

Chapter 2

Dynamics of Perfect Fluids

As discussed in the previous chapter, the viscosity of fluids induces tangential stresses in relatively moving fluids. A familiar example is water being poured into a rotating cylindrical container. Although the container is circular, water is pulled toward the container wall as it begins its movement. This example highlights that real fluids are viscous in nature.

In contrast, inviscid fluids are called perfect or ideal fluids. Apart from extremely special cases, such as the superfluid phenomenon, perfect fluids do not exist in reality; however, they are abstract entities that simplify the theoretical development of fluid dynamics. Indeed, the inviscid assumption is extremely useful to theoretical analysis, and fortunately, water and air encountered in daily use have low viscosity. Thus, the actual flow characteristics are explained well by the dynamics of perfect fluids. Conversely, by investigating the motion characteristics of perfect fluids, we can accurately understand the effect of viscosity and other properties of real fluids. Hence, this chapter discusses the dynamics and analytical method of perfect fluids.

2.1 Lagrange's Vortex Theorem

In Sect. 1.10, the vorticity equation for general viscous fluids was derived. Vorticity ($\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$) relates to the angular velocity of the rigid rotation of fluid particles; thus, it is expected that the laws analogous to the law of angular momentum conservation in the dynamics of particle systems should exist in those of perfect fluids. Actually, an extremely useful vortex theorem can be derived from the assumption of perfect fluids; thus, we will discuss this subject in this chapter.

First, we assume Euler's equation of perfect fluids with barotropic flow; that is, density ρ and pressure p are related through $\rho = f(p)$ and external forces are assumed as conservative forces. Rearranging, we obtain

$$\left. \begin{aligned} \frac{\partial \mathbf{u}}{\partial t} &= -\nabla \left(\frac{1}{2} q^2 + P + \Pi \right) + \mathbf{u} \times \boldsymbol{\omega}, \\ q &= |\mathbf{u}|, \quad P = \int^P \frac{dp'}{\rho} = \int^P \frac{dp'}{f(p')} \end{aligned} \right\} \quad (2.1)$$

Here Π is the potential of external forces. For incompressible fluids moving in a uniform gravitational field, P and Π in (2.1) are, respectively, reduced to

$$P = \frac{p}{\rho}, \quad \Pi = g z, \quad (2.2)$$

where g is the gravitational acceleration, and z is the vertical coordinate (assuming upwards as the positive direction).

Applying the $\nabla \times$ operator to both sides of (2.1), we obtain $\nabla \times \mathbf{u} = \boldsymbol{\omega}$, and given that $\nabla \times \nabla F = 0$ for an arbitrary scalar function F , we immediately obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}). \quad (2.3)$$

Alternatively, by rearranging the equation in Sect. 1.10 (Exercise 2.1), we obtain

$$\frac{D}{Dt} \left(\frac{\boldsymbol{\omega}}{\rho} \right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla \right) \mathbf{u}. \quad (2.4)$$

Equation (2.3) or (2.4) is called the vorticity equation for perfect fluids.

According to this equation, the temporal evolution of vorticity is unrelated to external force and pressure. This can be understood as follows: considering an infinitesimally small spherical fluid element, the external force operates on the center of mass of the sphere, while the pressure operates normal to the surface of the sphere. Thus, no rotation moment is generated around the center of mass, and the angular momentum of an infinitesimally small spherical fluid element remains unchanged during motion. This implies that if an infinitesimally small sphere is not rotating at a given instant, it will never rotate; conversely, if it is rotating initially, it will rotate forever. Since the angular velocity of a minute fluid element is half its vorticity, vortices can neither arise nor cease in a perfect fluid. This concept underlies **Lagrange's vortex theorem**. More simply, Lagrange's vortex theorem is called the theory of non-arising and non-ceasing vortices.

Using (2.4), let us explain Lagrange's vortex theorem from the perspective of a differential equation. If $\boldsymbol{\omega} = 0$ at $t = 0$, the material derivative of the fluid particle, from (2.4), is

$$\left[\frac{D}{Dt} \left(\frac{\boldsymbol{\omega}}{\rho} \right) \right]_{t=0} = 0. \quad (2.5)$$

An infinitesimally small time instant later ($t = \delta t$), $\boldsymbol{\omega}$ of the fluid particle remains 0 because we have

$$\left(\frac{\omega}{\rho}\right)_{t=\delta t} = \left(\frac{\omega}{\rho}\right)_{t=0} + \left[\frac{D}{Dt}\left(\frac{\omega}{\rho}\right)\right]_{t=0} \cdot \delta t = 0. \quad (2.6)$$

Considering that time progresses in infinitesimal increments, $\omega = 0$ at any arbitrary time.

In contrast, if $\omega \neq 0$ at $t = 0$, the above reasoning stipulates that $\omega \neq 0$ forever. Suppose that we can progress backward from some time at which $\omega = 0$. Following the above argument, $\omega = 0$ at $t = 0$, contradicting our assumption that $\omega \neq 0$ at $t = 0$. This implies that when an inviscid barotropic fluid moves under conservative forces, no vortices are generated or dissipated.

However, as discussed in the opening paragraph, real fluids are viscous in nature. If viscosity is significant, vortices are generated in the vicinity (boundary layer) of a body surface. This is because even if $\omega = 0$ in an area far from the body, by virtue of non-slip boundary conditions at the body surface, the $\omega = 0$ solution cannot be established. However, if the boundary layer breaks away from the body, the vortex forming the boundary layer will enter the main flow without disappearing, although new vortices rarely form within the fluid. This indicates that the perfect fluid assumption is fulfilled with satisfactory accuracy in regions slightly distant from body surfaces.

Note 2.1 Equation (2.3) was described as a vorticity equation; however, an analogous relationship is observed in magnetohydrodynamics. Specifically, when a fluid is a perfect conductor (with zero electrical resistance), and there is a magnetic field in this fluid, the magnetic flux density B satisfies the equation where ω in (2.3) is replaced by B . Therefore, the magnetic flux density has properties similar to vorticity in perfect fluids, and the magnetic field is neither generated nor dissipated inside the fluid of a perfect electrical conductor.

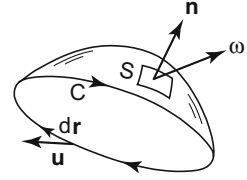
Now, when $\omega = \nabla \times \mathbf{u} = 0$ in a certain region of the flow, the flow is said to be irrotational in that region. As an identical equation in vector analysis, $\nabla \times \nabla \Phi = 0$ is established at all times relative to an arbitrary scalar function Φ . Thus, in an irrotational flow, there exists a scalar function Φ such that

$$\mathbf{u} = \nabla \Phi. \quad (2.7)$$

As the gradient of this function is velocity, Φ is called the **velocity potential**. In three-dimensional problems, a vector $\mathbf{u}$ with three unknowns can be calculated from a single scalar function Φ , which significantly simplifies the mathematical treatment. Generally, a flow that can be described in terms of velocity potential is called a potential flow, and an irrotational flow is a permanent potential flow. Velocity potential will be explained in detail in Sect. 2.5.

Exercise 2.1 Derive (2.1) and (2.4).

Fig. 2.1 Relationship between circulation and vorticity



2.2 Circulation and Vorticity

Consider any closed curve C in a flow and integrate $\mathbf{u}$ along C .

$$\Gamma(C) = \oint_C \mathbf{u} \cdot d\mathbf{r} = \oint_C u_s ds. \quad (2.8)$$

Here u_s is the tangential component of C intercepting flow velocity vector $\mathbf{u}$, and $\Gamma(C)$ is called the **circulation** along C .

Using Stokes' theorem of vector analysis (see Fig. 2.1 and Note 1.6), the above equation can be rearranged as

$$\Gamma(C) = \iint_S (\nabla \times \mathbf{u}) \cdot \mathbf{n} dS = \iint_S \boldsymbol{\omega} \cdot \mathbf{n} dS = \iint_S \omega_n dS. \quad (2.9)$$

Equation (2.9) relates circulation to vorticity. Specifically, circulation $\Gamma(C)$ is equal to the integral of the normal component ω_n of the vorticity $\boldsymbol{\omega}$ on a curved surface S surrounded by a closed curve C .

Consider an extremely thin vortex tube with a cross-sectional area σ . Taking the orthogonal section of the vortex tube as the integral surface in (2.9), $\omega_n = \omega$ can be considered fixed within the cross-sectional surface, and the circulation becomes

$$\Gamma(C) = \iint_S \omega_n dS = \omega \iint_S dS = \omega \sigma. \quad (2.10)$$

This result is independent of the position along the vortex tube (i.e., does not depend on the choice of C), and is hence called the strength of the vortex tube.

Note that if a single vortex tube is considered, the strength of the vortex tube is constant regardless of the position; the thinner the tube, the larger the vorticity ω , and thus the larger the rotational angular velocity of the fluid particles.

A wide vortex tube can be treated as a collection of many thin vortex tubes and expressed as

$$\Gamma(C) = \iint_S \omega_n dS = \sum_{i=1}^N \omega_i \sigma_i. \quad (2.11)$$

According to (2.11), the circulation along an arbitrary closed curve C equals the sum of the strengths of the vortex tubes passing through C .

We now relate the circulation $\Gamma(C)$ to the velocity potential Φ . Substituting (2.7) and (2.8) into (2.11) gives

$$\Gamma(C) = \oint_C \nabla \Phi \cdot d\mathbf{r} = \oint_C \frac{\partial \Phi}{\partial x_j} dx_j = [\Phi]_C. \quad (2.12)$$

In other words, the circulation along an arbitrary closed curve C inside a fluid region is the change in Φ along one circuit of C . Since the circulation is non-zero, the velocity potential is multi-valued. However, the flow velocity is a single-valued function (that is, at a given location, the flow velocity settles to a fixed value). Hence, when obtaining the flow velocity from $\mathbf{u} = \nabla \Phi$, the multi-valued property disappears.

2.3 Circulation Theorem and Vortex Theorem

2.3.1 Kelvin's Circulation Theorem

Section 2.2 only describes the properties of the vortex tube at a given time. Here we consider how the vortex tube changes over time. Hence, we evaluate the temporal change in circulation along a closed curve C , which moves with the flow. In terms of the material derivative (2.8), we have

$$\frac{D}{Dt} \Gamma(C) = \oint_C \frac{D}{Dt} (\mathbf{u} \cdot d\mathbf{r}) = \oint_C \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{r} + \oint_C \mathbf{u} \cdot \frac{D}{Dt} d\mathbf{r}. \quad (2.13)$$

Here, from the Euler equation and (2.1),

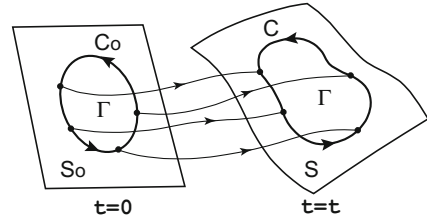
$$\left. \begin{aligned} \frac{D\mathbf{u}}{Dt} &= -\nabla(P + \Pi), \\ \mathbf{u} \cdot \frac{D}{Dt} d\mathbf{r} &= \mathbf{u} \cdot d\mathbf{u} = d\left(\frac{1}{2} q^2\right). \end{aligned} \right\} \quad (2.14)$$

Therefore, we have

$$\frac{D}{Dt} \Gamma(C) = \left[\frac{1}{2} q^2 - P - \Pi \right]_C. \quad (2.15)$$

Here $[\cdot \cdot \cdot]_C$ indicates the change along one circuit of the closed curve C . As the flow velocity q and pressure function P are physical quantities, (2.15) is a single-valued function of location. In general, the potential Π of external forces is also a single-valued function; hence, the right-hand side of (2.15) is 0. In other words, when an inviscid barotropic fluid moves under conservative forces, the circulation along a closed curve C formed by the fluid particles is constant in time. This concept is called **Kelvin's circulation theorem**.

Fig. 2.2 A vortex surface (tube) moves as a vortex surface (tube), and its strength is constant



It should be noted that, as circulation and vorticity are related by (2.10), Lagrange's vortex theorem can be explained in terms of Kelvin's circulation theorem.

2.3.2 Helmholtz Vortex Theorem

A curved surface created from vortex lines in a flow is called a vortex surface. Here we focus on one such vortex surface S_0 at $t = 0$ (see Fig. 2.2). Considering a closed curve C_0 on S_0 , no vortex line passes through C_0 ; hence, $\Gamma(C_0) = 0$.

Now suppose that C_0 at a certain time t forms a closed curve C because of the movement of fluid particles. By Kelvin's circulation theorem, we have $\Gamma(C) = \Gamma(C_0) = 0$. However, according to the concept of non-arising and non-ceasing vortices, any existing vortex surface should never disappear. Thus, $\Gamma(C) = 0$ indicates that a closed curve C can be defined on the vortex surface. It now becomes evident that "through fluid motion, a vortex surface remains a vortex surface." The vortex tube discussed in Sect. 2.2 is one such vortex surface; moreover, as the strength of a given vortex tube is fixed throughout space, the strength of the vortex tube is invariably maintained in space and time. This is called the **Helmholtz vortex theorem**.

Now consider an extremely thin vortex tube with a cross-sectional area σ . Vortex tubes move with fluid; hence, if we consider an infinitesimally small length δs of the vortex tube, $\rho\sigma\delta s$ is fixed in time by the mass conservation law. Moreover, the strength $\Gamma = \omega\sigma$ of the vortex tube defined by (2.10) is constant by the Helmholtz circulation theorem; thus, $\rho\sigma\delta s/\omega = \text{constant}$ must be true. In particular, as ρ is constant in an incompressible fluid, $\omega \propto \delta s$.

From this discussion, it follows that if a vortex tube is extended or contracted owing to temporal changes in the fluid, the vorticity will proportionately increase or decrease, respectively.

2.3.3 Flow and Circulation Around a Wing Section

To enhance our understanding of the abovementioned circulation and vortex theorems, we demonstrate the theorems on flow around a wing section. Consider that

Fig. 2.3 Explanation of Kelvin's circulation theorem part 1: **a** $\Gamma = 0$ for a wing section in a static fluid, **b** Even when the wing section moves with a velocity U , $\Gamma = 0$ as a whole

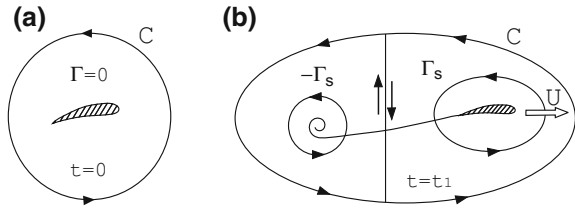
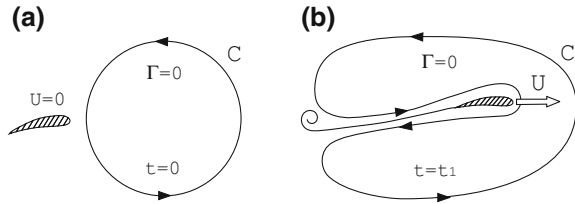


Fig. 2.4 Demonstration of Kelvin's circulation theorem part 2: **a** $\Gamma = 0$ in the fluid region in front of the wing section, **b** $\Gamma = 0$ as long as the same fluid region is considered



the wing is placed in a static fluid. Most areas of the fluid are irrotational; thus, the circulation is zero (see Fig. 2.3a). If the wing starts to move with velocity U in a real fluid, a thin boundary layer develops around the wing surface because of viscosity. However, the boundary layer formed beneath the wing section cannot reach the upper surface by fully traveling around the trailing edge and instead separates from the trailing edge to form a vortex. A short time later, a separation point on the wing section's upper surface moves to the trailing edge. When the flow becomes smooth, this vortex eventually flows toward the rear of the wing section, where it becomes a clockwise free vortex, as shown in Fig. 2.3. Suppose that the circulation of this vortex (called a starting vortex) is $-\Gamma_s$. According to Kelvin's circulation theorem, the circulation is initially 0; hence, the circulation along the closed curve surrounding both the starting vortex and wing section is also 0. In other words, a circulation Γ_s of the same strength as the starting vortex exists in the opposite direction around the wing section (see Fig. 2.3b). This circulation generates a **lift**, known as the **Kutta–Joukowski theorem**:

$$L = \rho U \Gamma_s. \quad (2.16)$$

The forces operating on bodies will be explained in Sect. 2.10.

Kelvin's circulation theorem can be understood from the closed curves shown in Fig. 2.4. We first focus on the fluid region in front of the wing section at $t = 0$. The circulation around the closed curve C surrounding this region is 0. Once the wing starts to move, after a short time, the wing section will intercept the curve C , deforming it (Fig. 2.4b). In Kelvin's circulation theorem, the curve encloses the same fluid region at all subsequent times; thus, the circulation along the closed curve shown in Fig. 2.4b certainly remains at 0.

2.4 Bernoulli's Theorem

We now introduce Bernoulli's theorem, one of the most important theorems in fluid dynamics. As we shall see, Bernoulli's theorem is based on the law of conservation of energy in fluid dynamics.

2.4.1 Irrotational Flows

As discussed above, the velocity potential Φ exists in an irrotational flow with $\omega = 0$. Substituting (2.7) into Euler's equation (2.1), we have

$$\nabla \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} q^2 + P + \Pi \right) = 0. \quad (2.17)$$

Therefore,

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} q^2 + P + \Pi = f(t). \quad (2.18)$$

Here $f(t)$ is a function of t only. For functions of time, $\nabla f(t) = 0$; thus, $f(t)$ can be included in the definition of velocity potential, and the right-hand side of (2.18) can be set to 0. However, in applications, it is more convenient to leave $f(t)$ as a pressure function in (2.18).

In any case, (2.18) is used to determine the pressure from the velocity potential, and it is known as **Bernoulli's pressure equation**. If the flow is irrotational, (2.18) can be applied to the whole flow field. Of course, the flow can be non-steady. The reader should note that Bernoulli's pressure equation is derived from the law of conservation of momentum (Euler's equation), as is clearly shown in the derivation.

2.4.2 Steady Flows

In steady flows, the following is immediately obtained from (2.1):

$$\nabla \left(\frac{1}{2} q^2 + P + \Pi \right) \equiv \nabla H = \mathbf{u} \times \boldsymbol{\omega}, \quad (2.19)$$

$$H = \frac{1}{2} q^2 + P + \Pi. \quad (2.20)$$

A surface with $H = \text{constant}$ is called a **Bernoulli surface** (see Fig. 2.5).

The vector $\mathbf{u} \times \boldsymbol{\omega}$ is perpendicular to both $\mathbf{u}$ and $\boldsymbol{\omega}$ and (as seen in (2.19)) parallel to ∇H . Put another way, both $\mathbf{u}$ and $\boldsymbol{\omega}$ are parallel to a Bernoulli surface, and both streamlines and vortex lines reside on Bernoulli surfaces. Thus, it can be concluded that, on one streamline, $H = \text{const}$. This is called **Bernoulli's theorem**, and flow can be established even in the presence of vortices. However, it is different from the pressure equation (2.18), and the flow must be steady.

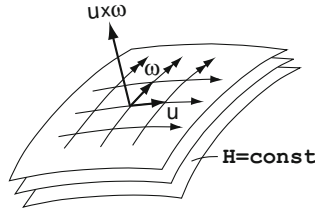


Fig. 2.5 Bernoulli surfaces

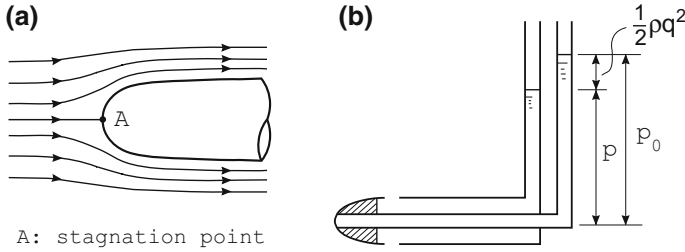


Fig. 2.6 Principle of the Pitot tube; **a** Pressure at the stagnation point A is p_0 , **b** Dynamic pressure is the difference between p_0 and the pressure at a point parallel to the flow (denoted p)

Bernoulli's theorem was obtained by integrating Euler's equation, which is based on the law of conservation of momentum. However, (2.20) is remarkably similar to the energy conservation law in a system of particles. The first term on the right-hand side of (2.20) can be interpreted as the kinetic energy per unit mass, the third term can be interpreted as the potential energy due to external forces, and the second term can be interpreted as the potential energy reserved as work done by the pressure. As H is fixed on streamlines, Bernoulli's theorem can be interpreted as an energy conservation law of fluid dynamics.

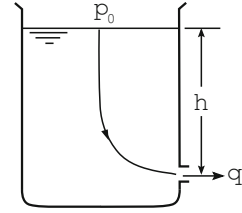
There are many fluid phenomena that can be explained by Bernoulli's theorem, and there are also many measurement devices that use Bernoulli's theorem. Here we introduce the **Pitot tube**, a device that measures flow velocities.

If a body with a rounded front surface is placed in the flow, a point forms in front of the body where the fluid velocity $u = 0$. This point (indicated as point A in Fig. 2.6a) is called a **stagnation point**. Let the pressure at the stagnation point (called the stagnation pressure) be p_0 . Applying Bernoulli's theorem to the streamlines passing through the stagnation point and ignoring the height difference (corresponding to the potential of external forces), we have

$$p + \frac{1}{2} \rho q^2 = \text{const} = p_0. \quad (2.21)$$

Consider $(1/2)\rho q^2$ as a type of pressure and call it **dynamic pressure**. Consider pressure p as **static pressure** in relation to this; then, the total pressure is the sum

Fig. 2.7 Torricelli's theorem and its application



of $(1/2)\rho q^2$ and p . Along the streamline, the total pressure is fixed and equal to the stagnation pressure p_0 .

If a tube such as that shown in Fig. 2.6b faces the direction of the flow, and the total and static pressures p_0 and p are measured through a hole in the front and side of the tube, respectively, the flow velocity q can be given by rearranging (2.21) to

$$q = \sqrt{\frac{2(p_0 - p)}{\rho}}. \quad (2.22)$$

A tube to measure the flow velocity based on this principle is called a Pitot tube.

Exercise 2.2 (Torricelli's theorem)

Consider the velocity when fluid inside a large container flows out through a hole made in the wall. In Fig. 2.7, suppose that the hole is made at $z = 0$, and the water surface is at $z = h$. Assuming that the surface area of the water is large relative to the cross-sectional area of the hole, the flow velocity at the water surface is approximately 0. Moreover, the pressure at the water surface equals the atmospheric pressure p_0 . Using Bernoulli's theorem, determine the velocity of the fluid flowing through the hole. The flow velocity can be computed from Torricelli's equation $q = \sqrt{2gh}$.

Exercise 2.3 A cylindrical water tank with water surface area A is filled with fluid to height H . The fluid is then released through a small hole of area a at the bottom of the tank. Assuming that $A/a \geq 1$, and that atmospheric pressure acts on the liquid surface and the fluid that has flowed out, calculate

1. The time for the depth H to decrease to $H/2$, $T_{1/2}$.
2. The ratio of $T_{1/2}$ and the time taken for H to become 0, T_1 .

2.5 Velocity Potential

Here we present a more detailed analysis of irrotational flow in incompressible perfect fluids. As shown in (2.7), irrotational flow is associated with a velocity potential such that

$$\mathbf{u} = \nabla \Phi. \quad (2.23)$$

2.5.1 The Laplace Equation

In Sect. 1.4, we derived the following continuity equation for an incompressible fluid:

$$\nabla \cdot \mathbf{u} = 0. \quad (2.24)$$

From (2.23) and (2.24), we have

$$\nabla^2 \Phi = 0. \quad (2.25)$$

In other words, velocity potential is governed by the **Laplace equation**.

The boundary condition of the velocity potential Φ can be obtained by substituting (2.23) into the boundary condition of the velocity $\mathbf{u}$ shown in Sect. 1.5. When the boundary surface is a rigid wall,

$$\mathbf{n} \cdot \nabla \Phi = \frac{\partial \Phi}{\partial n} = \mathbf{n} \cdot \mathbf{u}_b \equiv V_n. \quad (2.26)$$

Here $\mathbf{n}$ is a vector normal to the boundary surface, and V_n is the normal component of the velocity of the boundary surface $\mathbf{u}_b$.

In this way, problems seeking the velocity field in an incompressible irrotational flow result in problems that seek solutions to the Laplace equation that fulfills the given boundary conditions (harmonic functions). The harmonic function theory is mathematically complete, and its results are extensively used in fluid dynamics. (An analysis in spherical coordinates is shown below). If we can define the velocity potential Φ , the flow velocity and pressure function ($P = p/\rho$) can be computed from (2.23) and Bernoulli's pressure equation (2.18), respectively.

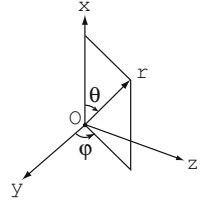
Note that the Laplace equation determining the flow velocity field is the same in both steady and non-steady motions. This indicates that the velocity field is determined at each moment and is unrelated to the history of the fluid motion. Therefore, regardless of whether the flow is steady or unsteady, the flow pattern will be identical provided that the boundary conditions remain the same. However, as shown by Bernoulli's pressure equation, pressure differs between steady and non-steady flows, and the fluid forces acting on bodies (obtained by integrating the pressure) depend on the history of the fluid motion.

Note 2.2 The Laplace equation expresses the mass conservation law; hence, the surface integral in

$$\iiint_V \nabla^2 \Phi \, dV = \iint_S \frac{\partial \Phi}{\partial n} \, dS = 0 \quad (2.27)$$

is a generalized expression that is valid in any coordinate system. Applying (2.27) to an infinitely small volume element in an arbitrary Cartesian coordinate system, we can express the Laplace equation in that coordinate system without requiring calculations for complicated coordinate transformations. For example, consider the

Fig. 2.8 Spherical coordinate



spherical coordinate system (Fig. 2.8).

$$\left. \begin{aligned} x &= r \cos \theta, \\ y &= r \sin \theta \cos \varphi, \\ z &= r \sin \theta \sin \varphi. \end{aligned} \right\} \quad (2.28)$$

An infinitesimally small line element in three-dimensional Cartesian coordinates is described by $(\delta x)^2 + (\delta y)^2 + (\delta z)^2$. Using (2.28), its representation in spherical coordinates is

$$\delta s_1 = \delta r, \quad \delta s_2 = r \delta \theta, \quad \delta s_3 = r \sin \theta \delta \varphi. \quad (2.29)$$

In terms of these coordinates, the surface integral of (2.27) can be thus rewritten:

$$\begin{aligned} \frac{\partial}{\partial s_1} \left(\frac{\partial \Phi}{\partial s_1} \delta s_2 \delta s_3 \right) \delta s_1 + \frac{\partial}{\partial s_2} \left(\frac{\partial \Phi}{\partial s_2} \delta s_3 \delta s_1 \right) \delta s_2 \\ + \frac{\partial}{\partial s_3} \left(\frac{\partial \Phi}{\partial s_3} \delta s_1 \delta s_2 \right) \delta s_3 = 0. \end{aligned} \quad (2.30)$$

Substituting (2.29) into (2.30), dividing the whole result by $\delta r \delta \theta \delta \varphi$, and rearranging, we obtain

$$\nabla^2 \Phi = \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \varphi^2} = 0. \quad (2.31)$$

2.5.2 Sources and Sinks

Imagine that uniform flow is output in all directions by a “source point” within an infinite fluid region. This flow is spherically symmetric with the source point at the center, and the flow velocity is zero at infinity.

This type of spherically symmetric flow can be solved by the Laplace equation. In spherical coordinates (r, θ, φ) as shown in Fig. 2.8, the velocity potential Φ becomes a function of r alone, and (2.31) is reduced to

$$\nabla^2 \Phi = \frac{d}{dr} \left(r^2 \frac{d\Phi}{dr} \right) = 0. \quad (2.32)$$

This general solution can be easily obtained by integration. With m and C as integration constants, it is given as

$$\Phi = -\frac{m}{r} + C. \quad (2.33)$$

The flow velocity is also a function of r alone, and it is given by

$$u_r = \frac{\partial \Phi}{\partial r} = \frac{m}{r^2}. \quad (2.34)$$

If $m > 0$, $u_r > 0$, and hence, the flow radiates from the origin. In fluid dynamics, this flow is described as originating from a **source**¹ at the origin. In contrast, if $m < 0$, the flow converges to the origin. This is called flow caused by the **sink**.

We now calculate the amount of flow Q released from the source point. Because flow is conserved, the amount of flow passing through an arbitrary closed curved surface S containing the origin is independent of the choice of the curved surface S . Assuming a spherical surface S with a radius r , Q is calculated as

$$Q = \iint_S \frac{\partial \Phi}{\partial n} dS = \iint_S u_r dS = \int_0^{2\pi} d\varphi \int_0^\pi \frac{m}{r^2} r^2 \sin \theta d\theta = 4\pi m. \quad (2.35)$$

Therefore, the constant m in (2.33) is related to the amount of flow released from the source point, known as the strength² of the source (If the origin is a sink, its strength is given by $-m = |m|$).

The constant C in (2.33) makes no contribution to the spatial differential value; hence, it can be arbitrarily selected. Setting $C = 0$ and substituting the result from (2.35), the velocity potential due to the source becomes

$$\Phi = -\frac{m}{r} = -\frac{Q}{4\pi} \left(\frac{1}{r} \right). \quad (2.36)$$

Equation (2.36) satisfies the Laplace equation at points other than the source ($r = 0$) but is singular at the origin. Let us consider the mass conservation law in fluid regions containing singular source points that release flow into the region. From (2.27), the amount of flow is given by

$$Q = \iint_S \frac{\partial \Phi}{\partial n} dS = \iiint_V \nabla^2 \Phi dV. \quad (2.37)$$

¹A source can also be referred to as an outlet.

²Sometimes, it is appropriate to define the flow volume Q , instead of m , as the strength of the source.

However, to satisfy the mass conservation law (expressed by a continuum equation), we must have

$$\nabla^2 \Phi = Q \delta(\mathbf{r}) \quad (2.38)$$

(see Note 2.3). Here $\delta(r)$ is the Dirac delta function that becomes 0 everywhere except at $r = 0$.

In summary, the velocity potential due to a source at the origin (2.36) satisfies (2.38). Mathematically, $Q = 1$ in (2.36) is called the fundamental solution³ of the Laplace equation.

If several (N) sources of different strengths exist at locations other than the origin, the total velocity potential is the sum of the N individual potentials. Furthermore, in the limiting case of a continuous distribution of sources within a region V , the summed velocity potential becomes an integral. Mathematically, the velocity potential is then given by

$$\Phi_P = -\frac{1}{4\pi} \sum_{k=1}^N \frac{Q_k}{r_k} \rightarrow -\frac{1}{4\pi} \iiint_V \frac{\sigma_Q}{r} dV ; \quad r = |\mathbf{r}_P - \mathbf{r}_Q|. \quad (2.39)$$

In (2.39), the subscripts P and Q represent a point and the integral point of interest in the fluid, respectively. In a continuous distribution, (2.38) becomes

$$\nabla \cdot \mathbf{u} = \nabla^2 \Phi = \sigma(\mathbf{r}) \quad (2.40)$$

where $\sigma(\mathbf{r})$ is called the density of the source distribution. Equation (2.40) is known as **Poisson's equation**.

Note 2.3 Dirac's delta function $\delta(x)$ can be defined in several ways; however, it is defined here as a generalized function that extracts the value at a given point.

If $F(x)$ is a continuous test function that disappears beyond a certain finite interval, the delta function possesses properties where the following is true:

$$\int_{-\infty}^{\infty} F(x) \delta(x - x_0) dx = \int_{-\infty}^{\infty} F(x + x_0) \delta(x) dx = F(x_0). \quad (2.41)$$

In other words, if an integral contains the delta function, the argument of the delta function is a generalized function (symbol function) that extracts the value of the test function $F(x_0)$ at $x = 0$.

Therefore, expanding in three dimensions as $x_0 = 0$, for any continuous function $F(r)$, we have

$$\iiint_V F(\mathbf{r}) \delta(\mathbf{r}) dV = F(0). \quad (2.42)$$

³Also known as the principal solution of the Laplace equation.

Let us reconsider (2.37). Since $\nabla^2 \Phi = 0$ if the origin ($r = 0$) is outside the integral region V , the integral of (2.37) is 0. Conversely, if the integral region encloses the origin, the integral of (2.37) gives the amount of flow Q . Comparing (2.37) with $F(\mathbf{r}) = Q$ (constant) and (2.42),

$$Q = \iiint_V Q \delta(\mathbf{r}) dV \quad (2.43)$$

we can obtain (2.38), that is, $\nabla^2 \Phi = Q \delta(\mathbf{r})$.

Note 2.4 We use the variable separation method to solve (2.31), assuming symmetry about the x -axis (i.e., no φ dependence). As Φ is a function of r and θ , we substitute $\Phi = R(r) \Theta(\theta)$ into (2.31) to obtain

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) \frac{1}{R} = - \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) \frac{1}{\Theta}. \quad (2.44)$$

The left-hand side of (2.44) is a function of r only, while the right-hand side is a function of θ only. For both sides to be equal, they must each equal a constant. Writing $n(n+1)$ for convenience, we get

$$\frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) - n(n+1) R = 0, \quad (2.45)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + n(n+1) \Theta = 0. \quad (2.46)$$

Since (2.45) is a differential equation of the same order, its solution is easily obtained. Denoting $\cos \theta$ by a new variable, z , (2.46) becomes a Legendre differential equation. Therefore, we obtain

$$\left. \begin{aligned} R(r) &= A r^n + \frac{B}{r^{n+1}}, \\ \Theta(\theta) &= C P_n(\cos \theta) + D Q_n(\cos \theta). \end{aligned} \right\} \quad (2.47)$$

However, A , B , C , and D are arbitrary constants, and P_n and Q_n are Legendre functions. Assuming that no singularities exist at $\cos \theta = \pm 1$, $D = 0$ and n must be integer values, $n = 0, 1, 2, \dots$. Furthermore, by adding another condition where $\nabla \Phi \rightarrow 0$ at infinity, $A = 0$ must be true, and we obtain

$$\Phi = \sum_{n=0}^{\infty} B_n \frac{P_n(\cos \theta)}{r^{n+1}}, \quad (2.48)$$

which is the solution to the Laplace equation.

Here $P_n(z)$ is called **Legendre's polynomial**. If no singularities exist at any θ of $z = \cos \theta$, $P_n(z)$ is given by

$$P_n(z) = \frac{1}{2^n n!} \frac{d^n}{dz^n} (z^2 - 1)^n, \quad (2.49)$$

$$P_0(\cos \theta) = 1, \quad P_1(\cos \theta) = \cos \theta, \quad \dots \quad (2.50)$$

When $n = 0$ in (2.48), the velocity potential reduces to (2.36) already derived from the source. When $n = 1$, (2.48) gives the velocity potential due to a doublet, as shown later.

Note 2.5 (2.48) can also be expressed as follows:

$$\Phi = \sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left(\frac{1}{r} \right). \quad (2.51)$$

To prove this mathematically, we use the relation equation of the Legendre function:

$$\frac{1}{\sqrt{(x - \xi)^2 + y^2 + z^2}} = \frac{1}{\sqrt{r^2 - 2r\xi \cos \theta + \xi^2}} = \sum_{n=0}^{\infty} \frac{P_n(\cos \theta)}{r^{n+1}} \xi^n. \quad (2.52)$$

Setting the left-hand side to $f(\xi)$, we obtain the following series expansion in ξ

$$f^{(n)}(0) = \left. \frac{\partial^n f(\xi)}{\partial \xi^n} \right|_{\xi=0} = (-1)^n \left. \frac{\partial^n f(\xi)}{\partial x^n} \right|_{\xi=0} = (-1)^n \frac{\partial^n}{\partial x^n} \left(\frac{1}{r} \right),$$

from which we find that

$$f(\xi) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \xi^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} \left(\frac{1}{r} \right) \xi^n. \quad (2.53)$$

Comparing (2.52) with (2.53), it becomes apparent that (2.48) and (2.51) are equal.

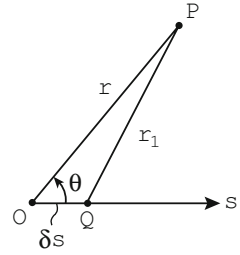
2.5.3 Doublet

Let us consider a situation where a source and a sink of the same strength are infinitely close to each other. In other words, consider a sink of strength m at origin O and a source of strength m at a point Q at a distance δs away (Fig. 2.9). From (2.36), we have

$$\Phi = \frac{m}{r} - \frac{m}{r_1}, \quad (2.54)$$

where $r_1 = \sqrt{r^2 - 2r\delta s \cos \theta + (\delta s)^2}$.

Fig. 2.9 Limiting values of source and sink



As $\delta s \rightarrow 0$, the limit approaches a fixed value $m\delta s \rightarrow \mu$. Here r_1 is equal to the left-hand side of (2.52), with $\xi = \delta s$. Alternatively, we can Taylor expand around $1/r_1$. As $\delta s \rightarrow 0$, both approaches give

$$\frac{1}{r_1} \sim \frac{1}{r} + \frac{\cos \theta}{r^2} \delta s + O((\delta s)^2). \quad (2.55)$$

Substituting (2.55) into (2.54), and denoting $m\delta s = \mu$, we obtain

$$\Phi = -\mu \frac{\cos \theta}{r^2}. \quad (2.56)$$

Equation (2.56) specifies the velocity potential of a **doublet** with the axis oriented along $\overrightarrow{OQ}$, and μ denotes its strength. This is none other than the solution to $n = 1$ in (2.48) shown in Note 2.4 (with $-\mu$ equivalent to B_1 in (2.48)). Equation (2.56) can be rewritten as

$$\Phi = -\frac{\mu}{r^2} \frac{\mathbf{r}}{r} \cdot \mathbf{e} = \mu \mathbf{e} \cdot \nabla \left(\frac{1}{r} \right) = -\mu \frac{\partial}{\partial s} \left(\frac{-1}{r} \right). \quad (2.57)$$

Here $\mathbf{e}$ is a unit vector in the direction $\overrightarrow{OQ}$. The vector $\mu \mathbf{e}$ is called the moment of the doublet. If $\mathbf{e}$ is oriented along the x -axis, (2.57) gives the solution to $n = 1$ in (2.51) shown in Note 2.5.

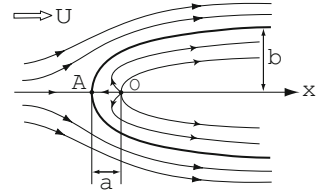
Equation (2.57) shows that the velocity potential of a doublet is the negative of the velocity potential $(-1/r)$ of a unit strength source differentiated along the direction of the doublet axis (in this case, differentiated with respect to s).

Differentiating (2.57) with respect to s , we obtain the velocity potential of two infinitely close doublets, i.e., the potential of a **quadrupole** source. Iterating this procedure, (2.51) can be physically interpreted as the velocity potential of 2^n overlapping sources (**multipoles**) oriented in the direction of the x -axis.

The above relationship between the multipole and unit strength source velocity potentials also holds in two-dimensional theory, as will be shown in Sect. 2.9.

Exercise 2.4 Consider a flow generated by the overlap of a uniform flow and a source at the origin O . This flow occurs when a uniform flow meets a cylindrical

Fig. 2.10 Flow around a semi-infinite body



body of semi-infinite length with a round end (semi-infinite body; see Fig. 2.10). Point A is a stagnation point.

The uniform flow has velocity U and moves in the positive direction of the x -axis. Considering the strength of the source as m , the velocity potential is

$$\Phi = Ux - \frac{m}{r}. \quad (2.58)$$

Obtain the distance a between the origin O and the stagnation point A and the cylindrical radius b under the infinite downstream produced by the branching streamlines from point A .

2.5.4 Flow Around a Sphere

Now consider a sphere of radius a moving through a static fluid with velocity U . The center of the sphere O is taken as the origin, and the propagation direction is the x -axis (see Fig. 2.11). In spherical coordinates (r, θ, φ) , considering the polar axis as the x -axis, the boundary condition equation for the velocity potential at the sphere's surface is (from 2.26)

$$\frac{\partial \Phi}{\partial r} = U \cos \theta \quad \text{at } r = a. \quad (2.59)$$

Given that $\nabla \Phi \rightarrow 0$ at $r = \infty$, the general solution satisfying this condition is (2.48) or (2.51), as shown above. Thus, the coefficient B_n that satisfies (2.59) is

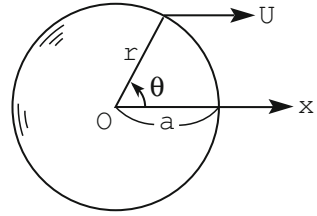
$$B_0 = 0, \quad B_1 = -\frac{1}{2} U a^3, \quad B_n = 0 \quad (n \geq 2). \quad (2.60)$$

Essentially,

$$\Phi = -\frac{U a^3 \cos \theta}{2 r^2} \quad (2.61)$$

is the velocity potential of the flow field around the sphere. As evident from (2.56), this represents the flow when there is a doublet of strength $U a^3/2$ in the x -axis direction. In other words, the flow induced by a sphere moving through a static fluid

Fig. 2.11 Sphere moving parallel to the x -axis with velocity U



is equal to the flow due to a doublet whose axis aligns in the direction of motion of the center of the sphere.

Next, we consider a sphere of radius a fixed in a uniform flow with velocity U . In this case, a uniform flow with flow velocity U in the negative x -axis direction can be superimposed onto the flow field disturbed by the sphere, as shown in Fig. 2.11. The velocity potential of this uniform flow is

$$\Phi = -Ux = -Ur \cos \theta. \quad (2.62)$$

Hence, the velocity potential for the whole is given by the sum of (2.61) and (2.62):

$$\Phi = -U \left(r + \frac{a^3}{2r^2} \right) \cos \theta, \quad (2.63)$$

and the corresponding flow velocity is

$$\left. \begin{aligned} u_r &= \frac{\partial \Phi}{\partial r} = -U \left(1 - \frac{a^3}{r^3} \right) \cos \theta, \\ u_\theta &= \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = U \left(1 + \frac{a^3}{2r^3} \right) \sin \theta. \end{aligned} \right\} \quad (2.64)$$

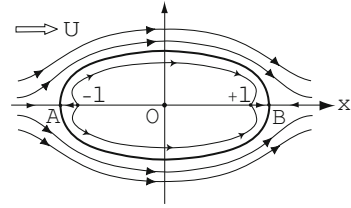
Clearly, $u_r = 0$ at the spherical surface ($r = a$). Therefore, the magnitude of the flow velocity at this surface is given by

$$q = |u_\theta| = \frac{3}{2} U |\sin \theta|. \quad (2.65)$$

The flow velocity is maximized at $\theta = \pi/2$, with magnitude $q_{\max} = 1.5U$. Moreover, at the stagnation points, which should lie on the x -axis (at $\theta = 0, \pi$), $q = 0$ from (2.65).

Exercise 2.5 Consider a uniform flow with velocity U with an overlapping source of strength m at $x = -\ell$ and a sink of strength m at $x = +\ell$. There are two stagnation points, A and B, as shown in Fig. 2.12. The curved surface created by diverging streamlines passing through these points is called the Rankine ovoid. Note that, in this way, when the sum of the strengths of the source and sink is 0, the diverging

Fig. 2.12 Flow around the Rankine ovoid



streamlines create a closed curved surface equivalent to a body in a uniform flow. The doublet expressing the flow around the sphere is an example of this.

Assume that the coordinates a of stagnation points due to the Rankine ovoid are functions of m and ℓ . Obtain the maximum radius b and the maximum flow velocity $u_{\max}$ of the diverging streamlines as functions of m , ℓ , and U .

2.6 Vector Potential

The previous section considered irrotational flows in the presence of source distribution within the flow field (including isolated sources). In other words,

$$\nabla \cdot \mathbf{u}_1 = \sigma, \quad \nabla \times \mathbf{u}_1 = 0. \quad (2.66)$$

This section focuses on rotational flow created by vorticity ω . In other words, consider a velocity field $\mathbf{u}_2$ satisfying

$$\nabla \cdot \mathbf{u}_2 = 0, \quad \nabla \times \mathbf{u}_2 = \omega. \quad (2.67)$$

Once $\mathbf{u}_1$ and $\mathbf{u}_2$ are obtained for (2.66) and (2.67), in a general case containing sources and vortices, it can be obtained from overlapping $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$.

As previously discussed, the velocity potential is derived from the condition $\nabla \times \mathbf{u} = 0$ on the basis of the identity equation $\nabla \times \nabla \Phi = 0$ of vector analysis. Similarly, another identity equation of vector analysis, $\nabla \cdot (\nabla \times \mathbf{A}) = 0$, can be established; thus, from the condition $\nabla \cdot \mathbf{u} = 0$ in (2.67), we can derive a **vector potential** $\mathbf{A}$ that satisfies

$$\mathbf{u} = \nabla \times \mathbf{A}. \quad (2.68)$$

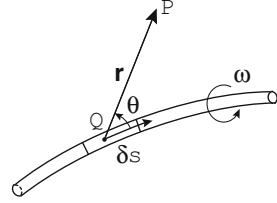
Since the condition $\nabla \cdot \mathbf{u} = 0$ holds for incompressible fluids, the vector potential can also be defined for viscous fluids.

Now, substituting (2.68) into the second equation of (2.67), we obtain

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \omega. \quad (2.69)$$

For convenience, we specify another condition for $\mathbf{A}$:

Fig. 2.13 Velocity at point P induced by a vortex filament



$$\nabla \cdot \mathbf{A} = 0 \quad (2.70)$$

(Whether the obtained solution actually satisfies this condition will be confirmed later (see Note 2.6).)

From (2.69) and (2.70), we have

$$\nabla^2 \mathbf{A} = -\boldsymbol{\omega} \quad (2.71)$$

as the governing equation on $\mathbf{A}$. Equation (2.71) is a Poisson equation of the same shape as (2.40); thus, its solution can be found in the same way as for (2.39):

$$\mathbf{A}_P = \frac{1}{4\pi} \iiint_V \frac{\boldsymbol{\omega}_Q}{r} dV; \quad r = |\mathbf{r}_P - \mathbf{r}_Q|. \quad (2.72)$$

Substituting (2.72) into (2.68), we get

$$\mathbf{u} = \frac{1}{4\pi} \iiint_V \nabla_P \times \frac{\boldsymbol{\omega}_Q}{r} dV = \frac{1}{4\pi} \iiint_V \boldsymbol{\omega}_Q \times \frac{\mathbf{r}}{r^3} dV, \quad (2.73)$$

which can be used to calculate the velocity induced by the vortex.

To investigate the meaning of (2.73), we consider the contribution from an infinitesimally small part of the vortex filament (Sect. 1.10) with the length of δs and cross-sectional area of σ . This filament passes through point Q (see Fig. 2.13). Since $dV = \sigma \delta s$ in this case, the flow velocity induced by point P based on (2.73) is

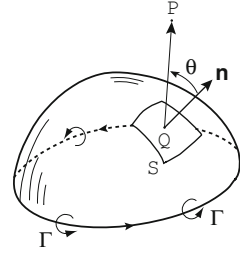
$$\delta \mathbf{u} = \frac{\boldsymbol{\omega}_Q \times \mathbf{r}}{4\pi r^3} \sigma \delta s = \frac{\omega_Q \sigma}{4\pi r^2} \left(\frac{\boldsymbol{\omega}_Q}{\omega_Q} \delta s \times \frac{\mathbf{r}}{r} \right) = \frac{\Gamma}{4\pi} \frac{\delta \mathbf{s} \times \mathbf{r}}{r^3}. \quad (2.74)$$

Here $\Gamma = \omega_Q \sigma$ is the strength of the vortex filament (the circulation indicated in (2.10)), and $\delta \mathbf{s} = (\omega_Q / \omega_Q) \delta s$ is the linear element vector along the axis of the vortex filament.

In (2.74), Γ (strength of the current) and $\mathbf{u}$ (magnetic field vector) are the same as those in the **Biot–Savart principle** of electromagnetism. In this regard, the vorticity-contributed velocity field behaves similar to an electromagnetic field.

The vector potential of rotating flows is comparable with the velocity potential of irrotational flows. However, the vector potential is still a vector; thus, it is the

Fig. 2.14 Relationship between vortex filament and doublet



same as the original velocity component as an unknown value. Therefore, the vector potential is not used as generally as the velocity potential.

Note 2.6 For (2.72), it is important to show that $\nabla \cdot \mathbf{A} = 0$:

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \frac{1}{4\pi} \iiint_V \omega_Q \cdot \nabla_P \left(\frac{1}{r} \right) dV = -\frac{1}{4\pi} \iiint_V \omega_Q \cdot \nabla_Q \left(\frac{1}{r} \right) dV \\ &= -\frac{1}{4\pi} \iiint_V \nabla_Q \cdot \left(\omega_Q \frac{1}{r} \right) dV \quad (\leftarrow \nabla \cdot \omega = 0) \\ &= -\frac{1}{4\pi} \iint_S \frac{\mathbf{n} \cdot \omega}{r} dS = 0 \quad (\leftarrow \mathbf{n} \cdot \omega = 0) \end{aligned}$$

Note 2.7 When a vortex filament of strength $\Gamma = \omega \sigma$ creates a closed curve C , it induces a velocity field equivalent to that of a doublet distribution on a curved surface S bounded by perimeter C .

The velocity field induced by a vortex filament of strength Γ , as shown in Fig. 2.14, is determined from (2.74) as

$$\mathbf{u}_P = \frac{\Gamma}{4\pi} \oint_C \frac{ds_Q \times \mathbf{r}}{r^3} = \frac{\Gamma}{4\pi} \oint_C ds_Q \times \nabla_Q \left(\frac{1}{r} \right). \quad (2.75)$$

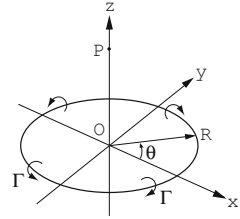
Applying Stokes' theorem to (2.75), and rearranging while paying careful attention to $\nabla_Q(1/r) = -\nabla_P(1/r)$, we obtain

$$\begin{aligned} \mathbf{u}_P &= -\frac{\Gamma}{4\pi} \iint_S (\mathbf{n} \times \nabla_Q) \times \nabla_P \left(\frac{1}{r} \right) dS \\ &= \nabla_P \left[-\frac{\Gamma}{4\pi} \iint_S \frac{\partial}{\partial n_Q} \left(\frac{1}{r} \right) dS \right]. \end{aligned} \quad (2.76)$$

This is identical to the velocity field in

$$\left. \begin{aligned} \mathbf{u} &= \nabla \Phi \\ \Phi &= -\frac{\Gamma}{4\pi} \iint_S \frac{\partial}{\partial n_Q} \left(\frac{1}{r} \right) dS = -\frac{\Gamma}{4\pi} \iint_S \frac{\cos \theta}{r^2} dS \end{aligned} \right\} \quad (2.77)$$

Fig. 2.15 Velocity induced by a circular vortex



This Φ can be interpreted as the velocity potential due to the uniform distribution of a doublet with the axis pointing normal to the curved surface S calculated by (2.56) or (2.57). Here the surface area S is introduced through Stokes' theorem; therefore, (2.77) holds for any curved surface bounded by a closed curve with perimeter C .

Exercise 2.6 Using the Biot–Savart principle, show that the velocity induced by an infinitely long and straight vortex filament is given by

$$u = \frac{\Gamma}{2\pi h} \quad (2.78)$$

where h is the perpendicular distance from point P to the straight vortex filament.

Exercise 2.7 Consider a circular vortex filament of radius R and strength Γ as shown in Fig. 2.15. Obtain the velocity w induced in the z -axis direction at point P(0, 0, z) as a function of Γ , R , and z using the following two methods:

1. Biot–Savart analogy (2.74)
2. Velocity potential of the doublet distribution (2.77)

Then, show that the solutions from both methods are the same.

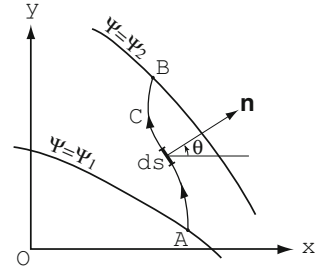
2.7 Stream Function

As explained in Sect. 2.6, the vector potential is a less useful concept than the velocity potential. However, in two- or three-dimensional problems with axial symmetry, it becomes a scalar function and can then be used analogously to the velocity potential.

2.7.1 Two-Dimensional Flow

We first consider a two-dimensional problem. In two-dimensional problems, $u = (u, v, 0)$ is a function of (x, y) alone. The vector potential then takes the form $\mathbf{A} = (0, 0, \Psi)$. Here

Fig. 2.16 Relationship between streamline, flow amount, and flow velocity



$$\mathbf{u} = \nabla \times \mathbf{A} = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x}, 0 \right). \quad (2.79)$$

Note that Ψ is none other than the **stream function**.⁴ This is because the curve, $\Psi = \text{const}$, equals the streamline. For example, from (2.79), we have

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = -v dx + u dy = 0. \quad (2.80)$$

Therefore,

$$\frac{dx}{u} = \frac{dy}{v}. \quad (2.81)$$

Clearly, when $\Psi = \text{const}$, the streamline line element ds is parallel to the flow velocity $\mathbf{u}$.

Note that the only condition for the stream function is $\nabla \cdot \mathbf{u} = 0$; thus, it can be defined for incompressible viscous fluids. Actually, it is easy to verify that $\nabla \cdot \mathbf{u} = 0$ is automatically satisfied by (2.79).

Now consider two streamlines $\Psi = \Psi_1$ and $\Psi = \Psi_2$ as shown in Fig. 2.16. Take an arbitrary curve C joining point A on Ψ_1 and point B on Ψ_2 and specify a normal vector $\mathbf{n} = (n_x, n_y)$ whose positive direction is left to right. The line element $ds = (dx, dy)$ on C is described by

$$dx + i dy = ds e^{i(\theta + \pi/2)} = i ds (n_x + i n_y). \quad (2.82)$$

From (2.79) and (2.82), the rate of change in Ψ along ds is given by

$$\begin{aligned} \frac{\partial \Psi}{\partial s} &= \frac{\partial \Psi}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial \Psi}{\partial y} \frac{\partial y}{\partial s} = -\frac{\partial \Psi}{\partial x} n_y + \frac{\partial \Psi}{\partial y} n_x \\ &= v n_y + u n_x = \mathbf{u} \cdot \mathbf{n} = u_n. \end{aligned} \quad (2.83)$$

⁴Often called a flow function or streamline function.

Therefore,

$$[\Psi]_A^B = \int_C d\Psi = \int_C u_n ds = Q. \quad (2.84)$$

In other words, $\Psi_2 - \Psi_1$ specifies the amount of flow between the two streamlines. Moreover, (2.83) indicates that when seeking a directional component of velocity $\mathbf{u}$, the derivative of Ψ in the counterclockwise right-angled direction should be calculated. In two-dimensional polar coordinates, the velocity components are given by

$$u_r = \frac{\partial \Psi}{r \partial \theta}, \quad u_\theta = -\frac{\partial \Psi}{\partial r}. \quad (2.85)$$

We now seek the stream function (vector potential) induced by two-dimensional vortices. The vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ only has a z -component (i.e., perpendicular to the x - y plane), and

$$\boldsymbol{\omega} = (0, 0, \omega), \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \quad (2.86)$$

Thus, substituting (2.79), we can see that

$$\nabla^2 \Psi = -\omega \quad (2.87)$$

is a governing equation and is identical to (2.71). Now, by paying attention to the fact that the solution in the two-dimensional problem that is equivalent to the solution, $1/4\pi r$, in the three-dimensional problem is $-(1/2\pi) \log r$ (see (2.120) below), similar to (2.72), we can obtain

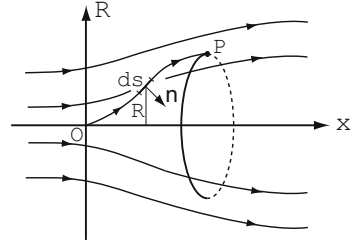
$$\Psi_P = -\frac{1}{2\pi} \iint_S \omega_Q \log r dS; \quad r = |\mathbf{r}_P - \mathbf{r}_Q|. \quad (2.88)$$

Moreover, if vortices are concentrated at the origin, and equivalently, if the flow contains vortex filaments (with cross-sectional area σ), then similar to (2.38), we can replace ω_Q with $\Gamma \delta(\mathbf{r}_Q)$. From the property of the delta function shown in Note 2.3, the following expressions apply:

$$\Psi_P = -\frac{\Gamma}{2\pi} \log r, \quad \Gamma = \omega_Q \sigma. \quad (2.89)$$

Later, this result will be derived again from the complex velocity potential (see (2.122)).

Fig. 2.17 Three-dimensional axisymmetric flow



2.7.2 Three-Dimensional Axisymmetric Flow

Next, let us consider a three-dimensional **axisymmetric** flow.⁵ In a cylindrical coordinate system (x, R, θ) with symmetry about the x -axis, the flow field is independent of θ . Therefore, in a plane center containing the axis of symmetry (see Fig. 2.17), axisymmetric flow is seen to be similar to a two-dimensional flow. However, a real flow would rotate Fig. 2.17 around the x -axis; consequently, the vortex lines would circle in the plane perpendicular to the x -axis.

If a point P in the flow is connected to the origin O by an arbitrary planar curve C, then similar to the derivation of (2.84), the amount of flow passing from left to right of the rotation surface, obtained by rotating C around the x -axis, is given by

$$Q = [\Psi]_O^P = \int_C d\Psi = \int_C u_n 2\pi R ds. \quad (2.90)$$

For axisymmetric flows, the effect of rotation around the x -axis is considered, and it can be redefined as $\Psi = 2\pi \Psi_a$. Then, the velocity in the normal direction can be obtained with

$$u_n = \frac{1}{R} \frac{\partial \Psi_a}{\partial s}. \quad (2.91)$$

The velocity components along the x - and R -axes, (u_x, u_R) , are calculated from

$$u_x = \frac{1}{R} \frac{\partial \Psi_a}{\partial R}, \quad u_R = -\frac{1}{R} \frac{\partial \Psi_a}{\partial x}, \quad (2.92)$$

as the δs components of u_x and u_R are δR and $-\delta x$, respectively.

$\Psi_a = \Psi/2\pi$ defined in (2.91) is equivalent to the stream function in a two-dimensional flow and is called **Stokes' stream function**. It is worth reiterating that, in three-dimensional problems, such scalar functions are limited to axisymmetric flows.

⁵Here axisymmetric flow only refers to flow with a zero velocity component in the θ -direction.

Exercise 2.8 The above-described flows around semi-infinite bodies (see Fig. 2.10), spheres (see Fig. 2.11), and the Rankine ovoid (see Fig. 2.12) are all three-dimensional axisymmetric flows. Obtain the Stokes' stream function Ψ_a in each of these flows.

2.8 Complex Velocity Potential

In this section, we restrict our discussion to incompressible, irrotational flows in two-dimensional problems. As discussed in Sect. 2.5, a velocity potential exists since it is irrotational flow, and it can be expressed from (2.23) and (2.40):

$$(u, v) = \left(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y} \right), \quad (2.93)$$

$$\nabla^2 \Phi = \sigma(x, y). \quad (2.94)$$

Here, $\sigma(x, y)$ is the source distribution density within the fluid.

On the other hand, as mentioned in the Sect. 2.7, the continuity equation gives rise to a stream function (two-dimensional vector potential) Ψ , and from (2.79) and (2.87), we can obtain

$$(u, v) = \left(\frac{\partial \Psi}{\partial y}, -\frac{\partial \Psi}{\partial x} \right), \quad (2.95)$$

$$\nabla^2 \Psi = -\omega(x, y). \quad (2.96)$$

Here $\omega(x, y)$ is the component of the vorticity vector perpendicular to the x - y plane.

In the fields in which the source distribution $\sigma(x, y)$ and vorticity distribution $\omega(x, y)$ are isolated on a certain point, σ and ω behave as delta functions, and in fluid regions outside those point, we have

$$\nabla^2 \Phi = 0, \quad \nabla^2 \Psi = 0. \quad (2.97)$$

In other words, Φ and Ψ are both harmonic functions. Now, from (2.93) and (2.95), we have

$$\left. \begin{aligned} u &= \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \\ v &= \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}. \end{aligned} \right\} \quad (2.98)$$

Equation (2.98) is the **Cauchy–Riemann relation** derived from the complex function theory. That is,

$$f(z) = \Phi(x, y) + i \Psi(x, y), \quad z = x + i y. \quad (2.99)$$

Equation (2.98) expresses the necessary and sufficient conditions for $f(z)$ to be an analytic function of z . This $f(z)$ is called a **complex velocity potential**.

Note 2.8 If $f(z)$ is analytic (holomorphic) at $z = z_0$, $f(z)$ is by definition differentiable at $z = z_0$ and its vicinity. In the case of real functions, in order for the derivative $f'(x)$ to exist at $x = x_0$, the right- and left-hand limits must be the same. In complex functions, this type of one-dimensional approach cannot work, and it has to be unrelated to the direction of approach. Instead, the required conditions are the Cauchy–Riemann relations, as elaborated below.

First, the differential of a complex function $f(z)$ is defined as

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}, \quad h = \delta x + i \delta y. \quad (2.100)$$

Here from (2.99),

$$f(z+h) - f(z) = \frac{\partial \Phi}{\partial x} \delta x + \frac{\partial \Phi}{\partial y} \delta y + i \left(\frac{\partial \Psi}{\partial x} \delta x + \frac{\partial \Psi}{\partial y} \delta y \right) + O(h^2).$$

Therefore,

$$f'(z) = \lim_{h \rightarrow 0} \frac{(\Phi_x + i \Psi_x) \delta x + (\Phi_y + i \Psi_y) \delta y}{\delta x + i \delta y}. \quad (2.101)$$

As the method of $h \rightarrow 0$ is arbitrarily obtained, when $\delta y = 0$ or $\delta x = 0$, we have

$$f'(z) = \Phi_x + i \Psi_x = \Psi_y - i \Phi_y. \quad (2.102)$$

We have stated (2.102) without proof, but the equation is clearly satisfied by (2.98). In other words, a derivative function $f'(z)$ exists if (2.98) holds.

Next, if (2.98) is assumed in (2.101), we have

$$\begin{aligned} f'(z) &= \lim_{h \rightarrow 0} \frac{(\Phi_x + i \Psi_x) \delta x + (-\Psi_x + i \Phi_x) \delta y}{\delta x + i \delta y} \\ &= \lim_{h \rightarrow 0} \frac{(\Phi_x + i \Psi_x)(\delta x + i \delta y)}{\delta x + i \delta y}. \end{aligned} \quad (2.103)$$

Regardless of the value of $h = \delta x + i \delta y$, the limit approaches a fixed value $f'(z) = \Phi_x + i \Psi_x$ as $h \rightarrow 0$. In other words, (2.98) is a sufficient condition for differentiability of $f(z)$.

Equations (2.98) and (2.99) can be rearranged similar to (2.103) to give

$$\frac{df}{dz} = \frac{\partial \Phi}{\partial x} + i \frac{\partial \Psi}{\partial x} = u - i v = q e^{-i \theta} \quad (2.104)$$

(see Fig. 2.18.) This quantity is called **complex velocity**.

Fig. 2.18 Relationship between complex velocity, and q and θ

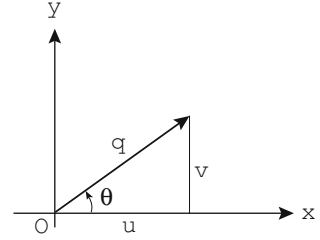
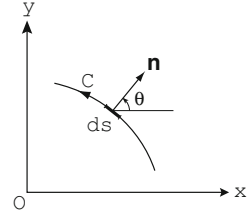


Fig. 2.19 Relationship between the normal line, and dz and ds



Now consider a contour integral of the complex velocity along an arbitrary closed curve C on the z -plane.

$$\oint_C \frac{df}{dz} dz = \oint_C df = [f]_C = [\Phi]_C + i [\Psi]_C. \quad (2.105)$$

As shown with (2.82), we have $dz = dx + i dy = i ds(n_x + i n_y)$ (see Fig. 2.19). Together with (2.104), this gives

$$\begin{aligned} \oint_C \frac{df}{dz} dz &= \oint_C (u dx + v dy) + i \oint_C (u n_x + v n_y) ds \\ &= \oint_C \mathbf{u} \cdot d\mathbf{r} + i \oint_C u_n ds = \Gamma(C) + i Q(C). \end{aligned} \quad (2.106)$$

The relationship equations obtained from (2.105) and (2.106) are same as those described in (2.12) and (2.84). In other words, $[\Phi]_C$ is the same as the circulation along C , $\Gamma(C)$, and $[\Psi]_C$ is the same as the amount of flow passing from left to right through C , $Q(C)$.

As already noted in (2.94) and (2.96), vortex filaments and isolated sources concentrated in the flow field are permitted in incompressible, irrotational flows, and we considered fluid region excluding these features. Therefore, if such concentrated activities occur within a closed curve C , neither $\Gamma(C)$ nor $Q(C)$ is 0. In other words, $f(z) = \Phi + i \Psi$ must be a multi-valued function. However, as this polyvalence is a constant, it is lost when differentiating the function and the complex velocity becomes a single-valued function.

2.9 Simple Two-Dimensional Potential Flows

In Sect. 2.5, for three-dimensional problems, we discussed the velocity potential induced by a source or a doublet and the velocity potential that expresses the flows around a sphere. Here we discuss the two-dimensional flows corresponding to these complex velocity potentials and evaluate their characteristics.

2.9.1 Uniform Flow

We denote the uniform flow velocity by U (> 0). If the direction of the flow and the x -axis form the angle, α , from (2.104), we obtain

$$f(z) = U e^{-i\alpha} z. \quad (2.107)$$

Equation (2.107) can be interpreted as the rotation of the coordinate axis. In other words, as shown in Fig. 2.20, if the O - xy coordinate system ($z = x + i y$) is rotated counterclockwise through an angle α to yield the O - XY coordinate system ($Z = X + i Y$), the systems are related by

$$Z = z e^{-i\alpha}. \quad (2.108)$$

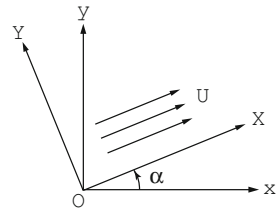
The complex velocity potential of the uniform flow in the positive direction of the x -axis in the O - XY coordinate system is $f(Z) = UZ$. When this is expressed with the O - xy coordinate system, it is clear that (2.108) can be substituted to obtain (2.107).

2.9.2 Flow Around a Corner

Consider

$$f(z) = A z^n \quad (A > 0, n > 0) \quad (2.109)$$

Fig. 2.20 Uniform flow parallel to the x -axis



As already discussed, when $n = 1$, the flow is uniform. Substituting $z = r \exp(i\theta)$ in (2.109), we obtain

$$f(z) = A r^n e^{in\theta} = A r^n (\cos n\theta + i \sin n\theta), \quad (2.110)$$

which can be separated into real and complex parts:

$$\Phi = A r^n \cos n\theta, \quad \Psi = A r^n \sin n\theta. \quad (2.111)$$

$\Psi = \text{constant}$ represents the streamlines as shown in (2.81); thus, setting $\Psi = C$ ($C > 0$) in (2.111) gives

$$r = \left(\frac{C}{A}\right)^{\frac{1}{n}} \frac{1}{(\sin n\theta)^{1/n}}. \quad (2.112)$$

The special case of $C = 0$ is an asymptote of these streamlines. As $\sin n\theta = 0$, from (2.111), we have

$$\theta = k \frac{\pi}{n}, \quad (k = 0, \pm 1, \pm 2, \dots). \quad (2.113)$$

These terms describe radial lines passing through the origin, obtained by successively rotating the positive component of the x -axis ($k = 0$) through π/n in the positive and negative directions. A group of straight lines can be regarded as a fixed wall.

From (2.109), the complex velocity is given as follows:

$$\frac{df}{dz} = A n z^{n-1} = A n r^{n-1} e^{i(n-1)\theta}. \quad (2.114)$$

Thus, the magnitude of velocity is

$$q = \left| \frac{df}{dz} \right| = A n r^{n-1}. \quad (2.115)$$

If $n > 1$ in (2.113), the angle encountered by the flow is less than π , as shown in Fig. 2.21, and the origin ($r = 0$) is a stagnation point. However, at infinity ($r \rightarrow \infty$), the flow velocity is infinitely large. On the other hand, if $1/2 < n < 1$, the flow passes around a convex corner (see Fig. 2.22) and its velocity at the origin is infinite. According to Bernoulli's theorem (2.18), an infinitely large flow velocity indicates that the pressure is infinitely negative ($p = -\infty$), clearly contradicting real phenomena. Therefore, fluid flows around convex angles cannot be assumed as inviscid and incompressible.

The special case where $n = 1/2$ indicates flow along a plate of semi-infinite length extending from the origin in the x -axis direction. Flow around a plate will be discussed in Sect. 2.11.

Fig. 2.21 Flow around a corner formed by two walls ($n > 1$)

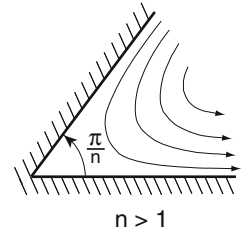
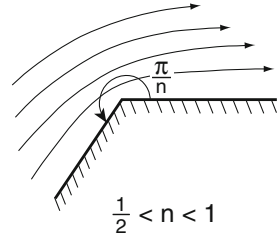


Fig. 2.22 Flow going at a convex angle ($1/2 < n < 1$)



2.9.3 Sources and Sinks

Consider the following

$$f(z) = m \log z \quad (m: \text{real number}). \quad (2.116)$$

Setting $z = r \exp(i\theta)$ in (2.116) gives

$$f(z) = m \log(r e^{i\theta}) = m(\log r + i\theta).$$

Splitting this expression into real and complex components, we obtain

$$\Phi = m \log r, \quad \Psi = m \theta. \quad (2.117)$$

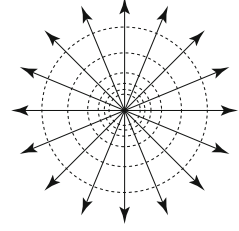
Since $\Psi = \text{constant}$ represents a streamline, setting $\theta = \text{constant}$ specifies a group of lines radiating from the origin.

Thus, the flow velocity in the radial direction is given by

$$u_r = \frac{\partial \Phi}{\partial r} = \frac{\partial \Psi}{r \partial \theta} = \frac{m}{r}, \quad (2.118)$$

from which we see that if $m > 0$, then $u_r > 0$ (see Fig. 2.23). Therefore, similar to the three-dimensional problem, the flow when $m > 0$ is called a flow by a two-dimensional source of the strength m . If $m < 0$, the flow of strength m is drawn into a sink.

Fig. 2.23 Streamlines emitted by a source



The change in Ψ throughout a single cycle about the origin calculated from (2.105) and (2.106) is

$$Q(C) = [\Psi]_C = m [\theta]_C = 2\pi m. \quad (2.119)$$

In other words, the flow amount Q is a 2π multiple of the strength of the source⁶ m .

Therefore, the velocity potential due to a two-dimensional source, given by (2.117), is

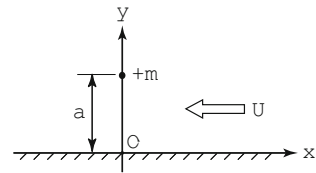
$$\Phi = m \log r = \frac{Q}{2\pi} \log r = -\frac{Q}{2\pi} \log\left(\frac{1}{r}\right). \quad (2.120)$$

It should be noted that (2.36) is the corresponding three-dimensional representation of (2.120). In other words, setting $Q = 1$ in (2.120) yields the basic solution of the two-dimensional Laplace equation (this argument was used in the explanation of (2.88)).

Exercise 2.9 Assume that the x -axis is a rigid wall surface (floor). Consider a two-dimensional source (strength is m) above the origin O at a distance a , and an uniform flow passing through this system from right to left at a speed U (Fig. 2.24). In this situation,

1. Obtain the complex velocity potential.
2. Obtain the pressure distribution along the x -axis, and calculate the point where the pressure will be the lowest.

Fig. 2.24 Source and uniform flow on a rigid wall surface



⁶Similar to the three-dimensional case, Q is a measure of the strength of the source.

2.9.4 Vortex Filaments

Consider the following:

$$f(z) = -i\kappa \log z \quad (\text{where } \kappa \text{ is a real number}). \quad (2.121)$$

Similar to (2.117), we have

$$f(z) = -i\kappa(\log r + i\theta).$$

Therefore,

$$\Phi = \kappa\theta, \quad \Psi = -\kappa \log r. \quad (2.122)$$

As $\Psi = \text{constant}$ is defined by $r = \text{const}$, the streamlines of this flow form a group of concentric circles centered at the origin (Fig. 2.25).

The circumferential component of the velocity is given by (2.122), (2.85), and (2.98) as

$$u_\theta = \frac{\partial \Phi}{r \partial \theta} = -\frac{\partial \Psi}{\partial r} = \frac{\kappa}{r} = \frac{\Gamma}{2\pi r}. \quad (2.123)$$

If $\kappa > 0$, the flow is counterclockwise (see Fig. 2.24). Moreover, (2.123) is identical to (2.78) calculated from the Biot–Savart principle.

Performing a contour integration along an arbitrary closed curve C enclosing the origin, the circulation along C is determined from (2.105) and (2.106) as follows:

$$\Gamma(C) = [\Phi]_C = \kappa [\theta]_C = 2\pi\kappa. \quad (2.124)$$

Equation (2.124) describes the flow when the vorticity is focused at the origin (a vortex filament). The stream function described by (2.122) and (2.124) is consistent with (2.88), previously obtained by solving the Poisson equation.

Exercise 2.10 Figure 2.26 illustrates a vortex filament with a counterclockwise circulation Γ at distance a above the origin O in place of an outlet. The conditions are similar to those in Exercise 2.9. In this situation,

1. Obtain the complex velocity potential.

Fig. 2.25 Streamlines induced by a vortex filament

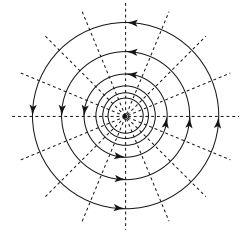


Fig. 2.26 Vortex filament and uniform flow on rigid surface

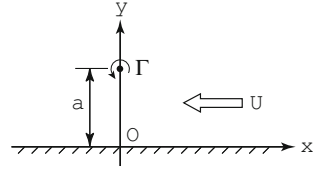
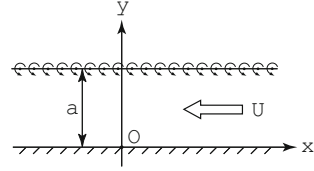


Fig. 2.27 Continuous vortex layer and uniform flow on a rigid surface



- Obtain the condition (value of Γ) under which the vortex filament settles at the original position $(0, a)$.

Exercise 2.11 Consider an infinitely long, continuous vortex layer (strength per unit length is γ) parallel to and at a distance a above the x -axis (Fig. 2.27). Furthermore, assume that the x -axis is a rigid wall surface. If uniform flows passing through this system from right to left overlap, obtain the velocity distribution along the y -axis and the condition (value of γ) under which the velocity at the wall is 0.

2.9.5 Doublet

Similar to the three-dimensional problem, a doublet is formed when a source becomes infinitely close to a sink of identical strength m .

When a sink of strength m at the origin O is separated from a source of strength m by distance δs (Fig. 2.28), the complex velocity potential is given by

$$\begin{aligned} f(z) &= m \log \frac{z - z_0}{z} = m \log \left(1 - \frac{z_0}{z} \right) \\ &= -m \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{z_0}{z} \right)^n. \end{aligned} \quad (2.125)$$

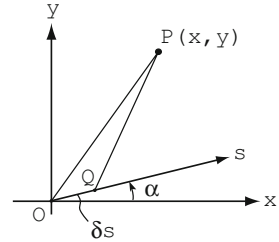
Here $z_0 = \delta s \exp(i\alpha)$, and $\delta s \rightarrow 0$.

Instead, if we consider the limiting case $m \delta s \rightarrow \bar{\mu}$, the velocity potential becomes

$$f(z) = -\bar{\mu} \frac{e^{i\alpha}}{z} \quad (2.126)$$

$$= -\bar{\mu} \frac{d}{dz} (\log z), \quad (2.127)$$

Fig. 2.28 Limiting value of the two-dimensional source and sink



where $\mu = \bar{\mu} e^{i\alpha}$. Equation (2.127) is equivalent to (2.57) in the three-dimensional problem. That is, the complex velocity potential of a doublet is obtained by differentiating the complex velocity potential of a unit strength source with respect to z and reversing its sign.

Again, analogous to the three-dimensional problem, differentiating (2.126) with respect to z yields the multipole complex velocity potential, which is the general solution to the two-dimensional Laplace equation. In other words, the solutions corresponding to (2.51) or (2.48) in the three-dimensional problem are given by

$$f(z) = k_0 \log z + \sum_{n=1}^{\infty} \frac{k_n}{z^n} \quad (2.128)$$

$$= k_0 \log z + \sum_{n=1}^{\infty} k_n \frac{(-1)^{n+1}}{(n-1)!} \frac{d^n}{dz^n} (\log z) \quad (2.129)$$

The negative power series term in (2.128) is infinitely holomorphic and is known as Laurent's series in complex function theory. (A positive power series represents flow around the corner of two walls as shown in (2.109) and infinite flow velocity at infinite distance.)

Note 2.9 Let us solve the two-dimensional Laplace equation by the method of separation of variables. The Laplace equation in polar coordinates (r, θ) is

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = 0. \quad (2.130)$$

When assuming $\Phi = R(r)\Theta(\theta)$ and substituting into (2.130), we can rearrange as follows:

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) \frac{1}{R} = - \frac{d^2 \Theta}{d\theta^2} \frac{1}{\Theta} = n^2 \text{ (a constant)}. \quad (2.131)$$

Thus, these can be rewritten as two ordinary differential equations

$$r \frac{d}{dr} \left(r \frac{dR}{dr} \right) - n^2 R = 0, \quad (2.132)$$

$$\frac{d^2 \Theta}{d\theta^2} + n^2 \Theta = 0. \quad (2.133)$$

Separately considering $n = 0$ and $n \neq 0$, the solution for which $\nabla \Phi \rightarrow 0$ at infinity is given by

$$\Phi = A_0 \log r + B_0 \theta + \sum_{n=1}^{\infty} \frac{1}{r^n} (A_n \cos n\theta - B_n \sin n\theta). \quad (2.134)$$

This is the same expression as the one obtained by substituting $z = r \exp(i\theta)$ and $k_n = A_n - iB_n$ into the Laurent series (2.128) and considering the real part ($\Phi = \text{Re } f$).

Exercise 2.12 Similar to the three-dimensional problem, the flow around a two-dimensional semi-infinite body can be obtained by overlapping a uniform flow with a source at origin O. Obtain the complex velocity potential of such a flow. Calculate the distance a between the origin and stagnation point and the height b of a semi-infinite body placed infinitely downstream (see Fig. 2.10). Compare with the results of the three-dimensional problems and discuss the three-dimensional effect.

2.9.6 Flow Around a Cylinder

Equation (2.128) is the generalized two-dimensional complex velocity potential for which $\nabla \Phi \rightarrow 0$ at infinity. We now use this to determine uniform flows around a cylinder.

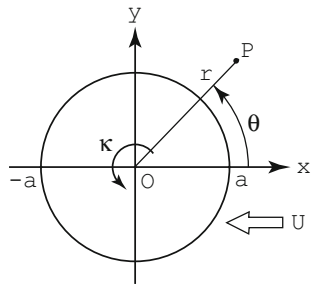
Suppose that a cylinder of radius a is placed in a uniform flow along the x -axis in the negative direction (Fig. 2.29). Overlapping the velocity potential of a uniform flow on (2.128), we obtain the complex velocity potential:

$$f(z) = -Uz + k_0 \log z + \sum_{n=1}^{\infty} \frac{k_n}{z^n}. \quad (2.135)$$

By (2.26), the body surface condition is $V_n = 0$. This condition is equivalent to $\Psi = \text{Im } f = \text{const}$ when setting $z = a \exp(i\theta)$, that is, the cylinder surface is a streamline, we thus have

$$\Psi = -Ua \sin \theta + \kappa \log a + m \theta - \sum_{n=1}^{\infty} \frac{1}{a^n} (B_n \cos n\theta + A_n \sin n\theta), \quad (2.136)$$

Fig. 2.29 Two-dimensional flow around a cylinder



where $k_0 = m - i\kappa$ and setting $k_n = A_n - iB_n$ have been substituted. The condition of $\Psi = \text{constant}$ gives the following conditions:

$$m = 0, \quad A_1 = -Ua^2, \quad B_1 = 0, \quad k_n = 0 \quad (n \geq 2) \quad (2.137)$$

Substituting these conditions into (2.135), the complex velocity potential is given by

$$f(z) = -U \left(z + \frac{a^2}{z} \right) - i\kappa \log z. \quad (2.138)$$

Same as in the three-dimensional problem, it indicates the flow introduced by a cylinder because of a doublet of strength Ua^2 whose axis opposes the flow direction (i.e., whose axis is oriented along the positive x -axis). This solution differs from that of the three-dimensional problem in the presence of the counterclockwise vortex filament $-i\kappa \log z$ at the origin. It should be noted that, the strength κ of this vortex filament is unknown at this moment. In other words, to determine κ , aside from the dynamic surface condition of a body such as normal velocity $= 0$, another condition is necessary. This extra condition will likely be imposed by viscous effects, such as a specific stagnation point on the body surface. Kelvin's circulation theorem states that the circulation around a body is related to the initial vortices shed from boundary layers close to the surface of the body. The magnitude of this circulation is presumably determined by viscous effects inside the boundary layer. Later, we will demonstrate that the circulation around a two-dimensional wing moves the stagnation point toward the back edge of the wing, a condition known as **Kutta's condition**.

We now determine the flow velocity at the cylinder surface. Substituting $z = a \exp(i\theta)$ into (2.138), we obtain

$$\Phi = -2Ua \cos \theta + \kappa \theta, \quad (2.139)$$

$$u_\theta = \frac{\partial \Phi}{a \partial \theta} = 2U \sin \theta + \frac{\kappa}{a}. \quad (2.140)$$

In the absence of circulation ($\kappa = 0$), the flow velocity is given by

$$q = |u_\theta| = 2U |\sin \theta|, \quad (2.141)$$

whose maximum value is $q_{\max} = 2U$. On a three-dimensional spherical surface, the maximum flow velocity is 1.5 times the uniform velocity U (see (2.65)). As the flow is restricted to the x - y plane in two-dimensional problems, the increase in speed around the body is greater than that in a three-dimensional flow field. To obtain the pressure distribution, we substitute (2.140) into Bernoulli's pressure equation to get

$$\begin{aligned} p &= p_0 - \frac{1}{2} \rho q^2 = p_0 - \frac{1}{2} \rho u_\theta^2 \\ &= \rho \left(U^2 \cos 2\theta - 2U \frac{\kappa}{a} \sin \theta \right) + \text{const}. \end{aligned} \quad (2.142)$$

The fluid force acting on the cylinder is obtained by integrating the pressure. As the positive direction of the normal is outward from the cylinder, it is given by $n_x + i n_y = \exp(i\theta)$, and we obtain

$$\left. \begin{aligned} X &= - \int_0^{2\pi} p \cos \theta a d\theta = 0, \\ Y &= - \int_0^{2\pi} p \sin \theta a d\theta = \rho U \Gamma, \end{aligned} \right\} \quad (2.143)$$

where $\Gamma = 2\pi\kappa$ is the (counterclockwise) circulation around the cylinder.

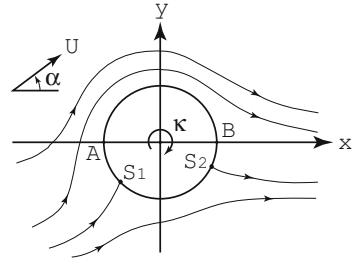
The force components acting parallel and perpendicular to the flow are called **drag** and **lift**, respectively. According to (2.143), the drag is 0, which contradicts observed phenomena. This situation, called **d'Alembert's paradox**, arises because viscosity generates a large wake behind the cylinder, whereas the streamlines of inviscid flows are right-left symmetric; hence, the pressure is also right-left symmetric.

On the other hand, the lift force according to the Kutta–Joukowski theorem (introduced in (2.16)) is $\rho U \Gamma$. If a counterclockwise circulation coexists with uniform right-left flow, the flow is accelerated more on the upper side of the cylinder than on the lower side, and the pressure is reduced by Bernoulli's theorem. This pressure difference generates lift. The same principle, called the **Magnus effect**, operates when rotation is added to a baseball curve or drop throw.

Exercise 2.13 Obtain the complex velocity potential of a uniform flow around a cylinder of radius a , tilted by angle α above against the x -axis (Fig. 2.30). Assume that the strength of the clockwise vortex filament is κ . Consider two stagnation points (S_1 and S_2) on the surface of the cylinder and show what happens when κ is determined under the condition that point S_2 matches the back end (point B).

Exercise 2.14 A cylinder of radius a is intercepted by a uniform flow in the positive x -axis direction (flow velocity U). Show that the point at which the flow velocity is cU lies on the curve $(c^2 - 1)r^4 + 2a^2 r^2 \cos 2\theta = a^4$. Moreover, show that the point at which the flow velocity forms an angle α with the x -axis lies on the curve $r^2 \sin \alpha = a^2 \sin(\alpha + 2\theta)$.

Fig. 2.30 Calculation of circulation flow around a cylinder



2.10 Forces Acting on a Body

In the previous section, (2.143) was solved for flows around a cylinder. However, it can be shown that (2.143) holds for more general, arbitrary bodies in two dimensions. The force acting on a body in a flow $\mathbf{F} = (X, Y)$ is obtained by integrating the pressure along the body surface; hence,

$$\mathbf{F} = - \oint_{C_B} p \mathbf{n} ds. \quad (2.144)$$

Here C_B is a closed curve representing the body surface, and the positive direction of the normal points away from the body. The pressure p is

$$p = -\rho \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} q^2 \right). \quad (2.145)$$

As shown in (2.82), the boundary surface satisfies

$$dz = dx + i dy = i ds (n_x + i n_y),$$

i.e.,

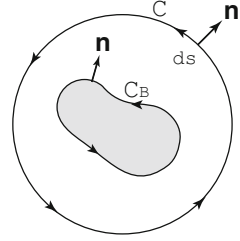
$$(n_x + i n_y) ds = -i dz. \quad (2.146)$$

From the above expressions, we obtain the following complex equation:

$$X + i Y = -i \frac{\rho}{2} \oint_{C_B} \left\{ \frac{\partial}{\partial t} (f + f^*) + \left| \frac{df}{dz} \right|^2 \right\} dz, \quad (2.147)$$

where f^* is the complex conjugate of f . The moment acting on a body can be calculated similarly. Denoting the moment around the origin as M , from (2.146), $n_x ds = dy$, $n_y ds = -dx$; hence,

Fig. 2.31 Replacement of the integral path



$$\begin{aligned}
 M &= - \oint_{C_B} p (x n_y - y n_x) ds \\
 &= \oint_{C_B} p (x dx + y dy) = \frac{1}{2} \oint_{C_B} p d(z z^*). \quad (2.148)
 \end{aligned}$$

Substituting (2.145) into (2.148), we obtain

$$M = -\frac{\rho}{4} \oint_{C_B} \left\{ \frac{\partial}{\partial t} (f + f^*) + \left| \frac{df}{dz} \right|^2 \right\} d(z z^*). \quad (2.149)$$

2.10.1 The Case of Steady Motion

In steady flows, $\Psi = \text{constant}$ at the surface of the body, and we have $df = d\Phi = df^*$. Additionally,

$$\left| \frac{df}{dz} \right|^2 dz^* = \frac{df}{dz} \frac{df^*}{dz^*} dz^* = \frac{df}{dz} \frac{df}{dz} dz = \left(\frac{df}{dz} \right)^2 dz \quad (2.150)$$

Thus, the complex conjugate of (2.147) is

$$X - iY = i \frac{\rho}{2} \oint_{C_B} \left(\frac{df}{dz} \right)^2 dz. \quad (2.151)$$

Furthermore, since the derivative function df/dz is holomorphic in the fluid region, the integral path on the body surface C_B can be replaced by an arbitrary closed curve C surrounding the body (see Fig. 2.31). Thus, we have

$$X - iY = i \frac{\rho}{2} \oint_C \left(\frac{df}{dz} \right)^2 dz \quad (2.152)$$

Equation (2.152) is called **Blasius' first formula**.

By an arrangement similar to that of (2.150), we obtain the following expression for the moment:

$$\left| \frac{df}{dz} \right|^2 d(z z^*) = \frac{df}{dz} \frac{df^*}{dz^*} (z dz^* + z^* dz) = 2 \operatorname{Re} \left\{ \left(\frac{df}{dz} \right)^2 z dz \right\}. \quad (2.153)$$

Moreover, $(df/dz)^2 z$ is holomorphic with respect to z ; hence, the line integral along C_B can be replaced by an integral along an arbitrary closed curve C . Therefore,

$$M = -\frac{\rho}{2} \operatorname{Re} \oint_C \left(\frac{df}{dz} \right)^2 z dz. \quad (2.154)$$

Equation (2.154) is called Blasius' second formula.

In determining the contour integrals in (2.152) and (2.154), it is convenient to introduce the following Laurent series. In other words, we can use the following formula

$$\left. \begin{aligned} \text{When } F(z) &= \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \\ \oint_C F(z) dz &= 2\pi i a_{-1}. \end{aligned} \right\} \quad (2.155)$$

Therefore, we need to only retain the $1/z$ terms in the integrands of (2.152) and (2.154).

Extracting the $1/z$ terms contributing to (2.152) from the general solution (2.135), we can write

$$\left(\frac{df}{dz} \right)^2 = -2U \frac{(m - i\kappa)}{z} + \dots, \quad (2.156)$$

which, together with (2.152), (2.155), and (2.156), gives

$$\begin{aligned} X - iY &= -i \frac{\rho}{2} 2U (m - i\kappa) 2\pi i \\ &= \rho U Q - i \rho U \Gamma. \end{aligned} \quad (2.157)$$

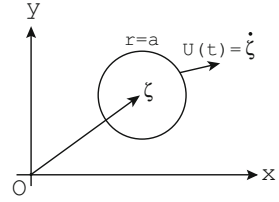
Here $Q = 2\pi m$ and $\Gamma = 2\pi\kappa$.

This result is independent of the body shape, as evidenced from the integral path in (2.152). Moreover, as observed in the Rankine ovoid, a body with closed streamlines implies that the sum of the sources and sinks is 0, that is, $Q = 0$. Under these circumstances, we encounter d'Alembert's paradox, that is, the drag acting on a body with ordinary rigid boundary surfaces is always zero in a perfect fluid. Of course, if the body itself ejects fluid, it exerts a thrust of $X = \rho U Q$.

2.10.2 The Case of Non-steady Motion

Equation (2.157) provides a general solution to flows intercepted by steadily moving bodies. In this section, we consider a cylinder of radius a moving non-steadily through

Fig. 2.32 Non-steady motion around a cylinder. $U(t)$ is time dependent



a static fluid with velocity $U(t)$. The coordinates at the center of the cylinder are given by $z = \zeta$ in a spatially fixed coordinate system (O, x, y) .

In this case, the complex velocity potential is given by (2.138) with the uniform flow component being removed; that is,

$$f(z) = -\frac{Ua^2}{z - \zeta} - i\kappa \log(z - \zeta). \quad (2.158)$$

The first term Ua^2 , in terms of fluid dynamics, is the strength of a doublet. As in (2.127), here we regard U in (2.158) as a complex number specifying the direction of progression.

The velocity of this cylinder is given by

$$\frac{d\zeta}{dt} = \dot{\zeta} = U(t) \quad (2.159)$$

(see Fig. 2.32).

Now, from (2.158), we obtain

$$\frac{\partial f(z)}{\partial t} = -\frac{a^2 \dot{U}}{z - \zeta} - \frac{a^2 U^2}{(z - \zeta)^2} + i\kappa \frac{U}{z - \zeta}, \quad (2.160)$$

$$\frac{df(z)}{dz} = \frac{Ua^2}{(z - \zeta)^2} - \frac{i\kappa}{z - \zeta}. \quad (2.161)$$

Equations (2.160) and (2.161) can be used to calculate (2.147). The contour integral of (2.147) assumes the center of the cylinder $z = \zeta$ as the origin. Moreover, z and its complex conjugate z^* are considered as independent variables. Thus, by the residue theorem of (2.155), we have

$$\oint_{C_B} \frac{\partial}{\partial t} \{f + f^*\} d(z - \zeta) = (-a^2 \dot{U} + i\kappa U) 2\pi i, \quad (2.162)$$

$$\begin{aligned}
\oint_{C_B} \left| \frac{df}{dz} \right|^2 d(z - \zeta) &= \oint_{C_B} \left(\frac{Ua^2}{z^2} - \frac{i\kappa}{z} \right) \left(\frac{Ua^2}{z^{*2}} + \frac{i\kappa}{z^*} \right) dz \\
&= \oint_{C_B} \left(\frac{Ua^2}{z^2} - \frac{i\kappa}{z} \right) \left(\frac{U}{a^2} z^2 + \frac{i\kappa}{a^2} z \right) dz \\
&= i\kappa U 2\pi i
\end{aligned} \tag{2.163}$$

where we use $zz^* = a^2$. Substituting these results into (2.147) and $2\pi\kappa = \Gamma$ as the result, we obtain

$$X + iY = i \frac{\rho}{2} \left\{ i 2\pi a^2 \dot{U} + 2 U \Gamma \right\} = -\rho \pi a^2 \dot{U} + i \rho U \Gamma. \tag{2.164}$$

Please note that this equation considers U as a complex number. Thus, it is not necessarily true that the first and second items on the right side correspond to the X and Y of fluid force, respectively.

The second term on the right side, $\rho U \Gamma$, is the same as that in (2.143) in its shape, and is the lift in the Kutta–Joukowski theorem. It is indicated that this lift moves in a vertical direction (iU) relative to the direction of the movement U .

On the other hand, the first term on the right side is inertial flow strength proportional to acceleration $\dot{U}$. The fact that its magnitude is negative indicates that it is working as inertial resistance in the direction opposite to that of the movement. This magnitude is called **added mass**, and for a cylinder, it is expressed as follows:

$$m = \rho \pi a^2. \tag{2.165}$$

In other words, it is equal to the mass of the fluid displaced by the cylinder.

Generally, when a body moves through fluid, it must displace the fluid by a certain amount, which increases the inertia of the body. The inertial increase is the added mass. Added mass can be defined as the amount of fluid particles accelerated by the motion of the body. However, the range of fluid particles accelerated by the motion of the body must extend infinitely. Therefore, to calculate the added mass, all fluid particles accelerated by the motion should be proportionally weighted and integrated over the displaced mass.

For an arbitrarily shaped body, the number of accelerated fluid particles, and hence the added mass, will depend on the direction of motion of the body. This aspect of fluid dynamics starkly contrasts with the inertia of a body in space, which is unrelated to its direction of motion. To improve our understanding of added mass, let us consider the kinetic energy of irrotational flow. To generalize the discussion, we discuss the three-dimensional problem. The kinetic energy within a region V surrounded by a closed surface S is given by

$$E = \frac{1}{2} \iiint_V \rho q^2 dV = \frac{\rho}{2} \iiint_V \nabla \Phi \cdot \nabla \Phi dV. \tag{2.166}$$

Since Φ satisfies the Laplace equation, by applying Gauss' theorem, we obtain

$$E = \frac{\rho}{2} \iiint_V \nabla \cdot (\Phi \nabla \Phi) dV = -\frac{\rho}{2} \iint_S \mathbf{n} \cdot (\Phi \nabla \Phi) dS = -\frac{\rho}{2} \iint_S \Phi \frac{\partial \Phi}{\partial n} dS \quad (2.167)$$

The negative sign of the surface integral indicates the positive direction of the inward normal of the fluid from the boundary surface.

Equation (2.167) shows that kinetic energy can be solely calculated from Φ at the boundary surface and normal flow velocity $\partial \Phi / \partial n$. From the boundary condition (2.26), $\partial \Phi / \partial n$ on the surface of a body in non-steady motion equals normal velocity V_n and thus can be expressed as follows:

$$\frac{\partial \Phi}{\partial n} = V_n = \mathbf{V}(t) \cdot \mathbf{n}(\mathbf{r}). \quad (2.168)$$

Here the velocity potential Φ can be expressed as an overlap of components linearly separated in the direction of motion, as

$$\Phi(\mathbf{r}, t) = \mathbf{V}(t) \cdot \boldsymbol{\phi}(\mathbf{r}) = V_j(t) \phi_j(\mathbf{r}), \quad (2.169)$$

where $\phi_j(\mathbf{r})$ is the velocity potential induced by a body moving at unit velocity in the j th direction j ($j = 1 \sim 6$). The function $\phi_j(\mathbf{r})$ depends only on the body's shape. For example, for a sphere of radius a moving in the x -axis direction,

$$\phi_1 = -\frac{a^3 \cos \theta}{2 r^2} = -\frac{a^3 x}{2 r^3}, \quad (2.170)$$

as can be understood from (2.61). From (2.158), a cylinder moving in the same direction induces a potential

$$\phi_1 = -a^2 \frac{\cos \theta}{r} = -a^2 \frac{x}{r^2}. \quad (2.171)$$

Note that (r, θ) of (2.170) is expressed in spherical coordinates, whereas (r, θ) of (2.171) is expressed in two-dimensional polar coordinates.

Substituting (2.169) into (2.167), we obtain

$$E = \frac{1}{2} \left[-\rho \iint_S \phi_j \frac{\partial \phi_j}{\partial n} dS \right] V_j^2 \equiv \frac{1}{2} m_{jj} V_j^2. \quad (2.172)$$

However, the kinetic energy of the fluid can be expressed as follows:

$$m_{jj} = -\rho \iint_S \phi_j \frac{\partial \phi_j}{\partial n} dS = -\rho \iint_S \phi_j n_j dS. \quad (2.173)$$

Equation (2.173) gives the energy of the fluid particles accelerated by the non-steady motion of the body. Comparing this expression to the kinetic energy of the material at some point in space, m_{jj} of (2.173) can be defined as the added mass in direction j of the motion. In the case of a cylinder, from (2.171) we have

$$m_{11} = \rho a \int_0^{2\pi} \cos \theta a \cos \theta d\theta = \rho a^2 \int_0^{2\pi} \cos^2 \theta d\theta = \rho \pi a^2, \quad (2.174)$$

which equals the result from (2.165).

On the other hand, using (2.170), the mass added by a moving sphere is given as

$$m_{11} = \frac{\rho a}{2} \iint_S \cos^2 \theta dS = \frac{\rho a}{2} \int_0^{2\pi} d\varphi \int_0^\pi \cos^2 \theta a^2 \sin \theta d\theta = \rho \frac{2}{3} \pi a^3 \quad (2.175)$$

In other words, in terms of the proportion of the mass of the fluid replaced by the body of interest (coefficient of added mass), it is 1.0 for a cylinder and 0.5 for a sphere. As seen, flows disturbed by bodies in a three-dimensional system will spread in three orthogonal directions; hence, the coefficient of the added mass is generally smaller than that for two-dimensional bodies.

2.11 Flow Around a Flat Plate

If a mapping function is used, flows around a two-dimensional flat plate can be analyzed relatively easily using our previously gained knowledge of flow fields around a cylinder. The analytical method is described below.

Consider a uniform flow of velocity U parallel to the x -axis on the z -plane ($z = x + iy$). The complex velocity potential of this flow is

$$f(z) = Uz. \quad (2.176)$$

This situation is equivalent to the plate of zero thickness aligned parallel to the x -axis.

Now consider a cylinder of radius a placed in the uniform flow with the velocity U oriented parallel to the ξ -axis in the ζ -plane ($\zeta = \xi + i\eta$) separate from the z -plane (in which circulation is absent). Using (2.138), the complex velocity potential can be expressed as follows:

$$f(\zeta) = U \left(\zeta + \frac{a^2}{\zeta} \right). \quad (2.177)$$

Defining

$$z = \zeta + \frac{a^2}{\zeta} \quad (a > 0) \quad (2.178)$$

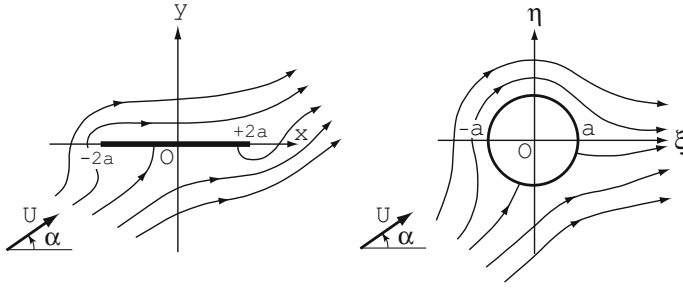


Fig. 2.33 Flow around a flat plate (*left*) and flow around a cylinder (*right*) are related by the Joukowski transformation (2.178)

as an analytical mapping function relating the z -plane to the ζ -plane, we can establish the relationship between the flow around a cylinder and the flow around a flat plate through a mapping function. Equation (2.178) is called the **Joukowski transformation**. Setting $\zeta = a \exp(i\theta)$ in (2.178), we get

$$z = x + iy = 2a \cos \theta, \quad (2.179)$$

which maps the cylindrical surface in the z -plane to a linear section (flat plate) of length $4a$ on the x -axis.

On the other hand, we can express ζ in (2.178) in terms of z :

$$\zeta = \frac{1}{2} \left(z + \sqrt{z^2 - 4a^2} \right). \quad (2.180)$$

We have only selected the positive sign because the outside region of the cylinder is considered to correspond to the z -plane (in other words, $\zeta \rightarrow \infty$ and $z \rightarrow \infty$ correspond to one another, and in the long distance limit, $\zeta \simeq z$).

We now consider a uniform flow tilted by an angle α from the real axis (ξ -axis) on the ζ -plane (Fig. 2.33). Rotating the coordinate axis does not change the fact that it is a flow around a cylinder. Thus, according to (2.180), we can replace ζ by $\zeta e^{-i\alpha}$, and from (2.177),

$$f(\zeta) = U \left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta} e^{i\alpha} \right). \quad (2.181)$$

Inserting (2.180) into (2.181), we immediately obtain the flow when a uniform flow in the direction that forms an angle α with the x -axis on a flat plate is placed on the x -axis:

$$\begin{aligned} f(z) &= U \left(\frac{z + \sqrt{z^2 - 4a^2}}{2} e^{-i\alpha} + \frac{z - \sqrt{z^2 - 4a^2}}{2} e^{i\alpha} \right) \\ &= U \left(z \cos \alpha - i \sqrt{z^2 - 4a^2} \sin \alpha \right). \end{aligned} \quad (2.182)$$

As a special case of (2.182), if $\alpha = -\pi/2$, a uniform flow of velocity U from the positive y -direction is perpendicular to the flat plate. In this case, the flow is described by a complex velocity potential, given by

$$f(z) = iU\sqrt{z^2 - 4a^2}. \quad (2.183)$$

The second term on the right-hand side of (2.182) represents the flow introduced by the flat plate. Around the edge of the flat plate ($z = \pm 2a$), the flow becomes equivalent to that of $n = 1/2$ in (2.109), which describes flow around a corner.

The velocity of flow around the flat plate can be directly determined by differentiating (2.182) with respect to z . Alternatively, we can consider ζ as a parameter, and using the calculation on the ζ -plane and a mapping function gives

$$\frac{df}{dz} = \frac{df}{d\zeta} \frac{d\zeta}{dz} = \frac{df}{d\zeta} / \frac{dz}{d\zeta}. \quad (2.184)$$

In any case, when discussing flows around a convex angle, we established that the flow velocity $q = \infty$ at $z = \pm 2a$ (i.e., at the ends of the flat plate).

Now consider that the flow described by (2.181) is overlapped with a clockwise circulation $\Gamma (= 2\pi\kappa)$, representing a flow around a cylinder (This flow is considered in Exercise 2.13). This complex velocity potential is

$$f = U\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta} e^{i\alpha}\right) + i\kappa \log \zeta. \quad (2.185)$$

Using (2.184), the flow velocity around the flat plate is given by

$$\frac{df}{dz} = \frac{U(e^{-i\alpha} - a^2 e^{i\alpha}/\zeta^2) + i\kappa/\zeta}{1 - a^2/\zeta^2}. \quad (2.186)$$

If we impose a limit on the flow velocity at $\zeta = a$, corresponding to the rear end of the flat plate ($z = +2a$), we can set the numerator in (2.186) to 0:

$$U(e^{-i\alpha} - e^{i\alpha}) + i \frac{\kappa}{a} = 0.$$

Therefore,

$$\kappa = 2a U \sin \alpha \quad (\Gamma = 4\pi a U \sin \alpha) \quad (2.187)$$

As noted in (2.138), the circulation could not be solely determined from the kinetic boundary conditions at the body surface. However, in deriving (2.187), we imposed an additional condition that the flow velocity is limited at the end of the flat plate. This condition, called **Kutta's condition**, enables the determination of the circulation.

The condition (2.187) can also be directly calculated on the z -plane after substituting (2.180) into (2.185). That is,

$$f(z) = U(z \cos \alpha - i\sqrt{z^2 - 4a^2} \sin \alpha) + i\kappa \log \frac{z + \sqrt{z^2 - 4a^2}}{2}, \quad (2.188)$$

$$\frac{df}{dz} = U \cos \alpha + \frac{i}{\sqrt{z^2 - 4a^2}} (\kappa - z U \sin \alpha). \quad (2.189)$$

It shows that (2.187) is the necessary condition for the flow to be limited at $z = +2a$.

Next, we consider the force acting on a flat plate. Equation (2.157) is applicable to an arbitrarily shaped body; hence, the lift on a flat plate is given by

$$L = \rho U \Gamma = 4\pi \rho a U^2 \sin \alpha \quad (2.190)$$

and is oriented perpendicular to the uniform flow. This result is known as the Kutta–Joukowski theorem.

Let us obtain this result from calculation on the z -plane. Although the flow velocity and pressure need to be obtained for above and below the flat plate, note that $\sqrt{z^2 - 4a^2}$ is a two-valued function and equals $\pm i\sqrt{4a^2 - x^2}$ on the top (positive) and bottom (negative) surfaces of the flat plate. Hence, from (2.189), the flow velocity and pressure are given by

$$u_{\pm} = U \cos \alpha \pm \frac{(\kappa - xU \sin \alpha)}{\sqrt{4a^2 - x^2}}, \quad v = 0 \quad (2.191)$$

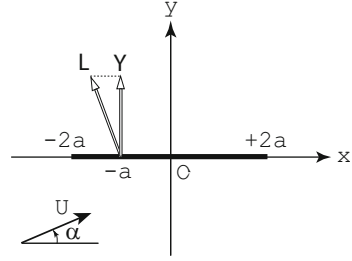
$$p_{\pm} = -\frac{1}{2} \rho \left[U^2 \cos^2 \alpha + \frac{(\kappa - xU \sin \alpha)^2}{4a^2 - x^2} \pm 2U \cos \alpha \frac{(\kappa - xU \sin \alpha)}{\sqrt{4a^2 - x^2}} \right] \quad (2.192)$$

where the subscripts $+$ and $-$ indicate the upper and lower sides of the flat plate, respectively. From (2.192), the fluid force acting in the y -axis direction becomes

$$\begin{aligned} Y &= - \oint_{S_B} p n_y ds = \int_{-2a}^{2a} (p_- - p_+) dx \\ &= 2\rho U \cos \alpha \int_{-2a}^{2a} \frac{(\kappa - xU \sin \alpha)}{\sqrt{4a^2 - x^2}} dx \\ &= 2\pi \rho U \kappa \cos \alpha \\ &= 4\pi \rho a U^2 \sin \alpha \cos \alpha \end{aligned} \quad (2.193)$$

Equation (2.193) is the $\cos \alpha$ -component of the lift L ; that is, it specifies the component of the lift in the direction of the y -axis. Moreover, the moment around the origin (counterclockwise) is calculated as

Fig. 2.34 Lift force acts at a point $1/4$ unit distant from the front end of the flat plate



$$\begin{aligned}
 M &= - \oint_{S_B} p x n_y ds = \int_{-2a}^{2a} (p_- - p_+) x dx \\
 &= 2\rho U \cos \alpha \int_{-2a}^{2a} \frac{(\kappa - xU \sin \alpha)}{\sqrt{4a^2 - x^2}} x dx = -4\pi\rho a^2 U^2 \sin \alpha \cos \alpha.
 \end{aligned} \tag{2.194}$$

Therefore, the lift is applied at $M/Y = -a$, i.e., at $1/4$ of the plate's full length ($4a$) from the front end of the flat plate (see Fig. 2.34).

Note 2.10 Let us consider the force acting on the flat plate in the x -axis direction. Since the lift L acts perpendicular to the uniform flow, the x -axis component of the lift is given by

$$X = -L \sin \alpha = -4\pi\rho a U^2 \sin^2 \alpha \tag{2.195}$$

This result can be obtained by directly integrating the pressure on the flat plate. Although the integration on the flat plate is difficult, it can be considerably simplified by applying Blasius' first formula (2.152) to the distant flow field. First, from (2.189), we have

$$\left(\frac{df}{dz}\right)^2 = U^2 \cos^2 \alpha - \frac{(\kappa - zU \sin \alpha)^2}{z^2 - 4a^2} + 2iU \cos \alpha \frac{(\kappa - zU \sin \alpha)}{\sqrt{z^2 - 4a^2}}. \tag{2.196}$$

In the far field, we can make the following approximation:

$$\frac{1}{\sqrt{z^2 - 4a^2}} \sim \frac{1}{z} \left(1 + \frac{2a^2}{z^2}\right), \quad \frac{1}{z^2 - 4a^2} \sim \frac{1}{z^2} \left(1 + \frac{4a^2}{z^2}\right). \tag{2.197}$$

Removing the $1/z$ terms, we get

$$\left(\frac{df}{dz}\right)^2 \sim \frac{2\kappa U}{z} (\sin \alpha + i \cos \alpha) + \dots \tag{2.198}$$

Applying (2.144), we obtain

$$X - iY = i \frac{\rho}{2} 2\kappa U (\sin \alpha + i \cos \alpha) 2\pi i = -4\pi\rho a U^2 (\sin \alpha + i \cos \alpha) \sin \alpha. \quad (2.199)$$

From (2.199), it is easily seen that

$$\left. \begin{aligned} X &= -4\pi\rho a U^2 \sin \alpha \sin \alpha = -L \sin \alpha \\ Y &= 4\pi\rho a U^2 \sin \alpha \cos \alpha = L \cos \alpha \end{aligned} \right\}. \quad (2.200)$$

While on the topic, let us also calculate the moment around the origin using Blasius' second formula (2.154). From (2.196), removing the $1/z$ terms in $(df/dz)^2 z$, we obtain

$$\left(\frac{df}{dz}\right)^2 z \sim \frac{(2aU \sin \alpha)^2 - \kappa^2}{z} - i \frac{(2aU)^2}{z} \cos \alpha \sin \alpha + \dots \quad (2.201)$$

Substituting (2.201) into (2.154) and applying (2.155), the moment is given by

$$M = -\frac{\rho}{2} (-i 4 a^2 U^2 \cos \alpha \sin \alpha) 2\pi i = -4\pi\rho a^2 U^2 \cos \alpha \sin \alpha. \quad (2.202)$$

Equation (2.202) is identical to (2.194) obtained by integrating the pressure.

Exercise 2.15 As in Fig. 2.35, consider a uniform flow of velocity $U = 1$ from the positive direction of the y -axis intercepting a flat plate of length $2a$ (moving in the positive direction of the y -axis at unit speed).

From (2.83), the complex velocity potential is given by

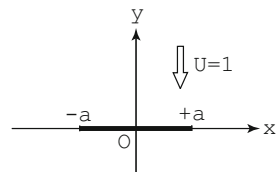
$$f(z) = i\sqrt{z^2 - a^2}.$$

To extract the flow alone, when a flat plate moves in the direction of y -axis, the complex velocity potential $f(z) = -iz$ of a uniform flow in the positive direction of the y -axis ($U = 1$) should be overlapped. Thus,

$$f(z) = i(\sqrt{z^2 - a^2} - z). \quad (2.203)$$

Obtain the velocity potential in this situation at the flat plate's upper surface ($y = 0_+$) and the lower surface ($y = 0_-$), then use (2.173) to calculate the added mass and hence show that

Fig. 2.35 A flat plate in a uniform flow



$$m = \rho\pi a^2. \quad (2.204)$$

Note that this result is the same as the added mass (2.174) for a cylinder of radius a (a circle circumscribed by a flat plate of length $2a$).

Exercise 2.16 Consider Joukowski's transformation as a mapping function between the physical surface $z = x + iy$ and a mapped plane $\zeta = \xi + i\eta$:

$$z = \zeta + \frac{a^2}{\zeta} \quad (a > 0).$$

Here are a circle of radius R ($R > a$) with a center at the origin of the ζ -plane is mapped to an ellipse with a longer axis $A = R + a^2/R$ and a shorter axis $B = R - a^2/R$. Obtain added mass when this elliptic cylinder is moving in the direction of the long axis (x -axis) and the short axis (y -axis).

Note 2.11 (*Literature for further study*)

For further studies, textbooks [2, 3, 6, 8–10] may be referred. The advanced readers may also refer some other textbooks [1, 4, 5, 7, 11, 12], although they are written in Japanese.

References

1. Adachi, C.: Vectors and Tensors. Baifukan (1957)
2. Aris, R.: Vectors, Tensors, and the Basic Equations of Fluid Mechanics. Prentice-Hall, Englewood Cliffs (1962)
3. Batchelor, G.K.: An Introduction to Fluid Dynamics. Cambridge University Press, Cambridge (1967)
4. Hino, M.: Fluid Dynamics. Asakura Publishing, Japan (1992)
5. Imai, I.: Fluid Dynamics. Shokabo, Tokyo (1973)
6. Jeffreys, H.: Cartesian Tensors. Cambridge University Press, Cambridge (1969)
7. Kambe, T. (ed.): Fluid Dynamics. Shokabo, Tokyo (1995)
8. Landau, L.D., Lifshitz, E.M.: Fluid Mechanics. Butterworth-Heinemann, Oxford (1987)
9. Newman, J.N.: Marine Hydrodynamics. The MIT Press, Cambridge (1977)
10. Prandtl, L., Tietjens, O.G.: Fundamentals of Hydro- and Aeromechanics (translated by Rosenhead, L.). Dover, New York (1957)
11. Tatsumi, T.: Fluid Dynamics. Baifukan, Tokyo (1982)
12. Tomita, Y.: Introduction to Fluid Dynamics. Yokendo, Hongo (1975)

Fluid Dynamics for Global Environmental Studies

Ichikawa

2017, XX, 368 p. 167 illus., 5 illus. in color., Softcover

ISBN: 978-4-431-56497-3