Preface

The aim of this book is to present an elementary introduction to the theory of *noncausal stochastic calculus*[1] that arises as a natural alternative to the standard theory of stochastic calculus founded by Prof. Kiyoshi Itô. To be more precise, we are going to study in this book a noncausal theory of stochastic calculus based on the noncausal integral that was introduced by the author in 1979. We would like to show not only the necessity of such a theory of noncausal stochastic calculus but also its growing possibility as a tool for modelling and analysis in every domain of mathematical sciences.

It was around 1944 that late Prof. Kiyoshi Itô first introduced a stochastic integral (with respect to Brownian motion), called nowadays the Itô integral in his name, and originated the theory of stochastic differential equations. Then not immediately but after the Second World War, the potential importance of his theory on stochastic calculus and stochastic differential equations was recognized by mathematicians worldwide and also by physicists and engineers. They all together welcomed Itô's theory of stochastic differential equations and joined to develop the theory extensively in various directions following their disciplines. Since then the theory has been continuously developed and by virtue of their contributions the theory has grown to be one of the standard languages in mathematical sciences and engineering.

However, during these 70 years of its history some problems were recognized to be out of the range of the standard theory of Itô calculus and some people have become aware of the necessity to develop an alternative theory to cover those irregular situations. Those are the problems of a noncausal nature. Let us remember that in the theory of Itô calculus, to fix our discussion let us take the calculus with respect to Brownian motion: the random functions $f(t,\omega)$ are supposed to be "causal" in the sense that, for each fixed "t", the random variable $f(t,\omega)$ is not affected by the future behaviour of Brownian motion after "t". But when we are not sure whether the functions satisfy this condition, we call such a situation "noncausal"

[1] The caligraphy presented in the previous page is the title in Japanese, given by Michi Ogawa.

or “anticipative” and the problem involving such noncausal functions or events the author used to call “noncausal problems”. The causality being in the base of Itô’s theory, it can hardly apply to those genuine noncausal problems. We will show some typical noncausal problems in Chap. 1.

As a solution to the necessity of such a stochastic calculus that is free from the causality constraint, there were attempts to develop the so-called noncausal (or anticipative) calculus. *Grosso modo* we have had two alternatives as follows: one is the calculus due to A. Skorokhod and other is the noncausal calculus introduced by S. Ogawa, the author of the present book. For the sake of distinction between them we like to call the former “anticipative calculus” and the latter “noncausal calculus”.

The anticipative calculus was founded by A. Skorokhod in 1967 [54] on the basis of the so-called Skorokhod integral. This line of new calculus has been developed by a group of ex-Soviet mathematicians around him, including A. Seveljakov ([51], [52]), Yu. Daletskii and S. Paramanova ([2]), and later their studies were followed by many illustrative mathematicians such as; M. Zakai and D. Nualart ([60], [61]), S. Ustunel ([56]), P. Imkeller, E. Pardoux and P. Protter ([45]), M. Pontier ([46]), F. Russo ([48]) among others. The *calculs de variation* originated by P. Malliavin might be classified on this line.

A little bit later from the introduction of the anticipative calculus, in 1979 S. Ogawa published a note in *Comptes Rendus* [26] on a probabilistic approach to Feynman’s path integral ([26], [25]), where he had introduced the idea of the noncausal stochastic integral and shown the possibility of an alternative way to stochastic calculus. That was indeed the beginning of the noncausal theory of stochastic calculus, the main subject of the present book. Unfortunately this attempt to the noncausal calculus had not attracted the attention of many people and during a certain period the research had been carried out by the author alone. However, after such a silent period the study has also begun to attract the interest of other mathematicians, among them the paper [60] by M. Zakai and D. Nualart is to be noted, where they introduced the author’s noncausal integral and referred to the relation with the Skorokhod integral. And now the research on our noncausal calculus is in a state of steady progress.

These two attempts to develop a new calculus, “anticipative” or “noncausal”, were aiming at the same purpose of providing a stochastic calculus that can work without the assumption of causality, but as we will see in this book they are quite different from each other in many phases, for instance in their mathematical backgrounds, materials and domains of applications. The Skorokhod integral, and so also the anticipative calculus, is established in the framework of N. Wiener’s theory of homogeneous chaos [57], which was refined by K. Itô [13] to be the theory of multiple Wiener–Itô integrals. Hence it was quite natural as we saw in the epoch-making article by M. Zakai and D. Nualart ([60], 1987) that the research on the anticipative calculus was soon joined with the study on so-called Malliavin calculus and is still now on that path.

On the other hand, another alternative calculus due to S. Ogawa, namely “the noncausal calculus”, is built on his stochastic integral of noncausal type which does not rely on the theory of homogeneous chaos and so is quite different from

Skorokhod's as tools of calculus. This means that these two theories of calculus constructed on different stochastic integrals should be different from each other. To explain the difference one would often like to say that the Skorokhod integral is a generalization of the Itô integral, while the author's noncausal integral stands as a generalization of the symmetric integrals (e.g. the $I_{1/2}$-integral and Stratonovich–Fisk integral). But this is not a good explication. Tools should be compared by their usefulness; in which situations and to what purposes do they work better. By the difference between two theories, we mean to say that the application domains for each of these theories should not be the same.

Anyhow these alternative theories of stochastic calculus have together grown up to cover many problems of noncausal nature and become recognized as the important counterparts to the standard theory. Hence for its growing importance the author thought of the necessity of providing an introductory textbook on this subject, more precisely a book with special emphasis on the "noncausal calculus". The reasons for this idea are twofold as follows: (1) Our main interest in the application of the noncausal theory is the analysis of various functional equations arising as models for stochastic phenomena in physics or engineering and our *noncausal calculus* is better fit to such an objective than the *anticipative calculus* since, as we will see in this book, the mapping defined by the noncausal integral exhibits a natural linearity while the one by the Skorokhod integral does not. (2) All books published up to the present time concern Skorokhod and Malliavin calculus while few regard the "noncausal calculus". In such situation we believe that the existence of a book with special emphasis on *the noncausal calculus* is itself desirable. We would add at this stage that what we intend to prepare here is not an exhaustive guidebook to general theory of noncausal stochastic calculus, but the first and introductory textbook on that subject. The author intends to achieve the purpose by presenting the theory with historical sketches and also various applications that are missed or hardly treated in the standard theory of causal calculus, as well as in the "anticipative calculus". Such is the basic idea that exists behind the present book.

The original project for the book was conceived in 2006, at the Workshop of Probability and Finance held at Firenze University where the author had met Dr. Catriona Byrne of Springer Verlag. There we talked about the plan of writing a monograph on the present subject. By her suggestion the author began to prepare the manuscript but the plan did not proceed in a straight way because of various difficult situations, official or personal, that the author would be facing in the coming years. It is only in the past two years that he has finally found time to accomplish the plan.

Hence for the achievement of the book the author is grateful to his friends, Profs. Gerard Kerkyacharian, Dominique Picard, Huyen Pham at University Paris 7, with whom he could have many valuable discussions, especially during 2007 when he stayed at LPMA (Laboratoire de Probabilités et Modèlea Alétoires) of University Paris-7 and 6 for one year on his sabbatical leave from Ritsumeikan University. Special thanks go to the late Professor Paul Malliavin for his valuable advice and continual encouragement and to Prof. Chii-Ruey Hwang at Mathematical Institute of Academia Sinica in Taiwan who kindly invited the author many times to Academia Sinica for discussions and offered him many opportunities to give seminar talks about the main subject of the present book. The author wishes to thank Prof. Hideaki Uemura of Aichi-Kyoiku University for his collaboration in doing research on stochastic Fourier transformation, the contents of Chap. 8. He also wishes to thank the referees for the careful reading of the draft and many valuable comments and suggestions.

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