Chapter 2
Basics for a Single-Degree-of-Freedom Rotor

Abstract This chapter specifies the definitions, calculation and measurement of basic vibration properties: natural frequency, modal damping, resonance and Q-value (Q-factor).

Basic properties featuring in a vibrating system, which are obtained from the free vibration waveform, are:

– Natural frequency f_n [Hz], or natural angular (or, circular) frequency $\omega_n = 2\pi f_n$ [rad/s]
– Damping ratio ζ [–], or logarithmic decrement $\delta = 2\pi\zeta$ [–]

Using these parameters, the resonance caused by forced excitation can be predicted with

– Resonance frequency (critical speed in unbalanced vibration) = natural frequency f_n [Hz]
– Resonance sensitivity $Q = 1/(2\zeta)$ [–]

Since separation of resonance or reduction of the Q-value are fundamental requirements in the vibration design of rotating machinery, the placements of a natural frequency and the damping ratio are very important design indices.

Keywords Single-dof · Natural frequency · Damping ratio · Equivalent mass · Bode plot · Nyquest plot · Q-value

2.1 Free Vibrations

2.1.1 Natural Frequency

Undamped free vibration in a single-dof (degree-of-freedom) system consisting of a mass m [kg] and a spring constant k [N/m] is featured by the natural angular (circular) frequency ω_n, given by

$$\omega_n = \sqrt{k/m}\,[\mathrm{rad/s}] \tag{2.1}$$

O. Matsushita et al., *Vibrations of Rotating Machinery*,
Mathematics for Industry 16, DOI 10.1007/978-4-431-55456-1_2

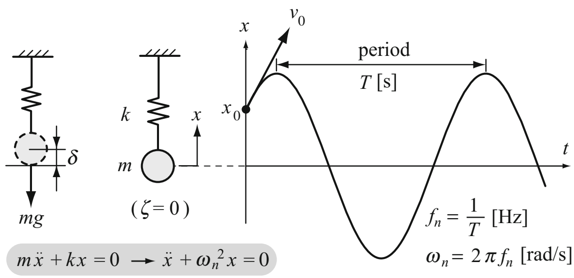


Fig. 2.1 (Undamped) Free vibration wave form

which can be converted to the natural frequency by

$$f_n = \omega_n/(2\pi) \text{ [Hz]} \tag{2.2}$$

The term "natural frequency" will be also used hereafter for the accurate term "natural circular frequency" according to the convention in industry (Fig. 2.1).

2.1.2 *Calculation of Spring Constant*

The spring constant k [N/m] is the reciprocal of deflection per unit load, and is determined from a static deflection calculation involving the strength of the material. Several formulae [9] to obtain spring constants are summarized in Table 2.1.

Table 2.1 Example of spring constant [9]

shaft system	shaft system
(1) cantilever: $k = \frac{3EI}{l^3}$ (a) circular cross-section: $I = \pi d^4/64$ (b) rectangular cross-section: $I = bh^3/12$	(3) thrust of bar: $k_a = \frac{EA}{l}$
	(4) simple supported beam at both sides: $k = \frac{3EIl}{l_1^2 l_2^2}$ $l_1 = l_2 = l/2$ cantilever $k = \frac{48EI}{l^3}$
(2) torsion of bar: $k_t = \frac{GJ}{l}$ (a) $J = \frac{\pi d^4}{32}$ (b) $J = \left[\frac{1}{3} - 0.2\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right] ab^3$	(5) overhang: $k = \frac{3EI}{a^2(l+a)}$

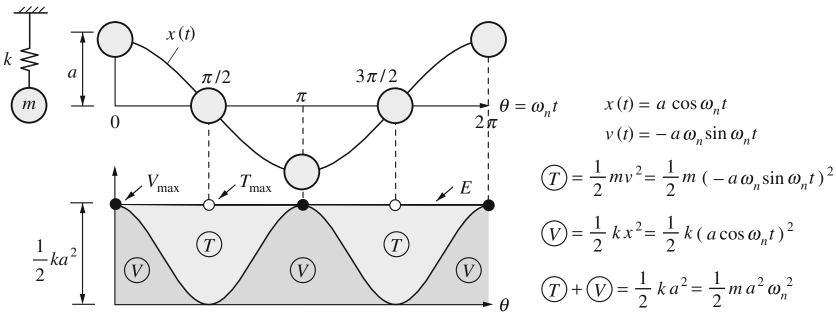


Fig. 2.2 The conservation of energy

2.1.3 Conservation of Energy

A system of moving objects is called conservative if the sum E of the kinetic energy T and potential energy (strain energy) V of the objects remains constant, i.e., $T + V =$ constant. Figure 2.2 shows that the kinetic energy and potential energy in a conservative system is complementary and their sum is thus constant. The maximum kinetic energy $T_{\max}$ is equal to the maximum potential energy $V_{\max}$:

$$T_{\max} = V_{\max} \tag{2.3}$$

This relationship determines the natural circular frequency of the system:

$$2T_{\max} = m(a\omega_n)^2,\ 2V_{\max} = ka^2 \rightarrow \omega_n = \sqrt{k/m} \tag{2.4}$$

Since free vibration is a movement with respect to a static equilibrium point and may be termed as dynamic behavior, the energy changes are represented as the deviations from the energy at that point. The equilibrium point in Fig. 2.2 is the point where gravity balances the reaction of the spring via the expansion of the spring. Therefore, the waveform of the free vibration under the gravity would remain the same even under zero gravity.

2.1.4 Mass Effects of Spring on Natural Frequency

Calculation of natural frequency is straightforward under an ideal condition of a massless spring. In actual cases, however, the natural frequency is lower than the ideal value due to the added mass of the spring. Ignoring this effect in design is dangerous because it often results in optimistic solutions having higher natural frequency than in reality.

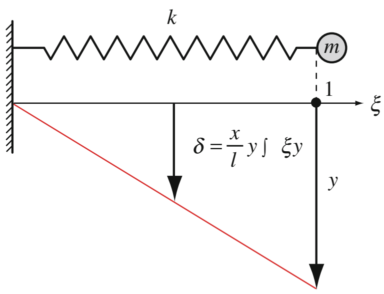


Fig. 2.3 Example of spring mass effect

In the mass m and spring k system shown in Fig. 2.3, let m_s be the mass of the coil spring and y the displacement of the tip of the spring. The displacement δ of any part of the spring can be represented by the linear equation

$$\delta = (x/l)y \equiv \xi y \tag{2.5}$$

The kinetic energy of the system is the sum of that for the mass at the tip and the distributed mass of the spring mass with a line density (mass per unit length) ρ_l:

$$T = \frac{m}{2}\dot{y}^2 + \frac{1}{2}\int_0^1 \rho_1 \dot{\delta}^2 dx = \dot{y}^2\left(\frac{m}{2} + \frac{m_s}{2}\int_0^1 \xi^2 d\xi\right) = \frac{1}{2}\left(m + \frac{m_s}{3}\right)\dot{y}^2 \tag{2.6}$$

Therefore, the formula for natural angular frequency ω_n becomes

$$\omega_n = \sqrt{\frac{k}{m + m_s/3}} \tag{2.7}$$

Thus, one third of the spring mass is added to reduce the natural frequency.

Example 2.1 Figure 2.4 shows several examples of added mass effect, when considering the spring stiffness of a uniform bar. Confirm the factor as the added mass of spring for each case.

Note: (a) $\delta(\xi) = \xi^2(3 - \xi)2y$, (b) $\delta(\xi) = \xi y$, (c) $\delta(\xi) = \left[\xi\left(3 - 4\xi^2\right) - (1 - 2\xi)^3 U(\xi - 12)\right]y$, (d) $\delta(\xi) = 4\left[(3 - 4\xi)\xi^2 - (1 - 2\xi)^3 U(\xi - 12)\right]y$,
where the step function is $U(t) = 0$ for $t < 0$ and $U(t) = 1$ for $t \geq 0$.

Example 2.2 Figure 2.5 shows a cantilever, l in length and m_s in mass. Find the equivalent mass m_{eq} of the point located at a distance al from the supporting point for $a = \{1, 0.9, 0.8, 0.7\}$.

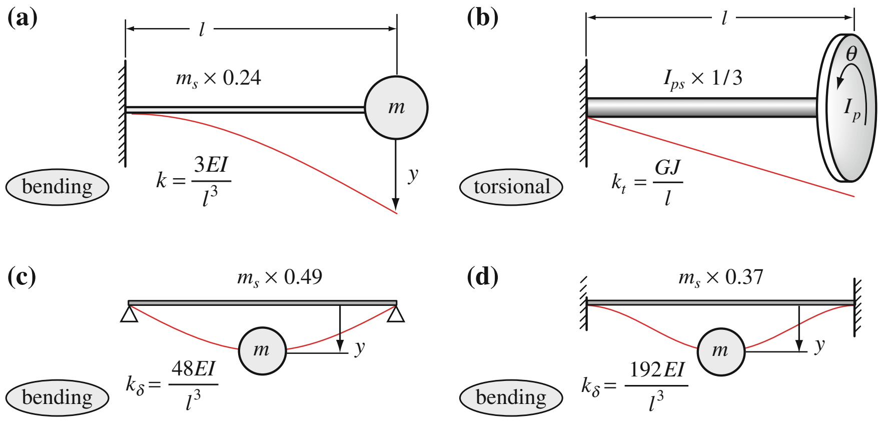
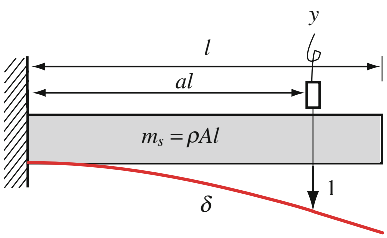


Fig. 2.4 Added mass of spring (equivalent mass)

Fig. 2.5 Deflection curve



Answer

If the displacement is unity at the point al, the deflection curve δ is defined:

$$\delta = \frac{\xi^2(3a-\xi)}{2a^3} + \frac{(\xi-a)^3}{2a^3}U(\xi-a) \quad \left(\xi = \frac{x}{l}\right) \tag{2.8}$$

The deflection curve is third order up to the point al and linear beyond it. The equivalent mass m_{eq} is given by

$$m_{eq} = m_s \int_0^1 \delta^2 d\xi \tag{2.9}$$

Thus m_{eq} = {0.25, 0.33, 0.47, 0.70}m_s for the conditions given.

2.2 Damped Free Vibration

2.2.1 Mass-Spring-Viscous Damped System

If a viscous damping coefficient c [Nm/s] is added to a mass-spring system as shown in Fig. 2.6, the system becomes non-conservative and its vibration is observed as a damped vibration waveform. For example, the impulse response is given by

$$x(t) = a\,e^{-\zeta\omega_n t}\sin qt \tag{2.10}$$

The damping is evaluated by the damping ratio ζ, defined as the viscous damping coefficient c [Ns/m] divided by the critical viscous damping coefficient $c_c = 2\sqrt{mk}$ [Ns/m]:

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} = \frac{c}{2m\omega_n}[-] \tag{2.11}$$

Figure 2.7 compares damped free vibration waveforms for different damping ratios.

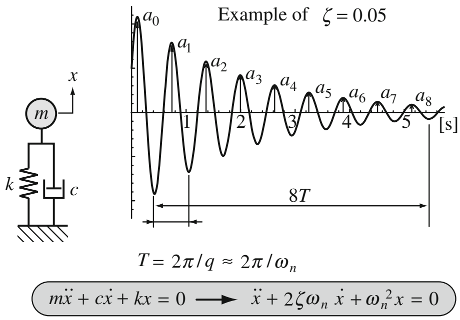


Fig. 2.6 Damped vibration system

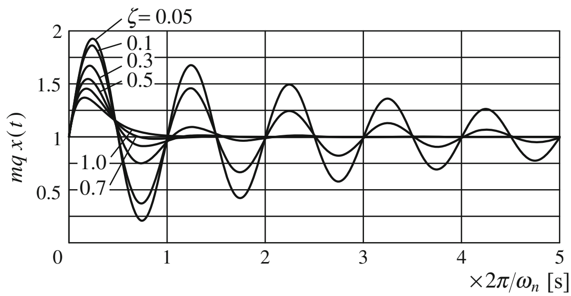


Fig. 2.7 Damped vibration waveform in impulse test

The frequency of the damped free vibration is the damped natural angular frequency q given by

$$q = \omega_n \sqrt{1 - \zeta^2} \tag{2.12}$$

which is a value close to the undamped natural angular frequency ω_n when the damping ratio is not too large.

If Eq. (2.10) is rewritten using the characteristic root (complex eigenvalue) λ as

$$x(t) = a\,\mathrm{Im}[e^{\lambda t}] \tag{2.13}$$

the characteristic root is given by solving the characteristic equation $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$:

$$\lambda \equiv \alpha \pm jq = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \tag{2.14}$$

In design practice, the characteristic root $\lambda = \alpha + jq$ is first obtained by complex eigenvalue analysis of a multi-degree-of-freedom system representing the rotor system. The vibration characteristics (natural circular frequency ω_n and damping ratio ζ) are then calculated from the characteristic root:

$$\begin{aligned} \omega_n &= \sqrt{\alpha^2 + q^2} = |\lambda| \\ \zeta &= -\alpha/|\lambda| \end{aligned} \tag{2.15}$$

The system is stable when the real part of the complex eigenvalue $\alpha = -\zeta\omega_n$ is negative, and unstable if it is positive. A waveform for each case is shown in Fig. 2.8.

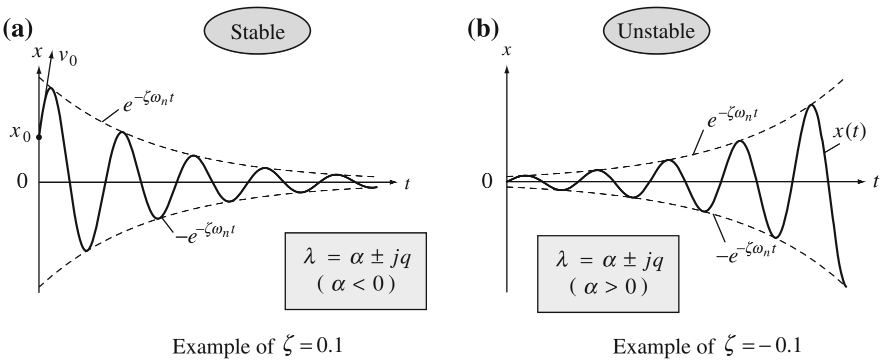


Fig. 2.8 Stable/unstable system

2.2.2 Measurement of Damping Ratio

The damping ratio ζ is an important parameter (dimensionless) in a vibrating system as well as the natural circular frequency ω_n. It can be estimated by observing the damped free vibration waveform. The amplitude envelope $\vec{A}$ of damped vibration (2.10) is represented by

$$\vec{A}(t) = e^{-\zeta\omega_n t} \tag{2.16}$$

Since the frequency of the vibration is q and the period $2\pi/q$, as indicated in Fig. 2.6,

$$\vec{A}(0) = 1, \quad \vec{A}(2\pi/q) = e^{-\zeta\omega_n 2\pi/q} \tag{2.17}$$

The ratio of the two terms is

$$\vec{A}(0)/\vec{A}(2\pi/q) = e^{\zeta\omega_n 2\pi/q} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{2.18}$$

This means that the ratio of amplitudes in consecutive cycles is a function of damping ratio ζ only. The latter can therefore be identified by calculating the former.

The series of peak heights a_1, a_2, a_3, …, measured for consecutive cycles as shown in Fig. 2.6, is geometric:

$$1 < \frac{a_0}{a_1} = \frac{a_1}{a_2} = \frac{a_2}{a_3} = \cdots = \frac{a_n}{a_{n+1}} = \cdots = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}} \tag{2.19}$$

The damping ratio is obtained by taking the natural logarithms of each term, considering that is generally small enough to be approximated as follows::

$$\text{Logarithmic decrement}: \delta = \ln\frac{a_n}{a_{n+1}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \approx 2\pi\zeta \tag{2.20}$$

$$\text{Damping ratio}: \zeta = \frac{\delta}{2\pi} = \frac{1}{2\pi}\ln\frac{a_n}{a_{n+1}} \tag{2.21}$$

In practice, a semi-logarithmic plot is constructed for the series of peak heights a_i measured for consecutive cycles as shown in Fig. 2.9a, and the damping ratio ζ is calculated from the gradient m_a of the straight line fitted to the plot as follows:

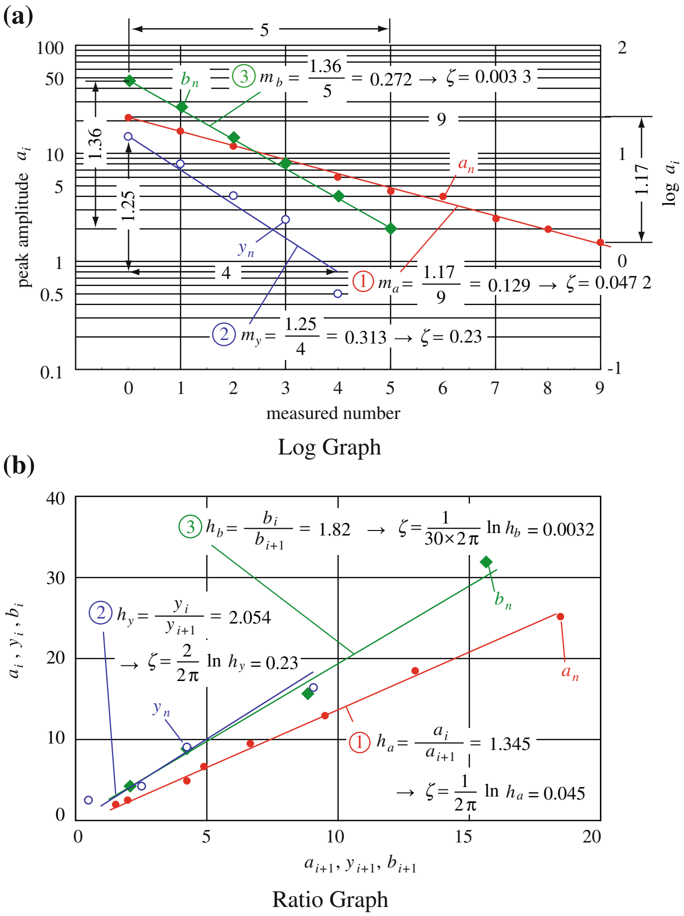


Fig. 2.9 Data plot

$$\zeta = \frac{\log(a_n/a_{n+1})}{2\pi \log e} = \frac{m_a}{2.73} \tag{2.22}$$

where m_a is the gradient for a cycle. Note that the vertical axis $\log a_i$ is indicated on the right ordinate of Fig. 2.9a, hence

$$m_a = \log a_n - \log a_{n+1} = \frac{\log a_0 - \log a_n}{n}$$

In Fig. 2.9b, another description of damped waveforms is drawn, where the ordinate axis indicates a_i and the abscissa axis a_{i+1}. Since the slope is $h \equiv a_n/a_{n+1}$, the damping ratio ζ is then given by

$$\zeta = \frac{\delta}{2\pi} = \frac{1}{2\pi}\ln\frac{a_n}{a_{n+1}} = \frac{1}{2\pi}\ln h \tag{2.23}$$

To improve the accuracy, the average value of the slope is recommended, which is known by the straight-line fitting. Note that it is not necessary for this straight line to pass through the origin, because the offset from the origin depends on the friction effect.

Example 2.3 Find the damping ratio by measuring the peak amplitudes in Fig. 2.6 with a ruler.

Answer

The peak amplitude readings for each cycle are: 21.5, 16, 11.5, 8.5, …, which are plotted as ① in Fig. 2.9. In Fig. 2.9a, the gradient m_a of the line fitted is 1.17/9 = 0.129, leading to an identified value $\zeta = 0.129/2.73 = 0.047$.

In the case of Fig. 2.9b, the slope is approximated as h_a = 1.345, thus giving $\delta = \ln(h_a) = 0.296 \rightarrow \zeta = \delta/(2\pi) = 0.0472$.

Example 2.4 Find the damping ratio for a strongly damped vibration waveform shown in Fig. 2.10 by reading peak heights y_0, y_1, y_2, … for consecutive half cycles.

Answer

The peak amplitude readings for each half cycle are: 14.5, 8, 4, 2.5, …, which are plotted as ② in Fig. 2.9. In Fig. 2.9a, the gradient m_a of the line fitted is 1.25/(4/2) = 0.63, leading to an identified value $\zeta = 0.63/2.73 = 0.23$.

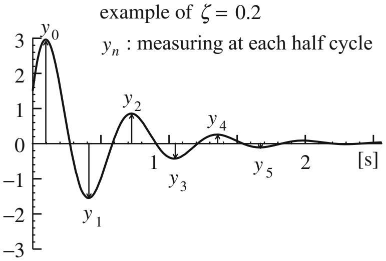
In the case of Fig. 2.9b, the slope is approximated as h_y = 2.054, thus giving $\delta = 2\ln(h_a) = 1.44 \rightarrow \zeta = \delta/(2\pi) = 0.23$.

Example 2.5 Find the damping ratio for a poorly damped vibration shown in Fig. 2.11 by reading the amplitude envelope height b_1, b_2, b_3, … at a constant time interval Δt [s] in (a) of the figure and determine the natural frequency f_n using the magnified time domain waveform in (b).

Answer

The magnified graph (b) gives f_n = 30 Hz. The readings of amplitude envelope height at an interval of t = 1 s in graph (a) are: 46, 26, 14, 8, …, which are plotted as ③ in Fig. 2.9.

Fig. 2.10 Large damping vibration waveform



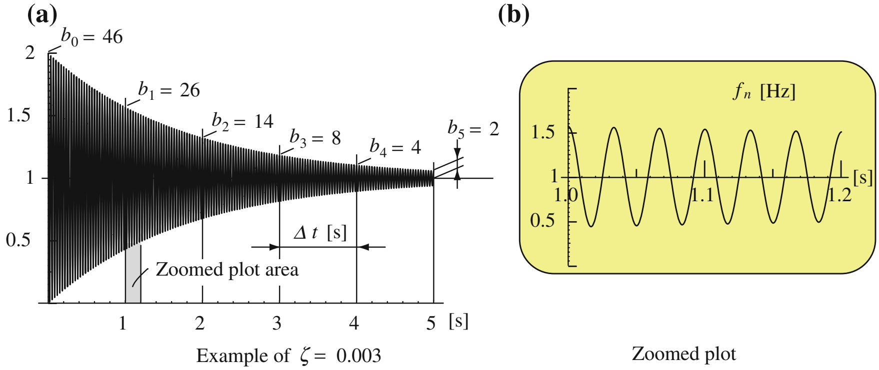
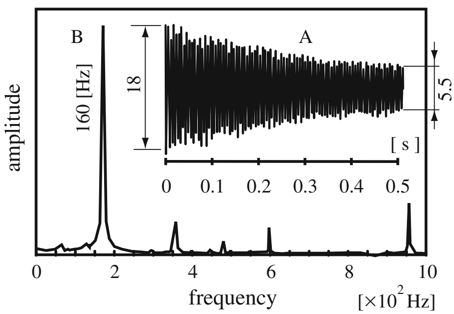


Fig. 2.11 Small damping vibration waveform

Fig. 2.12 Estimation of damping ratio



In Fig. 2.9a, the gradient $m_a = 1.36/(5 \times \Delta t \times f_n) = 0.009$ gives an identified value $\zeta = 0.009/2.73 = 0.00033$.

In the case of Fig. 2.9b, the slope is approximated as $h_b = 1.82$, thus giving $\delta = \ln(h_a)/30 = 0.02 \rightarrow \zeta = \delta/(2\pi) = 0.0032$.

Example 2.6 Find the damping ratio of the system using its impulse response waveform A and FFT analysis result B in Fig. 2.12.

Answer

The amplitude (p-p) $2a_0 = 18$ at 0 s and $2a_1 = 5.5$ at $\Delta t = 0.5$ s, and the natural frequency $f_n = 160$ Hz are read. The damping ratio is identified as follows:

$$\delta = \frac{1}{n}\ln\left(\frac{a_0}{a_1}\right) = \frac{1}{\Delta t f_n}\ln\left(\frac{a_0}{a_1}\right) = \frac{1}{0.5 \times 160}\ln\left(\frac{18}{5.5}\right) = 0.015 \rightarrow \zeta = \frac{\delta}{2\pi} = 0.0024$$

2.2.3 Phase Lead/Lag Corresponding to Damping Ratio

Control force (e.g. electromagnetic force) via a control unit in response to a displacement of a mass is generated, for example, by the feedback system of mechatronic equipment. In the block diagram of Fig. 2.13a, the mass having the plant transfer function G_p is the controlled object, and the controller transfer function, G_r, represents the characteristics from the displacement input x through to the control force u.

The displacement $x(t)$ as input and control force $u(t)$, i.e., reaction force, as output are expressed as follows in terms of the natural frequency ω_n:

$$\begin{aligned} x(t) &= a\cos\omega_n t \\ u(t) &= f_0\cos(\omega_n t + \phi) \end{aligned} \tag{2.24}$$

As shown in Fig. 2.13b, the phase difference ϕ between the input and output signals is positive in phase-lead control and negative in phase-lag control. Considering that

$$u(t) = f_0\cos(\omega_n t + \phi) = f_0\cos\omega_n t\cos\phi - f_0\sin\omega_n t\sin\phi \tag{2.25}$$

if $u(t)$ is equivalent to the force exerted by the spring k and viscous damping c

$$u(t) \equiv kx(t) + c\dot{x}(t) = ka\cos\omega_n t - ca\omega_n\sin\omega_n t \tag{2.26}$$

comparison of Eqs. (2.25) and (2.26) gives

$$ka = f_0\cos\phi\,,\ ca\omega_n = f_0\sin\phi \tag{2.27}$$

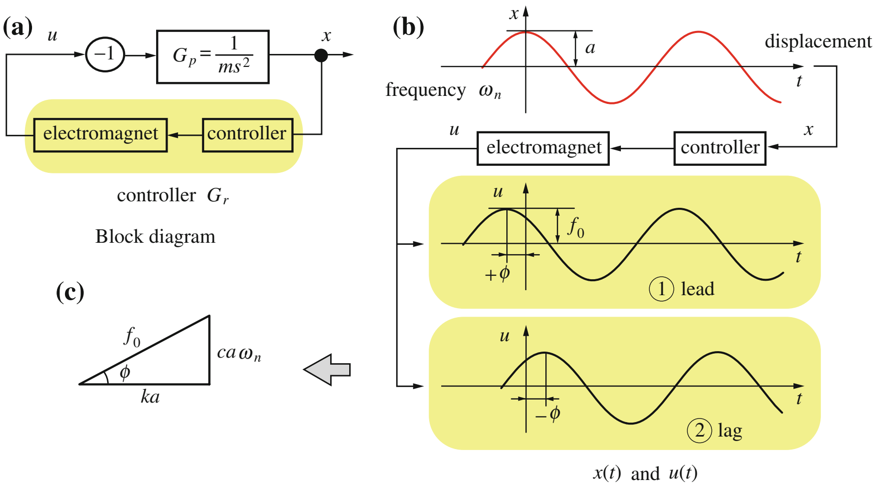


Fig. 2.13 Phase lead/lag

It is clear from Fig. 2.13c or the second equation of Eq. (2.27) that phase lead/lag means positive/negative damping, and $\phi > 0$ and $\phi < 0$ relate to stable damping vibration and unstable self-excited vibration, respectively. The stability of the system is thus dependent on the phase lead/lag of the reaction force $u(t)$ with respect to the natural frequency vibration $x(t)$.

Using Eq. (2.27), the damping ratio of a stable system may be directly estimated from the phase lead $\phi > 0$:

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{c\omega_n}{2k} = \frac{f_0 \sin\phi}{2f_0 \cos\phi} = \frac{1}{2}\tan\phi \tag{2.28}$$

Note that phase lead/lag means positive/negative damping and that the damping ratio may be directly estimated from the phase lead $\phi > 0$.

Example 2.7 If the displacement signal $x(t)$ at the natural frequency and the corresponding control force signal are measured as shown in Fig. 2.13 ①, what damping ratio ζ is expected?

Answer
The figure ① gives $\phi = +40°$, which leads to $\zeta = 1/2 \tan 40° = 0.4$.

2.3 Unbalance Vibration of a Rotating Shaft

2.3.1 Complex Displacement and Equation of Motion

Consider a rotating shaft system, vertically supported by bearings at both ends, having a rotating disk at the midpoint as shown in Fig. 2.14a. Since the rotor spins in a horizontal plane with the rotating speed Ω, gyroscopic moment are negligible and the effect of gravity are considered to be small. Assume that the shaft has stiffness, but no mass.

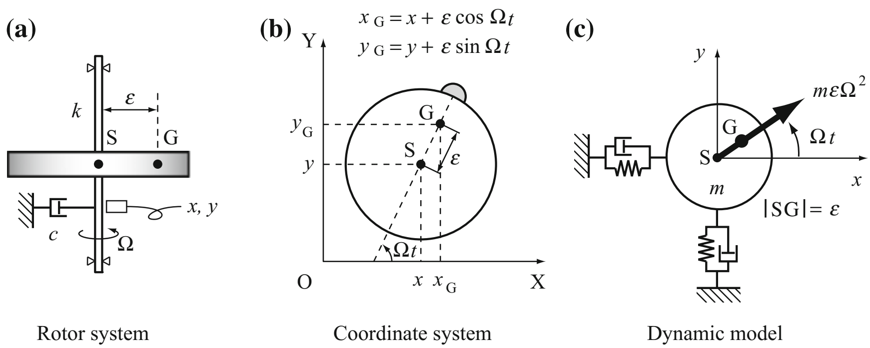


Fig. 2.14 Unbalanced rotor

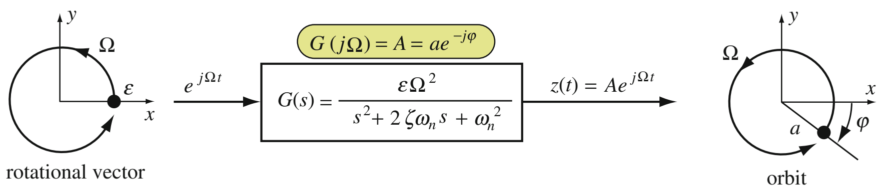


Fig. 2.15 Block diagram of unbalance vibration

Consider an inertial coordinate system O-XY fixed in the space with the origin coinciding with the shaft center at rest as shown in Fig. 2.14b. The shaft is fixed at the disk centroid S. Let the vibration displacement of the shaft, as measured by displacement sensors, be (x, y). The gravity center G of the disk is at a distance ε from the centroid S. For a rotational angular velocity Ω, the angle formed by the S-G axis, fixed to the shaft, and the O-X axis is the angle of rotation Ωt.

Taking viscous damping c into account, a single-dof model consisting of mass, spring and viscous damping as shown in Fig. 2.14c, can be constructed for the rotor equation of motion derived from Newton's second law. For the center of gravity of the rotor $\{x_G = x + \varepsilon\cos\Omega t,\ y_G = y + \varepsilon\sin\Omega t\}$, the reaction forces of spring and damping are proportional to the displacement of the centroid S (x, y). Therefore,

$$\begin{aligned} m\ddot{x}_G &= -kx - c\dot{x} \\ m\ddot{y}_G &= -ky - c\dot{y} \end{aligned} \tag{2.29}$$

which can be rewritten as

$$\begin{aligned} m\ddot{x}_G + c\dot{x} + kx &= m\varepsilon\Omega^2\cos\Omega t \\ m\ddot{y}_G + c\dot{y} + ky &= m\varepsilon\Omega^2\sin\Omega t \end{aligned} \tag{2.30}$$

Using the complex displacement $z = x + jy$ for simplicity, the equation of motion is written as

$$m\ddot{z} + c\dot{z} + kz = m\varepsilon\Omega^2 e^{j\Omega t} \rightarrow \ddot{z} + 2\zeta\omega_n\dot{z} + \omega_n^2 z = \varepsilon\Omega^2 e^{j\Omega t} \tag{2.31}$$

where $\omega_n^2 = k/m$, $2\zeta\omega_n = c/m$.

A block diagram (Fig. 2.15) can be constructed with the rotation vector $e^{j\Omega t}$ as the input and the unbalance vibration as the output. The transfer function $G(s)$ is defined from the above equation as shown in the figure.

2.3.2 *Complex Amplitude of Unbalance Vibration*

An unbalance vibration is observed as a forward whirling motion synchronized with the rotation angular velocity Ω including an orbit radius a and a phase lag angle φ

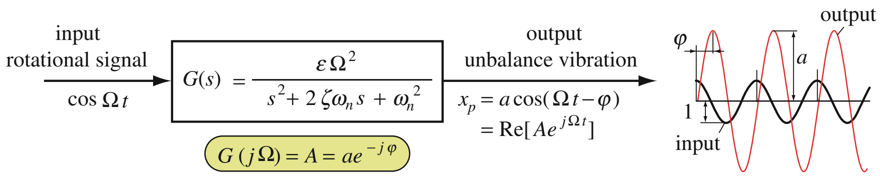


Fig. 2.16 Unbalance vibration waveform

behind the rotating unbalance direction (see Fig. 2.15 for the definition of a and φ). This motion is described by

$$z(t) \equiv Ae^{j\Omega t} \equiv ae^{-j\varphi}e^{j\Omega t} = ae^{j(\Omega t-\varphi)} \tag{2.32}$$

where $A \equiv ae^{-j\varphi} \equiv a\angle-\varphi$ is the complex amplitude, which is equal to the value of the transfer function $G(s)$ with $s = j\Omega$:

$$A = G(j\Omega) = \frac{\varepsilon\Omega^2}{-\Omega^2+\omega_n^2+2j\zeta\omega_n\Omega} = \frac{\varepsilon p^2}{1-p^2+2j\zeta p} \tag{2.33}$$

where $p = \Omega/\omega_n$ is the dimensionless rotating speed.

When observing the orbit from one stationary direction, the whirling motion appears as a repetition of approaching and leaving motion, i.e., vibration. For example, the vibrations in the X and Y directions are thus represented by

$$\begin{aligned} x(t) &= a\cos(\Omega t-\varphi) = \mathrm{Re}[\mathrm{A}e^{\mathrm{j}\Omega\mathrm{t}}] \\ y(t) &= a\sin(\Omega t-\varphi) = \mathrm{Im}[\mathrm{A}e^{\mathrm{j}\Omega\mathrm{t}}] \end{aligned} \tag{2.34}$$

The block diagram of Fig. 2.16 corresponds to the motion in the X direction. The observed phase lag angle φ represents the phase difference between the cosine input waveform and the output waveform $x(t)$.

2.3.3 Resonance Curves

Graphs of the vibration amplitude a and phase difference $-\varphi$ of unbalance vibration versus rotational speed are called resonance curves or Bode plots. Figures 2.17a, b show examples with various damping ratios ζ.

(1) **Amplitude**

As seen in Fig. 2.17a, the amplitude approaches zero at low rotational speeds $p = \Omega/\omega_n \ll 1$, while it approaches the mass eccentricity ε (which is relatively small) at high rotational speeds $p = \Omega/\omega_n \gg 1$. The amplitude increases rapidly as Ω approaches ω_n. The peak amplitude is infinite if $\zeta = 0$, for an undamped system,

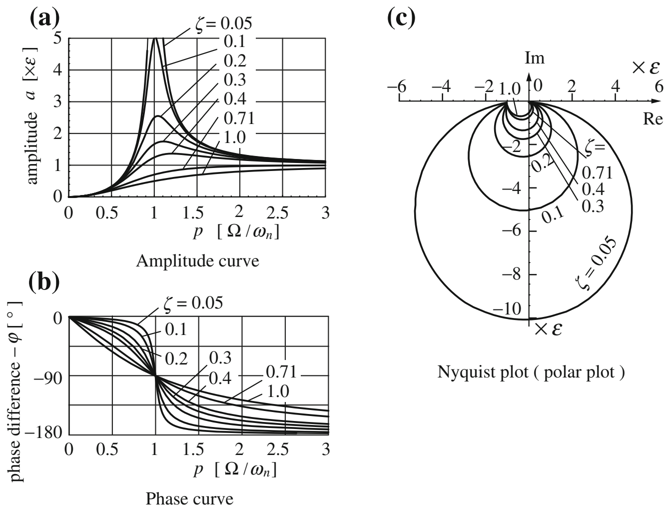


Fig. 2.17 Unbalance vibration response

but decreases as the damping ratio ζ increases. The peak amplitude is called the resonance amplitude a_p, and the corresponding rotating frequency is called the critical speed. They are approximated by:

$$Q\text{-value (or } Q\text{-factor)} : Q = \frac{a_p}{\varepsilon} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \approx \frac{1}{2\zeta} \tag{2.35}$$

$$\text{Critical speed: } p = \frac{\Omega}{\omega_n} = \frac{1}{\sqrt{1-2\zeta^2}} \approx 1 \tag{2.36}$$

(2) **Phase difference**

The curves in Fig. 2.17b show that the phase difference $-\varphi$ proceeds from 0° to $-180°$ as the rotational speed $p = \Omega/\omega_n$ increases. The phase lag φ at the critical speed $p = 1(\Omega = \omega_n)$ is always 90°. The phase lag changes slowly in strongly damped systems around the resonance point, but it changes rapidly in poorly damped systems.

2.3.4 Nyquist Plot

A Nyquist plot is a graphical representation of the complex amplitude of Eq. (2.33) in the complex plane. It is also called a vector locus or polar plot, because it displays the complex number as a vibration vector (amplitude $a \angle$ phase $-\varphi$) in polar coordinates.

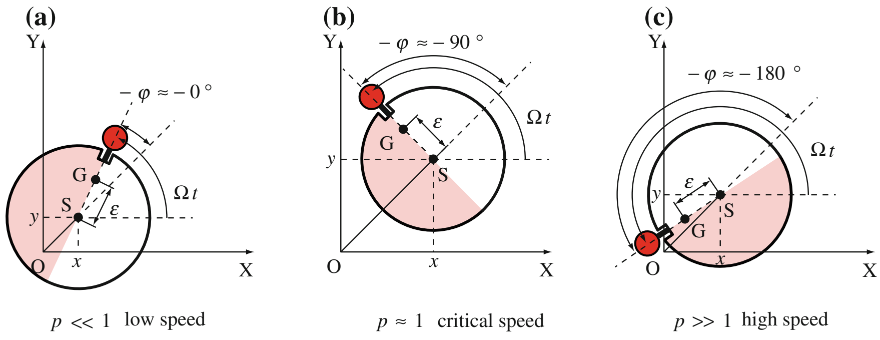


Fig. 2.18 Unbalance whirl motion

The plot in Fig. 2.17c shows that the amplitude gradually increases while drawing a clockwise circle tangent to the real axis after starting at the origin. This starting vector direction is set to the real axis (0°), because the phase difference of the vibration response with respect to the unbalance force is zero, i.e., they both have the same phase.

At low rotational speeds ($p = \Omega/\omega_n \ll 1$), a slight lag of the vibration response after the unbalance force appears in the vibration vector pointing to the 0 degree direction, i.e., the real axis. At the critical speed ($p = \Omega/\omega_n \approx 1$), the vibration vector lags by 90° and passes the zenith of the circular trajectory. At high rotational speeds ($p = \Omega/\omega_n \gg 1$), the amplitude decreases again, lagging by 180°, and approaches the point $(-\varepsilon, 0)$ from the negative direction of the real axis.

Example 2.8 Whirling motion

Figure 2.18 shows three instantaneous states of a rotor whirling by unbalance vibration. Find the complex amplitudes in (a) to (c) in the figure.

Answer

Since the complex amplitude = $(\mathrm{OS/OG})\varepsilon\angle-\varphi$, (a) $1.0\varepsilon\angle-18°$, (b) $3.2\varepsilon\angle-90°$ and (c) $2\varepsilon\angle-169°$.

Example 2.9 Waveform of unbalance vibration

Figure 2.19 shows comparisons of the input cosine waves and the corresponding unbalance vibration waveforms measured by an oscilloscope and their magnified views for measuring phase differences. Find the complex amplitudes in (a)–(c) in the figure.

Answer

Since the complex amplitude = $(a/\varepsilon)\varepsilon\angle-\varphi$, (a) $0.3\varepsilon\angle-15°$, (b) $5.9\varepsilon\angle-90°$ and (c) $1.5\varepsilon\angle-175°$

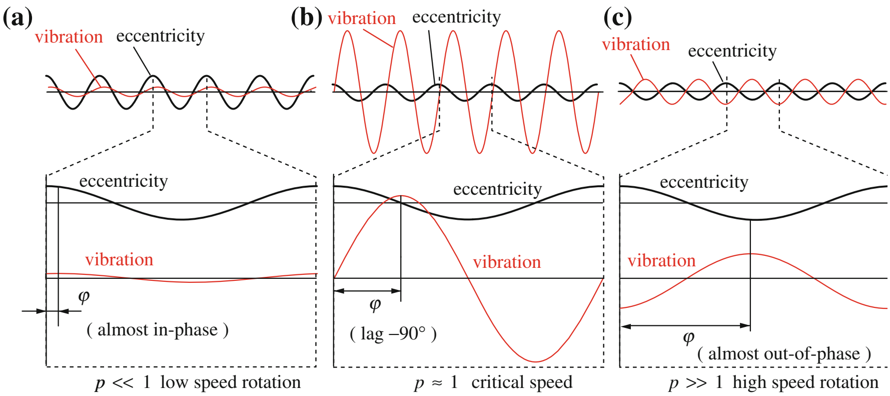


Fig. 2.19 Amplitude and phase related to unbalance eccentricity

2.3.5 Bearing Reaction Force at Resonance

The spring-viscous damping model of Fig. 2.14c can also be applied to a bearing, and permits one to describe how the unbalance force is balanced with bearing force at resonance. At the critical speed where the phase difference $-\varphi \approx -90^\circ$, the rotor vibration is represented by

$$z(t) \equiv A_p e^{j\Omega t} \equiv a_p e^{-j90^\circ t} e^{j\Omega t} = -ja_p e^{j\Omega t} \tag{2.37}$$

Substituting this into Eq. (2.31),

$$(k - m\Omega^2)A_p + j\Omega c A_p = m\varepsilon\Omega^2 \tag{2.38}$$

Considering $A_p = -ja_p$, the equation for static balance is obtained by setting the unbalance force to the imaginary axis as shown in Fig. 2.20b:

$$jm\varepsilon\Omega^2 - j\Omega c a_p + (-k + m\Omega^2)a_p = 0 \tag{2.39}$$

Figure 2.20a shows the instantaneous rotor position farthest from the origin through the X axis in a whirling orbit at the critical speed. The phase of the vibration vector A_p = OS is lagging by 90° behind the unbalance direction SG.

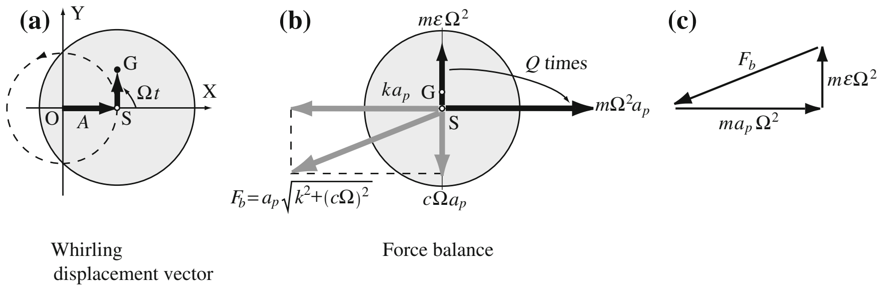


Fig. 2.20 Force balance at resonance point

Figure 2.20b illustrates the balance of forces at this point in time.

$$\begin{aligned}
&\text{Horizontal direction (real axis)} : -k + m\Omega^2 = 0 \rightarrow \Omega_c = \sqrt{k/m} = \omega_n \\
&\text{Vertical direction (imaginary axis)} : m\varepsilon\Omega^2 - \Omega c a_p = 0 \rightarrow a_p = \frac{m\varepsilon\omega_n}{c} = \frac{\varepsilon}{2\zeta} \\
&\text{Bearing reaction force} |F_b| = \left|(k + j\Omega c)A_p\right| = a_p\sqrt{k^2 + (c\Omega)^2} \\
&\qquad = m a_p \omega_c^2 \sqrt{1 + 4\zeta^2} \approx m a_p \Omega_c^2
\end{aligned} \tag{2.40}$$

The balance in the horizontal direction gives the critical speed and that in the vertical direction the resonance amplitude. Figure 2.20 indicates that the input unbalance force is balanced by the damping force at resonance. Increasing the viscous damping coefficient reduces accordingly the resonance amplitude. It can be stated that

- The resonance amplitude a_p is approximately Q times as great as the mass eccentricity ε, and
- The bearing reaction force at resonance F_b is approximately Q times as great as the unbalance force $m\varepsilon\Omega^2$.

Example 2.10 The force balance is represented by a closed arrangement of vectors as shown in Fig. 2.20c. Confirm that the three states of Fig. 2.18 can similarly be represented by the closed form as in Fig. 2.21.

Answer
The angle formed by the amplitude vector a (reset on real axis) and the eccentricity vector ε is the phase lag φ. Since the force balance is $m\Omega^2 a + m\varepsilon\Omega^2 e^{j\varphi} + F_b = 0$, the bearing reaction force vector F_b is added to the sum of the two vectors points as being back to the origin, as shown in Fig. 2.21.

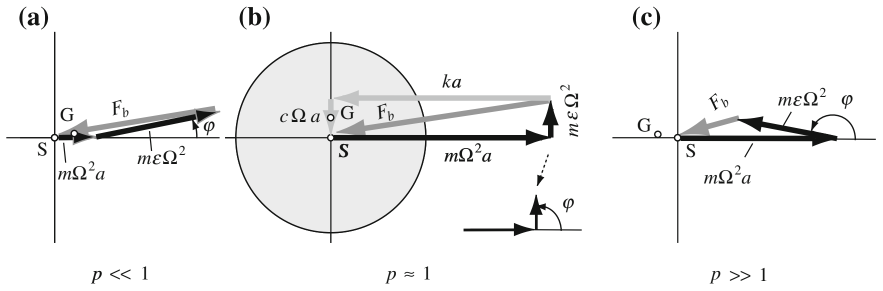


Fig. 2.21 Force balance

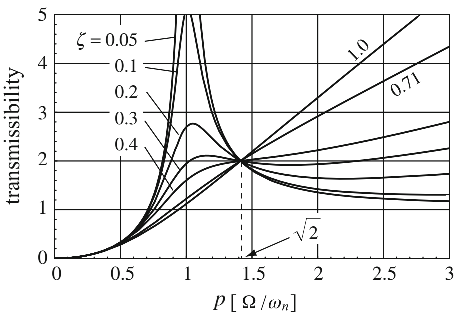
2.3.6 Transmissibility of Unbalance Vibration to Foundation

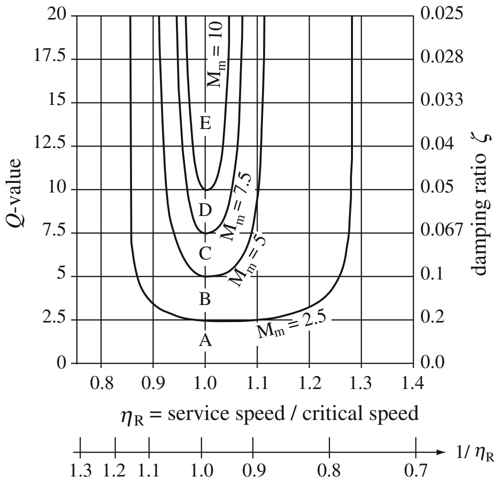
The transmissibility T that is the ratio of unbalance vibration to the foundation of the machine, normalized with the unbalance force at the critical speed $m\varepsilon\omega_n^2$, is written, considering Eq. (2.33), as

$$T \equiv \frac{|F_b|}{m\varepsilon\omega_n^2} = \frac{|kx + c\dot{x}|}{m\varepsilon\omega_n^2} = \frac{p^2|1+2j\zeta p|}{|1-p^2+2j\zeta p|} = \frac{p^2\sqrt{1+(2\zeta p)^2}}{\sqrt{(1-p^2)^2+(2\zeta p)^2}} \tag{2.41}$$

This ratio T is shown graphically in Fig. 2.22. When passing through the critical speed, both of the amplitude and transmission ratio are large and the peak value is $T \approx Q$. T always has the value 2 at $p = \Omega/\omega_n = \sqrt{2}$. It should be noted that at high rotational speeds T increases with increasing speed even if the amplitude decreases. Reduction of this transmissibility to the foundation is only possible by decrease of ε, i.e., by having a rotor that is well-balanced.

Fig. 2.22 Transmissibility of unbalance force to base





Modal sensitivity range		Expected running conditions
A	Very low sensitivity	Very smooth resonant speed; difficult to detect
B	Low sensitivity	Smooth, low and stable vibration
C	Moderate sensitivity	Acceptable, moderate and slightly unsteady vibration
D	High sensitivity	Sensitive to unbalance; Field balancing may be required
E	Very high sensitivity	Too sensitive to unbalance; to be avoided

Fig. 2.23 Modal sensitivity (Q-value) criteria

2.4 Evaluation of Q-Value

2.4.1 Q-Value Criterion

The Q-value is a universal and measurable index for evaluation of the vibration response characteristics of a rotating machine. It is very important to ascertain how low the Q-value can be maintained. The ISO 10814 [10] standard provides guidelines for Q-value design for various rotors. An example is shown in Fig. 2.23. The abscissa is rated rotational speed, non-dimensionalized with the critical speed, and the ordinate is the tolerated Q-value. The chart is usually used to evaluate the Q-value of the resonance mode near the rated speed.

As well as ISO's Q-value evaluation, the API standard [11] is strong in the oil and gas field of turbo-machinery industry.

Example 2.11 Confirm that a machine with a rated rotational speed of 11,000 rpm and a critical speed of 10,000 rpm requires approximately $Q \leq 9$ for a B-zone specification. Note that (rated rotational speed)/(critical speed) is 1.1.

Example 2.12 If the rotational speed of the machine increases slowly, so that the abscissa in Fig. 2.23 is traced very slowly from left to right through the critical speed, the machine may be considered to always be in the steady state, and the Q-value at (rotational speed)/(critical speed) = 1 should be evaluated in the whole range. Confirm that the machine requires $Q \leq 5$ for a B-zone specification.

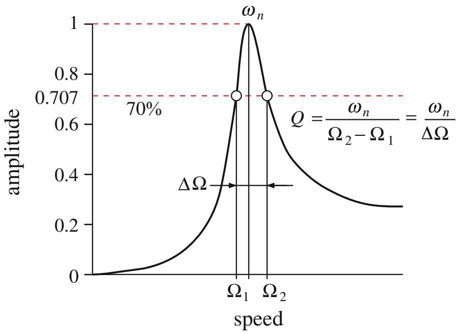


Fig. 2.24 Measuring Q-value (ISO 10814)

2.4.2 Measurement of Q-Value by the Half Power Point Method

The ISO standard specifies procedures to measure the Q-value from a resonance curve for forced vibration by the half power point method. Consider a measured resonance curve shown in Fig. 2.24, where the maximum amplitude is $a_{\max}$ at the resonance point ω_n. The difference $\Delta\Omega$ of the frequencies Ω_1 and Ω_2 of the half-power points, i.e., the intersect to give the half power amplitudes

$$a_{70} = a_{\max}/\sqrt{2} \approx 0.7a_{\max}$$

which permits determination of the Q-value by

$$Q = \frac{1}{2\zeta} = \frac{\omega_n}{\Delta\Omega} \tag{2.42}$$

Example 2.13 Measure the Q-value for the resonance curve shown in Fig. 2.24.

Answer
Draw a horizontal line through the 70 % amplitude. Use a length scale to obtain ω_n = 20 mm and $\Delta\Omega = 4\,\text{mm}$. Q = 20/4 = 5 and $\zeta = 0.1$ are thus obtained.

2.4.3 Measurement of Q-Value Using a Nyquist Plot

This is also a method specified in the ISO standard. Suppose a vector trajectory that describes a counterclockwise circle (Fig. 2.25). Connect the starting point of the circle, i.e., the origin, and the point ω_n corresponding to the peak amplitude. Draw two lines from the origin on both sides of this line at angles of 45° each, and let the intersections with the vector trajectory be Ω_{45} and Ω_{135}. The Q-value is given by

Fig. 2.25 Measuring Q-value (ISO 10814)

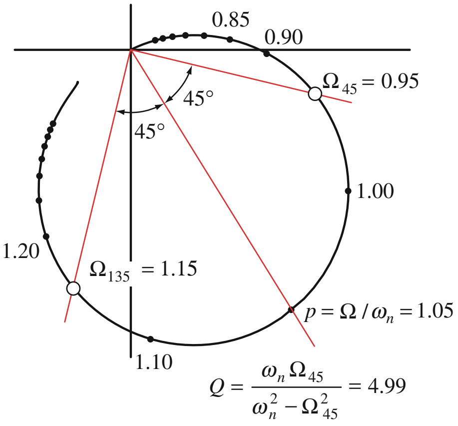
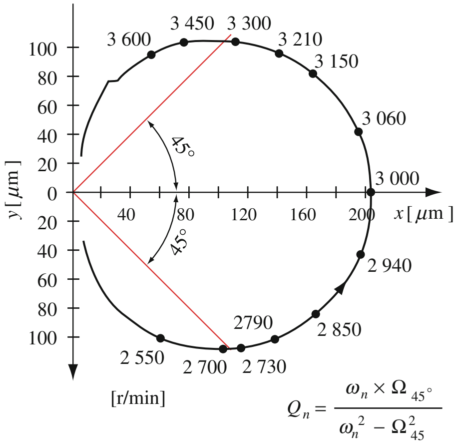


Fig. 2.26 Measuring Q-value (ISO 10814)



N_1	$N_{45°}$	Q_1
3 000 r/min	2 710 r/min	4.91

$$Q = \frac{1}{2\zeta} = \frac{\omega_n}{\Omega_{135} - \Omega_{45}} = \frac{\omega_n \Omega_{45}}{\omega_n^2 - \Omega_{45}^2} \tag{2.43}$$

Note: The phase angle is theoretically concerned with phase lead. The trajectory of the Nyquist plot proceeds clockwise due to the phase lag of the response. However, measuring instrument (called vector monitor) usually operates on the basis of a phase lag, and gives a counterclockwise trajectory as shown in Fig. 2.26.

Example 2.14 Measure the Q-value for the Nyquist plot shown in Fig. 2.25.

Answer

Interpolation based on $p = \Omega/\omega_n$ in the figure gives $\Omega_{45} = 0.95$ and $\Omega_{135} = 1.15$. Therefore $Q = 1/(1.15 - 0.95) = 5$.

Example 2.15 Measure the Q-value for the Nyquist plot shown in Fig. 2.26.

Answer
With the critical speed $\omega_n = 3,000\,\text{rpm}$ and $\Omega_{45} = 2,710\,\text{rpm}$, Eq. (2.43) gives Q = 4.91.

2.4.4 Re-evaluation of Q-Value for Rapid Acceleration

The Q-value of Fig. 2.23 evaluated from the resonance curve may be used for cases in which the critical speed is passed so slowly such that the angular acceleration is negligible and that the system is almost in steady state. In contrast, if the critical speed is passed rapidly, there is not enough time for the vibration to attain a steady state. This means that the peak amplitude is smaller in such a case. In other words, the Q-value apparently decreases by rapid acceleration of rotation. ISO provides a chart (Fig. 2.27) to find equivalent damping ratios by re-evaluating the apparent decrease in Q-value.

Example 2.16 Determination of zone according to the Q-value criterion
Fill in the blanks [?] below.

(1) If a rotating machine, with a rated rotational speed of 3,000 rpm, 1st critical speed of 2,730 rpm and damping ratio $\zeta = 0.04$, is slowly accelerated (and start/stop is infrequent), zone [?] is assigned to it according to Fig. 2.23, because Q = 12.5 and $\eta_R = 3,000/2,730 = 1.1$.

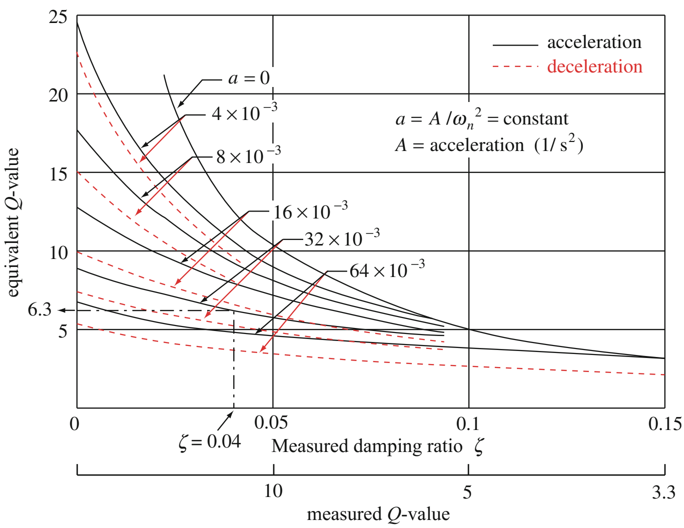


Fig. 2.27 Equivalent Q-value in acceleration and deceleration (ISO 10814)

(2) If the same machine is slowly accelerated up to 3,000 rpm, η_R should be regarded as 1 because operation at the critical speed cannot be excluded. According to Fig. 2.23, zone [?] should be assigned in this case, which notes that the operation at the critical speed is very dangerous.

(3) If the same machine is accelerated rapidly from 1,000 to 30,000 rpm over the critical speed 2,730 rpm, where $\zeta = 0.04$ and Q = 12.5,

$$\text{Acceleration: } A = 2\pi(30000 - 1000)/60/1.161 = 2615\, s^{-2}, \text{ and}$$

$$\text{Constant} : a = \frac{2615}{(2730 \times 2\pi/60)^2} = 32 \times 10^{-3}.$$

An equivalent Q-value of 6.3 is re-evaluated according to Fig. 2.27, as indicated by arrows. Since $\eta_R = 1$ because the machine passes the critical speed during the acceleration, zone [?] is assigned to the machine for Q = 6.3 according to Fig. 2.23.

Answer
(1) C, (2) E, (3) C

Example 2.17 In Fig. 2.28a, consider a rotor of a total length l = 750 mm, consisting of a disk of mass m = 40 kg fixed to the right end, and uniform shaft of mass

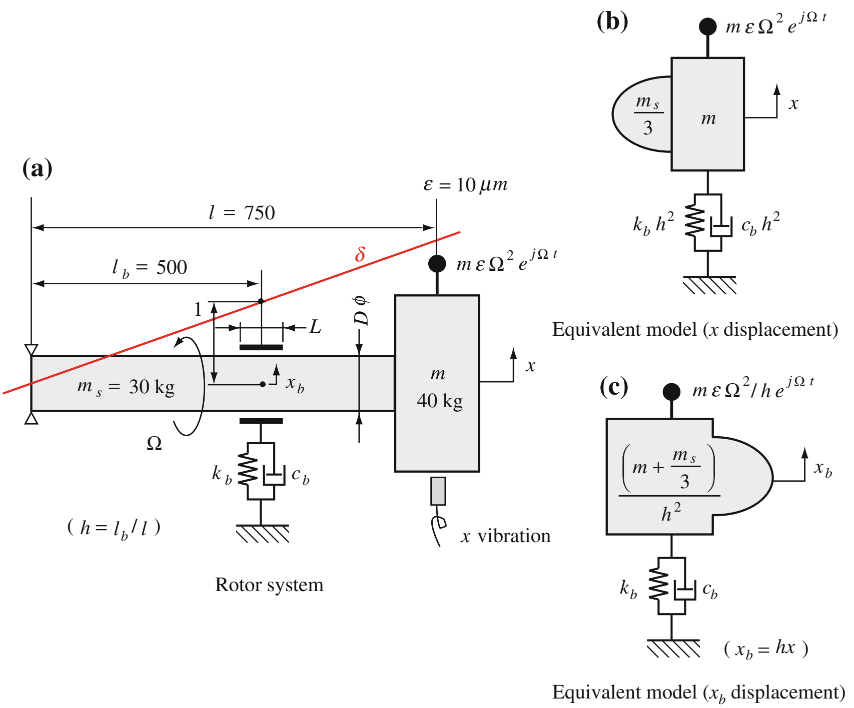


Fig. 2.28 Rotor and equivalent model system

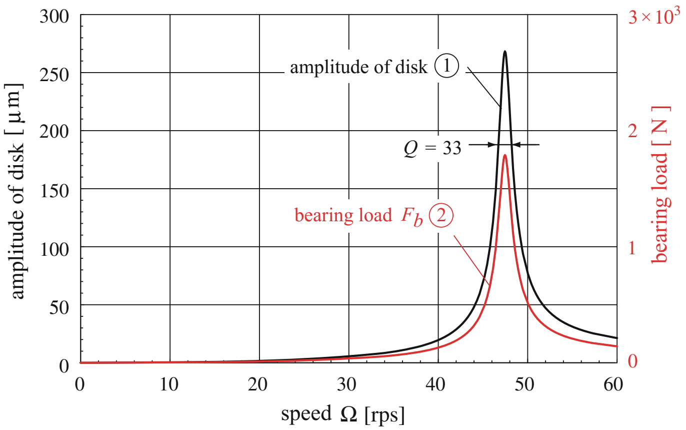


Fig. 2.29 Unbalance vibration

m_s = 30 kg. The shaft has a pinned support at the left end, and a bearing support at a distance of l_b = 500 mm from the left end (spring constant k_b = 10 MN/m and viscous damping coefficient c_b = 10.310^2 Ns/m). The resonance curve ① and bearing reaction force ② for the unbalance vibration of the rotor are shown in Fig. 2.29. Answer the following questions.

(1) The single-dof model equivalent to the system shown in Fig. 2.28a is shown in Figs. 2.28b, c. The former model with respect to the displacement x of the shaft tip includes 1/3 times of the shaft mass. The latter is another model with respect to the displacement x_b of the bearing portion ($x_b = hx$). Obtain the equation of motion for both models.
(2) Measure the critical speed Ω_c, the peak amplitude a_{peak} of the disk, and the Q-value of the system in the resonance curve ①.
(3) Find the equivalent mass m_{eq} using the displacement x_b of the bearing portion.
(4) Find the peak displacement a_p of vibration at the bearing portion.
(5) Find the peak load F_b on the bearing.
(6) Assuming that the journal diameter D = 100 mm and length L = 50 mm, find the journal surface pressure P at resonance.
(7) Find the acceleration A necessary to halve the surface pressure by rapid acceleration.

Answer

(1) The equation of motion of Fig. 2.28a: $I\ddot{\theta} + c_b l_b^2 \dot{\theta} + k_b l_b^2 \theta = m\varepsilon\Omega^2 l e^{j\Omega t}$,

$$\text{where } I = ml^2 + \int_0^l \rho A z^2 dz = \left(m + \frac{\rho A l}{3}\right) l^2 = \left(m + \frac{m_s}{3}\right) l^2$$

The equation of motion of Fig. 2.28b: Introduce $\theta = x/l$ into the above equation and divide it by l;

$$(m + m_s/3)\ddot{x} + (c_b\dot{x} + k_b x)l_b^2/l^2 = m\varepsilon\Omega^2 e^{j\Omega t}$$

The equation of motion of Fig. 2.28c: Introduce $\theta = x_b/l_b$ into the above equation and divide it by l_b;

$$(m + m_s/3)l_b^2/l^2\ddot{x}_b + (c_b\dot{x}_b + k_b x_b) = l/l_b m\varepsilon\Omega^2 e^{j\Omega t}$$

(2) $\Omega_c = 47\,\text{Hz}$, $a_{peak} = 267\,\text{mm}$, $Q = 33$, $\zeta = 0.015$.

(3) $m_{eq} =$ (the modal mass of the linear mode, when setting the bearing journal displacement $x_b = 1$) $= (750/500)^2(m + m_s/3) = (m + m_s)/h^2 = 112.5$ kg.

(4) The peak displacement a_p of vibration at the bearing portion $a_p = 267 \times 500/750 = 178$ m.

(5) The peak load F_b on the bearing; $|F_b| = m_{eq}a_p\Omega^2 = 112.5 \times 178 \times 10^{-6} \times (2\ \pi\ 47)^2 = 1744$ N, which corresponds to the peak of the bearing reaction force curve ②.

(6) The surface pressure $P = |F_b/(DL)| = 1744/(0.1 \times 0.05) = 3.4 \times 10^5$ Pa = 0.34 MPa.

(7) Aiming at a point ($Q = 15$ on y-axis, $\zeta = 0.015$ on x-axis) on Fig. 2.27, $a = 6 \times 10^{-3}$ may be chosen; this leads to an acceleration rate $A = a\omega_n^2 = 522[1/\text{s}^2] = 83\,\text{Hz/s}$, meaning that the critical speed must be passed through in approximately 50/83 ≈ 0.5 s.

2.4.5 *Vibration in Passing Through a Critical Speed*

Unlike in the steady state, reduced resonance peak amplitude and transient amplitude changes are observed when passing the critical speed rapidly. For example, calculation predicts the vibration shown in Fig. 2.30 for a system with

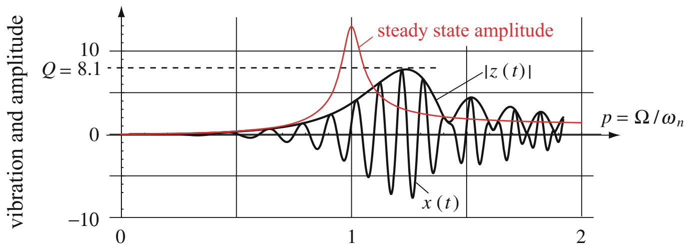


Fig. 2.30 Unbalance vibration wave form in acceleration ($\zeta = 0.04, a = 16 \times 10^{-3}, \varepsilon = 1$)

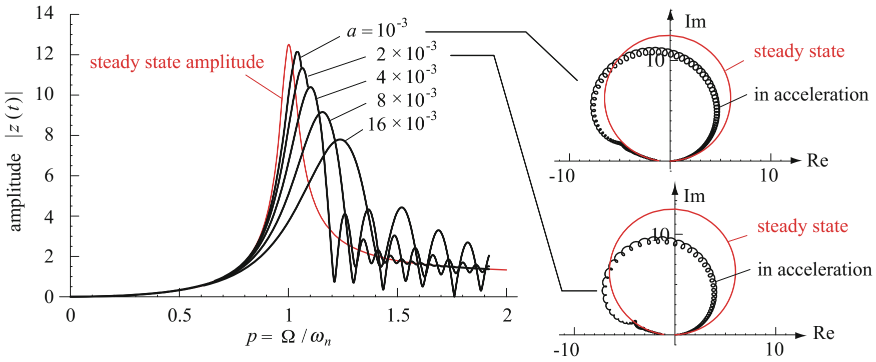


Fig. 2.31 Constant a and unbalance vibration envelope and polar plot

$\zeta = 0.04$ and acceleration $a = 16 \times 10^{-3}$ [1/s^2]. The x-directional vibration waveform $x(t)$ and the radius of whirling $|z(t)|$ shown in the figure along with the amplitude in steady state. The peak at the critical speed indicates that Q = 8.1, which is in accordance with the ISO guideline shown in Fig. 2.27.

Figure 2.31 compares envelopes and polar plots for different values of constant a. The amplitude approaches the steady state amplitude in slow acceleration, but the polar plot remains considerably different from that for the steady state: it resembles a circle deformed downwards, and corresponds to smaller amplitude. Acceleration for balancing purposes should be as slow as possible because an accurate polar plot is needed.

http://www.springer.com/978-4-431-55455-4

Vibrations of Rotating Machinery
Volume 1. Basic Rotordynamics: Introduction to Practical Vibration Analysis
Matsushita, O.; Tanaka, M.; Kanki, H.; Kobayashi, M.; Keogh, P.
2017, XIII, 360 p. 318 illus., 203 illus. in color., Hardcover
ISBN: 978-4-431-55455-4