Chapter 2
Basic Tools of Riemannian Geometry

In this chapter, omitting proofs we collect some material such as fundamental identities concerning Riemannian manifolds equipped with metric structures, and our choice of material presented is made for the applications to Carleman estimates and inverse problems in the succeeding chapters, in particular in Chaps. 3, 4 and 6. In these chapters, we will use the identities mainly for systematic calculations. As for details, we refer to monographs, for example, Jost [66].

2.1 Manifolds

The concept of a manifold is a bit complicated, but its definitions starts with the notion of a coordinate chart.

Definition 2.1 Let M be a topological space. Then a pair (U, φ) is called a chart (coordinate system), if

$$\varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^n$$

is a homeomorphism from an open subset U of M onto an open set $\varphi(U)$ in $\mathbb{R}^n$. The coordinate functions on U are defined as $x_j : U \to \mathbb{R}$, so that

$$\varphi(a) = (x_1(a), \dots, x_n(a)), \quad a \in U$$

and n is called the dimension of the coordinate system.

Definition 2.2 A topological space M is called Hausdorff if for every two distinct points $a_1, a_2 \in$ M there are open sets $U_1, U_2 \subset$ M such that

$$a_1 \in U_1, \quad a_2 \in U_2, \quad U_1 \cap U_2 = \emptyset.$$

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We now want to consider the case where M is covered by charts and satisfies some consistency conditions.

Definition 2.3 An n-dimensional atlas on a topological space M is defined as a collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ such that:

- M is covered by $\{U_\alpha\}_{\alpha\in I}$;
- $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$ for all $\alpha, \beta \in I$;
- the map
$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$
 is differentiable for all $\alpha, \beta \in I$.

Definition 2.4 Two atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\left\{(V_\beta, \psi_\beta)\right\}$ are called to be compatible if their union is an atlas.

The set of atlases compatible with a given atlas can be ordered by inclusion. The maximal element is called the complete atlas.

We define a manifold:

Definition 2.5 An n-dimensional differentiable manifold M is defined as a Hausdorff space M with a complete atlas.

Example 2.1

1. $\mathbb{R}^n$ and any finite-dimensional vector space are differentiable manifolds.
2. An open subset of a differentiable manifold is a differentiable manifold.

Definition 2.6 A differentiable n-dimensional manifold with boundary is called a Hausdorff space together with an open cover $\{U_\alpha\}$ and homeomorphisms $\varphi_\alpha : U_\alpha \to \widetilde{U}_\alpha$ such that each $\widetilde{U}_\alpha$ is an open set in $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n\,;\ x_n \geq 0\}$ and $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$ is a differentiable map if $U_\alpha \cap U_\beta$ is nonempty.

If M is a manifold with boundary, then we say that x is a boundary point if $\varphi(x) \in \{x \in \mathbb{R}^n\,;\ x_n = 0\}$ for some chart φ, and an interior point if $\varphi(x) \in \{x \in \mathbb{R}^n\,;\ x_n > 0\}$ for some chart φ. By ∂M and M$^{\text{int}}$, respectively, we denote the set of the boundary points and the set of all the interior points of M.

Let $m \in \mathbb{N}$ or $m = \infty$.

In this book we deal with C^m-manifolds, which means that $\varphi_\beta \circ \varphi_\alpha^{-1}$ are C^m-maps. The functions
$$\varphi_\alpha(x) = (x_1(x), \ldots, x_n(x)) \in \mathbb{R}^n$$
are called local coordinates on U_α. When there is no danger of misunderstanding, we also write $x = (x_1, \ldots, x_n)$, identifying a point $x \in$ M with its representation in some local coordinates. All manifolds in this book are assumed to be compact and connected.

2.2 C^m-Functions and Tangent Vectors

In this section we introduce the notion of the tangent space $T_a\mathrm{M}$ of a differentiable manifold M at a point $a \in \mathrm{M}$. This is a vector space of the same dimensions as M.

Definition 2.7 A function $f : \mathrm{M} \to \mathbb{R}$ is said to be C^m if for every chart $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in I}$ on M, the function $f \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is of C^m for any $\alpha \in I$. The set of all C^m-functions on the manifolds M is denoted by $C^m(\mathrm{M})$.

Definition 2.8 Let M be a differentiable manifold and $a \in \mathrm{M}$ be a point on M. A tangent vector at point $a \in \mathrm{M}$ is defined as a map $X_a : C^m(\mathrm{M}) \to \mathbb{R}$ such that

- X_α is $\mathbb{R}$-linear:

$$X_a(f_1 + f_2) = X_a(f_1) + X_a(f_2), \quad X_a(\lambda f_1) = \lambda X_a(f_1)$$

  for all $\lambda \in \mathbb{R}$ and $f_1, f_2 \in C^m(\mathrm{M})$.
- X_α satisfies the Leibniz rule

$$X_a(f_1 f_2) = X_a(f_1) f_2(a) + f_1(a) X_a(f_2) \quad \text{for all} \quad f_1, f_2 \in C^m(\mathrm{M}).$$

The set of all tangent vectors at $a \in \mathrm{M}$ is denoted by $T_a\mathrm{M}$ and is called the tangent space at a.

We see that $T_a\mathrm{M}$ is a vector space of dimensions n. In a given coordinate system $(x_1, \ldots, x_n)$, every vector $X_a \in T_a\mathrm{M}$ can be written as

$$X_a = \sum_{i=1}^{n} X_{ai}\partial_i. \tag{2.1}$$

Henceforth when we fix local coordinates, we can identify $X_a \in T_a\mathrm{M}$ with $(X_{a1}, \ldots, X_{an}) \in \mathbb{R}^n$.

Definition 2.9 Let $f : \mathrm{M} \to \mathbb{R}$ be a C^m-function. Then for each $a \in \mathrm{M}$, the differential of f is defined by the linear map

$$df_a(X_a) = X_a(f), \quad df_a : T_a\mathrm{M} \to \mathbb{R}$$

for all $X_a \in T_a\mathrm{M}$.

Definition 2.10 Let M be differentiable manifold. We set

$$T\mathrm{M} = \bigcup_{x\in\mathrm{M}} T_x\mathrm{M},$$

and we call $T\mathrm{M}$ the tangent bundle of M.

Next let us introduce the notion of vector field on manifolds, which assigns an element of $T\mathrm{M}$ to every point on M.

Definition 2.11 Let $X : M \longrightarrow T\mathrm{M}$ be a map. For each $f \in C^m(\mathrm{M})$, we define a function Xf by

$$(Xf)(a) := X(a)f \in \mathbb{R}, \quad a \in \mathrm{M}.$$

Here we note that $X(a) \in T\mathrm{M}$ and so $X(a)f \in \mathbb{R}$ for each $f \in C^m(\mathrm{M})$. We say that $X \in C^m(\mathrm{M}, T\mathrm{M})$ if $Xf \in C^m(\mathrm{M})$ for each $f \in C^m(\mathrm{M})$ and call X a vector field of class C^m.

In a local coordinates system $(x_1, \dots, x_n)$, any C^m-vector field can be uniquely written as

$$X = \sum_{i=1}^{n} X_i \partial_i,$$

where $X_i : \mathrm{M} \to \mathbb{R}$ is a real C^m-function, and X_i is called the components of X. Note that X is differentiable if X_i, $1 \le i \le n$, are differentiable.

Definition 2.12 Let $X, Y \in C^m(\mathrm{M}, T\mathrm{M})$ be vector fields on a manifold M. Then the Lie bracket $[X, Y]$ is defined as the vector field

$$[X, Y] = \sum_{i,j=1}^{n} \left(X_j \partial_j Y_i - Y_j \partial_j X_i\right) \partial_i, \quad X = \sum_{i=1}^{n} X_i \partial_i, \quad Y = \sum_{j=1}^{n} Y_j \partial_j.$$

We say that the vector fields X and Y commute if $[X, Y] = 0$.

Lemma 2.1 *The Lie bracket* $[\cdot, \cdot]$ *is bilinear over* $\mathbb{R}$. *For any differentiable function* f *we have*

$$[X, Y] f = X(Y(f)) - Y(X(f)).$$

Furthermore, the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

holds for any three vector fields X, Y, Z.

Proof In local coordinates, writing $X = \sum_{i=1}^{n} X_i \partial_i$ and $Y = \sum_{i=1}^{n} Y_i \partial_i$, we have

$$[X, Y] f = \sum_{i,j=1}^{n} \left(X_j \partial_j Y_i - Y_j \partial_j X_i\right) \partial_i f,$$

and this is linear in f, X, Y. This implies the first claim. The Jacobi identity follows by direct computation.

2.3 Riemannian Metric

In this section we introduce the notion of a Riemannian manifold (M, g). The metric g provides us with an inner product on each tangent space and thus can be used to measure the length of curves on the manifold. It defines a distance function and turns the manifold into a metric space in a natural way.

Definition 2.13 Let M be a C^m-manifold. A Riemannian metric g on M is defined as a map which associates to any C^m-vector fields X and Y on M a C^m-function $\mathrm{g}(X, Y)$ on M such that

$$\mathrm{g}(X_1 + X_2, Y) = \mathrm{g}(X_1, Y) + \mathrm{g}(X_2, Y), \quad \mathrm{g}(X, Y_1 + Y_2) = \mathrm{g}(X, Y_1) + \mathrm{g}(X, Y_2),$$

$$\mathrm{g}(fX, Y) = f\mathrm{g}(X, Y) = \mathrm{g}(X, fY), \quad \mathrm{g}(X, Y) = \mathrm{g}(Y, X),$$

for all real-valued C^m-functions f and vector fields X, X_1, X_2, Y, Y_1, Y_2, and

$$\mathrm{g}(X, X) > 0 \quad \text{whenever} \quad X \neq 0.$$

A Riemannian manifold (M, g) is defined as a manifold M with metric g.

In local coordinates, g is given by a positive definite and symmetric C^m-matrix function $\mathrm{g} = (\mathrm{g}_{jk})_{1 \le j,k \le n}$:

$$\mathrm{g}_{jk} = \mathrm{g}(\partial_j, \partial_k).$$

Definition 2.14 For $x \in \mathrm{M}$, the inner product $\langle X, Y \rangle$ and the norm $|X|$ for $X, Y \in T_x\mathrm{M}$ are defined by

$$\langle X, Y \rangle_{\mathrm{g}} = \langle X, Y \rangle := \sum_{j,k=1}^{n} \mathrm{g}_{jk} X_j Y_k$$

and

$$|X|_{\mathrm{g}} = |X| := \langle X, X \rangle_{\mathrm{g}}^{1/2}$$

for $X = \sum_{i=1}^{n} X_i \partial_i$ and $Y = \sum_{i=1}^{n} Y_i \partial_i$.

Later for a vector field in the form of $\nabla_{\mathrm{g}} u$, we simply write $|\nabla_{\mathrm{g}} u|$ in place of $|\nabla_{\mathrm{g}} u|_{\mathrm{g}}$.

We write also

$$\langle X, Y \rangle = \mathrm{g}(X, Y).$$

Let (M, g) be a Riemannian manifold. It is easy to see that the coefficients g_{ij} of the metric g have the following properties:

1. For all i, j the function g_{ij} is of C^m on M.
2. The matrix $(g_{ij}(x))$ is symmetric for any $x \in$ M.
3. The matrix $(g_{ij}(x))$ is positive definite for any $x \in$ M.

Henceforth $(g^{ij}(x))$ denotes the inverse matrix to $(g_{ij}(x))$.

2.4 Connection

In local coordinates, we set

$$\Gamma_{ij}^k(x) = \frac{1}{2}\sum_{p=1}^{n} g^{kp}(x)\left(\partial_i g_{jp} + \partial_j g_{ip} - \partial_p g_{ik}\right), \quad 1 \le i, j, k \le n. \tag{2.2}$$

We call Γ_{ij}^k the Christoffel symbols in a coordinate system under consideration.

Next we set

$$\nabla_X Y = \sum_{\ell,p=1}^{n} X_p \left(\partial_p Y_\ell + \sum_{q=1}^{n} \Gamma_{pq}^{\ell} Y_q\right)\partial_\ell. \tag{2.3}$$

This is called the Levi-Civita connection.

Then we can prove [66].

Theorem 2.1 (The fundamental theorem of Riemannian geometry)

$$Z(\langle X, Y\rangle) = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \quad \textit{for all } X, Y, Z \in C^m(\mathrm{M}, T\mathrm{M}) \tag{2.4}$$

and

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle &= X(\langle Y, Z\rangle) - Z(\langle X, Y\rangle) + Y(\langle Z, X\rangle) \\ &- \langle X, [Y, Z]\rangle + \langle Z, [X, Y]\rangle + \langle Y, [Z, X]\rangle, \quad X, Y, Z \in C^m(\mathrm{M}, T\mathrm{M}). \end{aligned} \tag{2.5}$$

Here and henceforth we note that Xf is defined in Definition 2.11 for a vector field X and $f \in C^m(\mathrm{M})$, and $\langle Y, Z\rangle, |Y|_g^2 \in C^m(\mathrm{M})$ for vector fields Y and Z.

2.5 Laplace-Beltrami Operator and Hessian on Riemannian Manifolds

In local coordinates, we define

$$(\nabla_g f)_j = \sum_{i=1}^{n} g^{ij}\partial_i f. \tag{2.6}$$

Then we set $\nabla_{\mathrm{g}} f = \sum_{j=1}^{n} (\nabla_{\mathrm{g}} f)_j \partial_j$. We can identify

$$\nabla_{\mathrm{g}} f = ((\nabla_{\mathrm{g}} f)_1, \ldots, (\nabla_{\mathrm{g}} f)_n)^T. \tag{2.7}$$

Definition 2.15 We define the divergence $\mathrm{div}_{\mathrm{g}}\, X$ of a vector field X by

$$\mathrm{div}_{\mathrm{g}}\, X = \sum_{i=1}^{n} \left(\partial_i X_i + \sum_{j=1}^{n} \Gamma_{ij}^{i} X_j \right) = \frac{1}{\sqrt{\det \mathrm{g}}} \sum_{i=1}^{n} \partial_i \left(\sqrt{\det \mathrm{g}}\, X_i \right) \tag{2.8}$$

for $X = \sum_{i=1}^{n} X_i \partial_i$.

We note that $\mathrm{div}\, X$ is a real-valued function. If $f \in C^1(\mathrm{M})$ and $X \in C^1(\mathrm{M}, T\mathrm{M})$, then

$$\mathrm{div}_{\mathrm{g}}\, (f X) = \langle \nabla_{\mathrm{g}} f, X \rangle + f \mathrm{div}_{\mathrm{g}}\, X. \tag{2.9}$$

Definition 2.16 Let (M, g) be a Riemannian manifold. The Laplace-Beltrami operator is given by

$$\Delta_{\mathrm{g}} f = \mathrm{div}_{\mathrm{g}}(\nabla_{\mathrm{g}} f), \quad f \in C^2(\mathrm{M}).$$

In local coordinates, Δ_{g} is given by

$$\Delta_{\mathrm{g}} = \frac{1}{\sqrt{\det \mathrm{g}}} \sum_{j,k=1}^{n} \partial_j \left(\sqrt{\det \mathrm{g}}\, \mathrm{g}^{jk} \partial_k \right). \tag{2.10}$$

Here (g^{jk}) is the inverse of the matrix of the metric g.

Let ψ and f be C^2- functions on M. Applying (2.9) with $X = \nabla_{\mathrm{g}} \psi$, we obtain

$$\mathrm{div}_{\mathrm{g}}(f \nabla_{\mathrm{g}} \psi) = f \Delta_{\mathrm{g}} \psi + \langle \nabla_{\mathrm{g}} f, \nabla_{\mathrm{g}} \psi \rangle. \tag{2.11}$$

Definition 2.17 For any $f \in C^2(\mathrm{M})$, we call $\nabla_{\mathrm{g}}^2 f = \nabla_{\mathrm{g}}(\nabla_{\mathrm{g}} f)$ the Hessian of the function f.

Then, in local coordinates, the Hessian of $f \in C^2(\mathrm{M})$ with respect to the metric g is given by

$$\nabla_{\mathrm{g}}^2 f(X, X) = \sum_{i,j=1}^{n} X_i \left(\sum_{\ell=1}^{n} (\partial_i (\nabla_{\mathrm{g}} f)_\ell) \mathrm{g}_{\ell j} + \sum_{k,\ell=1}^{n} (\nabla_{\mathrm{g}} f)_k \mathrm{g}_{\ell j} \Gamma_{ik}^{\ell} \right) X_j \tag{2.12}$$

for $X = \sum_{i=1}^{n} X_i \partial_i$. Here we recall that

$$(\nabla_g f)_\ell = \sum_{i=1}^{n} g^{i\ell}\partial_i f, \quad \ell = 1, \dots, n,$$

and Γ_{ik}^{ℓ} are the Christoffel symbols defined by (2.2).

In the succeeding chapters, we use the following lemma.

Lemma 2.2 *(i) Let z and $\psi \in C^2(\mathrm{M})$. The following identity holds:*

$$\langle \nabla_g z, \nabla_g \left(\langle \nabla_g z, \nabla_g \psi\rangle\right)\rangle = \nabla_g^2 \psi \left(\nabla_g z, \nabla_g z\right) + \frac{1}{2}\langle \nabla_g \psi, \nabla_g \left(|\nabla_g z|^2\right)\rangle.$$

(ii)

$$\langle \nabla_g u, \nabla_g v\rangle = \sum_{k,\ell=1}^{n} g^{k\ell}(\partial_\ell u)\partial_k v$$

for $u, v \in C^1(\mathrm{M})$.

Proof Applying (2.4) with $Z = X = \nabla_g z$ and $Y = \nabla_g \psi$, we obtain

$$\begin{aligned}\langle \nabla_g z, \nabla_g \left(\langle \nabla_g z, \nabla_g \psi\rangle\right)\rangle &= \nabla_g z \left(\langle \nabla_g z, \nabla_g \psi\rangle\right)\\ &= \langle \nabla_{\nabla_g z}\nabla_g z, \nabla_g \psi\rangle + \langle \nabla_g z, \nabla_{\nabla_g z}\nabla_g \psi\rangle\\ &= \nabla_g^2 \psi \left(\nabla_g z, \nabla_g z\right) + \nabla_g^2 z \left(\nabla_g z, \nabla_g \psi\right).\end{aligned}$$

Here we have used that (see [66], p. 145 (3.3.48))

$$\nabla_g^2 f(X, Y) = \langle \nabla_X \nabla_g f, Y\rangle, \quad f \in C^1(\mathrm{M}),\ X, Y \in C^1(\mathrm{M}, T\mathrm{M}).$$

Moreover

$$\begin{aligned}\langle \nabla_g \psi, \nabla_g \left(|\nabla_g z|^2\right)\rangle &= \nabla_g \psi \left(\langle \nabla_g z, \nabla_g z\rangle\right)\\ &= \langle \nabla_{\nabla_g \psi}\nabla_g z, \nabla_g z\rangle + \langle \nabla_g z, \nabla_{\nabla_g \psi}\nabla_g z\rangle\\ &= 2\nabla_g^2 z \left(\nabla_g z, \nabla_g \psi\right).\end{aligned}$$

Hence

$$\langle \nabla_g z, \nabla_g \left(\langle \nabla_g z, \nabla_g \psi\rangle\right)\rangle = \nabla_g^2 \psi \left(\nabla_g z, \nabla_g z\right) + \frac{1}{2}\langle \nabla_g \psi, \nabla_g \left(|\nabla_g z|^2\right)\rangle.$$

This completes the proof of (i).

(ii) Since $\sum_{j=1}^{n} g_{kj} g^{jp} = \delta_{kp}$, $(\nabla_g u)_k = \sum_{\ell=1}^{n} g^{k\ell}\partial_\ell u$ and $(\nabla_g v)_j = \sum_{p=1}^{n} g^{jp}\partial_p v$, we directly calculate:

$$\langle \nabla_g u, \nabla_g v\rangle = \sum_{k,j=1}^{n} g_{kj}(\nabla_g u)_k(\nabla_g v)_j$$

$$
\begin{aligned}
&= \sum_{k,j=1}^{n} \sum_{\ell,p=1}^{n} g_{kj} g^{k\ell} g^{jp} (\partial_\ell u) \partial_p v \\
&= \sum_{k,\ell,p=1}^{n} \left(\sum_{j=1}^{n} g_{kj} g^{jp} \right) g^{k\ell} (\partial_\ell u) \partial_p v \\
&= \sum_{k,\ell,p=1}^{n} \delta_{kp} g^{k\ell} (\partial_\ell u) \partial_p v = \sum_{k,\ell=1}^{n} g^{k\ell} (\partial_\ell u) \partial_k v.
\end{aligned}
$$

Thus the proof of (ii) is complete.

2.6 Green's Formula

The metric tensor g induces a Riemannian volume form, which is an n-form defined locally by

$$dx = (\det g)^{1/2}\, dx_1 \cdots dx_n.$$

We denote by $L^2(\mathrm{M})$ the completion of $C^\infty(\mathrm{M})$ with respect to the usual inner product

$$(f_1, f_2)_{L^2(D)} = (f_1 . f_2) = \int_{\mathrm{M}} f_1(x) f_2(x)\, dx, \qquad f_1, f_2 \in L^2(\mathrm{M}),$$

where f_1, f_2 are real-valued. For complex-valued functions f_1, f_2, we set

$$(f_1, f_2) = \int_{\mathrm{M}} f_1(x) \overline{f_2(x)} dx,$$

where $\overline{f_2}$ denotes the complex conjugate but in this book we mainly discuss real-valued functions.

The Sobolev space $H^1(\mathrm{M})$ is the completion of $C^\infty(\mathrm{M})$ with respect to the norm $\| \cdot \|_{H^1(\mathrm{M})}$:

$$\|f\|^2_{H^1(\mathrm{M})} = (f, f) + (\nabla_g f, \nabla_g f).$$

The normal derivative is

$$\partial_\nu u = (\nabla_g u \cdot \nu) := \sum_{j,k=1}^{n} g^{jk} \nu_j \partial_k u,$$

where $\nu = (\nu_1, \cdots \nu_n)^T$ is the unit outward normal vector to $\partial\mathrm{M}$ in the sense of $\sum_{j=1}^{n} \nu_j^2 = 1$.

In terms of $\nabla_{\mathrm{g}} u$ defined by (2.6) and (2.7), we can write

$$\partial_\nu u = (\nabla_{\mathrm{g}} u \cdot \nu)$$

in local coordinates.

Here we show convenient relations for $u \in C^1(\mathrm{M})$ satisfying $u|_{\partial \mathrm{M}} = 0$, which are used later:

$$\begin{cases} \partial_N u = \frac{1}{(\mathrm{g}^{-1}\nu\cdot\nu)}\partial_\nu u, \qquad \nabla_{\mathrm{g}} u = (\partial_N u)\mathrm{g}^{-1}\nu, \\ |\nabla_g u|^2 := \langle \nabla_{\mathrm{g}} u, \nabla_{\mathrm{g}} u\rangle = \frac{1}{(\mathrm{g}^{-1}\nu\cdot\nu)}(\partial_\nu u)^2, \\ \langle \nabla_{\mathrm{g}} u, \nabla_{\mathrm{g}} \psi\rangle = \frac{1}{(\mathrm{g}^{-1}\nu\cdot\nu)}(\partial_\nu u)\partial_\nu \psi \quad \text{on } \partial\mathrm{M} \text{ for } u, \psi \in C^1(\mathrm{M}) \text{such that } u|_{\partial\mathrm{M}} = 0. \\ \langle H, \nabla_{\mathrm{g}} u\rangle = \frac{1}{(\mathrm{g}^{-1}\nu\cdot\nu)}(H\cdot\nu)\partial_\nu u \quad \text{on } \partial\mathrm{M} \text{ for a vector field } H. \end{cases} \tag{2.13}$$

Here we set

$$\partial_N u = (\nabla u \cdot \nu).$$

In particular, we see that there exists a constant $C > 0$, which is independent of choices of u, such that

$$C^{-1}|\partial_\nu u(x)| \le |\nabla_{\mathrm{g}} u(x)| \le C|\partial_\nu u(x)|, \quad x \in \partial\mathrm{M}$$

if $u = 0$ on ∂M.

Proof of (2.13). By $u|_{\partial\mathrm{M}} = 0$, we have $\nabla u = (\partial_N u)\nu$ on ∂M. Therefore

$$(\nabla_{\mathrm{g}} u)_i = \sum_{k=1}^n \mathrm{g}^{ik}\partial_k u = \sum_{k=1}^n \mathrm{g}^{ik}(\partial_N u)\nu_k = (\partial_N u)(\mathrm{g}^{-1}\nu)_i, \qquad i = 1, \cdots, n,$$

so that the second equality of (2.13) is proved. Next

$$\partial_\nu u = (\nabla_{\mathrm{g}} u \cdot \nu) = (\partial_N u)\sum_{i=1}^n (\mathrm{g}^{-1}\nu)_i\nu_i = (\partial_N u)(\mathrm{g}^{-1}\nu\cdot\nu),$$

which proves the first equality of (2.13). Since $\sum_{i=1}^n \mathrm{g}_{ij}\mathrm{g}^{ik} = \delta_{jk}$, we have

$$\begin{aligned} |\nabla_{\mathrm{g}} u|^2 &= \sum_{i,j=1}^n \mathrm{g}_{ij}(\nabla_{\mathrm{g}} u)_i(\nabla_{\mathrm{g}} u)_j = \sum_{i,j,k,l=1}^n \mathrm{g}_{ij}\mathrm{g}^{ik}(\partial_N u)\nu_k \mathrm{g}^{jl}(\partial_N u)\nu_l \\ &= (\partial_N u)^2 \sum_{j,l=1}^n \nu_j\nu_l \mathrm{g}^{jl} = (\partial_N u)^2(\mathrm{g}^{-1}\nu\cdot\nu), \end{aligned}$$

which proves the third equality by the first equality. We have

$$\langle \nabla_g u, \nabla_g \psi \rangle = \sum_{i,j=1}^{n} g_{ij} (\nabla_g u)_i (\nabla_g \psi)_j = \sum_{i,j,k,l=1}^{n} g_{ij} g^{ik} (\partial_N u) \nu_k g^{jl} (\partial_l \psi)$$
$$= \sum_{j,l=1}^{n} (\partial_N u) \nu_j g^{jl} (\partial_l \psi) = (\partial_N u) \partial_\nu \psi,$$

which proves the fourth equality by the first. Finally

$$\langle H, \nabla_g u \rangle = \sum_{i,j=1}^{n} g_{ij} (\nabla_g u)_i H_j = (\partial_N u) \sum_{i,j,k=1}^{n} g_{ij} g^{ik} \nu_k H_j$$
$$= (\partial_N u) \sum_{j=1}^{n} \nu_j H_j = \frac{1}{(g^{-1}\nu \cdot \nu)} (H \cdot \nu) \partial_\nu u.$$

Thus the proof of (2.13) is complete.

If X is a vector field, then the divergence formula reads

$$\int_{M} \operatorname{div}_g X \, dx = \int_{\partial M} \langle X, \nu \rangle \, ds,$$

where ∂M is the boundary of M and ds is the area element on ∂M.

For $f \in H^1(M)$, Green's formula reads

$$\int_{M} (\operatorname{div}_g X) \, f \, dx = -\int_{M} \langle X, \nabla_g f \rangle \, dx + \int_{\partial M} \langle X, \nu \rangle f \, ds.$$

Then for $f \in H^1(M)$ and $w \in H^2(M)$, we have Green's formula:

$$\int_{M} (\Delta_g w) f \, dx = -\int_{M} \langle \nabla_g w, \nabla_g f \rangle \, dx + \int_{\partial M} (\partial_\nu w) f \, ds.$$

We conclude this chapter with a lemma which is used in Chap. 3.

Lemma 2.3 *Let* (M, g) *be a* C^m*-Riemannian manifold with compact boundary* ∂M. *Then there exists a* C^{m-1}*-vector field* N *such that*

$$N(x) = \nu(x), \quad x \in \partial M, \quad \text{and} \quad |N(x)| \le 1, \quad x \in M,$$

where ν *is the unit outward normal vector to* ∂M.

Proof Since ∂M is of C^m, for every $x_* \in \partial M$ there exist, using a chart, an open neighborhood V of x_* in $\mathbb{R}^n$ and a function $\theta \in C^m(V)$ such that

$$\nabla_g \theta(x) \neq 0, \quad x \in V, \quad \theta(x) = 0, \quad x \in V \cap \partial \mathrm{M}.$$

Replacing θ by $-\theta$ if needed, we can assume that

$$(\nu(x_*) \cdot \nabla_g \, \theta(x_*)) > 0.$$

Then the function $\mu : V \to \mathbb{R}^n$ given by

$$\mu(x) = \left(\sum_{j=1}^{n} (\nabla_g \theta(x))_j^2 \right)^{-1/2} \nabla_g \theta(x), \quad x \in V,$$

is of C^{m-1}. We show that $\mu = \nu$ on $V \cap \partial \mathrm{M}$. In fact, since $\theta = 0$ on $V \cap \partial \mathrm{M}$, we have

$$\nabla_g \theta(x) = (\nabla_g \theta \cdot \nu)\nu + (\nabla_g \theta \cdot \tau)\tau = (\partial_\nu \theta)\nu,$$

which implies that μ, $\nabla_g \theta$, and ν are parallel to each other on $V \cap \partial \mathrm{M}$. This together with $|\mu| = |\nu| = 1$ shows that $\mu = \nu$ on $V \cap \partial \mathrm{M}$.

Since $\partial \mathrm{M}$ is compact, it can be covered by a finite number of neighborhoods $V_1, \ldots, V_q$. Each of them plays the role of V in the earlier reasoning. By μ_i, $i = 1, \ldots, q$ denoting the corresponding functions of V_i, we have

$$\partial \mathrm{M} \subset V_1 \cup \cdots \cup V_q$$

and

$$\mu_i = \nu \quad \text{on} \quad V_i \cap \partial \mathrm{M}, \quad i = 1, \ldots, q.$$

Fix an open set V_0 such that

$$\mathrm{M} \subset V_0 \cup V_1 \cup \cdots \cup V_q, \quad \text{and} \quad V_0 \cap \partial \mathrm{M} = \emptyset,$$

and define $\mu_0 : V_0 \to \mathbb{R}^n$ by $\mu_0(x) = 0$ in V_0. Let $\psi_0, \ldots, \psi_q$ be a smooth partition of unity corresponding to the covering $V_0, \ldots, V_q$ of M:

$$\psi_i \in C_0^\infty(V_i), \quad \text{and} \quad 0 \leq \psi_i \leq 1, \quad i = 0, 1, \ldots, q,$$

and

$$\psi_0 + \psi_1 + \cdots + \psi_q = 1 \quad \text{on M}.$$

It is obvious that

$$N = \sum_{i=0}^{q} \psi_i \mu_i$$

is the required vector field.

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