left(A\right) dA \Longleftrightarrow k < 0.91176$$

Note that in our example, $k \in (0.68182, 0.688)$. This implies

$$k \in (0.68182, 0.688) \Longrightarrow Exp\left[W^*_{auction}\left(A, \frac{1}{10}\right)\right] < Exp\left[W^*_{lottery}\left(A, \frac{1}{10}\right)\right].$$

Here also in equilibrium, in the first stage the government will allocate the right to operate in market 2 through lottery.

2.6 Conclusion

Any policy decision regarding allocation of rights to firms to use scarce resources to serve new markets is a very serious one. It would appear from the recent observations of the Supreme Court of India (and also from the report of the 'Comptroller and Auditor General' of India) that auctions should be preferred over other options as a means of allocating the right to scarce natural resources. In fact, of late the Government of India has been allocating scarce resources like spectrum or coal through auctions only.[10]

In this chapter we analyzed a model to discuss possible options in allocating a scarce resource in an emerging economy like India. We did so in the context of a multimarket oligopoly. We demonstrated that when there are diseconomies of scope (costs are interrelated across markets) or when there are strict capacity constraints, then allocating the right of access to the new market through auctions need not always fetch a better outcome as compared to the case when such a right is allocated through a lottery. That is, allocations through auctions need not lead to higher total welfare. This is especially true when the size of market 2 is relatively small. This has serious policy implications. In a country like India, often new markets (market 2 in our model) are in smaller towns and their sizes tend to be small. Also, most firms that participate in bidding operate in other markets as well. In our exercise we have tried to capture this multimarket aspect.

Note that when the right is allocated through an auction a firm with the highest type wins just as in a symmetric independent private value model. When the right is allocated randomly, the firm with the highest type wins with probability half. It would appear that allocative efficiency is not ensured when the right is allocated randomly. That is, total welfare would be higher if the right is allocated through an auction. It may however be noted here this would be indeed true if there was a single market. Since, in our exercise we have *multimarket* oligopoly, when costs are quadratic or when there are strict capacity constraints, the choice of output in one market affects the strategic decisions of the players in the other market. Consequently, total welfare (which indicates efficiency) need not be higher when the right is allocated through an auction.

We know that market quality is essentially a multidimensional concept: it encompasses efficiency, fairness and the principle of non-discrimination. Since we assumed that there is no corruption in allocation (whether in auction or in the random allocation), "fairness in pricing" is automatically guaranteed. Since both auctions and lottery are non-discriminatory in our set-up, the 'principle of non-discrimination'

[10]The "common good" which was mentioned by the Supreme Court in its order may be interpreted to be total welfare (consumer surplus plus producer surplus plus government revenue).

is automatically satisfied. Consequently, in our model, total welfare is a very good proxy for market quality.

The following research questions arise.

1. With corruption, will our results still hold? It may appear that auctions are less prone to corruption than lotteries are (The Indian experience seems to suggest that). However, political interference even in the auction process (example, politician deciding on the rules of the auction) is possible. We know that random allocation processes have suffered from corruption (the first-come-first-serve basis allocation in 2G spectrum case in India is an example of that). Factoring in corruption in our model, that lead to 'unfair allocations' would be an interesting exercise.
2. Will the results change if instead of Cournot competition there is Bertrand competition in market 1 in the third stage? There are papers in oligopoly theory that show that equilibrium outcomes depend crucially on the nature of competition (Cournot or Bertrand). It would be interesting to recast this exercise with price competition in the third stage.[11]

[11] See Vives (1999) for a succinct summary of the classic results around this point. Dastidar (2015a) and Alipranti et al. (2014) provide some recent results. Also, see Dastidar (1996, 1997).

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