Preface

For a long time, wave equations in general and Maxwell's equations in particular were solved in the time-harmonic domain[1] by finite element methods (FEM). Equations in time domain were solved by using second-order finite difference methods (FDM) whose outstanding representative is the Yee scheme (also called FDTD) used for electromagnetic waves since 1966 and still alive.

Unfortunately, second-order FDM found their limits in modeling long time propagation which often occur in physical problems, either because the wavelength is small compared to the size of the efficient domain, or because we have trapped waves in cavities. Actually, the number of points required to get an accurate solution grows with the interval of resolution in time. A palliative to this drawback seemed to be the use of higher order FDM which enable us to increase the size of the space-step while keeping a satisfactory accuracy. However, this technique is very troublesome to model complex geometries because of the large size of the grid cells.

People were nevertheless reluctant to use FEM in the time domain[2] (called FETD in the electromagnetic community), which could ensure a good approximation of these geometries. The reason came from the presence of a mass matrix which is naturally diagonal for FDM, but n-diagonal for FEM, with n increasing with the dimension of space and the order of the method. This matrix requires to be inverted at each time-step and substantially slows down the performance of the method, even when using iterative algorithms of inversion.

An answer to this difficult problem was given in two ways. A first way, introduced by Cohen et al. [1] for wave equations, was based on mass lumping of FEM on quadrilateral and hexahedral meshes with Gauss–Lobatto points. Actually, this idea was first used for reservoir simulation [2] and neutronics [3] and was later

[1] We recall that time-harmonic problems are derived from the transient wave equations by Fourier transform in time which replaces time by frequency. Their solutions then depend on the frequency.

[2] Although resolution in the time domain became attractive since the sources became polychromatic and even broadband.

extended to other problems under the name of "spectral elements" [4]. This method was then combined to mixed formulations first for electromagnetic (using $H(curl)$-L^2 elements) [5] and then for acoustic (using H^1-L^2 elements) [6, 7] wave equations and led to efficient and low-storage approximations. These approximations were called "mixed spectral element methods."

A second way to overcome the inversion of an n-diagonal mass matrix was proposed by Hesthaven et al. [8, 9]. It was based on the use of high-order discontinuous Galerkin methods (DGM) on triangular and tetrahedral meshes and led to block-diagonal mass matrices. The use of high-order is here essential for Maxwell's equations since these equations produce parasitic waves which must be suppressed by adding a penalty term. This term being dissipative, higher order methods substantially reduce the dissipation. On the other hand, Hesthaven dramatically decreased the storage of DGM, providing a reasonable increase of computation time, by reconstructing the stiffness matrix at each time-step. Moreover, he computed quasi-optimal interpolation points to avoid the Runge phenomenon for high-order polynomial approximations [10].

This new point of view suggested to construct DGM by using spectral elements and mixed formulation. This new approach was attractive for Maxwell's equations which presented parasitic modes even when using edge ($H(curl)$) elements [11]. However, Cohen et al. [12] noticed that the penalty term used for DGM could be also used for the mixed spectral element approach of Maxwell's equations which led to an even more efficient method.

The main advantage of mixed spectral element methods is to produce very sparse mass and stiffness matrices, which leads to very fast algorithm. Unfortunately, it has an important drawback: hexahedral meshes are difficult to construct for very complex geometries (which cannot be regarded as deformation of a cube).[3] For cons, tetrahedral meshes are much easier to produce, which gives an important advantage to Hesthaven's DGM. However, matrices involved in this method are not very sparse and substantially slow down its performance compared to hexahedral mixed spectral elements.

The above remarks naturally led to an interest in hybrid meshes mostly composed of hexahedra and using tetrahedra around the complex domains. The problem then was to stick the two types of elements. A natural way to do it is to use pyramidal and even prismatic elements. The second elements are deduced from triangular and spectral elements by the use of Cartesian product. The pyramidal elements, in turn, were much less obvious to construct since they used rational functions. This was achieved in a general form [13] and then extended to edge elements [14].

Another and important step was to get error estimates for hexahedral and quadrilateral elements which are not obvious because of the presence of a

[3]On the other hand, as we shall see in this book, mass-lumped triangular and tetrahedral elements are difficult to construct and not really efficient.

nonconstant Jacobian matrix. A new approach was developed, which enabled us to analyze discontinuous and continuous spectral element approaches [15–17].

The scope of this book—which is essentially the continuation of G. Cohen's previous book [18]—is to provide a survey of all these methods and to describe how to implement them in physical models. It is divided into eight chapters.

The first chapter recalls the continuous formulations of the main wave equations (acoustics, Maxwell, linear elastodynamics) and the mathematical concepts needed to understand their approximation. Moreover, the important Hille–Yosida theorem is given and applied to the different equations.

The second chapter provides an exhaustive presentation of different finite elements classically used for wave equations approximations. These elements are of different shapes and are generally defined at any order.

The third chapter covers an approximation of the acoustics equation and the linear elastodynamics system by mixed spectral element methods. After their description, a plane wave analysis is given and error estimates are developed for the acoustics equation.

The fourth chapter is devoted to DGM approximations using spectral elements or Hesthaven's method. The method is first defined in a general framework containing both approaches and the three wave equations. Then the construction of the different methods for each equation is given in detail. Finally, plane wave analysis is done.

The fifth chapter deals with different approximations of Maxwell's equations and the important and not obvious problem of parasitic modes. The last section contains error estimates of spectral DGM on hexahedra.

The sixth chapter introduces the treatment of unbounded domains by absorbing boundary conditions (ABC) and perfectly matched layers (PML) for the three wave equations.

The seventh chapter defines different time approximation algorithms, first with a constant time-step, then by local time stepping methods which lead to a dramatic gain of computational time.

The eighth, and the last, chapter presents three more complex equations with specific properties: The linearized Euler equations (LEE) which model acoustics in flow and contains a convective term, the Cauchy–Poisson problem which models gravity waves and whose evolution equation is on a boundary, and two models of wave propagation in thin plates which are dispersive equations.

This book is the result of a fruitful (and harmonious) collaboration between G. Cohen and S. Pernet.[4] Although G. Cohen's contribution was mainly focused on construction of the methods and S. Pernet's work was mainly focused on their analysis, the book was written in a very interactive way which enriched both contributions.

On the other hand, this book could not have been written without the contributions of people with whom the authors worked and to whom they are grateful.

[4]Who is a former Ph.D. student of G. Cohen.

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