Chapter 2
Linear Connections and Riemannian Geometry

In Sects. 2.1 and 2.2, we present the general theory of linear connections together with the reduction theory of the underlying frame bundle to some Lie subgroup of the general linear group. These reductions are usually referred to as H-structures.[1] They lead to a unified view on possible geometric structures manifolds may be endowed with. Using this framework, we discuss almost complex, pseudo-Riemannian, conformal, almost Hermitean and almost symplectic structures including a discussion of the corresponding compatible connections. Thus, from the perspective of H-structures, Riemannian geometry is an important special example. In Sects. 2.3 and 2.5, we continue to study H-structures by investigating torsion-free compatible connections. We ask which holonomy groups may occur for such connections. This fundamental question has been first systematically studied by Berger. In this delicate analysis, the central object to be studied is the curvature mapping of the connection under consideration. In Sect. 2.3, we study the class of connections which are not locally symmetric with emphasis on the metric case, where the H-structure defines a pseudo-Riemannian manifold. For that case, we formulate the classification result of Berger without giving a proof. We also comment on the classification in the non-metric case. In Sect. 2.5, we study the case of locally symmetric connections. This leads us to the theory of symmetric spaces. We present the basics of this theory in a fairly consistent manner including a number of important classes of examples. Next, in Sect. 2.6, we extend our discussion of compatible connections to vector bundles with emphasis on Hermitean bundles and holomorphic structures. In Sect. 2.7, we present the basics of Hodge Theory[2] including a detailed study of Weitzenboeck-type formulae. Finally, in Sect. 2.8, we discuss properties of Riemannian manifolds which are special in dimension four.

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[1] Also called G-structures in the older literature.

[2] But, the proof of the Hodge Decomposition Theorem is postponed to Chap. 5.

G. Rudolph and M. Schmidt, *Differential Geometry and Mathematical Physics*,
Theoretical and Mathematical Physics, DOI 10.1007/978-94-024-0959-8_2

2.1 Linear Connections

Let M be an n-dimensional differentiable manifold and let $L(M)$ be its bundle of linear frames, cf. Example 1.1.14. Recall that a linear frame at $m \in M$ is an ordered basis $u = (u_1, \ldots, u_n)$ in $\mathrm{T}_m M$ and that $\pi : L(M) \to M$, $\pi(u) = m$, is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle. The free right action of GL(n, $\mathbb{R}$) on $L(M)$ is given by

$$L(M) \times \mathrm{GL}(n, \mathbb{R}) \to L(M)\,, \quad (u, a) \mapsto ua\,. \tag{2.1.1}$$

Here, $ua = (u_i a^i{}_1, \ldots, u_i a^i{}_n)$.

In the sequel, the basic representation of $\mathrm{GL}(n, \mathbb{R})$ given by matrix multiplication of elements of $\mathbb{R}^n$ from the left will be denoted by σ_n^0. Thus, $\sigma_n^0(a)\mathbf{x} = a\mathbf{x}$.

Definition 2.1.1 A principal connection Γ on the frame bundle $L(M)$ will be referred to as a linear connection on M.[3]

Given a linear connection on M, it induces connections on all tensor bundles over M. To see this, it is enough to show that all tensor bundles over M are vector bundles associated with $L(M)$. For the proof, take the basic representation σ_n^0 of $\mathrm{GL}(n, \mathbb{R})$ and the corresponding associated bundle $E := L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$. Define

$$\varphi \colon E \to \mathrm{T}M\,, \quad \varphi([(u, \mathbf{x})]) := x^i u_i\,, \tag{2.1.2}$$

where x^i are the components of $\mathbf{x} \in \mathbb{R}^n$ in the standard basis $\{\mathbf{e}_i\}$ of $\mathbb{R}^n$. It is easy to show that φ is an isomorphism of vector bundles (Exercise 2.1.1). Thus,

$$\mathrm{T}M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n\,. \tag{2.1.3}$$

Via the dual of the basic representation, this induces an isomorphism

$$\mathrm{T}^*M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathbb{R}^n)^* \tag{2.1.4}$$

and, thus,

$$\mathbb{T}^k_l M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{T}^k_l \mathbb{R}^n\,. \tag{2.1.5}$$

Remark 2.1.2 Often, a frame $u \in L(M)$ will be viewed as an isomorphism

$$u : \mathbb{R}^n \to \mathrm{T}_{\pi(u)} M\,, \quad u(\mathbf{x}) := x^i u_i\,.$$

By (2.1.2), we have

$$\varphi \circ \iota_u = u\,. \tag{2.1.6}$$

♦

[3] As in the general theory, Γ is a horizontal distribution on $L(M)$. Below, it will become clear why it is reasonable to speak of a connection on the base manifold M.

Now we can start discussing the theory of linear connections. First, we exhibit a structure which distinguishes frame bundles from general principal fibre bundles.

Definition 2.1.3 The differential form $\theta \in \Omega^1(L(M), \mathbb{R}^n)$ defined by

$$\theta(X) := u^{-1}(\pi'(X)) , \quad X \in \mathrm{T}_u L(M) , \tag{2.1.7}$$

is called the canonical $\mathbb{R}^n$-valued 1-form on $L(M)$, or, the soldering form.

Proposition 2.1.4 *The soldering form θ is a horizontal 1-form of type σ_n^0,*

$$\Psi_a^* \theta = a^{-1} \circ \theta , \quad a \in \mathrm{GL}(n, \mathbb{R}) .$$

Proof By definition, θ is horizontal. Let $u \in L(M)$ and $a \in \mathrm{GL}(n, \mathbb{R})$. If we view u as a mapping $\mathbb{R}^n \to \mathrm{T}_{\pi(u)} M$, then to $\Psi_a(u)$ there corresponds the mapping

$$u \circ a : \ \mathbb{R}^n \xrightarrow{a} \mathbb{R}^n \xrightarrow{u} \mathrm{T}_{\pi(u)} M .$$

Thus, for any $X \in \mathrm{T}_u L(M)$,

$$\begin{aligned}
(\Psi_a^* \theta)_u(X) &= \theta_{\Psi_a(u)}(\Psi_a' X) \\
&= (\Psi_a(u))^{-1}(\pi' \circ \Psi_a'(X)) \\
&= (u \circ a)^{-1}(\pi'(X)) \\
&= a^{-1} \theta_u(X) .
\end{aligned}$$

■

Remark 2.1.5 By Proposition 1.2.12, via the isomorphism (2.1.2), to θ there corresponds a unique 1-form $\hat{\theta} \in \Omega^1(M, \mathrm{T}M)$ given by

$$\hat{\theta}_m(X) = u \circ \theta(X^*) = u \circ u^{-1} \circ \pi'(X^*) = X ,$$

where $\pi(u) = m$, $X \in \mathrm{T}_m M$ and $X^* \in \mathrm{T}_u L(M)$ fulfilling $\pi'(X^*) = X$. Thus, $\hat{\theta}(X) = X$. That is why $\hat{\theta}$ is usually called the tautological 1-form. ♦

Now, let Γ be a linear connection on M and let ω be its connection form on $L(M)$. Then, any $\mathbf{x} \in \mathbb{R}^n$ defines a Γ-horizontal vector field $B(\mathbf{x})$ on $L(M)$ by assigning to $u \in L(M)$ the unique Γ-horizontal lift of $u(\mathbf{x}) \in \mathrm{T}_{\pi(u)} M$ to the point u .

Definition 2.1.6 The vector field $B(\mathbf{x})$ is called the horizontal standard vector field defined by $\mathbf{x} \in \mathbb{R}^n$.

Proposition 2.1.7 *For any $\mathbf{x} \in \mathbb{R}^n$, the horizontal standard vector field fulfils*

1. $\theta(B(\mathbf{x})) = \mathbf{x}$,
2. $\Psi_{a*} B(\mathbf{x}) = B(a^{-1}\mathbf{x})$, $a \in \mathrm{GL}(n, \mathbb{R})$,
3. *if $\mathbf{x} \neq 0$, then $B(\mathbf{x})$ vanishes nowhere.*

Proof 1. We calculate

$$\theta_u(B(\mathbf{x})) = u^{-1}(\pi'(B(\mathbf{x})_u)) = u^{-1}(u(\mathbf{x})) = \mathbf{x}\,.$$

2. By Proposition 2.1.4 and point 1, we have

$$\theta(\Psi_{a*}B(\mathbf{x})) = \Psi_a^*\theta(B(\mathbf{x})) = a^{-1}\theta(B(\mathbf{x})) = a^{-1}\mathbf{x}\,,$$

and, thus, $\pi'(\Psi_{a*}B(\mathbf{x})) = u(a^{-1}\mathbf{x})$. Since $\Psi_{a*}B(\mathbf{x})$ is horizontal, the assertion follows from the uniqueness of the horizontal lift.

3. Clearly, $B(\mathbf{x})_u = 0$ iff $u(\mathbf{x}) = 0$ and, thus, iff $\mathbf{x} = 0$, because $u : \mathbb{R}^n \to \mathrm{T}_{\pi(u)}M$ is a vector space isomorphism. ■

Remark 2.1.8 Let $\{\mathbf{e}_i\}$ be the standard basis in $\mathbb{R}^n$. Then, the horizontal standard vector fields $B_i = B(\mathbf{e}_i)$ span the horizontal distribution defined by Γ. Moreover, $B(\mathbf{x})$ is uniquely determined by the conditions

$$\theta(B(\mathbf{x})) = \mathbf{x}\,, \quad \omega(B(\mathbf{x})) = 0\,. \tag{2.1.8}$$

♦

Lemma 2.1.9 *Let A_* be the Killing vector field on $L(M)$ generated by $A \in \mathfrak{gl}(n,\mathbb{R})$ and let $\mathbf{x} \in \mathbb{R}^n$. Then,*

$$[A_*, B(\mathbf{x})] = B(A\mathbf{x})\,. \tag{2.1.9}$$

Proof Let $a_t = \exp(tA)$. Using point 2 of Proposition 2.1.7, we obtain

$$[A_*, B(\mathbf{x})]_u = (\mathscr{L}_{A_*} B(\mathbf{x}))_u = \frac{\mathrm{d}}{\mathrm{d}t}\restriction_0 \Big(\big(\Psi_{a_t^{-1}}\big)_* B(\mathbf{x})\Big)_u = \frac{\mathrm{d}}{\mathrm{d}t}\restriction_0 B(a_t\mathbf{x})_u = B(A\mathbf{x})_u\,.$$

■

Definition 2.1.10 Let Γ be a linear connection on M and let ω be its connection form. The 2-form $\Theta \in \Omega^2(L(M),\mathbb{R}^n)$ defined by

$$\Theta := D_\omega\theta \tag{2.1.10}$$

is called the torsion form of Γ.

Clearly, Θ is a horizontal 2-form of type σ_n^0. The Structure Equation (1.4.9) for the curvature of a linear connection is supplemented by a structure equation involving the torsion form.

Proposition 2.1.11 (Structure Equations) *Let ω, Ω and Θ be, respectively, the connection, curvature and torsion forms of a linear connection Γ on M. Then, for any $X, Y \in \mathrm{T}_uL(M)$,*

$$\mathrm{d}\omega(X, Y) = -[\omega(X), \omega(Y)] + \Omega(X, Y)\,, \tag{2.1.11}$$
$$\mathrm{d}\theta(X, Y) = -\big(\omega(X)\theta(Y) - \omega(Y)\theta(X)\big) + \Theta(X, Y)\,. \tag{2.1.12}$$

Proof Equation (2.1.11) coincides with the Structure Equation (1.4.9) of the general theory. Since θ is a horizontal form, (2.1.12) follows immediately from formula (1.4.1), with σ being the basic representation. ■

Remark 2.1.12 Using

$$\omega \wedge \theta(X, Y) = \omega(X)\theta(Y) - \omega(Y)\theta(X)\,,$$

the Structure Equations may be rewritten as follows:

$$\mathrm{d}\omega = -\omega \wedge \omega + \Omega\,, \quad \mathrm{d}\theta = -\omega \wedge \theta + \Theta\,. \tag{2.1.13}$$

If we decompose the above forms with respect to the standard bases $\{\mathbf{e}_i\}$ in $\mathbb{R}^n$ and $\{E^i{}_j\}$ in $\mathfrak{gl}(n, \mathbb{R})$,

$$\theta = \theta^i \mathbf{e}_i\,, \quad \Theta = \Theta^i \mathbf{e}_i\,, \quad \omega = \omega^i{}_j E^j{}_i\,, \quad \Omega = \Omega^i{}_j E^j{}_i\,, \tag{2.1.14}$$

then we obtain the Structure Equations in the form

$$\mathrm{d}\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \Omega^i{}_j\,, \quad \mathrm{d}\theta^i = -\omega^i{}_j \wedge \theta^j + \Theta^i\,. \tag{2.1.15}$$

♦

The Bianchi identity for the curvature has a counterpart for the torsion.

Proposition 2.1.13 (Bianchi Identities) *Let ω, Ω and Θ be, respectively, the connection, curvature and torsion forms of a linear connection Γ on M. Then,*

$$D_\omega \Omega = 0\,, \tag{2.1.16}$$
$$D_\omega \Theta = \Omega \wedge \theta\,. \tag{2.1.17}$$

Proof Equation (2.1.16) coincides with the Bianchi Identity (1.4.10) of the general theory. Equation (2.1.17) is an immediate consequence of Proposition 1.4.12, with $\sigma = \sigma^0_n$. ■

Alternatively, (2.1.17) may be checked by direct inspection. It is obtained by differentiating the first of the two equations in (2.1.15) and by using both of these equations thereafter (Exercise 2.1.5).

Remark 2.1.14

1. The 1-forms ω and θ may be combined to the joint object

$$\omega + \theta \in \Omega^1(L(M), \mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n)\,.$$

Clearly, $\mathfrak{gl}(n, \mathbb{R}) \oplus \mathbb{R}^n$ is the Lie algebra of the affine group on $\mathbb{R}^n$. Its commutation relations are obtained by supplementing the commutation relations of $\mathfrak{gl}(n, \mathbb{R})$ by

$$[A, \mathbf{x}] = -[\mathbf{x}, A] = A\mathbf{x}\,, \quad [\mathbf{x}, \mathbf{y}] = 0\,, \quad A \in \mathfrak{gl}(n, \mathbb{R})\,, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\,.$$

Accordingly, we may pass from the bundle $L(M)$ of linear frames to the bundle $A(M)$ of affine frames. Clearly, $\omega + \theta$ defines a connection form on $A(M)$ which is called the affine connection form induced by ω. This explains why linear connection and affine connection are often used as synonyms in the literature. Obviously,

$$D_{\omega+\theta}(\omega + \theta) = \mathrm{d}(\omega + \theta) + \frac{1}{2}[\omega + \theta, \omega + \theta] = \Omega + \Theta\,,$$

that is, curvature and torsion constitute a joint object on $A(M)$, namely the curvature of $\omega + \theta$.

2. Let $\{\mathbf{e}_i\}$ and $\{E^j{}_i\}$ be the standard bases of $\mathbb{R}^n$ and $\mathfrak{gl}(n, \mathbb{R})$, respectively. Let B_i be the horizontal standard vector field with respect to a chosen connection Γ generated by $\mathbf{e}_i$ and let $E^j{}_{i*}$ be the Killing vector field generated by $E^j{}_i$. Since the $E^j{}_{i*}$ span the vertical subspace $V_u \subset \mathrm{T}_u L(M)$, for every $u \in L(M)$, and since the $\{B_i\}$ span the (complementary) Γ-horizontal subspace Γ_u, these $n^2 + n$ vector fields provide a global frame in the tangent bundle $\mathrm{T}L(M)$ which is, therefore, trivial. One says that the manifold $L(M)$ admits a global parallelism given by the vector fields B_i, $E^j{}_{i*}$. Moreover, the vector fields B_i, $E^j{}_{i*}$ are dual to the 1-forms θ^i, $\omega^i{}_j$,

$$\begin{aligned} \theta^k(B_i) &= \delta^k{}_i\,, \qquad \theta^k(E^j{}_{i*}) = 0\,, \\ \omega^k{}_l(B_i) &= 0\,, \qquad \omega^k{}_l(E^j{}_{i*}) = \delta^k{}_i \delta^j{}_l\,. \end{aligned} \tag{2.1.18}$$

Thus, $\mathrm{T}^*L(M)$ is trivial, too, and the 1-forms θ^i, $\omega^i{}_j$ provide a global frame of $\mathrm{T}^*L(M)$, or, in more abstract terms, the affine connection $\omega + \theta$ induces an absolute parallelism on $A(M)$. As a consequence, every horizontal k-form α on $L(M)$ may be expanded with respect to the 1-forms θ^i,

$$\alpha = \frac{1}{k!}\alpha_{i_1 \ldots i_k} \theta^{i_1} \wedge \cdots \wedge \theta^{i_k}\,. \tag{2.1.19}$$

In particular,

$$\Omega^i{}_j = \tfrac{1}{2}\Omega^i_{klj}\,\theta^k \wedge \theta^l\,, \quad \Theta^i = \tfrac{1}{2}\Theta^i_{jk}\,\theta^j \wedge \theta^k\,. \tag{2.1.20}$$

♦

Since both Ω and Θ are horizontal 2-forms on $L(M)$ of type Ad, respectively, they uniquely correspond to 2-forms on M with values in certain associated vector bundles. By Proposition 1.2.12 and by the isomorphism (2.1.3), to $\Theta \in \Omega^2(L(M), \mathbb{R}^n)$

there corresponds an element $\mathsf{T} \in \Omega^2(M, \mathrm{T}M)$ defined by

$$\mathsf{T}_m(X, Y) = u(\Theta_u(X^*, Y^*))\,, \tag{2.1.21}$$

where $X, Y \in \mathrm{T}_m M$, $\pi(u) = m$ and $X^*, Y^* \in \mathrm{T}_u L(M)$ fulfilling $\pi'(X^*) = X$ and $\pi'(Y^*) = Y$.[4] By Remark 1.4.7, to Ω there corresponds a 2-form on M with values in the adjoint bundle $\mathrm{Ad}(L(M))$. Since the differential of the basic representation σ_n^0 identifies $\mathfrak{gl}(n, \mathbb{R})$ naturally with $\mathrm{End}(\mathbb{R}^n)$, this 2-form may be identified with the curvature endomorphism form $\mathsf{R} \in \Omega^2(M, \mathrm{End}(\mathrm{T}M))$,

$$\mathsf{R}_m(X, Y) = u \circ \Omega_u(X^*, Y^*) \circ u^{-1}\,, \tag{2.1.22}$$

cf. (1.5.13). Since R takes values in $\mathrm{End}(\mathrm{T}M)$, we may apply it to any tangent vector $Z \in \mathrm{T}_m M$:

$$\mathsf{R}_m(X, Y)Z = u\big(\Omega_u(X^*, Y^*)(u^{-1}Z)\big)\,. \tag{2.1.23}$$

Definition 2.1.15 Let Γ be a linear connection on $L(M)$ and let Θ and Ω be its curvature and torsion forms. The 2-forms T and R defined by (2.1.21) and (2.1.22) are called the torsion tensor field associated with Θ and the curvature tensor field associated with Ω, respectively.

Remark 2.1.16 Since, for any $u \in L(M)$, the assignment $\mathbb{R}^n \to \Gamma_u\,,\ \mathbf{x} \mapsto B(\mathbf{x})$, is an isomorphism of vector spaces, we have an induced isomorphism

$$b(u) : \bigwedge^2 \mathbb{R}^n \to \bigwedge^2 \Gamma_u\,, \quad b(u)(\mathbf{x} \wedge \mathbf{y}) = B(\mathbf{x})_u \wedge B(\mathbf{y})_u\,.$$

Using this, we get yet another presentation of curvature and torsion, which will turn out to be useful. We define mappings

$$\mathscr{R} : L(M) \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathfrak{gl}(n, \mathbb{R})\,, \quad \mathscr{T} : L(M) \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

by

$$\mathscr{R}(u) := \Omega_u \circ b(u)\,, \quad \mathscr{T}(u) := \Theta_u \circ b(u)\,. \tag{2.1.24}$$

In the sequel, $\mathscr{R}$ and $\mathscr{T}$ will be referred to as the curvature and the torsion mappings, respectively. Using that Ω and Θ are horizontal forms of type Ad and σ_n^0, respectively, together with (1.2.3), one finds:

$$\mathscr{R}(\Psi_a(u))(\mathbf{x}, \mathbf{y}) = \mathrm{Ad}(a^{-1}) \circ \big(\mathscr{R}(u)(a\mathbf{x}, a\mathbf{y})\big)\,, \tag{2.1.25}$$

$$\mathscr{T}(\Psi_a(u))(\mathbf{x}, \mathbf{y}) = a^{-1} \circ \big(\mathscr{T}(u)(a\mathbf{x}, a\mathbf{y})\big)\,. \tag{2.1.26}$$

By Proposition 1.2.6, to $\mathscr{R}$ and $\mathscr{T}$, there correspond unique sections of the associated bundles

[4] Clearly, for X^* and Y^* we may take the horizontal lifts of X and Y with respect to Γ.

$$L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \left(\bigwedge^2(\mathbb{R}^n)^* \otimes \mathfrak{gl}(n,\mathbb{R})\right), \quad L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \left(\bigwedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n\right),$$

respectively. By (2.1.24), they are given by

$$m \mapsto u \circ \mathscr{R}(u) \circ u^{-1} = \mathsf{R}_u \circ \bigwedge^2 u\,, \quad m \mapsto u \circ \mathscr{T}(u) = \mathsf{T}_u \circ \bigwedge^2 u\,, \tag{2.1.27}$$

where $\bigwedge^2 u : \mathbb{R}^n \wedge \mathbb{R}^n \to \mathrm{T}_{\pi(u)}M \wedge \mathrm{T}_{\pi(u)}M$ and $m = \pi(u)$. ♦

Next, we discuss the covariant derivative of tensor fields and apply the Koszul calculus developed in Sect. 1.5 to the case under consideration. By Definition 1.5.2, the covariant derivative

$$\nabla^\omega = (\mathrm{d}_\omega)_{\restriction \Omega^0(M,E)} : \ \Gamma^\infty(E) \to \Gamma^\infty(\mathrm{T}^*M \otimes E)$$

on an associated bundle $E = P \times_G F$, induced from a connection form ω, is given by

$$(\nabla^\omega \Phi)_m(X) = \iota_p \circ (D_\omega \tilde{\Phi})_p(X^*)\,, \tag{2.1.28}$$

with $\pi(p) = m$ and $X^* \in \mathrm{T}_pP$ fulfilling $\pi'(X^*) = X$. Applying this to a section Y of $TM \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \mathbb{R}^n$, that is, to a vector field on M, we read off

$$(\nabla^\omega Y)_m(X) = u \circ (D_\omega \tilde{Y})_u(X^*)\,, \quad \pi(u) = m\,, \tag{2.1.29}$$

where $\tilde{Y} \in \mathrm{Hom}_{\mathrm{GL}(n,\mathbb{R})}(L(M), \mathbb{R}^n)$ is given by $Y(m) = u \circ \tilde{Y}(u)$. According to (1.5.10), we have an associated operator

$$\nabla^\omega_X : \Gamma^\infty(\mathrm{T}M) \to \Gamma^\infty(\mathrm{T}M)\,, \quad \nabla^\omega_X Y := (\nabla^\omega Y)(X)\,. \tag{2.1.30}$$

In the sequel, we assume that a connection has been chosen and, for simplicity, we write ∇ instead of ∇^ω.

Remark 2.1.17

1. By (1.5.3), formula (2.1.29) may be rewritten as $(\nabla_X Y)(m) = u(X^*_u(\tilde{Y}))$, where X^* is the horizontal lift of X. Thus, using

$$\theta_u(Y^*) = u^{-1} \circ \pi'(Y^*) = u^{-1} Y_m = \tilde{Y}_u\,,$$

we obtain

$$(\nabla_X Y)(m) = u(X^*_u(\theta(Y^*)))\,. \tag{2.1.31}$$

2. Clearly, the covariant derivative ∇_X given by (2.1.30) has all the properties listed in Proposition 1.5.8. Moreover, it induces covariant derivatives in all tensor bundles over M. A general formula is easily derived from (1.4.2) by taking for σ the tensor

product representation of p copies of σ_n^0 and q copies of its dual, cf. Exercise 2.1.2. If not otherwise stated, by ∇ we mean the covariant derivative in $\mathrm{T}M$. ♦

The proof of the following proposition is left to the reader (Exercise 2.1.3). It provides an axiomatic characterization of the covariant derivative of a tensor field.

Proposition 2.1.18 *Let Γ be a linear connection on a manifold M and let ∇ be its covariant derivative in $\mathrm{T}M$. Then, the covariant derivative*

$$\nabla_X : \Gamma^\infty(\mathrm{T}^r_s M) \to \Gamma^\infty(\mathrm{T}^r_s M)\,,$$

acting on tensor fields of type (r, s) is uniquely determined by the following properties.

1. $\nabla_X f = X(f)$, *for $f \in C^\infty(M)$.*
2. ∇_X *is a derivation of the tensor algebra.*
3. ∇_X *commutes with any contraction.*

We express the curvature and torsion tensor fields in terms of the covariant derivative.

Proposition 2.1.19 *Let ∇ be the covariant derivative of a linear connection Γ on M. Then, the curvature and the torsion tensor fields of Γ are given by*

$$\mathsf{R}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\,, \tag{2.1.32}$$

$$\mathsf{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]\,. \tag{2.1.33}$$

Proof Formula (2.1.32) follows from Proposition 1.5.11 as a special case. To prove formula (2.1.33), let X^*, Y^* be the horizontal lifts of X and Y. Then, $\Theta(X^*, Y^*) = \mathrm{d}\theta(X^*, Y^*)$. Using this, together with (2.1.31) and $\pi'([X^*, Y^*]) = [X, Y]$, we obtain

$$\begin{aligned} \mathsf{T}(X, Y)(m) &= u(\Theta_u(X^*, Y^*)) \\ &= u(X^*_u(\theta(Y^*)) - Y^*_u(\theta(X^*)) - \theta_u([X^*, Y^*])) \\ &= (\nabla_X Y - \nabla_Y X - [X, Y])(m)\,. \end{aligned}$$

■

Finally, we carry over the concept of parallel transport and holonomy as developed in Sect. 1.7 to the case of linear connections on M. In this way, for a given connection, we obtain the operation of parallel transport along curves in M both for the frame bundle $L(M)$ and for any associated tensor bundle $\mathrm{T}^r_s M$. Correspondingly, we obtain holonomy groups in all associated tensor bundles. As in the general theory, there is a deep relation between holonomy and curvature, provided by the Ambrose-Singer Theorem 1.7.15. This has tremendous consequences for the structure theory of (pseudo-)Riemannian manifolds, see Sect. 2.3.

Clearly, comparing with the general theory, the situation here is special in so far as the parallel transport operators apply to geometric objects living on the base manifold M. Related to this fact, there is a special class of curves which we discuss next. Applying the theory to the tangent bundle, for any curve $\gamma : I \to M$, we obtain a unique parallel transport of tangent vectors along γ. In the sequel, let $I \subset \mathbb{R}$ denote an open interval containing 0. Let $\dot{\gamma}$ be the tangent vector field of γ. By Example I/1.5.5, it is given by

$$\dot{\gamma}(t) = \gamma'_t \left(\frac{\mathrm{d}}{\mathrm{d}t}{\restriction_t} \right),$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the unit vector field on I. Applying the notions developed in Sect. 1.7, a vector field X on M is parallel (with respect to a connection Γ) along a curve γ if

$$\nabla^{\gamma}_{\frac{\mathrm{d}}{\mathrm{d}t}} X = 0\,. \tag{2.1.34}$$

Here, ∇^{γ} is the covariant derivative along the mapping γ and X must be viewed as a section of $\mathrm{T}M$ along γ.[5] In particular, since $\dot{\gamma}$ is certainly a section of $\mathrm{T}M$ along γ, we may consider the equation

$$\nabla^{\gamma}_{\frac{\mathrm{d}}{\mathrm{d}t}} \dot{\gamma} = 0 \tag{2.1.35}$$

and we may ask whether it admits solutions.

Definition 2.1.20 Let Γ be a linear connection. A curve $\gamma : I \to M\,,\ t \mapsto \gamma(t)\,$, is called a geodesic with respect to Γ if it fulfils equation (2.1.35).

The following proposition is left as an exercise to the reader (Exercise 2.1.4).

Proposition 2.1.21 *If a curve $\gamma : I \to M$ is a geodesic, then for any $\alpha, \beta \in \mathbb{R}$ the curve $t \mapsto \gamma(\alpha \cdot t + \beta)$ is a geodesic, too.* ■

Proposition 2.1.22 *Let Γ be a linear connection on M. Then, the projection under $\pi : L(M) \to M$ of any integral curve of a horizontal standard vector field is a geodesic. Conversely, every geodesic is obtained in this way.*

Proof Let $\mathbf{x} \in \mathbb{R}^n$. By definition, $B(\mathbf{x})_u$ is the unique Γ-horizontal lift of $u(\mathbf{x}) \in \mathrm{T}_{\pi(u)}M$ to $u \in L(M)$. Let $t \mapsto \tilde{\gamma}(t)$ be an integral curve of $B(\mathbf{x})$. Define $\gamma := \pi \circ \tilde{\gamma}$. Then, using the natural identification (2.1.2) and omitting φ,

$$\dot{\gamma}(t) = \pi' \circ \dot{\tilde{\gamma}}(t) = \pi'(B(\mathbf{x})_{\tilde{\gamma}(t)}) = \tilde{\gamma}(t)(\mathbf{x}) = \iota_{\mathbf{x}}(\tilde{\gamma}(t))\,,$$

where $\tilde{\gamma}(t) : \mathbb{R}^n \to \mathrm{T}_{\gamma(t)}M$ as usual. Thus, by (1.7.13) and (1.3.4), we have

$$\nabla^{\gamma}_{\frac{\mathrm{d}}{\mathrm{d}t}} \dot{\gamma} = \omega^E \big(\iota'_{\mathbf{x}}(\dot{\tilde{\gamma}}(t))\big) = 0\,.$$

Conversely, let $\gamma : I \to M$ be a geodesic. Let $u_0 \in L(M)$ be such that $\pi(u_0) = \gamma(0)$ and let $\mathbf{x} := u_0^{-1}(\dot{\gamma}(0)) \in \mathbb{R}^n$. Let $t \mapsto \tilde{\gamma}(t)$ be the horizontal lift of γ through u_0.

[5] That is, more precisely, we should write $X \circ \gamma$ instead of X.

If $\mathbf{x} = 0$, we are done. Thus, let $\mathbf{x} \neq 0$. Then, there exists a curve $t \to \sigma(t)$ in $L(M)$ such that $\dot{\gamma}(t) = \sigma(t)(\mathbf{x})$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\gamma}(t) = \iota'_{\mathbf{x}}\dot{\sigma}(t)\,.$$

Since γ is a geodesic, that is, $\frac{\mathrm{d}}{\mathrm{d}t}\dot{\gamma}(t) \in \Gamma^{\mathrm{T}M} \subset \mathrm{T}(\mathrm{T}M)$, this formula implies that $t \mapsto \sigma(t)$ is horizontal in $L(M)$. Since $\sigma(0) = u_0$ and $\pi \circ \sigma = \gamma$, uniqueness of the horizontal lift implies $\sigma = \tilde{\gamma}$. Thus, $\dot{\gamma}(t) = \tilde{\gamma}(t)(\mathbf{x})$ and, since $\tilde{\gamma}$ is horizontal,

$$\theta(\dot{\tilde{\gamma}}(t)) = \tilde{\gamma}(t)^{-1}(\pi'(\dot{\tilde{\gamma}}(t))) = \tilde{\gamma}(t)^{-1}(\dot{\gamma}(t)) = \mathbf{x}\,.$$

Thus, $t \mapsto \tilde{\gamma}(t)$ is an integral curve of $B(\mathbf{x})$. ∎

Corollary 2.1.23 *Let Γ be a connection on M. For every $m \in M$ and every $X \in \mathrm{T}_m M$, there exists a unique geodesic $\gamma : I \to M$ with initial conditions (m, X), that is, $\gamma(0) = m$ and $\dot{\gamma}(0) = X$.* ∎

We say that a linear connection Γ on M is complete if every geodesic of Γ may be extended to $I = \mathbb{R}$. Then, we have another corollary following immediately from Proposition 2.1.22.

Corollary 2.1.24 *A linear connection on M is complete iff every horizontal standard vector field on $L(M)$ is complete.* ∎

If M is endowed with a complete linear connection Γ, we may define the following mapping. For every $m \in M$ and every $X \in \mathrm{T}_m M$, we take the unique geodesic γ with initial conditions $(\gamma(0) = m, \dot{\gamma}(0) = X)$ and put

$$\exp : \mathrm{T}M \to M\,, \quad \exp(X) := \gamma(1)\,. \tag{2.1.36}$$

This mapping is called the exponential mapping of Γ.

Remark 2.1.25 If Γ is not complete, then exp may still be defined. In this case, one defines exp on a neighbourhood of the zero section in $\mathrm{T}M$. This way, one obtains a smooth mapping which, for every $m \in M$, yields a local diffeomorphism from a neighbourhood of the origin in $\mathrm{T}_m M$ onto a neighbourhood U_m of m in M, see Fig. 2.1. For details, we refer to Propositions 8.1 and 8.2 in Chap. III of [381]. ♦

In the remainder of this section, we describe the above structures locally. Thus, let

$$m \mapsto \mathfrak{e}(m) = (e_1(m), \ldots, e_n(m))$$

be a local section of $L(M)$, that is, a local frame of $\mathrm{T}M$, and let

$$m \mapsto \vartheta(m) = (\vartheta^i(m), \ldots \vartheta^n(m))$$

be its dual coframe. Recall that $\mathfrak{e}(m)(\mathbf{e}_i) = e_i(m)$ for the standard basis $\{\mathbf{e}_i\}$ of $\mathbb{R}^n$.

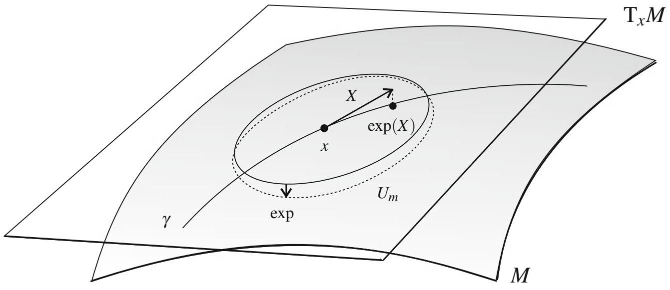

Fig. 2.1 Exponential mapping

Lemma 2.1.26 *For any local frame* $\mathfrak{e}$*,*

$$\mathfrak{e}^*\theta = \vartheta^i \otimes \mathbf{e}_i \,. \tag{2.1.37}$$

Proof For any $X \in \mathrm{T}_m M$, we calculate

$$(\mathfrak{e}^*\theta)_m(X) = \theta_{\mathfrak{e}(m)}(\mathfrak{e}'(X)) = (\mathfrak{e}(m))^{-1}(\pi' \circ \mathfrak{e}'(X)) = (\mathfrak{e}(m))^{-1}(X) \,.$$

Thus, decomposing $X = X^i e_i(m)$ and using $\mathfrak{e}(m)(\mathbf{e}_i) = e_i(m)$, we obtain

$$(\mathfrak{e}^*\theta)_m(X) = X^i(m)\mathbf{e}_i = \vartheta^i_m(X)\mathbf{e}_i \,. \qquad \blacksquare$$

Thus, for the components of θ with respect to the decomposition (2.1.14),

$$\mathfrak{e}^*\theta^i = \vartheta^i \,. \tag{2.1.38}$$

Next, the local representative $\mathscr{A} = \mathfrak{e}^*\omega$ of a linear connection Γ with connection form ω is a 1-form on M with values in $\mathfrak{gl}(n, \mathbb{R})$. Thus, it may be written as

$$\mathscr{A} = \mathscr{A}^i{}_k E^k{}_i = \Gamma^i{}_{jk}\, \vartheta^j \otimes E^k{}_i \,. \tag{2.1.39}$$

The coefficient functions $\Gamma^i{}_{jk}$ are called the Christoffel symbols of Γ in the local frame $\mathfrak{e}$.

Remark 2.1.27 Consider a change $\mathfrak{e} \to \mathfrak{e}'$ of the local frame.[6] Using (1.3.15), we obtain the following induced transformation formula for the Christoffel symbols (Exercise 2.1.6)

$$\Gamma'^l{}_{mn} = \Gamma^i{}_{jk}\, \rho^j{}_m \rho^k{}_n (\rho^{-1})^l{}_i + \rho^j{}_m \big(\partial_j\, \rho^i{}_n\big)(\rho^{-1})^l{}_i \,. \tag{2.1.40}$$

♦

[6] We emphasize the passive interpretation here, but formula (2.1.40) may also be interpreted actively.

Let us calculate the local representatives of curvature and torsion. For that purpose, we take the pullback of (2.1.20) under $\mathfrak{e}$,

$$\mathfrak{e}^*\Omega^i{}_j = \tfrac{1}{2}\left(\mathfrak{e}^*\Omega^i_{klj}\right)\vartheta^k \wedge \vartheta^l\,, \quad \mathfrak{e}^*\Theta^i = \tfrac{1}{2}\left(\mathfrak{e}^*\Theta^i_{jk}\right)\vartheta^j \wedge \vartheta^k\,, \tag{2.1.41}$$

and denote the local coefficient functions as follows:

$$\mathsf{R}^i{}_{klj} = \mathfrak{e}^*\Omega^i_{klj}\,, \quad \mathsf{T}^i{}_{jk} = \mathfrak{e}^*\Theta^i_{jk}\,.$$

To calculate them, we use the Structure Equations in the form given by (2.1.15). Taking the pullback of the first equation yields

$$\tfrac{1}{2}\mathsf{R}^i{}_{klj}\,\vartheta^k \wedge \vartheta^l = \mathrm{d}\mathscr{A}^i{}_j + \mathscr{A}^i{}_k \wedge \mathscr{A}^k{}_j\,.$$

Inserting (2.1.39) into this equation, we obtain (Exercise 2.1.7)

$$\mathsf{R}^i{}_{jkl} = e_j(\Gamma^i{}_{kl}) - e_k(\Gamma^i{}_{jl}) + \Gamma^m{}_{kl}\Gamma^i{}_{jm} - \Gamma^m{}_{jl}\Gamma^i{}_{km} - C^m{}_{jk}\Gamma^i{}_{ml}\,, \tag{2.1.42}$$

where the $C^i{}_{jk}$ are the structure functions of the local frame $\mathfrak{e}$ defined by

$$[e_j, e_k] = C^i{}_{jk} e_i\,. \tag{2.1.43}$$

In the same way, taking the pullback of the second equation in (2.1.15), we read off

$$\mathsf{T}^i{}_{jk} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj} - C^i{}_{jk}\,. \tag{2.1.44}$$

Next, by Proposition 1.5.3, the local version of the Koszul calculus is based upon the following formula. For a local frame $\mathfrak{e}$, we have

$$\nabla e_j = \Gamma^k{}_{ij}\vartheta^i \otimes e_k\,. \tag{2.1.45}$$

Correspondingly,

$$\nabla_{e_i} e_j = \Gamma^k{}_{ij}\, e_k\,. \tag{2.1.46}$$

Next, acting with ∇_{e_i} on the pairing $\vartheta^j(e_k) = \delta^j{}_k$ and using that the covariant derivative is a derivation of the tensor algebra, we obtain

$$\nabla_{e_i}\vartheta^j = -\Gamma^j{}_{ik}\,\vartheta^k\,. \tag{2.1.47}$$

Thus,

$$\nabla\vartheta^j = -\Gamma^j{}_{ik}\,\vartheta^i \otimes \vartheta^k\,. \tag{2.1.48}$$

Now, decomposing an arbitrary tensor field with respect to a local frame $\mathfrak{e}$ and its dual coframe ϑ and using (2.1.46) and (2.1.47), together with the properties of the covariant derivative, one can derive a local formula for the covariant derivative of any tensor field, see Exercise 2.1.7. In particular, for a vector field X and a 1-form α we obtain

$$\nabla_{e_i} X = \big(e_i(X^k) + \Gamma^k{}_{ij} X^j\big) e_k \,, \tag{2.1.49}$$

$$\nabla_{e_i} \alpha = \big(e_i(\alpha_j) - \Gamma^k{}_{ij} \alpha_k\big) \vartheta^j \,. \tag{2.1.50}$$

Using (1.5.8), we get $\nabla X = \vartheta^i \otimes \nabla_{e_i} X$ and $\nabla\alpha = \vartheta^i \otimes \nabla_{e_i}\alpha$. Clearly, the covariant derivative of any tensor field t may also be decomposed in this way,

$$\nabla t = \vartheta^i \otimes \nabla_{e_i} t \,, \tag{2.1.51}$$

in accordance with the fact that $\nabla t \in \Omega^1(M, \mathbb{T}^k_l(M))$.

Remark 2.1.28 By point 2 of Remark 1.2.15, it is clear that the local representatives of Ω and R, as well as the local representatives of Θ and T, coincide. Thus,

$$\mathsf{R}\left(e_j, e_k\right) e_l = \mathsf{R}^i{}_{jkl} e_i \,, \quad \mathsf{T}\left(e_j, e_k\right) = \mathsf{T}^i{}_{jk} e_i \,. \tag{2.1.52}$$

This can also be checked by direct inspection, inserting (2.1.46) into (2.1.32) and (2.1.33) and comparing with (2.1.42) and (2.1.44) (Exercise 2.1.8). ♦

Remark 2.1.29 (*Holonomic frame*) Let (U, κ) be a local chart of M and let x^i be the corresponding local coordinates. Then, $\{\partial_j\}$ is a local frame of $\mathrm{T}M$, called the induced holonomic frame of $\mathrm{T}M$ and $\{\mathrm{d}x^j\}$ is the dual coframe of T^*M. The name holonomic refers to the fact that $[\partial_i, \partial_j] = 0$, that is, the structure functions of a holonomic frame vanish. In such a frame, the formulae (2.1.39), (2.1.42), (2.1.44) and (2.1.45) take the following form:

$$\mathscr{A} = \Gamma^i{}_{jk}\, \mathrm{d}x^j \otimes E^k{}_i \,, \tag{2.1.53}$$

$$\mathsf{R}^i_{jkl} = \partial_j \Gamma^i{}_{kl} - \partial_k \Gamma^i{}_{jl} + \Gamma^m{}_{kl} \Gamma^i{}_{jm} - \Gamma^m{}_{jl} \Gamma^i{}_{km} \,, \tag{2.1.54}$$

$$\mathsf{T}^i_{jk} = \Gamma^i{}_{jk} - \Gamma^i{}_{kj} \,, \tag{2.1.55}$$

$$\nabla \partial_j = \Gamma^k{}_{ij}\, \mathrm{d}x^i \otimes \partial_k \,. \tag{2.1.56}$$

The change from one holonomic frame to another one is described by the Jacobi matrix of the coordinate transformation. Thus, here, the transition function is

$$x \mapsto \rho(x) = \left(\frac{\partial x^i}{\partial x'^l}\right)$$

and the transformation formula (2.1.40) reads

$$\Gamma'^l{}_{mn} = \Gamma^i{}_{jk} \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} \cdot \frac{\partial x'^l}{\partial x^i} + \frac{\partial^2 x^i}{\partial x'^m \partial x'^n} \frac{\partial x'^l}{\partial x^i} \,. \tag{2.1.57}$$

♦

It remains to analyze Eqs. (2.1.34) and (2.1.35) in local coordinates. Then, γ is given by $t \mapsto x^i(t)$ and, correspondingly, $X = X^i \partial_i$ and $\dot{\gamma} = \dot{x}^i \partial_i$. Using points 3 and 4 of Proposition 1.5.8 we calculate:

$$\nabla_{\dot{\gamma}} X = \nabla_{\dot{x}^i \partial_i} \left(X^j \partial_j \right) = \left(\dot{x}^i X^j \Gamma^k{}_{ij} + \partial_i \left(X^k \right) \dot{x}^i \right) \partial_k \,,$$

that is, Eq. (2.1.34) reads

$$\frac{dX^k}{dt} + \Gamma^k{}_{ij} \dot{x}^i X^j = 0 \,. \tag{2.1.58}$$

This is a system of first order ordinary differential equations, which according to standard theorems admits unique local solutions depending smoothly on the initial values $(t_0, X(t_0))$. The solution $t \mapsto X(t)$ provides the parallel transport

$$\hat{\gamma}_{\Gamma^{\mathrm{T}M}}(t) \colon \mathrm{T}_{\gamma(t_0)} M \to \mathrm{T}_{\gamma(t)} M \,. \tag{2.1.59}$$

Inserting $X^i = \dot{x}^i$ into (2.1.58), we obtain the local form of the geodesic equation:

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}t^2} + \Gamma^k{}_{ij} \dot{x}^i \dot{x}^j = 0 \,. \tag{2.1.60}$$

This is a system of second order ordinary differential equations, which admits unique local solutions depending smoothly on the initial conditions $(t_0, x^i(t_0), \dot{x}^i(t_0))$.

Remark 2.1.30

1. Consider the exponential mapping of a linear connection Γ on M, cf. equation (2.1.36) and Remark 2.1.25. Via the exponential mapping, any frame $u : \mathbb{R}^n \to \mathrm{T}_m M$ at $m \in M$ provides a local chart on $\mathrm{T}_m M$:

$$\varphi := \exp \circ u : \mathbb{R}^n \to U_m \,.$$

   This is a local diffeomorphism from a neighborhood of 0 in $\mathbb{R}^n$ onto a neighbourhood $U_m \subset M$ of m. Taking $\kappa := \varphi^{-1}$ we obtain a local chart (U, κ) centered at m which will be referred to as a local geodesic chart. The local coordinates x^i of that chart mapping will be called normal coordinates at m. In normal coordinates, any geodesic takes the form $x^i(t) = a^i \cdot t$. Thus, at m, we obviously have $\Gamma^k{}_{ij} + \Gamma^k{}_{ji} = 0$. That is, for vanishing torsion, the Christoffel symbols vanish at m (Exercise 2.1.9).
2. The parallel transport of a tangent vector along a closed curve yields a geometric interpretation of curvature. Note that this is in accordance with the Ambrose-Singer Theorem 1.7.15. We have (Exercise 2.1.9)

$$\triangle X^i = -\frac{1}{2} \mathsf{R}^i{}_{jkl} X^l \cdot f^{jk} \,, \tag{2.1.61}$$

   where f^{jk} is a bivector field characterizing the plane enclosed by γ.

3. The quantity

$$a^i := \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + \Gamma^i{}_{jk} \frac{\mathrm{d}x^j}{\mathrm{d}t} \frac{\mathrm{d}x^k}{\mathrm{d}t}$$

is the natural generalization of the notion of acceleration of a point particle to curved space. For $a^i = 0$, the particle moves on a geodesic. This occurs if the particle is not acted upon by additional (non-gravitational) external forces. ♦

Exercises

2.1.1 Prove that the mapping φ defined by (2.1.2) is an isomorphism of vector bundles.

2.1.2 Derive from (1.4.2) a formula for the covariant derivative of a tensor field t of type (r, s) by taking for σ the tensor product representation of s copies of σ_n^0 and r copies of its dual.

2.1.3 Prove Proposition 2.1.18.

2.1.4 Prove Proposition 2.1.21.

2.1.5 Prove equation (2.1.17) by a direct calculation using the Structure Equations.

2.1.6 Prove formula (2.1.40).

2.1.7 Prove the local formulae (2.1.42), (2.1.44), (2.1.49) and (2.1.50). Derive a local formula for the covariant derivative of an arbitrary tensor field t, cf. Exercise 2.1.2. Conclude that, in particular, in local coordinates the covariant derivative of t is given by

$$\nabla_{\partial_k} t^{i_1 \dots i_s}_{j_1 \dots j_r} = \partial_k \, t^{i_1 \dots i_s}_{j_1 \dots j_r} + \sum_l \Gamma^{i_l}_{km} t^{i_1 \dots i_l = m \dots i_s}_{j_1 \dots j_r} - \sum_l \Gamma^m_{k j_l} t^{i_1 \dots i_s}_{j_1 \dots j_l = m \dots j_r} \,.$$

2.1.8 Prove the statement of Remark 2.1.28.

2.1.9 Prove the statements of points 1 and 2 of Remark 2.1.30.

2.2 *H*-Structures and Compatible Connections

In the sequel, we will meet reductions of the frame bundle $L(M)$ to various Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$. The following concept allows for a unified treatment of all of them.

Definition 2.2.1 (*H-structure*) Let M be a smooth manifold.

1. A reduction P of the frame bundle $L(M)$ to a Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ is called an H-structure on M.
2. An H-structure P is called integrable if for every point $m \in M$ there exists a local chart (U, κ) with local coordinates x^j such that the induced holonomic frame $\{\partial_j\}$ is a local section of P. Such local coordinates are called admissible.
3. Let $\varphi : M \to M$ be a diffeomorphism. If $\varphi' : \mathrm{T}M \to \mathrm{T}M$ leaves P invariant, then φ is called an automorphism of the H-structure.

Clearly, the automorphisms of an H-structure form a group. By Corollary 1.6.5, reductions of $L(M)$ to a Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ are in one-to-one correspondence with smooth sections of the associated bundle

$$L(M) \times_{\mathrm{GL}(n,\mathbb{R})} (\mathrm{GL}(n, \mathbb{R})/H)\,, \tag{2.2.1}$$

or, equivalently, with elements of $\mathrm{Hom}_{\mathrm{GL}(n,\mathbb{R})}(L(M), \mathrm{GL}(n, \mathbb{R})/H)$. Thus, the existence of an H-structure on a manifold M is a topological problem which can be dealt with by applying methods of obstruction theory. In particular, if $\mathrm{GL}(n, \mathbb{R})/H$ is contractible, then an H-structure certainly exists. Note that, geometrically, an H-structure should be viewed as a bundle of distinguished frames on M.

Recall from Definition 1.6.11 the general notion of compatible connection.

Definition 2.2.2 A linear connection on M is called compatible with the H-structure P if it is reducible to P.

Next, recall Proposition 1.6.10 characterizing the reducibility of connections on principal bundles in terms of G-homomorphisms.

Proposition 2.2.3 *Let P be an H-structure on M and let*

$$\tilde{\Phi} : L(M) \to \mathrm{GL}(n, \mathbb{R})/H$$

be the $\mathrm{GL}(n, \mathbb{R})$-equivariant mapping defining P. Assume that $\mathrm{GL}(n, \mathbb{R})/H$ embeds into a $\mathrm{GL}(n, \mathbb{R})$-module F. Then, a linear connection ω on $L(M)$ is compatible with the H-structure P iff $\tilde{\Phi}$ is parallel with respect to ω, that is, iff

$$D_\omega \tilde{\Phi} = 0\,.$$

Proof By the proof of Proposition 1.6.2, $P = \Big\{u \in L(M) : \tilde{\Phi}(u) = [\mathbb{1}]\Big\}$. Thus, the restriction of $D_\omega \tilde{\Phi} = 0$ to P reads

$$\sigma'(\omega)[\mathbb{1}] = 0\,,$$

which holds iff ω restricted to P takes values in the Lie algebra of H. This is equivalent to being reducible to P. ■

Clearly, for a given H-structure P we may restrict the soldering form θ of $L(M)$ to P and, thus, for any connection ω on P we have a torsion 2-form Θ on P defined by (2.1.10). One says that ω is torsion-free if Θ vanishes.

Proposition 2.2.4 *If P is an integrable H-structure on M, then it admits a torsion-free connection.*

Proof Let $\pi : P \to M$ be the canonical projection. Let s be an integrable local section of P over $U \subset M$. Taking the tangent bundle of the graph of s and extending it using the right H-action to a distribution on P, we obtain a connection on $\pi^{-1}(U) \subset P$. Then, integrability implies $s^*\mathrm{d}\theta = 0$ (Exercise 2.2.1) and, thus, vanishing of the torsion. Next, we patch together these local connections to a connection on P using a partition of unity. Since torsion is additive this yields the assertion. ■

Since any other connection ω' on P differs from ω by a horizontal 1-form α on P with values in the Lie algebra $\mathfrak{h}$ of H,

$$\Theta' = \Theta + \alpha \wedge \theta \,.$$

By Remark 2.1.16, Θ and α may be identified with H-equivariant functions

$$\mathscr{T} : P \to \textstyle\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \,, \quad \tilde{\alpha} : P \to (\mathbb{R}^n)^* \otimes \mathfrak{h} \,,$$

respectively. Since $H \subset \mathrm{GL}(n, \mathbb{R})$, we have a natural inclusion

$$\iota_{\mathfrak{h}} : \mathfrak{h} \to \mathrm{End}(\mathbb{R}^n) \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^n \,.$$

Thus, under the above identification, $\alpha \wedge \theta$ is a function on P with values in $\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$. We claim that it coincides with the image of $\tilde{\alpha}$ under the mapping

$$\delta : (\mathbb{R}^n)^* \otimes \mathfrak{h} \to \textstyle\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \,, \quad \delta := (a \otimes \mathrm{id}_{\mathbb{R}^n}) \circ (\mathrm{id}_{(\mathbb{R}^n)^*} \otimes \iota_{\mathfrak{h}}) \,, \tag{2.2.2}$$

where $a : (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \to \bigwedge^2 (\mathbb{R}^n)^*$ is the anti-symmetrization mapping. Indeed, using $\tilde{\alpha}(u)(\mathbf{x}) = \alpha\big(B(\mathbf{x})\big)$, we calculate

$$(\alpha \wedge \theta)_u \big(B(\mathbf{x}), B(\mathbf{y})\big) = \big(\tilde{\alpha}(u)(\mathbf{x})\big)\mathbf{y} - \big(\tilde{\alpha}(u)(\mathbf{y})\big)\mathbf{x} = \big(\delta \circ \tilde{\alpha}(u)\big)(\mathbf{x}, \mathbf{y}) \,.$$

As a result,

$$\mathscr{T}' = \mathscr{T} + \delta(\tilde{\alpha}) \,. \tag{2.2.3}$$

Let

$$\mathrm{pr} : \textstyle\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \mathrm{coker}(\delta) = \Big(\bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n\Big) / \,\mathrm{im}(\delta)$$

be the natural projection.[7] Then, the mapping

$$\tau : P \to \mathrm{coker}(\delta)\,, \quad \tau(u) := \mathrm{pr}(\mathscr{T}(u))\,, \tag{2.2.4}$$

does not depend on the choice of the connection. This motivates the following definition.

Definition 2.2.5 The mapping τ is called the intrinsic torsion of the H-structure P. Moreover, P is called torsion-free if τ vanishes.

Clearly, τ yields the obstruction to the existence of a torsion-free connection on P.

Proposition 2.2.6 *Let P be an H-structure. Then, the following hold.*

1. *If ω and ω' are torsion-free connections on P and $\omega' = \omega + \alpha$, then $\tilde{\alpha}(u) \in \ker\delta$ for every $u \in P$. In particular, if $\ker(\delta) = 0$, then P admits at most one torsion-free connection.*
2. *P has a torsion-free connection iff it is torsion-free.*

Proof The first assertion follows immediately from (2.2.3). For the second one, if P has a torsion-free connection, then it is clearly torsion-free. We prove the converse: let ω be a connection with (non-vanishing) torsion Θ. By assumption, $\tau = 0$. Thus, $\mathscr{T}(u) \in \mathrm{im}(\delta)$ for every $u \in P$. That is, there exists an equivariant mapping $\tilde{\alpha} : P \to (\mathbb{R}^n)^* \otimes \mathfrak{h}$ such that $\mathscr{T} = \delta(\tilde{\alpha})$. Let α be the unique horizontal 1-form on P corresponding to $\tilde{\alpha}$. Then, $\omega' = \omega - \alpha$ is a torsion-free connection. ■

In particular, as an immediate consequence, we obtain

Corollary 2.2.7 *If δ is bijective, then P admits a unique torsion-free connection.* ■

Next, let us discuss a number of relevant examples.

Example 2.2.8 (*Orientation*) We take $H = \mathrm{GL}_+(n, \mathbb{R})$. Then, $\mathrm{GL}(n, \mathbb{R})/H \cong \mathbb{Z}_2$. According to Example 1.6.6, a section of the associated bundle (2.2.1) exists iff the manifold is orientable, that is, iff the first Stiefel-Whitney class[8] of M vanishes. In this case, the H-structure consists of those frames which are compatible with a chosen orientation. Note that this H-structure is integrable. Also note that automorphisms of this H-structure are exactly the orientation-preserving diffeomorphisms of M. ♦

Example 2.2.9 (*Volume form*) We consider $H = \mathrm{SL}(n, \mathbb{R})$. The basic representation of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ induces the following $\mathrm{GL}(n, \mathbb{R})$-action on $\bigwedge^n(\mathbb{R}^n)^*$:

$$\mathrm{GL}(n, \mathbb{R}) \times \textstyle\bigwedge^n(\mathbb{R}^n)^* \to \bigwedge^n(\mathbb{R}^n)^*\,, \quad (a, \mathsf{v}) \mapsto \det(a) \cdot \mathsf{v}\,.$$

[7] The mapping δ and its cokernel have an interpretation in terms of Spencer cohomology of $\mathfrak{h}$ which we suppress here. For details, see e.g. [569].

[8] See Sect. 4.2.

Restricted to $\bigwedge^n(\mathbb{R}^n)^* \setminus \{0\}$, this action is transitive and has the common stabilizer $\mathrm{SL}(n,\mathbb{R})$. Thus,

$$\mathrm{GL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{R}) \cong \bigwedge^n(\mathbb{R}^n)^* \setminus \{0\}\,.$$

Via the natural isomorphism $\bigwedge^n \mathrm{T}^*M \cong L(M) \times_{\mathrm{GL}(n,\mathbb{R})} \bigwedge^n(\mathbb{R}^n)^*$, the sections of the associated bundle (2.2.1) are in one-to-one correspondence with volume forms on M. The $\mathrm{SL}(n,\mathbb{R})$-structure corresponding to a given volume form v consists of those frames u fulfilling

$$\mathsf{v} = \mathsf{v}_0 \circ \bigwedge^n u\,,$$

where v_0 is the canonical volume form on $\mathbb{R}^n$. Since $\mathrm{GL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{R})$ is homotopy equivalent to $\mathrm{GL}(n,\mathbb{R})/\mathrm{GL}_+(n,\mathbb{R})$, M admits an $\mathrm{SL}(n,\mathbb{R})$-structure iff M is orientable. Moreover, it is easy to show that any $\mathrm{SL}(n,\mathbb{R})$-structure is integrable (Exercise 2.2.2). Finally, note that the automorphisms of this H-structure are the volume-preserving diffeomorphisms of M. ♦

Example 2.2.10 (*Almost complex structure*) Take $H = \mathrm{GL}(n,\mathbb{C})$ canonically embedded in $\mathrm{GL}(2n,\mathbb{R})$ via

$$a + ib \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad a, b \in \mathrm{GL}(n,\mathbb{R})\,, \tag{2.2.5}$$

and consider the canonical complex structure on $\mathbb{R}^{2n}$ given by

$$\mathsf{J}_0 = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}. \tag{2.2.6}$$

Since $\mathrm{End}(\mathbb{R}^{2n}) \cong (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$, the basic representation of $\mathrm{GL}(2n,\mathbb{R})$ induces a $\mathrm{GL}(2n,\mathbb{R})$-module structure on $\mathrm{End}(\mathbb{R}^{2n})$ given by

$$\mathrm{GL}(2n,\mathbb{R}) \times \mathrm{End}(\mathbb{R}^{2n}) \to \mathrm{End}(\mathbb{R}^{2n})\,, \quad (g, A) \mapsto g^{-1}Ag\,.$$

Since $\mathrm{End}(\mathbb{R}^{2n})$ is the Lie algebra of $\mathrm{GL}(2n,\mathbb{R})$, this is merely the adjoint representation. Now, by Proposition I/7.1.2, the induced action of $\mathrm{GL}(2n,\mathbb{R})$ on the subset of endomorphisms fulfilling $A^2 = -\,\mathrm{id}$ is transitive and the stabilizer of J_0 is

$$H_{\mathsf{J}_0} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathrm{GL}(n,\mathbb{R}) \right\} = \mathrm{GL}(n,\mathbb{C})\,. \tag{2.2.7}$$

Thus,

$$\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C}) \cong \left\{ A \in \mathrm{End}(\mathbb{R}^{2n}) : A^2 = -\,\mathrm{id} \right\}\,.$$

Thus, by (2.2.1), $\mathrm{GL}(n,\mathbb{C})$-structures are in one-to-one correspondence with sections J of $\mathrm{End}(\mathrm{T}M)$ fulfilling $\mathsf{J}_m^2 = -\,\mathrm{id}$ for every $m \in M$. A $\mathrm{GL}(n,\mathbb{C})$-structure will be referred to as an almost complex structure on M and (M, J) will be called an almost complex manifold. Since $\mathrm{End}(\mathbb{R}^{2n}) \cong (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}$, J may be viewed as a tensor field on M of type $(1, 1)$. The $\mathrm{GL}(n, \mathbb{C})$-structure defined by J will be denoted by $C(M, \mathsf{J})$ and will be referred to as the bundle of complex linear frames. Note that it consists of frames fulfilling

$$u \circ \mathsf{J}_0 = \mathsf{J}_m \circ u\,, \tag{2.2.8}$$

where $u : \mathbb{R}^{2n} \to \mathrm{T}_m M$ as usual. It is easy to show that every almost complex manifold is orientable (Exercise 2.2.4). For a discussion of the obstructions to the existence of almost complex structures we refer to [431].

Next, let us discuss integrability. By (2.2.8), an almost complex structure (M, J) is integrable if M has the structure of a complex manifold such that for any system of admissible local coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ we have

$$\mathsf{J}\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial y^k}\,, \quad \mathsf{J}\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}\,.$$

Then, $z^k := x^k + iy^k$ provide M with a local chart of complex coordinates. Conversely, we have

Proposition 2.2.11 *Viewed as a real C^∞-manifold, every complex manifold M carries a natural induced integrable almost complex structure.*

Proof Let $\{(U_i, \kappa_i)\}$ be a holomorphic atlas of M consisting of charts $\kappa_i : U_i \to \mathbb{C}^n$. For every i, we define an associated mapping $\tilde{\kappa}_i : U_i \to \mathbb{R}^{2n}$ given by

$$\tilde{\kappa}_i(m) := (\mathrm{Re}(\kappa_1(m)), \dots, \mathrm{Re}(\kappa_n(m)), \mathrm{Im}(\kappa_1(m)), \dots, \mathrm{Im}(\kappa_n(m)))\ ,$$

which clearly provides a C^∞-chart on U_i. Thus, $\{(U_i, \tilde{\kappa}_i)\}$ endows M with the structure of a real C^∞-manifold. Next, consider $\mathbb{R}^{2n}$ with the global coordinates $x^1, \dots, x^n, y^1, \dots y^n$. Then,

$$\mathsf{J}\left(\frac{\partial}{\partial x^k}\right) := \frac{\partial}{\partial y^k}\,, \quad \mathsf{J}\left(\frac{\partial}{\partial y^k}\right) := -\frac{\partial}{\partial x^k}\,,$$

clearly defines a complex structure on $\mathbb{R}^{2n}$. We transport this complex structure to M, viewed as a real manifold, via the local charts $\tilde{\kappa}_i$. The almost complex structure defined in this way is independent of the choice of the atlas, because the transition mappings are holomorphic and a mapping of an open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$ leaves an almost complex structure on $\mathbb{C}^n$ invariant iff it is holomorphic (Exercise 2.2.3). By construction, the above almost complex structure is integrable. Indeed,

$$(\mathbf{x}, \mathbf{y}) \mapsto \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}\right)$$

provides a local section of the $\mathrm{GL}(n, \mathbb{C})$-structure defined by J. ■

To summarize, an almost complex structure is integrable iff it is induced from a complex structure. The following notion provides a criterion for integrability.

Definition 2.2.12 Let (M, J) be an almost complex manifold. The Nijenhuis tensor of (M, J) is the tensor field $N \in \Gamma^\infty(\mathrm{T}^1_2(M))$ defined by

$$N(X, Y) := [\mathsf{J}X, \mathsf{J}Y] - [X, Y] - \mathsf{J}([X, \mathsf{J}Y]) - \mathsf{J}([\mathsf{J}X, Y]), \quad X, Y \in \mathfrak{X}(M).$$

The following deep theorem holds, see [485].

Theorem 2.2.13 (Newlander–Nirenberg) *An almost complex structure* J *is integrable iff the Nijenhuis tensor of* J *vanishes.* ■

Next, we show that J implies a natural splitting of tensor bundles over M. In particular, this will imply a variety of equivalent criteria for integrability. From now on, let $\mathrm{T} = \mathbb{R}^{2n}$ denote the basic $\mathrm{GL}(2n, \mathbb{R})$-module, let T^* be the dual (contragredient) module and let $\mathrm{T}_\mathbb{C}$ and $\mathrm{T}^*_\mathbb{C}$ be the complexifications of T and T^*, respectively. We extend J_0 to a $\mathbb{C}$-linear mapping of $\mathrm{T}_\mathbb{C}$ and decompose $\mathrm{T}_\mathbb{C}$ into eigenspaces $\mathrm{T}^{1,0}$ and $\mathrm{T}^{0,1}$ corresponding to the eigenvalues i and $-i$ of J_0:

$$\mathrm{T}_\mathbb{C} = \mathrm{T}^{1,0} \oplus \mathrm{T}^{0,1}. \tag{2.2.9}$$

Then,

$$\mathrm{T}^{1,0} = \{X - i\mathsf{J}_0 X : X \in \mathrm{T}\}, \quad \mathrm{T}^{0,1} = \{X + i\mathsf{J}_0 X : X \in \mathrm{T}\}. \tag{2.2.10}$$

On the other hand, recall from Sect. 7.5 of Part I that J_0 endows T with the structure of a complex vector space, denoted by V, via

$$(a + ib)X := aX + b\mathsf{J}_0 X, \quad a, b \in \mathbb{R}, \ X \in \mathrm{T}. \tag{2.2.11}$$

Clearly, $V \cong \mathbb{C}^n$ carries the basic $\mathrm{GL}(n, \mathbb{C})$-module structure. Let ι be the natural embedding of V into $\mathrm{T}_\mathbb{C}$. Via this mapping, a chosen basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ in V induces a basis $(\mathbf{e}_1, \mathsf{J}_0\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathsf{J}_0\mathbf{e}_n)$ in $\mathrm{T}_\mathbb{C}$. By (2.2.11), for $Z = (X^k + iY^k)\mathbf{e}_k$ we have

$$\iota(Z) = X^k\mathbf{e}_k + Y^k\mathsf{J}_0\mathbf{e}_k. \tag{2.2.12}$$

Note that ι is not complex linear. Next, let $\mathrm{pr}^{1,0} : \mathrm{T}_\mathbb{C} \to \mathrm{T}^{1,0}$ and $\mathrm{pr}^{0,1} : \mathrm{T}_\mathbb{C} \to \mathrm{T}^{0,1}$ be the canonical projections. Then,

$$\mathrm{pr}^{1,0} \circ \iota : V \to \mathrm{T}^{1,0}, \quad \mathrm{pr}^{0,1} \circ \iota : V \to \mathrm{T}^{0,1}_m, \tag{2.2.13}$$

are $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear vector space isomorphisms, respectively (Exercise 2.2.6). Next, recall the embedding $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$ given by (2.2.5). It extends to $\mathrm{T}_\mathbb{C}$ by

$$\rho : \mathrm{GL}(n, \mathbb{C}) \times \mathrm{T}_{\mathbb{C}} \to \mathrm{T}_{\mathbb{C}}\,, \quad \rho(g)\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} aX - bY \\ bX + aY \end{bmatrix}.$$

One easily checks (Exercise 2.2.6) that for any $Z \in V$,

$$\mathrm{pr}^{1,0} \circ \rho(g) \circ \iota(Z) = (a + ib)Z\,, \quad \mathrm{pr}^{0,1} \circ \rho(g) \circ \iota(Z) = (a - ib)\overline{Z}\,. \tag{2.2.14}$$

On the other hand, the subspaces $\mathrm{T}^{1,0}$ and $\mathrm{T}^{0,1}$ are invariant under the $\mathrm{GL}(n, \mathbb{C})$-action and, by (2.2.5), they carry the basic representation of $\mathrm{GL}(n, \mathbb{C})$ and its conjugate, respectively. It follows that V and $\mathrm{T}^{1,0}$ are isomorphic as $\mathrm{GL}(n, \mathbb{C})$-modules.

Next, note that, by duality, the decomposition (2.2.9) implies a decomposition

$$\mathrm{T^*}_{\mathbb{C}} = \mathrm{T}^{*1,0} \oplus \mathrm{T}^{*0,1}\,, \tag{2.2.15}$$

where $\mathrm{T}^{*1,0}$ and $\mathrm{T}^{*0,1}$ are the annihilators of $\mathrm{T}^{0,1}$ and $\mathrm{T}^{1,0}$, respectively. Thus, they carry the dual of the basic and the basic representation of $\mathrm{GL}(n, \mathbb{C})$, respectively. This decomposition induces the following decompositions:

$$\bigwedge^k \mathrm{T^*}_{\mathbb{C}} = \bigoplus_{p+q=k} \bigwedge^{p,q}\,, \quad \bigwedge^{p,q} = \bigwedge^p \mathrm{T}^{*1,0} \otimes \bigwedge^q \mathrm{T}^{*0,1}\,. \tag{2.2.16}$$

Clearly, in analogy to (2.2.9) and (2.2.15), J induces decompositions

$$\mathrm{T}_{\mathbb{C}}M = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M\,, \quad \mathrm{T^*}_{\mathbb{C}}M = \mathrm{T}^{*1,0}M \oplus \mathrm{T}^{*0,1}M\,. \tag{2.2.17}$$

Note that, as a complex vector bundle, $\mathrm{T}M$ is $\mathbb{C}$-linearly isomorphic to $\mathrm{T}^{1,0}M$ via (2.2.13). Corresponding to (2.2.16), we have

$$\bigwedge^k \mathrm{T^*}_{\mathbb{C}}M = \bigoplus_{p+q=k} \bigwedge^{p,q}M\,, \quad \bigwedge^{p,q}M = \bigwedge^p \mathrm{T}^{*1,0}M \otimes \bigwedge^q \mathrm{T}^{*0,1}M\,. \tag{2.2.18}$$

The spaces of sections of $\bigwedge^k \mathrm{T^*}_{\mathbb{C}}M$ and $\bigwedge^{p,q}M$ will be denoted by $\Omega^k_{\mathbb{C}}(M)$ and by $\Omega^{p,q}(M)$, respectively. Elements of $\Omega^{p,q}(M)$ are called differential forms of type (p, q). Let us denote the projection to $\Omega^{p,q}(M)$ by $\Pi^{p,q}$. Extending the exterior differential $\mathbb{C}$-linearly, we may define mappings $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ and $\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ via

$$\partial := \Pi^{p+1,q} \circ \mathrm{d}\,, \quad \overline{\partial} := \Pi^{p,q+1} \circ \mathrm{d}\,. \tag{2.2.19}$$

Proposition 2.2.14 *For an almost complex manifold, the following conditions are equivalent:*

1. $N(X, Y) = 0$ *for all* $X, Y \in \mathfrak{X}(M)$.
2. $\mathrm{T}^{1,0}M$ *is involutive.*
3. $\mathrm{d}\big(\Omega^{1,0}(M)\big) \subset \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$.

4. *For any* $\alpha \in \Omega^k_{\mathbb{C}}(M)$*, we have* $\mathrm{d}\alpha = \partial\alpha + \overline{\partial}\alpha$.

Proof Recall that, as a real vector space, $\mathrm{T}_{\mathbb{C}}$ decomposes as $\mathrm{T}_{\mathbb{C}} = \mathrm{T} + \mathrm{i}\mathrm{T}$. Correspondingly, we have real linear projections $\mathrm{Re}, \mathrm{Im} : \mathrm{T}_{\mathbb{C}} \to \mathrm{T}$ defined by $W = \mathrm{Re}(W) + \mathrm{i}\mathrm{Im}(W)$ for all $W \in \mathrm{T}_{\mathbb{C}}$. Now, for any $X, Y \in \mathfrak{X}(M)$, we calculate

$$\begin{aligned} N(X, Y) &= [\mathsf{J}X, \mathsf{J}Y] - [X, Y] - \mathsf{J}([X, \mathsf{J}Y]) - \mathsf{J}([\mathsf{J}X, Y]) \\ &= -\mathrm{Re}\big([X - i\mathsf{J}X, Y - i\mathsf{J}Y] + i\mathsf{J}[X - i\mathsf{J}X, Y - i\mathsf{J}Y]\big) \\ &= -8\mathrm{Re}\big([X^{1,0}, Y^{1,0}]^{0,1}\big)\,. \end{aligned}$$

Since for elements $W \in \mathrm{T}^{0,1}$ we have $\mathrm{Im}(W) = \mathsf{J}(\mathrm{Re}(W))$, points 1 and 2 are equivalent. For $\alpha \in \Omega^{1,0}(M)$ and $X, Y \in \Gamma^\infty(\mathrm{T}^{1,0}M)$,

$$\mathrm{d}\overline{\alpha}(X, Y) = X(\overline{\alpha}(Y)) - Y(\overline{\alpha}(X)) - \overline{\alpha}([X, Y]) = -\overline{\alpha}([X, Y])\,,$$

where $\overline{\alpha} \in \Omega^{0,1}(M)$ defined by $\overline{\alpha}(W) = \alpha(\overline{W})$ with $\overline{W}$ denoting the conjugation in $\mathrm{T}_{\mathbb{C}}$. This implies the equivalence of points 2 and 3. Clearly, point 4 implies point 3. Thus, it remains to prove the converse. We note that $\mathrm{d} = \partial + \overline{\partial}$ holds iff $\mathrm{d}\alpha \in \Omega^{p+1,q}(M) \oplus \Omega^{p,q+1}(M)$ for any $\alpha \in \Omega^{p,q}(M)$. Locally,

$$\alpha = f\vartheta^{i_1} \wedge \ldots \wedge \vartheta^{i_p} \wedge \varphi^{j_1} \wedge \ldots \wedge \varphi^{j_q}\,, \quad \vartheta^k \in \Omega^{1,0}(M)\,,\ \varphi^l \in \Omega^{0,1}(M)\,.$$

We have $\mathrm{d}f \in \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$, $\mathrm{d}\vartheta^k \in \Omega^{2,0}(M) \oplus \Omega^{1,1}(M)$. Since $\overline{\Omega^{1,0}(M)} = \Omega^{0,1}(M)$, point 3 implies $\mathrm{d}\varphi^l \in \Omega^{1,1}(M) \oplus \Omega^{0,2}(M)$ and the assertion follows. ■

Corollary 2.2.15 *If an almost complex structure* J *is integrable, then*

$$\partial^2 = 0\,, \quad \overline{\partial}^2 = 0\,, \quad \overline{\partial} \circ \partial + \partial \circ \overline{\partial} = 0\,. \tag{2.2.20}$$

Conversely, if $\overline{\partial}^2 = 0$*, then* J *is integrable.*[9]

Proof The first assertion is an immediate consequence of $\mathrm{d}^2 = 0$. The second assertion is left to the reader, see Exercise 2.2.7. ■

Let z^k be local coordinates on a complex manifold M. Then, any $\alpha \in \Omega^*_{\mathbb{C}}(M)$ locally reads[10] $\alpha = \alpha_{IJ}\mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J$ and

$$\partial\alpha = \frac{\partial\alpha_{IJ}}{\partial z^k}\mathrm{d}z^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J\,, \quad \overline{\partial}\alpha = \frac{\partial\alpha_{IJ}}{\partial \overline{z}^k}\mathrm{d}\overline{z}^k \wedge \mathrm{d}z^I \wedge \mathrm{d}\overline{z}^J\,.$$

[9] Using the operator $\overline{\partial}$, one can build a cohomology theory for complex manifolds, called the Dolbeault cohomology, see [336].

[10] We use the notation of Sect. 4.1 of Part I.

Finally, we note that a diffeomorphism $\varphi : M \to M$ is an automorphism of an almost complex structure J iff $\varphi' \circ \mathsf{J} = \mathsf{J} \circ \varphi'$. If J is integrable, then this means that φ is holomorphic. ♦

The following example is closely related to Example 1.6.6.

Example 2.2.16 (*Pseudo-Riemannian metric*) Denote the vector space of symmetric covariant tensors of second rank on $\mathbb{R}^n$ by $S^2\mathbb{R}^n$. Endow $\mathbb{R}^n$ with a pseudo-Euclidean metric $\eta \in S^2\mathbb{R}^n$ with signature (k, l) where $n = k + l$. The basic representation of $\mathrm{GL}(n, \mathbb{R})$ induces a $\mathrm{GL}(n, \mathbb{R})$-module structure on $S^2\mathbb{R}^n$ given by

$$\sigma : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Aut}(S^2\mathbb{R}^n), \quad \sigma(a) := (a^{-1})^T \otimes (a^{-1})^T . \tag{2.2.21}$$

As already noted under point 2 of Example 1.6.6, by the Sylvester Theorem, $\mathrm{GL}(n, \mathbb{R})$ acts transitively on the subspace $S^2_{(k,l)}\mathbb{R}^n \subset S^2\mathbb{R}^n$ consisting of elements with fixed signature, and the stabilizer of η is $\mathrm{O}(k, l)$, that is,

$$\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(k, l) \cong S^2_{(k,l)}\mathbb{R}^n .$$

Thus, by (2.2.1), $\mathrm{O}(k, l)$-structures are in one-to-one correspondence with pseudo-Riemannian metrics g on M and the $\mathrm{O}(k, l)$-structure corresponding to g coincides with the bundle $O(M)$ of frames which are orthonormal with respect to g. If (M, g) is oriented, then $O(M)$ further reduces to a principal $\mathrm{SO}(k, l)$-bundle, denoted by $O_+(M)$. Note that $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ is contractible. Thus, an $\mathrm{O}(n)$-structure, that is, a Riemannian metric, always exists. On the contrary, for an arbitrary signature, $\mathrm{O}(k, l)$-structures may not exist. E.g. the obstruction to the existence of a Lorentz-structure[11] on a 4-dimensional oriented manifold is given by the Euler class of the tangent bundle. Thus, for a non-compact M, there is no obstruction. Below, we will see that associated with a pseudo-Riemannian structure, there is a unique torsion-free connection. Then, point 1 of Remark 1.4.7 implies that an $\mathrm{O}(k, l)$-structure is integrable iff the curvature of this connection vanishes. Equivalently, a pseudo-Riemannian structure is integrable iff it is locally flat, that is, if for every point of M there exists a neighbourhood on which g is given by the Euclidean metric.

Clearly, a diffeomorphism $\varphi : M \to M$ is an automorphism of an $\mathrm{O}(k, l)$-structure iff φ is an isometry of the corresponding pseudo-Riemannian metric g, that is, $\varphi^*\mathsf{g} = \mathsf{g}$. It can be shown, see Theorem 3.4 in Chap. VI of [381], that the group of isometries carries a Lie group structure with respect to the compact-open topology. This Lie group will be denoted by $I(M)$. ♦

Example 2.2.17 (*Conformal structure*) For $n \geq 3$, consider the Lie subgroup

$$\mathrm{CO}(n) := \left\{a \in \mathrm{GL}(n, \mathbb{R}) : a^{\mathrm{T}} a = c\mathbb{1} , \ c \in \mathbb{R} , \ c > 0\right\} .$$

Clearly, $\mathrm{CO}(n) = \mathbb{R}_+ \times \mathrm{O}(n)$. By the previous example, $\mathrm{GL}(n, \mathbb{R})$ acts transitively on the space $S^2_{(k,l)}\mathbb{R}^n$. Thus, it also acts transitively on the set of conformal

[11] A pseudo-Riemannian structure with signature $(+, -, -, -)$.

equivalence classes of elements of $S^2_{(k,l)}\mathbb{R}^n$ defined by the relation $\eta \sim \eta'$ iff $\eta' = c\eta$ for some positive real number c. Clearly, the stabilizer of an element $[\eta]$ is $\mathrm{CO}(n)$. Thus, $\mathrm{CO}(n)$-structures are in one-to-one correspondence with conformal equivalence classes $[\mathsf{g}]$ of metrics on M, with the equivalence defined as follows: two metrics g_1 and g_2 are conformally equivalent iff they differ by a positive function. The $\mathrm{CO}(n)$-structure corresponding to class $[\mathsf{g}]$ is denoted by $CO(M)$ and is referred to as the bundle of conformal frames.

Since $\mathrm{CO}(n) = \mathbb{R}_+ \times \mathrm{O}(n)$, the representation theory of $\mathrm{CO}(n)$ is essentially obtained as an extension of the representation theory of the orthogonal group $\mathrm{O}(n)$. The irreducible representations of $\mathbb{R}_+$ on $\mathbb{R}$ are labeled by real numbers $r \in \mathbb{R}$ and are given by

$$\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}\,, \quad (t, x) \mapsto t^r x\,.$$

The number r is called the conformal weight of the representation under consideration. Let us denote the corresponding representation space by L^r (a copy of $\mathbb{R}$). Then, a typical CO-module is a tensor product of an $\mathrm{O}(n)$-module with L^r. Note that, with respect to the conformal structure $[\mathsf{g}]$, the tangent and the cotangent bundles can no longer be identified, because they correspond to representations containing the factors L^r and L^{-r}, respectively. Clearly, on the level of vector bundles over M, the additional factors L^r corresponds to building the tensor product with an associated line bundle characterized by r.

In close relation to the previous example, one can show that a conformal structure is integrable iff it is locally conformally flat, that is, iff for every point of M there exists a neighbourhood on which the metric is given by $\mathsf{g} = f^2\mathsf{g}_0$, where g_0 is the (flat) Euclidean metric and f is a nowhere vanishing function on that neighbourhood. If this condition holds globally, then one says that (M, g) is conformally flat or, equivalently, that $(M, [\mathsf{g}])$ is flat.

A diffeomorphism $\varphi : M \to M$ is an automorphism of a $\mathrm{CO}(n)$-structure iff there exists a nowhere vanishing function $f \in C^\infty(M)$ such that $\varphi^*\mathsf{g} = f^2\mathsf{g}$, where g is some representative of this structure. The automorphism group of a conformal structure $(M, [\mathsf{g}])$ is called the conformal group of $(M, [\mathsf{g}])$. It will be denoted by $\mathrm{C}(M, [\mathsf{g}])$. The following classical theorem may be found in [381].[12]

Theorem 2.2.18 *Let (M, g) be a connected n-dimensional Riemannian manifold with $n \geq 3$. Then, its conformal group $\mathrm{C}(M, [\mathsf{g}])$ is a Lie group of dimension at most $\frac{1}{2}(n+1)(n+2)$.* ■

For a systematic study of conformal geometry, we refer to [61, 119, 382, 492, 686, 608]. ♦

Example 2.2.19 (*Almost Hermitean structure*) Recall from Example I/7.5.5 that, in the standard embedding (2.2.6) of $\mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(2n, \mathbb{R})$, we have

$$\mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{GL}(n, C)\,. \tag{2.2.22}$$

[12] The authors of [381] outline a proof based upon results of Eisenhardt [183] and Palais [499].

Explicitly,

$$\mathrm{U}(n) = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : aa^{\mathrm{T}} + bb^{\mathrm{T}} = \mathbb{1}\,,\ ab^{\mathrm{T}} - ba^{\mathrm{T}} = 0\,,\ a, b \in \mathrm{GL}(n, \mathbb{R}) \right\}. \tag{2.2.23}$$

This shows that we may combine an almost complex structure $C(M)$ with the $\mathrm{SO}(2n)$-structure $O_+(M)$ of a $2n$-dimensional (oriented) Riemannian manifold by intersecting them. On the algebraic level, $\mathsf{J}_0^{\mathrm{T}} \eta \mathsf{J}_0 = \eta$. Thus, if we assume that J is an isometry, that is,

$$\mathsf{g}(\mathsf{J}X, \mathsf{J}Y) = \mathsf{g}(X, Y)\,, \quad X, Y \in \mathfrak{X}(M)\,, \tag{2.2.24}$$

then the intersection

$$U(M) := C(M) \cap O_+(M) \tag{2.2.25}$$

is a $\mathrm{U}(n)$-structure.[13] It is called the bundle of unitary frames. If (2.2.24) is fulfilled, we say that g is a Hermitean metric with respect to J. The triple $(M, \mathsf{g}, \mathsf{J})$ is called an almost Hermitean manifold. If, additionally, J is integrable, then $(M, \mathsf{g}, \mathsf{J})$ is called a Hermitean manifold. Note that

$$\beta(X, Y) := \mathsf{g}(X, \mathsf{J}Y) \tag{2.2.26}$$

is a non-degenerate 2-form on M. Thus, β^n is a nowhere vanishing $2n$-form, that is, an orientation of M. This shows that every almost Hermitean manifold is endowed with a canonical volume form. Existence and integrability criteria of almost Hermitean structures are obtained from Examples 2.2.10 and 2.2.16 above. Clearly, a diffeomorphism $\varphi : M \to M$ is an automorphism of a $\mathrm{U}(n)$-structure iff it is an automorphism of the $\mathrm{GL}(n, \mathbb{C})$- and of the $\mathrm{SO}(2n)$-structure.

We give an equivalent description of an almost Hermitean manifold $(M, \mathsf{g}, \mathsf{J})$. Viewing its tangent bundle $\mathrm{T}M$ as a complex vector bundle, each of its fibres carries a Hermitean scalar product, given by[14]

$$\mathsf{h}(X, Y) := \mathsf{g}(X, Y) + i\mathsf{g}(X, \mathsf{J}Y)\,. \tag{2.2.27}$$

Equivalently, by (2.2.26),

$$\mathsf{h}(X, Y) = \mathsf{g}(X, Y) + i\beta(X, Y) = \beta(\mathsf{J}X, Y) + i\beta(X, Y)\,. \tag{2.2.28}$$

Note that h is linear in the first and anti-linear in the second entry (Exercise 2.2.8). Thus, $(\mathrm{T}M, \mathsf{h})$ is a Hermitean vector bundle, cf. Definition 1.1.16. As usual, let $\tilde{\mathsf{h}}$, $\tilde{\mathsf{g}}$ and $\tilde{\mathsf{J}}$ be the equivariant mappings corresponding to h, g and J, respectively.

[13]It suffices to assume that $C(M)$ and $O_+(M)$ have a nonempty intersection over every point of M.

[14]See Sect. 7.5 of Part I. Note that we have changed conventions in order to be compatible with the standard literature.

Restricted to $U(M)$, $\tilde{\mathsf{g}}$ and $\tilde{\mathsf{J}}$ coincide with the Euclidean metric η and the standard complex structure J_0, respectively. Let h_0 be the Hermitean form defined by η and J_0 via (2.2.27). Since η is $\mathrm{SO}(2n)$-invariant and since J_0 commutes with the $\mathrm{U}(n)$-action, h_0 is $\mathrm{U}(n)$-invariant. This yields the following.

Proposition 2.2.20 *Relative to a given almost complex structure* J *on* M*,* $\mathrm{U}(n)$*-structures on* M *are in one-to-one correspondence with Hermitean fibre metrics on* $\mathrm{T}M$*.*[15] ■

Finally, we give a characterization of the above objects in terms of the decompositions (2.2.9), (2.2.15) and (2.2.16). Here, T may be viewed as the basic $\mathrm{SO}(2n)$-module and, by (2.2.22), the subspaces $\mathrm{T}^{1,0}$ and $\mathrm{T}^{0,1}$ carry the basic representation of $\mathrm{U}(n)$ and its conjugate, respectively. Thus, V and $\mathrm{T}^{1,0}$ are isomorphic as $\mathrm{U}(n)$-modules. For $k = 2$, the decomposition (2.2.16) takes the form

$$\bigwedge^2 \mathrm{T}^*{}_{\mathbb{C}} = \bigwedge^{2,0} \oplus \bigwedge^{1,1} \oplus \bigwedge^{0,2} . \tag{2.2.29}$$

By standard representation theory, the adjoint representation of $\mathrm{U}(n)$ is given by the tensor product of the basic representation and its dual. Thus, after intersecting with the real exterior product $\bigwedge^2 \mathrm{T}^*$, formula (2.2.29) corresponds to the decomposition $\mathfrak{o}(2n) = \mathfrak{u}(n) \oplus \mathfrak{u}(n)^\perp$, where

$$\mathfrak{u}(n) = \bigwedge^{1,1} \cap \bigwedge^2 \mathrm{T}^* , \quad \mathfrak{u}(n)^\perp = \left(\bigwedge^{2,0} \oplus \bigwedge^{0,2}\right) \cap \bigwedge^2 \mathrm{T}^* . \tag{2.2.30}$$

For a given basis $(\mathbf{e}_1, \mathsf{J}\mathbf{e}_1, \ldots, \mathbf{e}_n, \mathsf{J}\mathbf{e}_n)$ of T, let $(\vartheta^1, \varphi^1, \ldots, \vartheta^n, \varphi^n)$ be the dual basis in T^*. Clearly, the latter yields the bases

$$\{\vartheta^k \wedge \vartheta^l\} , \quad \{\vartheta^k \wedge \varphi^l\} , \quad \{\varphi^k \wedge \varphi^l\} , \quad k < l , \quad k, l = 1, \ldots n ,$$

in, respectively, $\bigwedge^{2,0}$, $\bigwedge^{1,1}$ and $\bigwedge^{0,2}$. In particular, for $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ we may choose the standard basis in $V \cong \mathbb{C}^n$. Since $\tilde{\mathsf{h}}$ takes values in the space of bilinear forms on $\mathrm{T}_{\mathbb{C}}$, we obtain (Exercise 2.2.9)

$$\tilde{\mathsf{h}}(u) = \sum_{k=1}^{n} (\vartheta^k \otimes \vartheta^k + \varphi^k \otimes \varphi^k) - i \sum_{k=1}^{n} \vartheta^k \wedge \varphi^k , \tag{2.2.31}$$

for any $u \in U(M)$. From (2.2.28), we read off

$$\tilde{\mathsf{g}}(u) = \sum_{k=1}^{n} (\vartheta^k \otimes \vartheta^k + \varphi^k \otimes \varphi^k) , \quad \tilde{\beta}(u) = -\sum_{k=1}^{n} \vartheta^k \wedge \varphi^k . \tag{2.2.32}$$

To summarize, for $u \in U(M)$,

---

[15] Clearly, this is consistent with Example 1.1.18, where we considered the orthonormal frame bundle of an arbitrary vector bundle carrying a fibre metric.

$$\tilde{\mathsf{h}}(u) \in \bigwedge^{1,1}, \quad \tilde{\mathsf{g}}(u) \in \left(\bigwedge^{2,0} \oplus \bigwedge^{0,2}\right) \cap S^2\mathrm{T}^*, \quad \tilde{\beta}(u) \in \bigwedge^{1,1} \cap \bigwedge^2\mathrm{T}^*. \tag{2.2.33}$$

Note that $\beta(u) \in \mathfrak{u}(n)$ is $\mathrm{U}(n)$-invariant. Thus, it spans a 1-dimensional invariant subspace in $\mathfrak{u}(n)$ and gives rise to the decomposition $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus i\mathbb{R}$. ♦

Example 2.2.21 (*Almost symplectic structure*) Consider $H = \mathrm{Sp}(n, \mathbb{R})$. Recall that this is the group of linear transformations of $\mathbb{R}^{2n}$ leaving the standard symplectic form (2.2.6) invariant.[16] Thus, $\mathrm{Sp}(n, \mathbb{R})$-structures are in one-to-one correspondence with 2-forms on M of maximal rank. Such structures are called almost symplectic. By the previous example, each almost Hermitean structure defines such a 2-form β. By Proposition I/7.5.3,

$$\mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n) = \mathrm{SO}(2n) \cap \mathrm{Sp}(n, \mathbb{R}),$$

and, thus, each pair built from the triple $(\mathsf{g}, \mathsf{J}, \beta)$ yields the same $\mathrm{U}(n)$-structure. Moreover, since $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$ contain $\mathrm{U}(n)$ as their maximal compact subgroup, M admits an almost symplectic structure iff it admits an almost complex structure. Clearly, by the Darboux Theorem I/8.1.5, an almost symplectic structure is integrable iff $\mathrm{d}\beta = 0$. Then (M, β) is called a symplectic manifold. A Hermitean manifold $(M, \mathsf{g}, \mathsf{J})$ such that the 2-form β defined by (2.2.26) is closed is called Kähler. For the discussion of existence, see Remark I/8.1.4.

Clearly, a diffeomorphism $\varphi : M \to M$ is an automorphism of an $\mathrm{Sp}(n, \mathbb{R})$-structure iff $\varphi^*\beta = \beta$. If (M, β) is symplectic, then φ is called a symplectomorphism. For the study of the group of symplectomorphisms see Sect. 8.8 in Part I. ♦

In the remainder of this section, we discuss compatible connections.

Example 2.2.22 (*Metric connection*) By Example 2.2.16, pseudo-Riemannian manifolds are in one-to-one correspondence with $\mathrm{O}(k, l)$-structures. Thus, let (M, g) be a pseudo-Riemannian manifold and let $O(M)$ be its $\mathrm{O}(k, l)$-structure. In terms of the corresponding equivariant mapping $\tilde{\mathsf{g}}$,

$$O(M) = \{u \in L(M) \colon \tilde{\mathsf{g}}(u) = \eta\}, \tag{2.2.34}$$

where η is the standard inner product on $\mathbb{R}^n$ with signature (k, l). By Proposition 2.2.3, a linear connection ω on M is compatible with the $\mathrm{O}(k, l)$-structure iff g is parallel with respect to ω. A linear connection fulfilling this condition is called metric. By (2.2.21), the metricity condition $D\tilde{\mathsf{g}} = \mathrm{d}\tilde{\mathsf{g}} + \sigma'(\omega)\tilde{\mathsf{g}} = 0$ reads

$$\mathrm{d}\tilde{\mathsf{g}} - \left(\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \omega^{\mathrm{T}}\right)(\tilde{\mathsf{g}}) = 0. \tag{2.2.35}$$

More explicitly, decomposing ω with respect to the basis $\{E^j{}_i\}$ in $\mathfrak{gl}(n, \mathbb{R})$ and $\tilde{\mathsf{g}}$ with respect to the basis in $S^2\mathbb{R}^n$ induced from the standard basis of $\mathbb{R}^n$, (2.2.35) takes the form

[16] Note the double role of J_0.

$$\mathrm{d}\tilde{\mathsf{g}}_{jk} - \tilde{\mathsf{g}}_{jl}\omega^l{}_k - \tilde{\mathsf{g}}_{kl}\omega^l{}_j = 0 . \tag{2.2.36}$$

But, on $O(M)$ we have $\tilde{\mathsf{g}}_{kl} = \eta_{kl}$ and, thus, $\mathrm{d}\tilde{\mathsf{g}}_{jk} = 0$. This shows that ω is metric iff its reduction to $O(M)$ fulfils

$$\omega_{jk} + \omega_{kj} = 0 ,$$

that is, iff this reduction takes values in the Lie algebra $\mathfrak{o}(k, l)$, indeed. Equivalently, the metricity condition is given by $\nabla \mathsf{g} = 0$. Since ∇_X is a derivation of the tensor algebra, the latter is equivalent to

$$X(\mathsf{g}(Y, Z)) = \mathsf{g}(\nabla_X Y, Z) + \mathsf{g}(Y, \nabla_X Z) , \tag{2.2.37}$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Remark 2.2.23 Let (V, q) be a quadratic vector space over $\mathbb{K}$. Assume that $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$ and that q is non-degenerate. Recall from Example I/5.2.6 that the Lie algebra $\mathfrak{o}(V, \mathsf{q})$ of the orthogonal group $\mathrm{O}(V, \mathsf{q})$ coincides with those endomorphisms of V which are anti-symmetric with respect to the symmetric bilinear form η of q. In the context of Clifford algebras, see Sect. 5.2, we will see that the following canonical isomorphism of Lie algebras holds:

$$\kappa : \mathfrak{o}(V, \mathsf{q}) \to \bigwedge^2 V , \quad \kappa(A) = \frac{1}{4} A(\mathbf{e}_i) \wedge \eta^{-1}(\vartheta^i) , \tag{2.2.38}$$

where $\{\mathbf{e}_i\}$ is a q-orthogonal basis in V and $\{\vartheta^j\}$ is the dual basis. Denoting $A_{kl} = \mathsf{g}(\mathbf{e}_k, A\mathbf{e}_l)$, we obtain

$$\kappa(A) = \frac{1}{4}\eta^{ij} A(\mathbf{e}_i) \wedge \mathbf{e}_j = \frac{1}{4} A^{ij} \mathbf{e}_i \wedge \mathbf{e}_j . \tag{2.2.39}$$

♦

Proposition 2.2.24 *Any* $\mathrm{O}(k, l)$*-structure has a unique torsion-free connection.*

Proof By Corollary 2.2.7, it is enough to show that the mapping δ given by (2.2.2) is bijective. In the case under consideration, $\mathfrak{h} = \mathfrak{o}(k, l) \cong \bigwedge^2 \mathbb{R}^n \cong \bigwedge^2 (\mathbb{R}^n)^*$. Thus,

$$\delta : (\mathbb{R}^n)^* \otimes \bigwedge^2 (\mathbb{R}^n)^* \to \bigwedge^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n .$$

Let $\alpha \in (\mathbb{R}^n)^* \otimes \bigwedge^2 (\mathbb{R}^n)^*$ and let α_{ijk} be the components of α in the basis induced from the standard basis of $\mathbb{R}^n$. Then, $\alpha_{ijk} = -\alpha_{ikj}$ and the components of $\delta(\alpha)$ are given by $\frac{1}{2}(\alpha_{ijk} - \alpha_{jik})$. Assume $\delta(\alpha) = 0$. Then,

$$\alpha_{ijk} = \alpha_{jik} = -\alpha_{jki} = -\alpha_{kji} = \alpha_{kij} = \alpha_{ikj} = -\alpha_{ijk} ,$$

that is $\ker(\delta) = 0$. Now, bijectivity follows from dimension counting. ■

A classical proof of Proposition 2.2.24 is obtained by using (2.2.37) and (2.1.33),

$$X(\mathsf{g}(Y,Z)) = \mathsf{g}(\nabla_X Y, Z) + \mathsf{g}(Y, \nabla_X Z)\,, \quad 0 = \nabla_X Y - \nabla_Y X - [X,Y]\,.$$

Then, by direct inspection (Exercise 2.2.10),

$$\begin{aligned} 2\mathsf{g}(\nabla_X Y, Z) = {} & X(\mathsf{g}(Y,Z)) + Y(\mathsf{g}(X,Z)) - Z(\mathsf{g}(X,Y)) \\ & + \mathsf{g}([X,Y],Z) + \mathsf{g}([Z,X],Y) + \mathsf{g}([Z,Y],X)\,. \end{aligned} \tag{2.2.40}$$

One easily checks that this equation defines a torsion-free connection. In the sequel, the unique torsion-free connection defined by g will be called the Levi-Civita connection.

Finally, we derive local formulae for the Levi-Civita connection. In contrast to general linear connections, here we have two natural types of local frames:

(a) local holonomic frames $\{\partial_j\}$ induced from arbitrary local charts (U_j, κ_j),
(b) local frames $\{e_j\}$ which are orthonormal with respect to g.

Since the formulae (2.1.42), (2.1.44), (1.5.8) and (2.1.46)–(2.1.50) hold true for any local frame, they clearly apply here. Let $\mathfrak{e}$ be an arbitrary local frame. By (2.2.40),

$$2\mathsf{g}(\nabla_{e_i} e_j, e_k) = e_i(\mathsf{g}_{jk}) + e_j(\mathsf{g}_{ik}) - e_k(\mathsf{g}_{ij}) + C^l{}_{ij}\mathsf{g}_{lk} + C^l{}_{ki}\mathsf{g}_{lj} + C^l{}_{kj}\mathsf{g}_{li}\,,$$

where $\mathsf{g}_{ij} = \mathsf{g}(e_i, e_j)$. Thus,

$$\begin{aligned} \Gamma^m{}_{ij} = {} & \frac{1}{2}\mathsf{g}^{mk}\left(e_i(\mathsf{g}_{jk}) + e_j(\mathsf{g}_{ik}) - e_k(\mathsf{g}_{ij})\right) \\ & + \frac{1}{2}\left(C^m{}_{ij} + \mathsf{g}^{km}\mathsf{g}_{lj}C^l{}_{ki} + \mathsf{g}^{km}\mathsf{g}_{li}C^l{}_{kj}\right)\,. \end{aligned} \tag{2.2.41}$$

For the case (a), we have $\mathsf{g}_{ij} = \mathsf{g}(\partial_i, \partial_j)$ and $C^i_{jk} = 0$. Thus,

$$\Gamma^m{}_{ij} = \frac{1}{2}\,\mathsf{g}^{mk}(\mathsf{g}_{jk,i} + \mathsf{g}_{ki,j} - \mathsf{g}_{ji,k})\,, \quad \Gamma^m{}_{ij} = \Gamma^m{}_{ji}\,. \tag{2.2.42}$$

For the case (b), we have $\mathsf{g}_{ij} = \eta_{ij}$ and, therefore,

$$\Gamma^m{}_{ij} = \frac{1}{2}\left(C^m{}_{ij} + \eta^{km}\eta_{lj}C^l{}_{ki} + \eta^{km}\eta_{li}C^l{}_{kj}\right)\,. \tag{2.2.43}$$

Thus, $\Gamma_{kij} = \eta_{km}\Gamma^m{}_{ij} = \frac{1}{2}(C_{kij} + C_{jki} + C_{ikj})$ and, consequently, for case (b) we have

$$\Gamma_{kij} = -\Gamma_{jik}\,, \quad \Gamma^k{}_{ik} = 0\,. \tag{2.2.44}$$

Using (2.1.46) and (2.2.43), we obtain

$$\mathrm{d}\vartheta^i(e_j, e_k) = -\vartheta^i([e_j, e_k]) = \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$$

and, thus,

$$\mathrm{d}\vartheta^i = -\Gamma^i{}_{jk}\vartheta^j \wedge \vartheta^k\,. \tag{2.2.45}$$

Comparing with (2.1.46), we read off

$$\mathrm{d}\vartheta^i = \vartheta^j \wedge \nabla_{e_j}\vartheta^i\,. \tag{2.2.46}$$

This implies the following useful formula (Exercise 2.2.11). For any $\alpha \in \Omega^k(M)$,

$$\mathrm{d}\alpha = \vartheta^j \wedge \nabla_{e_j}\alpha\,. \tag{2.2.47}$$

Since the operator d is intrinsically defined, this formula does not depend on the choice of the frame. It can be rewritten as

$$\mathrm{d}\alpha(e_0, \ldots, e_k) = \sum_j (-1)^j \left(\nabla_{e_j}\alpha\right)\left(e_0, \overset{j}{\check{\ldots}}, e_k\right)\,. \tag{2.2.48}$$

By the locality property of ∇ and by the multilinearity of α, we conclude

$$\mathrm{d}\alpha(X_0, \ldots, X_k) = \sum_j (-1)^j \left(\nabla_{X_j}\alpha\right)\left(X_0, \overset{j}{\check{\ldots}}, X_k\right)\,, \tag{2.2.49}$$

for any set of vector fields $X_0, \ldots, X_k$ on M. ♦

Example 2.2.25 (*Almost complex connection*) By Example 2.2.10, $\mathrm{GL}(n, \mathbb{C})$-structures on a manifold M are in one-to-one correspondence with sections J of $\mathrm{End}(\mathrm{T}M)$ fulfilling $\mathsf{J}_m^2 = -\operatorname{id}$ for every $m \in M$. By Proposition 2.2.3, a linear connection ω on M is compatible with a $\mathrm{GL}(n, \mathbb{C})$-structure iff J is parallel with respect to ω. A linear connection fulfilling this condition is called almost complex. Recall that the obstruction to integrability of an almost complex structure is given by the Nijenhuis tensor N.

Proposition 2.2.26 *An almost complex manifold* (M, J) *admits a torsion-free almost complex linear connection iff* J *is integrable.*

Proof We show that the intrinsic torsion vanishes iff J is integrable. Here, the mapping (2.2.2) takes the form

$$\delta : (\mathbb{R}^{2n})^* \otimes \mathfrak{gl}(n, \mathbb{C}) \to \bigwedge\nolimits^2 (\mathbb{R}^{2n})^* \otimes \mathbb{R}^{2n}\,.$$

We pass to the complexifications of both the domain and the target space of δ and use the decompositions (2.2.9), (2.2.15) and (2.2.29), together with the embedding (2.2.5). Then, the target space reads

$$
\begin{aligned}
\left(\bigwedge^2 \mathrm{T}^*_{\mathbb{C}}\right) \otimes \mathrm{T}_{\mathbb{C}} &= \left(\bigwedge^{2,0} \oplus \bigwedge^{1,1} \oplus \bigwedge^{0,2}\right) \otimes \left(\mathrm{T}^{1,0} \oplus \mathrm{T}^{0,1}\right) \\
&= \left(\bigwedge^{2,0} \otimes \mathrm{T}^{1,0}\right) \oplus \left(\bigwedge^{1,1} \otimes \mathrm{T}^{1,0}\right) \oplus \left(\bigwedge^{0,2} \otimes \mathrm{T}^{1,0}\right) \\
&\quad \oplus \left(\bigwedge^{2,0} \otimes \mathrm{T}^{0,1}\right) \oplus \left(\bigwedge^{1,1} \otimes \mathrm{T}^{0,1}\right) \oplus \left(\bigwedge^{0,2} \otimes \mathrm{T}^{0,1}\right),
\end{aligned}
$$

and for the image of δ we get

$$
\mathrm{im}(\delta) = \left(\left(\bigwedge^{1,1} \oplus \bigwedge^{0,2}\right) \otimes \mathrm{T}^{0,1}\right) \oplus \left(\left(\bigwedge^{2,0} \oplus \bigwedge^{1,1}\right) \otimes \mathrm{T}^{1,0}\right). \tag{2.2.50}
$$

The latter is obtained by a straightforward calculation, see Exercise 2.2.5. Thus, the intrinsic torsion takes values in

$$
\mathrm{coker}(\delta) = \left(\bigwedge^{0,2} \otimes \mathrm{T}^{1,0}\right) \oplus \left(\bigwedge^{2,0} \otimes \mathrm{T}^{0,1}\right).
$$

We give the argument for the first component. Let $\mathfrak{e} = (e_1, \ldots, e_n)$ be a holomorphic frame and let $(\vartheta^1, \ldots, \vartheta^n)$ be the dual coframe. Taking the pullback under $\mathfrak{e}$ of the Structure Equation for the torsion, cf. (2.1.15), we obtain

$$
\mathrm{T}^i = \mathrm{d}\vartheta^i + \mathscr{A}^i{}_j \wedge \vartheta^j .
$$

Evaluating the $(1, 0)$-component of this equation on $X_1, X_2 \in \Gamma^\infty(\mathrm{T}^{0,1}M)$, we obtain

$$
\mathrm{T}^i(X_1, X_2) = -\vartheta^i([X_1, X_2]) .
$$

We get the same equation for the $(0, 1)$-component evaluated on a pair of vector fields of type $(1, 0)$. Thus, the intrinsic torsion vanishes iff $\mathrm{T}^{1,0}M$ and $\mathrm{T}^{0,1}M$ are involutive. Now, point 2 of Proposition 2.2.14 yields the assertion. ■

By the above proof and point 1 of Proposition 2.2.14, the Nijenhuis tensor measures the torsion of an almost complex linear connection, see also Theorem 3.4 in Chap. IX of [381] for a classical proof. ♦

Example 2.2.27 (*Unitary connection*) Here, we take up Example 2.2.19. Thus, let $U(M)$ be a $\mathrm{U}(n)$-structure and let $(M, \mathsf{g}, \mathsf{J})$ be the corresponding $2n$-dimensional almost Hermitean manifold. Clearly, by Proposition 2.2.3, a linear connection ω on M is compatible with the $\mathrm{U}(n)$-structure iff both g and J are parallel with respect to ω. Such a connection will be called unitary.

Assume that there exists a torsion-free unitary connection ω on M. Since $U(M) = C(M) \cap O_+(M)$ and since the Levi-Civita connection of g is the unique torsion-free connection on $O_+(M)$, ω is necessarily obtained as a reduction of the Levi-Civita connection to $U(M)$. Thus, if it exists, it is necessarily unique.

Proposition 2.2.28 *Let $U(M)$ be a $\mathrm{U}(n)$-structure, let $(M, \mathsf{g}, \mathsf{J})$ be the corresponding almost Hermitean manifold and let β be the almost symplectic form defined* *by the pair* (g, J). *Then, the Levi-Civita connection* ω *of* g *is compatible with the* $\mathrm{U}(n)$*-structure iff* J *is integrable and* β *is symplectic.*

Proof Assume that ω is $\mathrm{U}(n)$-compatible. Then, both g and J are ω-parallel and, by Proposition 2.2.26, since ω is torsion-free and since J is parallel, J is integrable. Moreover, the parallelity of g and J imply the parallelity of β. Then, (2.2.49) yields $\mathrm{d}\beta = 0$. The converse statement follows immediately from the identity

$$2\mathsf{g}((\nabla_X \mathsf{J})Y, Z) = \mathrm{d}\beta(X, \mathsf{J}Y, \mathsf{J}Z) - \mathrm{d}\beta(X, Y, Z) + \mathsf{g}(N(Y, Z), \mathsf{J}X)\,, \tag{2.2.51}$$

where ∇ is the covariant derivative of ω and $X, Y, Z \in \mathfrak{X}(M)$, see Exercise 2.2.12. ■

Thus, ω is compatible with the $\mathrm{U}(n)$-structure iff $(M, \mathsf{g}, \mathsf{J})$ is Kähler. For a detailed description of Kähler structures in terms of local coordinates we refer to Sects. 4 and 5 of Chap. IX in [381].

Finally, by the discussion in Example 2.2.19, we obtain a characterization of unitary connections in terms of the Hermitean fibre metric h defined by g and J.

Proposition 2.2.29 *A linear connection* ω *on a Hermitean manifold* $(M, \mathsf{g}, \mathsf{J})$ *is unitary iff the Hermitean fibre metric* h *defined by* g *and* J *is parallel with respect to* ω.

According to (2.2.33), $\tilde{\mathsf{h}}(u) \in \bigwedge^{1,1}$. Explicitly, the $\mathrm{U}(n)$-module structure of $\bigwedge^{1,1}$ is given by

$$\sigma : \mathrm{U}(n) \to \mathrm{Aut}\left(\bigwedge\nolimits^{1,1}\right)\,, \quad \sigma(g) = \left(g^{-1}\right)^{\mathrm{T}} \otimes \overline{\left(g^{-1}\right)^{\mathrm{T}}}\,. \tag{2.2.52}$$

Thus, the metricity condition $D\,\tilde{\mathsf{h}} = \mathrm{d}\tilde{\mathsf{h}} + \sigma'(\omega)\tilde{\mathsf{h}} = 0$ restricted to $U(M)$ implies

$$\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \overline{\omega^{\mathrm{T}}} = 0\,. \tag{2.2.53}$$

Analyzing (2.2.53) in the standard basis as in Example 2.2.22, we obtain $\omega^{\dagger} + \omega = 0$, that is, ω takes values in the Lie algebra $\mathfrak{u}(n)$, indeed. ♦

Exercises

2.2.1 Show that integrability of a section s in an H-structure P implies $s^*\mathrm{d}\theta = 0$.

2.2.2 Prove that any $\mathrm{SL}(n, \mathbb{R})$-structure is integrable.

2.2.3 Prove that a mapping of an open subset of $\mathbb{C}^n$ to $\mathbb{C}^m$ is compatible with the natural almost complex structures iff it is holomorphic.

2.2.4 Prove that every almost complex manifold is orientable.

2.2.5 Prove formula (2.2.50). *Hint.* Let $\xi \in (\mathbb{R}^{2n})^*$ and $a \in \mathfrak{gl}(n, \mathbb{C}) \cong (\mathbb{C}^n)^* \otimes \mathbb{C}^n$. To calculate $\delta(\xi \otimes a)$, decompose both elements with respect to bases adapted to the decompositions (2.2.9) and (2.2.15) and calculate the image explicitly using (2.2.5).[17]

2.2.6 Prove that the mappings $\mathrm{pr}^{1,0} \circ \iota$ and $\mathrm{pr}^{0,1} \circ \iota$, defined by (2.2.13), are $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear, respectively. Show that (2.2.14) holds.

2.2.7 Prove the second assertion in Corollary 2.2.15. *Hint.* Use point 2 of Proposition 2.2.14.

2.2.8 Prove that h defined by (2.2.27) is linear in the first and anti-linear in the second entry.

2.2.9 Prove formula (2.2.31).

2.2.10 Give an alternative proof of Proposition 2.2.24 by using (2.2.37) and (2.1.33).

2.2.11 Prove formula (2.2.47).

2.2.12 Prove formula (2.2.51). *Hint.* Prove that $\mathsf{g}((\nabla_X \mathsf{J})Y, Z) = \mathsf{g}(\nabla_X(\mathsf{J}Y), Z) + \mathsf{g}(\nabla_X Y, \mathsf{J}Z)$ and rewrite the terms on the right hand side according to (2.2.40). Use formula I/4.1.6. Alternatively, the proof can be found in [381], see Proposition 4.2 in Chap. IX.

2.2.13 Prove that for $H = \mathrm{Sp}(n, \mathbb{R})$, the cokernel of the mapping (2.2.2) is isomorphic to $\bigwedge^3(\mathbb{R}^{2n})^*$. Show that the corresponding intrinsic torsion coincides with the exterior derivative of the almost symplectic form, cf. Example 2.2.21.

2.3 Curvature and Holonomy

In this section, we continue the discussion of connections compatible with H-structures. Here, we consider exclusively torsion-free connections and ask which holonomy groups may occur for such a connection. This question has first been studied systematically by Berger, see [68, 69].

At this point, the reader may wish to recall the basic notions from the general holonomy theory as presented in Sect. 1.7. For a linear connection Γ in $L(M)$, let $P_{u_0}(\Gamma)$ be the holonomy bundle of Γ with base point $u_0 \in L(M)$. By Proposition 1.7.12, Γ is reducible to $P_{u_0}(\Gamma)$ and thus, for any $u \in P_{u_0}(\Gamma)$, the curvature Ω of Γ takes values in the Lie algebra $\mathfrak{h}_{u_0}(\Gamma)$ of the holonomy group $\mathscr{H}_{u_0}(\Gamma) \subset \mathrm{GL}(n, \mathbb{R})$. By the Ambrose-Singer Theorem 1.7.15, we have

$$\mathfrak{h}_{u_0}(\Gamma) = \mathrm{span}\left\{\Omega_u(X, Y) : u \in P_{u_0}(\Gamma)\,,\;\; X, Y \in \Gamma_u\right\}\,. \tag{2.3.1}$$

It is the condition of torsion-freeness which makes the above question nontrivial. If we drop this assumption, then any closed Lie subgroup $H \subset \mathrm{GL}(n, \mathbb{R})$ may occur

[17] Cf. also Example 2.2.19.

as the holonomy group of a linear connection on some n-dimensional manifold M, see [283]. However, in general, such a connection will have a nontrivial torsion. By the Bianchi identity (2.1.17), vanishing of the torsion implies

$$\Omega \wedge \theta = 0\,, \tag{2.3.2}$$

and, by the Ambrose-Singer Theorem, this yields a nontrivial restriction on the holonomy. Now, let $P \subset L(M)$ be an H-structure on an n-dimensional manifold M, let ω be an H-compatible connection and let Ω be its curvature. For simplicity, let us denote $\mathbb{R}^n \equiv V$. By Remark 2.1.16, we may represent Ω equivalently by the curvature mapping

$$\mathscr{R} : P \to \bigwedge\nolimits^2 V^* \otimes \mathfrak{h} \tag{2.3.3}$$

fulfilling the equivariance condition (2.1.25) with respect to the natural representation $\sigma : H \to \mathrm{Aut}\left(\bigwedge^2 V^* \otimes \mathfrak{h}\right)$ given by

$$\sigma_a((\xi \wedge \tau) \otimes A) := \left((a^{-1})^{\mathrm{T}}\xi \wedge (a^{-1})^{\mathrm{T}}\tau\right) \otimes \mathrm{Ad}(a)A\,. \tag{2.3.4}$$

Since the exterior products of the components θ^i of θ span the spaces of horizontal forms, (2.3.2) implies that $\mathscr{R}$ takes values in the kernel $\mathfrak{K}(\mathfrak{h})$ of the mapping

$$\delta : \bigwedge\nolimits^2 V^* \otimes \mathfrak{h} \to \bigwedge\nolimits^3 V^* \otimes V\,, \quad \delta = (a \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \iota_{\mathfrak{h}})\,, \tag{2.3.5}$$

where a is the anti-symmetrization mapping, cf. (2.2.2). Clearly,

$$\mathfrak{K}(\mathfrak{h}) = \left\{F \in \bigwedge\nolimits^2 V^* \otimes \mathfrak{h} : F(\mathbf{x},\mathbf{y})\mathbf{z} + F(\mathbf{y},\mathbf{z})\mathbf{x} + F(\mathbf{z},\mathbf{x})\mathbf{y} = 0\,, \quad \mathbf{x},\mathbf{y},\mathbf{z} \in V\right\}.$$

The space $\mathfrak{K}(\mathfrak{h})$ is called the space of curvature mappings.

Lemma 2.3.1 *The subspace*

$$\underline{\mathfrak{h}} := \mathrm{span}\,\{F(\mathbf{x},\mathbf{y}) \in \mathfrak{h} : F \in \mathfrak{K}(\mathfrak{h})\,, \quad \mathbf{x},\mathbf{y} \in V\} \tag{2.3.6}$$

is an ideal of $\mathfrak{h}$.

Proof Let $F(\mathbf{x},\mathbf{y}) \in \underline{\mathfrak{h}}$ and let $A \in \mathfrak{h} \subset \mathrm{End}(V)$. Then, we may write

$$[F(\mathbf{x},\mathbf{y}), A] = \tilde{F}(\mathbf{x},\mathbf{y}) - F(A\mathbf{x},\mathbf{y}) - F(\mathbf{x}, A\mathbf{y})\,,$$

where

$$\tilde{F}(\mathbf{x},\mathbf{y}) = [F(\mathbf{x},\mathbf{y}), A] + F(A\mathbf{x},\mathbf{y}) + F(\mathbf{x}, A\mathbf{y})\,.$$

One checks by direct inspection that $\tilde{F} \in \mathfrak{K}(\mathfrak{h})$. ∎

Note that $\tilde{F}$ corresponds exactly to the action of A on F obtained by differentiating the equivariance condition (2.1.25).[18] Thus, by the Ambrose-Singer Theorem, for the Lie algebra $\mathfrak{h}_{u_0}(\Gamma)$ of the holonomy group of a torsion-free connection Γ, we have

$$\underline{\mathfrak{h}_{u_0}(\Gamma)} = \mathfrak{h}_{u_0}(\Gamma)\,.$$

We conclude that a Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{R})$ can occur as the Lie algebra of the holonomy group of a torsion-free connection only if it coincides with the ideal $\underline{\mathfrak{h}}$. This is commonly referred to as the first criterion of Berger. It yields a necessary condition for a Lie subalgebra to be the holonomy Lie algebra of a torsion-free connection.

Next, let us analyze the Bianchi identity (2.1.16) in terms of $\mathscr{R}$. The covariant derivative $D\mathscr{R} = \mathrm{d}\mathscr{R} + \sigma'(\omega)\mathscr{R}$ is a horizontal 1-form on P with values in $\mathfrak{K}(\mathfrak{h})$.

Definition 2.3.2 A torsion-free connection fulfilling $D\mathscr{R} = 0$ is called locally symmetric.

Decomposing $D\mathscr{R}$ with respect to the horizontal frame $\{\theta^i\}$, we obtain a function $D\mathscr{R} : P \to V^* \otimes \mathfrak{K}(\mathfrak{h})$. Using the fact that the commutators of horizontal standard vector fields corresponding to a torsion-free connection are vertical (Exercise 2.3.1), we calculate

$$\begin{aligned}
D\Omega(B(\mathbf{x}), B(\mathbf{y}), B(\mathbf{z})) &= \mathrm{d}\Omega(B(\mathbf{x}), B(\mathbf{y}), B(\mathbf{z})) \\
&= B(\mathbf{x})\big(\Omega(B(\mathbf{y}), B(\mathbf{z})\big) - \Omega([B(\mathbf{x}), B(\mathbf{y})], B(\mathbf{z})) + \text{cycl.} \\
&= \mathrm{d}\mathscr{R}(B(\mathbf{x}))(\mathbf{y} \wedge \mathbf{z}) + \text{cycl.} \\
&= D\mathscr{R}(\mathbf{x})(\mathbf{y} \wedge \mathbf{z}) + \text{cycl.}\ .
\end{aligned}$$

Thus, by the Bianchi identity $D\Omega = 0$, we conclude that the function $D\mathscr{R}$ takes values in the kernel of the mapping

$$\delta' : V^* \otimes \mathfrak{K}(\mathfrak{h}) \to \bigwedge^3 V^* \otimes \mathfrak{h}\,, \tag{2.3.7}$$

defined as the composition

$$V^* \otimes \mathfrak{K}(\mathfrak{h}) \to V^* \otimes \bigwedge^2 V^* \otimes \mathfrak{h} \to \bigwedge^3 V^* \otimes \mathfrak{h}$$

of the inclusion and the anti-symmetrization mappings. Clearly, the kernel of δ' is

$$\mathfrak{K}^1(\mathfrak{h}) := \left\{\Phi \in V^* \otimes \mathfrak{K}(\mathfrak{h}) : \Phi(\mathbf{x})(\mathbf{y}, \mathbf{z}) + \Phi(\mathbf{y})(\mathbf{z}, \mathbf{x}) + \Phi(\mathbf{z})(\mathbf{x}, \mathbf{y}) = 0\,,\ \ \mathbf{x}, \mathbf{y}, \mathbf{z} \in V\right\}\,.$$

Thus, if $\mathfrak{h}$ is the holonomy Lie algebra of a torsion-free linear connection that is not locally symmetric, then necessarily $\mathfrak{K}^1(\mathfrak{h}) \neq 0$. This is usually referred to as the second Berger criterion.

---

[18] Clearly, this is the action of the Killing vector field generated by A.

Definition 2.3.3 A Lie subalgebra $\mathfrak{h} \subset \mathrm{End}(V)$ is called a Berger algebra if $\underline{\mathfrak{h}} = \mathfrak{h}$. A Berger algebra is called symmetric if $\mathfrak{K}^1(\mathfrak{h}) = 0$ and non-symmetric otherwise. Correspondingly, a Lie subgroup $H \subset \mathrm{Aut}(V)$ is referred to as a (symmetric or non-symmetric) Berger group if its Lie algebra is a (symmetric or non-symmetric) Berger algebra.

By the above discussion, we have the following.

Proposition 2.3.4 (Berger) *Let $\mathfrak{h} \subset \mathrm{End}(V)$ be a Lie subalgebra. Then,*

1. *If $\mathfrak{h}$ is the Lie algebra of the holonomy group of a torsion-free connection on some manifold, then $\mathfrak{h}$ is a Berger algebra.*
2. *If $\mathfrak{K}^1(\mathfrak{h}) = 0$, then any torsion-free connection on a manifold whose holonomy Lie algebra is contained in $\mathfrak{h}$ must be locally symmetric.*

Based upon these criteria, Berger started to tackle the above classification problem. It is natural to distinguish between the following two classes:

(a) Lie subalgebras $\mathfrak{h}$ lying in some $\mathfrak{o}(\eta)$, where η is some non-degenerate bilinear form on V. In this case, the associated H-structure defines a pseudo-Riemannian manifold. Therefore, this is called the metric case.
(b) Lie subalgebras which are not contained in any orthogonal Lie algebra. This is called the non-metric case.

Within this general analysis, Berger obtained a list of candidates for Lie subalgebras of type (a) and also an (incomplete) list for type (b).[19] These lists where refined and completed by the work of Alekseevski [14], Bryant [108, 109], Chi [132], Merkulov and Schwachhöfer [569]. The final full classification of irreducible holonomies of torsion-free affine connections was obtained by Merkulov and Schwachhöfer [441]. For an exhaustive discussion, we refer to the reviews of Bryant [110] and Schwachhöfer [570] and to the textbooks of Besse [76], Joyce [353] and Salamon [555]. In [110], the reader can find the complete classification list (divided into four parts) together with a lot of information on methods for proving that a given group in the list really occurs as a holonomy. It turns out that every such group is realized at least locally.[20]

In the remainder of this section, we exclusively consider the metric case. That is, we consider (pseudo-)Riemannian manifolds (M, g), endowed with their unique torsion-free metric connection (the Levi-Civita connection). Under this assumption, the frame bundle reduces to the orthonormal frame bundle $O(M)$ and the whole theory may be described in terms of objects living on $O(M)$. Consequently, in the case under consideration, the holonomy group is a subgroup of the structure group $\mathrm{O}(k, l)$. If the Levi-Civita connection is locally symmetric, we call (M, g) locally symmetric.

[19] The list provided by Theorem 2.3.19 below is included in type (a).

[20] The appropriate method working for three of the above mentioned four tables is to describe torsion-free connections with a given holonomy as solutions to an exterior differential system and to apply Cartan's existence theorem.

Definition 2.3.5 Let (M, g) be a pseudo-Riemannian manifold. The curvature mapping

$$\mathscr{R} : O(M) \to \bigwedge^2 V^* \otimes \mathfrak{o}(k, l)$$

of the Levi-Civita connection of g is called the Riemann curvature mapping. Correspondingly, the curvature tensor R is called the Riemann curvature of (M, g).

Comparing with the general case, $\mathscr{R}$ has some additional properties coming from the fact that we may use the metric η to identify V with V^*. In particular, $\mathfrak{o}(k, l) \cong \bigwedge^2 V^*$, and thus

$$\mathscr{R}(u) \in \bigwedge^2 V^* \otimes \bigwedge^2 V^* , \tag{2.3.8}$$

for every $u \in O(M)$.

Proposition 2.3.6 *The Riemann curvature mapping $\mathscr{R}$ of a pseudo-Riemannian manifold has the following algebraic properties. For any $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in V$,*

$$\mathscr{R}(\mathbf{x}, \mathbf{y}) = -\mathscr{R}(\mathbf{y}, \mathbf{x}) , \tag{2.3.9}$$

$$\eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z}, \mathbf{w}) = -\eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{w}, \mathbf{z}) , \tag{2.3.10}$$

$$\eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z}, \mathbf{w}) = \eta(\mathscr{R}(\mathbf{z}, \mathbf{w})\mathbf{x}, \mathbf{y}) , \tag{2.3.11}$$

$$\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z} + \mathscr{R}(\mathbf{y}, \mathbf{z})\mathbf{x} + \mathscr{R}(\mathbf{z}, \mathbf{x})\mathbf{y} = 0 . \tag{2.3.12}$$

Proof Formulae (2.3.9) and (2.3.10) follow immediately from (2.3.8) and formula (2.3.12) is a direct consequence of the fact that $\mathscr{R}$ takes values in the kernel $\mathfrak{K}(\mathfrak{h})$ of the mapping (2.3.5). It remains to prove (2.3.11). For that purpose, we write down the following four versions of (2.3.12).

$$
\begin{aligned}
0 &= \eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{z}, \mathbf{w}) + \eta(\mathscr{R}(\mathbf{y}, \mathbf{z})\mathbf{x}, \mathbf{w}) + \eta(\mathscr{R}(\mathbf{z}, \mathbf{x})\mathbf{y}, \mathbf{w}) , \\
0 &= \eta(\mathscr{R}(\mathbf{y}, \mathbf{z})\mathbf{w}, \mathbf{x}) + \eta(\mathscr{R}(\mathbf{z}, \mathbf{w})\mathbf{y}, \mathbf{x}) + \eta(\mathscr{R}(\mathbf{w}, \mathbf{y})\mathbf{z}, \mathbf{x}) , \\
0 &= -\eta(\mathscr{R}(\mathbf{z}, \mathbf{w})\mathbf{x}, \mathbf{y}) - \eta(\mathscr{R}(\mathbf{w}, \mathbf{x})\mathbf{z}, \mathbf{y}) - \eta(\mathscr{R}(\mathbf{x}, \mathbf{z})\mathbf{w}, \mathbf{y}) , \\
0 &= -\eta(\mathscr{R}(\mathbf{w}, \mathbf{x})\mathbf{y}, \mathbf{z}) - \eta(\mathscr{R}(\mathbf{x}, \mathbf{y})\mathbf{w}, \mathbf{z}) - \eta(\mathscr{R}(\mathbf{y}, \mathbf{w})\mathbf{x}, \mathbf{z}) .
\end{aligned}
$$

Summation of these equations and using (2.3.9) and (2.3.10) yields the assertion. ■

Remark 2.3.7

1. By Proposition 2.3.6,

$$\mathscr{R} : O(M) \to S^2 \left(\bigwedge^2 V^*\right) , \tag{2.3.13}$$

where $S^2 \left(\bigwedge^2 V^*\right) = \bigwedge^2 V^* \overset{s}{\otimes} \bigwedge^2 V^*$ is the symmetrized tensor product. By (2.1.25), $\mathscr{R}$ has the following equivariance property, see Exercise 2.3.2,

$$\mathscr{R}(\Psi_a(u))(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \mathscr{R}(u)(a\mathbf{x}, a\mathbf{y}, a\mathbf{u}, a\mathbf{v}) , \tag{2.3.14}$$

for $a \in \mathrm{O}(k,l)$ and $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in V$.

2. By (2.1.27), the Riemann curvature R fulfils identities corresponding to (2.3.9)–(2.3.12) with $\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \in V$ replaced by $X, Y, Z, W \in \mathrm{T}_m M$ and η replaced by g. Thus, in particular, $\mathsf{R} \in \Gamma^\infty\big(S^2(\bigwedge^2 \mathrm{T}^* M)\big)$. For a local frame $\{e_i\}$, using (2.1.52) we write
$$\mathsf{R}_{ijkl} \equiv \mathsf{g}(\mathsf{R}(e_i, e_j)e_k, e_l) = \mathsf{R}^m{}_{ijk}\, \mathsf{g}_{ml} \,.$$
In this notation, the algebraic properties (2.3.9)–(2.3.12) read
$$\mathsf{R}_{ijkl} = -\mathsf{R}_{jikl}\,, \quad \mathsf{R}_{ijkl} = -\mathsf{R}_{ijlk}\,, \quad \mathsf{R}_{ijkl} = \mathsf{R}_{klij}\,, \tag{2.3.15}$$
$$\mathsf{R}_{ijkl} + \mathsf{R}_{jkil} + \mathsf{R}_{kijl} = 0 \tag{2.3.16}$$

♦

Using the above properties, the space of Riemann curvature mappings $\mathfrak{K}(\mathfrak{o}(k,l))$ may be characterized as follows. By standard representation theory of the group $\mathrm{O}(k,l)$, for $n \geq 4$, one obtains the following decompositions into $\mathrm{O}(k,l)$-irreducible modules [76, 555]:
$$\bigwedge^3 V^* \otimes V^* = \bigwedge^2 V^* \oplus \bigwedge^4 V^* \oplus U\,, \tag{2.3.17}$$
$$S^2\left(\bigwedge^2 V^*\right) = \mathbb{R} \oplus \Sigma^2_0 \oplus \bigwedge^4 V^* \oplus W\,, \tag{2.3.18}$$
where Σ^2_0 stands for the space of traceless endomorphisms of $\mathbb{R}^n$ (viewed as symmetric 2-tensors) and where U and W are orthogonal complements. By dimension counting, U and W are not isomorphic.

Proposition 2.3.8 *The space of Riemann curvature mappings is given by*
$$\mathfrak{K}(\mathfrak{o}(k,l)) = \ker\varphi \,\cap\, S^2\left(\bigwedge^2 V^*\right)\,, \tag{2.3.19}$$
where
$$\varphi : \bigwedge^2 V^* \otimes \bigwedge^2 V^* \to \bigwedge^4 V^*\,, \quad \varphi(\xi \otimes \tau) := \xi \wedge \tau\,. \tag{2.3.20}$$

Proof Under the identifications $\mathfrak{o}(k,l) \cong \bigwedge^2 V^*$ and $V \cong V^*$, $\mathfrak{K}(\mathfrak{o}(k,l))$ coincides with the kernel of the mapping $\chi : \bigwedge^2 V^* \otimes \bigwedge^2 V^* \to \bigwedge^3 V^* \otimes V^*$ given by
$$\chi(\alpha \otimes (\zeta \wedge \tau)) := (\alpha \wedge \zeta) \otimes \tau - (\alpha \wedge \tau) \otimes \zeta\,.$$
Now, consider the decompositions (2.3.17) and (2.3.18). Viewing χ as an $\mathrm{O}(k,l)$-intertwining mapping and using Schur's Lemma, together with the fact that χ is surjective, we conclude that χ must be zero on the irreducible subspaces $\mathbb{R}$, Σ^2_0 and W. By dimension counting, these subspaces span the kernel of χ. Moreover, restricted to $S^2(\bigwedge^2 V^*)$, χ maps onto $\bigwedge^4 V^*$ and coincides with φ. ■

Combining (2.3.19) and (2.3.18), for $n \geq 4$, we obtain[21]

$$\mathfrak{K}(\mathfrak{o}(k,l)) = \mathbb{R} \oplus \Sigma_0^2 \oplus W\,. \tag{2.3.21}$$

This yields a decomposition of the Riemann curvature into its irreducible components with respect to the action of $\mathrm{O}(k,l)$. The component Σ_0^2 corresponds to the contraction to $V^* \otimes V^*$ defined by taking the trace of the mapping $\mathbf{z} \mapsto \mathscr{R}(\mathbf{z},\mathbf{x})\mathbf{y}$ and restricting it to $S^2(V^*)$.

Definition 2.3.9 (*Ricci tensor*) Let (M, g) be a pseudo-Riemannian manifold and let $\mathscr{R}$ be its Riemann curvature mapping. The mapping

$$\widetilde{\mathsf{Ric}} : O(M) \to S^2(V^*)\,, \quad \widetilde{\mathsf{Ric}}(u)(\mathbf{x},\mathbf{y}) := \operatorname{tr}\{\mathbf{z} \mapsto \mathscr{R}(u)(\mathbf{z},\mathbf{x})\mathbf{y}\} \tag{2.3.22}$$

is called the Ricci curvature mapping. Correspondingly,

$$\mathsf{Ric} : \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}\,, \quad \mathsf{Ric}(X,Y) := \operatorname{tr}\{Z \mapsto \mathsf{R}(Z,X)Y\} \tag{2.3.23}$$

is called the Ricci tensor of (M, g).

Note that Ric is of the same geometric type as the metric. Thus, viewing it as a mapping $\mathrm{T}_m M \to \mathrm{T}^*_m M$ and using $\mathsf{g}^{-1} : \mathrm{T}^*_m M \to \mathrm{T}_m M$, we can define a scalar on M.

Definition 2.3.10 (*Scalar curvature*) Let (M, g) be a pseudo-Riemannian manifold and let Ric be its Ricci tensor. The function

$$\mathsf{Sc} : M \to \mathbb{R}\,, \quad \mathsf{Sc}(m) := \operatorname{tr}(\mathsf{g}^{-1} \circ \mathsf{Ric})(m) \tag{2.3.24}$$

is called the scalar curvature of (M, g). The corresponding equivariant function $\widetilde{\mathsf{Sc}} : O(M) \to \mathbb{R}$ is called the scalar curvature mapping.

The scalar curvature corresponds to the first component in the decomposition (2.3.21). The component corresponding to the third summand is called the Weyl tensor. In Sect. 2.8, the above decomposition will be discussed in detail for the case $n = 4$.

Remark 2.3.11 Denoting $\mathsf{R}_{ij} = \mathsf{Ric}(e_i, e_j)$, we obtain the following local expressions for the Ricci tensor and the scalar curvature,

$$\mathsf{R}_{ij} = \mathsf{g}^{kl}\,\mathsf{R}_{kijl}\,, \quad \mathsf{Sc} = \mathsf{g}^{ij}\mathsf{R}_{ij}\,. \tag{2.3.25}$$

In particular, for a holonomic frame, we obtain

$$\mathsf{R}_{ij} = \partial_i\,\Gamma^l{}_{jl} - \partial_j\,\Gamma^l{}_{il} + \Gamma^l{}_{jm}\Gamma^m{}_{il} - \Gamma^l{}_{im}\Gamma^m{}_{jl}\,. \tag{2.3.26}$$

[21] For $k + l = 3$, one obtains $\mathfrak{K}(\mathfrak{o}(k,l)) = \mathbb{R} \oplus \Sigma_0^2$. For $k + l = 4$, this result belongs to Singer and Thorpe [592].

For an orthonormal local frame, we have $\mathsf{R}_{ij} = \eta^{kl}\,\mathsf{R}_{kijl}$. This yields the following useful formula

$$\mathsf{Ric}(X, Y) = \eta^{kl}\mathsf{g}(\mathsf{R}(e_k, X)Y, e_l)\,, \quad X, Y \in \mathfrak{X}(M)\,. \tag{2.3.27}$$

♦

There is an important special class of Riemannian manifolds characterized by the fact that their curvature has a trivial Σ_0^2-component in the decomposition (2.3.21).

Definition 2.3.12 (*Einstein manifold*) A (pseudo-)Riemannian manifold (M, g) is called Einstein if its Ricci tensor is a constant multiple of the metric at each point of M.

Note that for an n-dimensional Einstein space (M, g) we have

$$\mathsf{Ric} = \frac{\mathsf{Sc}}{n}\,\mathsf{g}\,, \tag{2.3.28}$$

where Sc is constant. In Sect. 2.5, we will see a large class of Einstein manifolds.

In the next step, we show which impact the above additional structures have on the analysis of the Berger criteria in the metric case. For a chosen orthonormal frame $u_0 \in P_{u_0}(\Gamma)$, let us consider the holonomy bundle $P_{u_0}(\Gamma) \subset O(M)$. Let us denote

$$H = \mathscr{H}_{u_0}(\Gamma)\,, \quad \mathfrak{h} = \mathfrak{h}_{u_0}(\Gamma)\,.$$

On $P_{u_0}(\Gamma)$, the curvature takes values in $\mathfrak{h} \subset \mathfrak{o}(k, l) \cong \bigwedge^2(V^*)$. This fact, together with (2.3.19), implies the following.

Proposition 2.3.13 *For any point $u \in P_{u_0}(\Gamma)$, the Riemann curvature $\mathscr{R}(u)$ belongs to the space*

$$\mathfrak{K}(\mathfrak{h}) = \ker\varphi\, \cap\, S^2(\mathfrak{h})\,. \tag{2.3.29}$$

■

It turns out that for many subgroups $H \subset \mathrm{O}(k, l)$, the restriction of φ to $S^2(\mathfrak{h})$ is injective. This implies $\mathfrak{K}(\mathfrak{h}) = 0$ and, thus, $\underline{\mathfrak{h}} = 0$. Then, the first Berger criterion implies that, in this case, H cannot occur as a holonomy group.

In the same way, the covariant derivative $D\mathscr{R}$ may be dealt with. By the above discussion, we have the following.

Proposition 2.3.14 *For any point $u \in P_{u_0}(\Gamma)$, the covariant derivative of $\mathscr{R}(u)$ takes values in*

$$\mathfrak{K}^1(\mathfrak{h}) = \ker\delta'\, \cap\, \big(V^* \otimes \mathfrak{K}(\mathfrak{h})\big)\,, \tag{2.3.30}$$

where $\delta' : V^* \otimes \mathfrak{K}^1(\mathfrak{o}(k, l)) \to \bigwedge^3 V^* \times \mathfrak{o}(k, l)$, cf. formula (2.3.7). ■

As already mentioned above, the condition $\mathfrak{K}^1(\mathfrak{h}) = 0$ distinguishes a special class of possible candidates. By Proposition 2.3.4, in this case the Riemannian manifold is necessarily locally symmetric. We exclude this class of spaces for a while, postponing its presentation to Sect. 2.5.

Finally, we show that we may limit our attention to the case where the representation of the holonomy group H on $V \equiv \mathbb{R}^n$ is irreducible. We consider the Riemannian metric case and comment on the pseudo-Riemannian case at the end. Under this assumption, the holonomy group is a subgroup of $\mathrm{O}(n)$. Let us assume, on the contrary, that the representation of H is reducible, that is, there exists a proper subspace $W \subset V$ invariant under H. Since we assume that η be definite, there exists an invariant orthogonal complement $W^\perp \subset V$. Proceeding further in this manner, we obtain an invariant orthogonal decomposition

$$V = W_0 \oplus W_1 \oplus \ldots \oplus W_k \,, \tag{2.3.31}$$

with W_0 carrying the trivial representation[22] (acting as the identity) and W_k carrying nontrivial irreducible representations of H for all $k \geq 1$. The following theorem belongs to de Rham [150]. It simplifies the holonomy classification problem essentially.

Theorem 2.3.15 (de Rham Splitting Theorem) *Let (M, g) be a Riemannian manifold. If the holonomy group H acts reducibly on $\mathbb{R}^n$, then the restricted holonomy group[23] H^0 of (M, g) is isomorphic to a product,*

$$H^0 = \{e\} \times H_1 \times \ldots \times H_k \,,$$

and M is locally isomorphic to a product of Riemannian manifolds,

$$M_0 \times M_1 \times \ldots \times M_k \,,$$

with M_0 being flat.

Proof By the above discussion, $\mathscr{R} : O(M) \to \bigwedge^2 V^* \otimes \mathfrak{o}(n)$ and $\mathscr{R}(u)(\mathbf{x}, \mathbf{y})$ takes values in $\mathfrak{h} \equiv \mathfrak{h}_{u_0}(\Gamma)$, for any $u \in P_{u_0}(\Gamma)$ and any $\mathbf{x}, \mathbf{y} \in V$. Since the decomposition (2.3.31) is invariant, we have

$$\mathscr{R}(u)(\mathbf{x}, \mathbf{y})_{\restriction W_0} = 0 \,, \quad \mathscr{R}(u)(\mathbf{x}, \mathbf{y})_{\restriction W_i} \subset W_i \,, \tag{2.3.32}$$

for $1 \leq i \leq k$. We decompose $\mathbf{x} = \sum \mathbf{x}_i$ and $\mathbf{y} = \sum \mathbf{y}_i$ with respect to (2.3.31) and insert this decomposition into $\mathscr{R}(u)(\mathbf{x}, \mathbf{y})$. This yields

$$\mathscr{R}(u)(\mathbf{x}, \mathbf{y}) = \sum_i \mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i) + \sum_{i \neq j} \mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_j) \,.$$

[22] Clearly, W_0 may be zero.

[23] By Theorem 1.7.9, this is the identity connected component of H.

By (2.3.12) and (2.3.32), we have $\mathscr{R}(u)(W_i, W_j)W_k = 0$ for i, j and k pairwise distinct. Next, consider the case $i = j \neq k$. Then, again by (2.3.12),

$$\mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i)\mathbf{z}_k = 0\,, \quad \mathscr{R}(u)(\mathbf{y}_i, \mathbf{z}_k)\mathbf{x}_i = -\mathscr{R}(u)(\mathbf{z}_k, \mathbf{x}_i)\mathbf{y}_i\,.$$

The first of these equations implies $\mathscr{R}(u)(W_i, W_i)W_k = 0$ for $i \neq k$. Using (2.3.11), from the second equation we obtain

$$\eta\big(\mathscr{R}(u)(\mathbf{z}_k, \mathbf{x}_i)\mathbf{y}_i, \mathbf{x}_i\big) = \eta\big(\mathscr{R}(u)(\mathbf{z}_k, \mathbf{y}_i)\mathbf{x}_i, \mathbf{x}_i\big) = \eta\big(\mathscr{R}(u)(\mathbf{x}_i, \mathbf{x}_i)\mathbf{z}_k, \mathbf{y}_i\big)\,,$$

and the anti-symmetry of $\mathscr{R}$ implies $\mathscr{R}(u)(W_k, W_i)W_i = 0$ for $i \neq k$. We conclude

$$\mathscr{R}(u)(\mathbf{x}, \mathbf{y}) = \sum_i \mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i)\,.$$

Now, according to the equivariance of $\mathscr{R}$, as u ranges over $\pi^{-1}(m) \cap P_{u_0}(\varGamma)$ and $\mathbf{x}, \mathbf{y}$ over V, for every i, the mappings $\mathscr{R}(u)(\mathbf{x}_i, \mathbf{y}_i)$ span an ideal $\mathfrak{h}_i(m) \subset \mathrm{End}(W_i)$ of $\mathfrak{h}$. Finally, varying m yields ideals $\mathfrak{h}_i$ and, by (2.3.1), the decomposition

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k\,.$$

This proves the first assertion. To prove the second assertion, first note that the splitting (2.3.31) induces a splitting of the horizontal distribution $\varGamma$ on $P_{u_0}(\varGamma)$,

$$\varGamma = \varGamma_1 \oplus \ldots \oplus \varGamma_k\,, \quad \varGamma_i := \varGamma \cap \theta^{-1}(W_i)\,.$$

By H-equivariance, this splitting induces a family of distributions $D_i = \pi'(\varGamma_i)$ on M such that

$$\mathrm{T}M = D_1 \oplus \ldots \oplus D_k\,.$$

Moreover, corresponding to (2.3.31), let us decompose

$$\theta = \theta_1 + \ldots + \theta_k\,, \quad \omega = \omega_1 + \ldots + \omega_k\,, \quad \varOmega = \varOmega_1 + \ldots + \varOmega_k\,,$$

with $\theta_i \in \varOmega^1(P_{u_0}(\varGamma)) \otimes W_i$ and $\omega_i\,,\ \varOmega_i \in \varOmega^*(P_{u_0}(\varGamma)) \otimes \mathfrak{h}_i$. We define the distributions

$$\hat{\varGamma}_i := \varGamma_i \oplus V_i$$

on $P_{u_0}(\varGamma)$, with V_i being the vertical distribution spanned by the Killing vector fields generated from elements of $\mathfrak{h}_i$. Clearly, $\varGamma_i$ is spanned by the horizontal standard vector fields generated by any basis of W_i. Thus, $\hat{\varGamma}_i$ annihilates both θ_j, ω_j, and $\varOmega_j$ for any $j \neq i$ and, by point 2 of Remark 2.1.14 and (1.4.5), for every i the distribution $\hat{\varGamma}_i$ is involutive. Consequently, by the Frobenius Theorem, it is integrable and, for every i, we have

$$d\theta_i + \omega_i \wedge \theta_i = 0\,, \quad \Omega_i = d\omega_i + \frac{1}{2}[\omega_i, \omega_i]\,. \tag{2.3.33}$$

Let $P_i \subset P_{u_0}(\Gamma)$ be an integral manifold of $\hat{\Gamma}_i$. Integrability of $\hat{\Gamma}_i$ clearly induces integrability of D_i and the integral manifolds U_i of D_i fulfil $U_i = \pi(P_i) \subset M$. Moreover, for every i, the restriction $\pi_i : P_i \to U_i$ of π defines a principal H_i-bundle and, by (2.3.33), ω_i is a torsion-free connection on P_i with restricted holonomy group H_i.

To summarize, for every $m \in M$, there exists a neighbourhood $U \cong U_1 \times \ldots \times U_k$ of m in M, with the U_i being integral manifolds of D_i, and the Levi-Civita connection restricted to U being a product of the Levi-Civita connections on the components U_i. ■

Definition 2.3.16 A Riemannian manifold (M, g) which is locally isomorphic to a product of Riemannian manifolds is called locally reducible. It is called irreducible if it is not locally reducible.

Clearly, by Theorem 2.3.15, if (M, g) is irreducible, then the restricted holonomy group necessarily acts irreducibly. Under additional assumptions, de Rham [150] was able to prove the following global version of Theorem 2.3.15.

Theorem 2.3.17 (Global de Rham Splitting Theorem) *Let (M, g) be a geodesically complete simply connected Riemannian manifold and assume that the holonomy group*[24] *of the Levi-Civita connection acts reducibly. Then, (M, g) is the direct product of geodesically complete simply connected irreducible Riemannian manifolds (M_i, g_i),*

$$(M, \mathsf{g}) = (M_0, \mathsf{g}_0) \times (M_1, \mathsf{g}_1) \times \ldots \times (M_k, \mathsf{g}_k)\,.$$

Here, (M_0, g_0) is a Euclidean vector space whose dimension is possibly zero. ■

Remark 2.3.18 Both versions of the de Rham Splitting Theorem have been extended to the case of an indefinite metric by Wu [682, 683]. ♦

Summarizing our discussion, for finding the possible holonomy groups of a Riemannian manifold (M, g), it is reasonable to make the following assumptions:

(a) M is simply connected. This ensures that the holonomy group is connected and that it coincides with the restricted holonomy group.
(b) (M, g) is irreducible. This implies that the holonomy group acts irreducibly.
(c) (M, g) is not locally symmetric. This requires $\mathfrak{K}^1(\mathfrak{h}) \neq 0$.

Under these assumptions, for the Riemannian case, Berger obtained the following.

Theorem 2.3.19 (Berger) *Let (M, g) be an n-dimensional simply connected irreducible Riemannian manifold which is not locally symmetric. Then, its holonomy group H belongs to one of the following classes:*

[24] By Remark 1.7.11, if M is simply connected, then the holonomy group and the restricted holonomy group coincide.

1. $H = \mathrm{SO}(n)$, $n \geq 2$, *(generic Riemannian manifold)*
2. $H = \mathrm{U}(m)$, $n = 2m \geq 4$, *(generic Kähler manifold)*
3. $H = \mathrm{SU}(m)$, $n = 2m \geq 4$, *(special Kähler manifold)*
4. $H = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1)$, $n = 4m \geq 8$, *(quaternionic Kähler manifold)*
5. $H = \mathrm{Sp}(m)$, $n = 4m \geq 8$, *(Hyper-Kähler manifold)*
6. $H = G_2$, $n = 7$, *(special holonomy)*
7. $H = \mathrm{Spin}(7)$, $n = 8$, *(special holonomy).* ■

For the proof, which is beyond the scope of this book, we refer to [68, 69, 555].

Remark 2.3.20

1. An elegant proof of Theorem 2.3.19 is obtained from the following result of Simons [591]: if M is irreducible, then either the holonomy group H acts transitively on S^{n-1} or its identity component acts trivially on the space of curvature tensors $\mathfrak{K}(\mathfrak{h})$. Then, Theorem 2.3.19 is obtained by using the classification of simple Lie algebras and their representations.
2. According to Examples 2.2.22 and 2.2.27, it was clear from the beginning that the groups $\mathrm{SO}(n)$ and $\mathrm{U}(n)$ must occur in the above list. For a detailed discussion of examples for all the groups occuring in Theorem 2.3.19, we refer to [555]. ♦

Exercises

2.3.1 Show that the commutators of horizontal standard vector fields corresponding to a torsion-free connection are vertical.

2.3.2 Confirm the equivariance property (2.3.14). *Hint:* Under the identification $\mathfrak{o}(n) \cong (\mathbb{R}^n)^* \wedge (\mathbb{R}^n)^*$, the adjoint representation is mapped onto the second exterior power of the dual of the basic representation.

2.3.3 Show that, in terms of the Riemann curvature R, the Bianchi identity (2.1.16) reads

$$(\nabla_X \mathsf{R})(Y, Z) + (\nabla_Y \mathsf{R})(Z, X) + (\nabla_Z \mathsf{R})(X, Y) = 0\,. \tag{2.3.34}$$

2.4 Sectional Curvature

In this section, we discuss a generalization of the classical Gaussian curvature of surfaces in $\mathbb{R}^3$. It reduces the study of the Riemann curvature to the study of real valued functions. Let (M, g) be a pseudo-Riemannian manifold. Let $\Sigma_m \subset \mathrm{T}_m M$ be a 2-dimensional subspace such that $\mathsf{g}_{\restriction \Sigma_m}$ is non-degenerate. Let $\{X, Y\}$ be an arbitrary basis of Σ_m. We put

$$\mathsf{K}(\Sigma_m) := \frac{\langle \mathsf{R}(X, Y)Y, X\rangle}{\parallel X \parallel^2 \parallel Y \parallel^2 - \langle X, Y\rangle^2}\,, \tag{2.4.1}$$

where $\| \cdot \|^2$ and $\langle \cdot, \cdot \rangle$ are the quadratic form and the bilinear form, respectively, induced from g. It can be easily shown that $\mathsf{K}(\Sigma_p)$ is well defined, that is,

(a) the right hand side of (2.4.1) does not depend on the choice of the basis. This is a simple consequence of the symmetry properties of R given by point 2 of Remark 2.3.7 and is, thus, left to the reader (Exercise 2.4.1).
(b) Σ_m is non-degenerate iff $\| X \|^2 \| Y \|^2 - \langle X, Y \rangle^2 \neq 0$, (Exercise 2.4.2).

Note that K may be viewed as a mapping from the Graßmann manifold $G_2(\mathrm{T}_m M)$ to $\mathbb{R}$. Let $G_2^0(\mathrm{T}_m M) \subset G_2(\mathrm{T}_m M)$ be the subset of non-degenerate subspaces.

Definition 2.4.1 The mapping $\mathsf{K} : G_2^0(\mathrm{T}_m M) \to \mathbb{R}$ given by (2.4.1) is called the sectional curvature of the pseudo-Riemannian manifold at $m \in M$.

Clearly, in the Riemannian case, every 2-dimensional subspace of $\mathrm{T}_m M$ is non-degenerate.

Proposition 2.4.2 *The curvature tensor* R *is completely determined by the sectional curvature. If the mapping* K *is constant, that is,* $\mathsf{K}(\Sigma_m) = k(m)$ *for every* $\Sigma_m \in G_2^0(\mathrm{T}_m M)$*, then*

$$\mathsf{R}_m(X, Y)Z = k(m)\big(\langle Y, Z \rangle X - \langle X, Z \rangle Y\big) . \tag{2.4.2}$$

Conversely, if (2.4.2) *is fulfilled, then all non-degenerate planes have sectional curvature* $k(m)$.

Proof The proof of the first assertion is the consequence of the following simple polarization argument. Denote $\alpha(X, Y) := \langle \mathsf{R}(X, Y)X, Y \rangle$, for any $X, Y \in \mathrm{T}_m M$. Then, by direct inspection,

$$\begin{aligned} -6\langle \mathsf{R}(X, Y)Z, W \rangle = {} & \alpha(X + W, Y + Z) - \alpha(X + W, Y) - \alpha(X + W, Z) \\ & - \alpha(X, Y + Z) - \alpha(W, Y + Z) + \alpha(X, Z) + \alpha(W, Y) \\ & - \alpha(Y + W, X + Z) + \alpha(Y + W, X) + \alpha(Y + W, Z) \\ & + \alpha(Y, X + Z) + \alpha(W, X + Z) - \alpha(Y, Z) - \alpha(W, X) , \end{aligned}$$

showing that R is determined by α and, thus, by K. We prove the second statement. For that purpose, denote

$$\mathsf{R}_0(X, Y)Z := \langle Y, Z \rangle X - \langle X, Z \rangle Y .$$

Note that R_0 shares the symmetry properties (2.3.9), (2.3.10) and (2.3.12) of R.[25] Assume that $\mathsf{K}(\Sigma_m) = k(m)$ for all non-degenerate planes. If X, Y span a non-degenerate plane, then by (2.4.1),

$$\langle \mathsf{R}(X, Y)Y, X \rangle = k(m)\big(\langle Y, Y \rangle X - \langle X, Y \rangle Y\big) = \langle k(m)\mathsf{R}_0(X, Y)Y, X \rangle .$$

[25] It also shares the symmetry property (2.3.11), but this is not needed here.

Thus, the tensor $\hat{\mathsf{R}} := \mathsf{R} - k(m)\mathsf{R}_0$ has the above symmetry properties and fulfils

$$\langle \hat{\mathsf{R}}(X, Y)Y, X\rangle = 0\,. \tag{2.4.3}$$

If X and Y span a degenerate plane, we can choose sequences $X_n \to X$ and $Y_n \to Y$ of tangent vectors such that X_n and Y_n span non-degenerate planes for each n.[26] Then, $\langle \hat{\mathsf{R}}(X_n, Y_n)Y_n, X_n\rangle = 0$ for all n and, thus, (2.4.3) holds for degenerate planes as well. Finally, note that this equation is also true for pairs X, Y which are linearly dependent. We conclude that (2.4.3) holds for all $X, Y \in \mathrm{T}_m M$. Now, the assertion is a consequence of the following simple algebraic fact (Exercise 2.4.3): If

$$\tilde{\mathsf{R}} : \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}$$

is a quadrilinear mapping sharing the symmetry properties (2.3.9), (2.3.10) and (2.3.12) of R, then $\langle \tilde{\mathsf{R}}(X, Y)Y, X\rangle = 0$ implies $\tilde{\mathsf{R}} = 0$.

The converse statement is trivial. ■

Proposition 2.4.2 leads us to an important class of pseudo-Riemannian manifolds.

Definition 2.4.3 If $\mathsf{K}(\Sigma_m) = k(m)$ for every $\Sigma_m \in G_2^0(\mathrm{T}_m M)$, then we say that (M, g) is a space of constant curvature at m. Let k be a real number. We say that (M, g) is a space of constant curvature k if $\mathsf{K}(\Sigma_m) = k$ at every point $m \in M$.

Remark 2.4.4

1. By the proof of Proposition 2.4.2, for a space of constant curvature, we have

$$\mathsf{R}(X, Y)Z = k\big(\langle Y, Z\rangle X - \langle X, Z\rangle Y\big)\,, \quad k \in \mathbb{R}\,. \tag{2.4.4}$$

2. By a theorem of Schur, see Theorem 2.2. in Chap. V of [381], if (M, g) is a space of constant curvature at every point of M and $\dim M \geq 3$, then M is a space of constant curvature, that is, the mapping $m \to k(m)$ is constant.
3. It is not hard to construct models of spaces of constant curvature. The simplest Riemannian example is the n-sphere of radius r embedded in the standard way in $\mathbb{R}^{n+1}$. This is a space of constant curvature equal to $\frac{1}{r^2}$. The simplest pseudo-Riemannian model is the pseudo-Euclidean space $(\mathbb{R}^n_s, \mathsf{g}^n_s)$ with the signature $(n - s, s)$. It is easy to show that this is a space of constant curvature equal to 0. In Sect. 2.5, we will see a large class of spaces of constant curvature. For an exhaustive presentation of this subject we refer to [676].
4. In the indefinite case, there is a lot of subtleties and there is quite a number of classical papers on that subject, see [63, 145, 257, 395, 490] and further references therein. ♦

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[26] By property (b) above, in any fixed basis of $\mathrm{T}_m M$, $\| X \|^2 \| Y \|^2 - \langle X, Y\rangle^2$ is a polynomial in the components of X and Y whose zero set does not contain any open subset.

Exercises

2.4.1 Prove that (2.4.1) does not depend on the choice of the basis.

2.4.2 Show that the restriction of a pseudo-Riemannian metric to a 2-dimensional subspace $\Sigma_m \subset \mathrm{T}_m M$ is non-degenerate iff $\parallel X \parallel^2 \parallel Y \parallel^2 - \langle X, Y \rangle^2 \neq 0$.

2.4.3 Prove the following. If $\tilde{\mathsf{R}} : \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \times \mathrm{T}_m M \to \mathbb{R}$ is a quadrilinear mapping sharing the symmetry properties (2.3.9), (2.3.10) and (2.3.12) of R, then $\langle \tilde{\mathsf{R}}(X, Y)Y, X\rangle = 0$ implies $\tilde{\mathsf{R}} = 0$.

2.5 Symmetric Spaces

In this section, we take up the discussion from Sect. 2.3. We analyze the special case $\mathfrak{K}^1(\mathfrak{h}) = 0$, that is, we analyze the condition

$$D\mathscr{R} = 0\,, \tag{2.5.1}$$

defining locally symmetric manifolds, cf. Definition 2.3.2. Thus, we give up assumption (c) prior to Theorem 2.3.19, but we keep on assuming the following.

(a) M is simply connected, which ensures that the holonomy group H is connected and that it coincides with the restricted holonomy group.
(b) (M, g) is irreducible, which implies that H acts irreducibly.

Moreover, as above, we limit our attention to the Riemannian metric case, that is, $H \subset \mathrm{O}(n)$ is a compact Lie subgroup acting irreducibly on $V \equiv \mathbb{R}^n$. Then, by the Holonomy Principle, cf. Proposition 1.7.20, the space of parallel sections of

$$E = O(M) \times_{\mathrm{O}(n)} S^2 \left(\bigwedge\nolimits^2 V^*\right)$$

is in one-to-one correspondence with the space of holonomy-invariant vectors in $S^2 \left(\bigwedge^2 V^*\right)$ as follows. Any $\mathscr{R}$ satisfying (2.5.1) is constant on $P_{u_0}(\Gamma)$ and, restricted to $P_{u_0}(\Gamma)$, it takes values in $\mathfrak{K}(\mathfrak{h})$ given by (2.3.29). Thus, the Holonomy Principle assigns to $\mathscr{R}$ the H-invariant element

$$F := \mathscr{R}(u) \in \mathfrak{K}(\mathfrak{h}), \quad u \in P_{u_0}(\Gamma)\,. \tag{2.5.2}$$

Lemma 2.5.1 *Let $H \subset \mathrm{O}(n)$ be a closed subgroup and let $F \in \mathfrak{K}(\mathfrak{h})$ be an H-invariant element. Then, $\mathfrak{g} = \mathfrak{h} \oplus V$ carries the structure of a Lie algebra given by*

$$[A, \mathbf{x}] = -[\mathbf{x}, A] = A\mathbf{x}\,, \quad [\mathbf{x}, \mathbf{y}] = -F(\mathbf{x}, \mathbf{y})\,, \quad A \in \mathfrak{h}\,,\ \mathbf{x}, \mathbf{y} \in V\,.$$

Proof Bilinearity and anti-symmetry are obvious. We prove that the Jacobi identity holds. For that purpose, we have to consider three cases:
(a) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$. Since $F(\mathbf{x}, \mathbf{y}) \in \mathfrak{h} \subset \mathrm{End}(V)$, the definition of $\mathfrak{K}(\mathfrak{h})$ implies

$$[[\mathbf{x}, \mathbf{y}], \mathbf{z}] + [[\mathbf{y}, \mathbf{z}], \mathbf{x}] + [[\mathbf{z}, \mathbf{x}], \mathbf{y}] = 0 .$$

(b) Let $\mathbf{x}, \mathbf{y} \in V$. By the H-invariance of F, cf. (2.1.25), we have

$$F(\mathbf{x}, \mathbf{y}) = \mathrm{Ad}(a^{-1}) \circ F(a\mathbf{x}, a\mathbf{y}) , \quad a \in H \subset \mathrm{O}(n) .$$

Differentiating this equation, we obtain

$$[F(\mathbf{x}, \mathbf{y}), A] + F(A\mathbf{x}, \mathbf{y}) + F(\mathbf{x}, A\mathbf{y}) = 0$$

for any $A \in \mathfrak{h}$. This implies

$$[[\mathbf{x}, \mathbf{y}], A] + [[\mathbf{y}, A], \mathbf{x}] + [[A, \mathbf{x}], \mathbf{y}] = 0 .$$

(c) Let $\mathbf{x} \in V$ and $A, B \in \mathfrak{h}$. Then, by definition of the Lie bracket of $\mathfrak{h} \subset \mathrm{End}(V)$,

$$[A, B](\mathbf{x}) = A(B\mathbf{x}) - B(A\mathbf{x}) .$$

This proves the third case. ■

To make contact with the standard notation, we denote $V = \mathfrak{m}$. Then,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \tag{2.5.3}$$

and the commutation relations of $\mathfrak{g}$ fulfil:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} , \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} . \tag{2.5.4}$$

Moreover, by the Ambrose-Singer Theorem,

$$[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h} . \tag{2.5.5}$$

Associated with the decomposition (2.5.3), there is a linear mapping

$$\lambda : \mathfrak{g} \to \mathfrak{g} , \quad \lambda(A, \mathbf{x}) := (A, -\mathbf{x}) , \quad A \in \mathfrak{h} , \ \mathbf{x} \in \mathfrak{m} . \tag{2.5.6}$$

By (2.5.4), λ is an involutive Lie algebra homomorphism (Exercise 2.5.1). Conversely, we have the following.

Lemma 2.5.2 *Any involutive Lie algebra homomorphism λ of a Lie algebra $\mathfrak{g}$ induces a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ fulfilling (2.5.4).*

Proof Since $\lambda^2 = \mathrm{id}$, λ is diagonalizable and we may decompose $\mathfrak{g}$ into the eigenspaces $\mathfrak{h}$ and $\mathfrak{m}$ of λ corresponding to the eigenvalues $+1$ and -1, respectively. Now, the first relation in (2.5.4) is obvious. To check the remaining two, we calculate

$$\lambda([A, \mathbf{x}]) = [\lambda(A), \lambda(\mathbf{x})] = -[A, \mathbf{x}], \quad A \in \mathfrak{h}, \mathbf{x} \in \mathfrak{m},$$

that is, $[A, \mathbf{x}] \in \mathfrak{m}$. Similarly, $\lambda([\mathbf{x}, \mathbf{y}]) = [\mathbf{x}, \mathbf{y}] \in \mathfrak{h}$ for any $\mathbf{x}, \mathbf{y} \in \mathfrak{m}$. ■

Definition 2.5.3 Let $\mathfrak{g}$ be a Lie algebra and let λ be an involutive automorphism of $\mathfrak{g}$. Then, the pair $(\mathfrak{g}, \lambda)$ is called a symmetric Lie algebra. In addition,

1. if the set of fixed points $\mathfrak{h}$ of λ is a compactly embedded Lie subalgebra[27] of $\mathfrak{g}$, then $(\mathfrak{g}, \lambda)$ is called an orthogonal symmetric Lie algebra,
2. if $\mathfrak{h} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, then $(\mathfrak{g}, \lambda)$ is called effective.
3. if $(\mathfrak{g}, \lambda)$ is effective and $\mathrm{ad}([\mathfrak{m}, \mathfrak{m}])$ acts irreducibly on $\mathfrak{m}$, then $(\mathfrak{g}, \lambda)$ is called irreducible.

Proposition 2.5.4 *The Lie algebra $\mathfrak{g}$ constructed in Lemma 2.5.1, endowed with the involutive automorphism λ given by* (2.5.6)*, is an irreducible orthogonal symmetric Lie algebra.*

Proof By construction, $(\mathfrak{g}, \lambda)$ is symmetric. Since, by assumption, $H \subset \mathrm{O}(n)$ is a compact Lie subgroup acting faithfully on $\mathbb{R}^n$, $\mathrm{ad}(\mathfrak{h})$ is compact and, thus, $(\mathfrak{g}, \lambda)$ is orthogonal. Suppose $A \in \mathfrak{h} \cap \mathfrak{z}$. Then,

$$A\mathbf{x} = [A, \mathbf{x}] = 0$$

for every $\mathbf{x} \in \mathfrak{m}$ and, thus, $A = 0$. Thus, $(\mathfrak{g}, \lambda)$ is effective. Finally, by assumption, H acts irreducibly on $\mathfrak{m}$. Thus, $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$, too. This, together with (2.5.5) implies that $(\mathfrak{g}, \lambda)$ is irreducible. ■

In the sequel, the pair $(\mathfrak{g}, \lambda)$ constructed above will be called the canonical symmetric Lie algebra associated with the locally symmetric Riemannian manifold we started with. The decomposition (2.5.3) will be called the canonical decomposition of $(\mathfrak{g}, \lambda)$.

The following proposition characterizes irreducible symmetric Lie algebras.

Proposition 2.5.5 *Let $(\mathfrak{g}, \lambda)$ be an irreducible symmetric Lie algebra and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the decomposition induced by λ. Then, one of the following cases occurs:*

1. *$\mathfrak{g}$ is a simple Lie algebra.*
2. *$\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}$ with $\tilde{\mathfrak{g}}$ simple, fulfilling $\mathfrak{h} = \{(A, A) : A \in \tilde{\mathfrak{g}}\}$ and $\lambda(A, B) = (B, A)$ for any $A, B \in \tilde{\mathfrak{g}}$.*
3. *$[\mathfrak{m}, \mathfrak{m}] = 0$.*

For the proof we refer the reader to [381].[28]

[27] That is, the group of transformations of $\mathfrak{g}$ generated by $\mathrm{ad}(\mathfrak{h})$ is compact.

[28] Cf. Proposition 7.5 in Vol. 2, Chap. XI of [381].

Remark 2.5.6

1. Assume that either point 1 or point 2 of Proposition 2.5.5 holds. Then, since $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is an ideal in $\mathfrak{g}$, we have $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$. Thus, an effective symmetric Lie algebra is irreducible iff $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$, that is, iff $\mathfrak{g}$ is of the form described either by point 1 or by point 2. In particular, if $(\mathfrak{g}, \lambda)$ is irreducible, then $\mathfrak{g}$ is semisimple.
2. Conversely, if $(\mathfrak{g}, \lambda)$ is an orthogonal symmetric Lie algebra and $\mathfrak{g}$ is simple, then $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$, see Proposition 7.4 in Vol. 2, Chap. XI of [381]. ♦

Proposition 2.5.5 and property (2.5.5) imply that the canonical symmetric Lie algebra $(\mathfrak{g}, \lambda)$ is semisimple. Consequently, by Proposition I/5.4.10, the Killing form

$$\mathsf{k} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \quad \mathsf{k}(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)),$$

of $\mathfrak{g}$ is non-degenerate. Moreover, the relations (2.5.4) imply that the decomposition (2.5.3) is orthogonal with respect to k (Exercise 2.5.2). Equivalently, k is λ-invariant. This implies that the restrictions $\mathsf{k}^{\mathfrak{h}}$ and $\mathsf{k}^{\mathfrak{m}}$ of k to $\mathfrak{h}$ and $\mathfrak{m}$, respectively, are both non-degenerate and λ-invariant, too. Moreover, they have the following properties:

(a) By Corollary I/5.5.8, $\mathsf{k}^{\mathfrak{h}}$ is negative semidefinite and, since $(\mathfrak{g}, \lambda)$ is effective, it is negative definite.
(b) Since $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$ and since both $\mathsf{k}^{\mathfrak{m}}$ and the scalar product η on $\mathfrak{m}$ induced from the metric g are $\mathrm{ad}(\mathfrak{h})$-invariant, by Schur's Lemma, they must be proportional to each other,

$$\eta(\mathbf{x}, \mathbf{z}) = -c\, \mathsf{k}^{\mathfrak{m}}(\mathbf{x}, \mathbf{z}), \quad \mathbf{x}, \mathbf{z} \in \mathfrak{m}, \; c \in \mathbb{R}, c \neq 0. \tag{2.5.7}$$

Thus, since η is positive definite, $\mathsf{k}^{\mathfrak{m}}$ is either positive or negative definite.

Definition 2.5.7 An effective orthogonal symmetric Lie algebra $(\mathfrak{g}, \lambda)$ with $\mathfrak{g}$ semisimple is said to be of compact or of non-compact type, if the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}$ is, respectively, negative definite or positive definite.

Remark 2.5.8 Combining Proposition 2.5.5 with Propositions 7.4 and 7.5 in in Vol. 2, Chap. XI of [381], one can show that any irreducible orthogonal symmetric Lie algebra is either of compact or of non-compact type. ♦

Next, we show that, given an irreducible orthogonal symmetric Lie algebra $(\mathfrak{g}, \lambda)$, one can construct a special type of homogeneous Riemannian manifold.

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the decomposition induced from λ. Let $\tilde{G}$ be the connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\tilde{H}$ be the connected Lie subgroup corresponding to $\mathfrak{h}$. Then, the space of left cosets $M := \tilde{G}/\tilde{H}$ is a simply connected manifold endowed with the natural left $\tilde{G}$-action given by left translations. Let

$$\tilde{Z} = \left\{ g \in \tilde{G} : g(m) = m \;\text{ for all }\; m \in M \right\}$$

be the kernel of this action. Since, by assumption, $(\mathfrak{g}, \lambda)$ is effective, $\tilde{Z}$ must be discrete. Thus, M is an almost effective $\tilde{G}$-manifold. We pass to an effective action by setting $G := \tilde{G}/\tilde{Z}$ and $H := \tilde{H}/\tilde{Z}$. Then, $M = G/H$, G and H are connected, and we have the natural left effective action

$$\delta : G \times G/H \to G/H\,, \quad (a, [g]) \mapsto \delta_a([g]) := [ag]\,.$$

By point 4 of Example 1.1.4, the natural projection $\pi : G \to M$ endows G with the structure of a principal H-bundle P and the tangent mapping π' identifies $\mathfrak{m}$ and $\mathrm{T}_{[\mathbb{1}]}M$ as vector spaces. Under this identification, the isotropy representation

$$H \to \mathrm{Aut}(\mathrm{T}_{[\mathbb{1}]}M)\,, \quad h \mapsto (\delta_h)'_{\mathbb{1}}\,,$$

is given by $\mathrm{Ad}(H)$ acting on $\mathfrak{m}$, cf. point 1 of Remark I/6.2.10. Correspondingly,

$$G \times_{\mathrm{Ad}(H)} \mathfrak{m} \to \mathrm{T}M\,, \quad [(a, \mathbf{x})] \mapsto [\mathrm{L}'_a(\mathbf{x})], \tag{2.5.8}$$

is an isomorphism. Since $(\mathfrak{g}, \lambda)$ is orthogonal and irreducible, there exists an $\mathrm{Ad}(H)$-invariant scalar product η on $\mathfrak{m}$ which is unique up to a positive factor. Clearly, η induces an H-invariant scalar product on $\mathrm{T}_{[\mathbb{1}]}M$ which, using the left G-action δ, can be extended to a G-invariant Riemannian metric g on M. To summarize, we have constructed a simply connected transitive and effective G-manifold (M, g) with G acting by isometries.

Consider the bundle of orthonormal frames $O(M)$ of (M, g). Note that any η-orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ of $\mathfrak{m}$ induces via π' a g-orthonormal frame $(e_1, \ldots, e_n)$ at $[\mathbb{1}] \in M$ and, thus, an injective bundle morphism

$$\vartheta : P \to O(M)\,, \quad \vartheta(a) := (\delta'_a(e_1), \ldots, \delta'_a(e_n))\,, \tag{2.5.9}$$

projecting onto the identical diffeomorphism of M. The corresponding Lie group homomorphism $\tau : H \to O(n) \subset \mathrm{GL}(n, \mathbb{R}) \cong \mathrm{Aut}(\mathrm{T}_m M)$ is given by the adjoint action of H on $\mathfrak{m} \cong \mathrm{T}_{[\mathbb{1}]}M$. To summarize, P is a subbundle of $O(M)$.

Now, decompose the Maurer–Cartan form $\theta^G \in \Omega^1(G, \mathfrak{g})$ with respect to (2.5.3):

$$\theta^G = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}}\,.$$

By Example 1.3.19, $\theta_{\mathfrak{h}}$ coincides with the canonical G-invariant connection[29] ω^c on P. Recall that the corresponding horizontal distribution is generated by $\mathfrak{m}$, that is, by left invariant vector fields $a \mapsto (\mathbf{x}_*)_a = L'_a(\mathbf{x})$ with $\mathbf{x} \in \mathfrak{m}$.

Lemma 2.5.9 *Under the morphism* (ϑ, τ), $\theta_{\mathfrak{m}}$ *corresponds to the soldering form* θ *on* $O(M)$, *that is,* $\vartheta^*\theta = \theta_{\mathfrak{m}}$.

[29]Note that this is a special case of the canonical invariant connection defined in point 2 of Remark 1.9.14. It is obtained by setting $G = H$ and $\lambda = \mathrm{id}$ there.

Proof By $\mathfrak{m}$-valuedness of $\theta_{\mathfrak{m}}$ and horizontality of θ, both $\vartheta^*\theta$ and $\theta_{\mathfrak{m}}$ vanish on the left invariant vector fields generated by elements of $\mathfrak{h}$. Thus, let $\mathbf{x}_*$ be generated by $\mathbf{x} \in \mathfrak{m}$. Then, clearly $\theta_{\mathfrak{m}}(\mathbf{x}_*) = \mathbf{x}$. On the other hand,

$$(\vartheta^*\theta)_g(\mathbf{x}_*) = \vartheta(g)^{-1}(\rho' \circ \vartheta'(\mathbf{x}_*)) = \vartheta(g)^{-1}(\pi'(\mathbf{x}_*)) = \vartheta(g)^{-1}(\delta_g' \circ \pi'(\mathbf{x})) = \mathbf{x}\,,$$

where $\rho : O(M) \to M$ is the canonical projection. ■

Proposition 2.5.10 *The Riemannian manifold (M, g) has the following properties:*

1. *Under the morphism (ϑ, τ), the Levi-Civita connection ω^0 of (M, g) corresponds to the canonical connection ω^c, that is, $\vartheta^*\omega^0 = \omega^c$.*
2. *The Riemann curvature of (M, g) is constant and given by the linear mapping*

$$F : \bigwedge^2 \mathfrak{m} \to \mathfrak{h}\,, \quad F(\mathbf{x}, \mathbf{y}) = -[\mathbf{x}, \mathbf{y}]\,. \tag{2.5.10}$$

3. *The holonomy group based at $\vartheta(\mathbb{1})$ of ω^0 is H and the holonomy bundle coincides with P.*
4. *The Riemann curvature of (M, g) is parallel, that is, (M, g) is locally symmetric.*
5. *For any $\mathbf{x} \in \mathfrak{m}$, $t \mapsto \pi(L_g \exp(t\mathbf{x}))$ is a geodesic through $[g] \in M$. Conversely, every geodesic through $[g]$ is of this form. In particular, M is geodesically complete.*

Proof 1. We decompose the commutator $[\theta^G, \theta^G] \in \Omega^2(G, \mathfrak{g})$ with respect to (2.5.3). By (2.5.4),

$$[\theta^G, \theta^G]_{\mathfrak{h}} = [\theta_{\mathfrak{h}}, \theta_{\mathfrak{h}}] + [\theta_{\mathfrak{m}}, \theta_{\mathfrak{m}}]\,, \quad [\theta^G, \theta^G]_{\mathfrak{m}} = 2[\theta_{\mathfrak{h}}, \theta_{\mathfrak{m}}]\,. \tag{2.5.11}$$

Since the Levi-Civita connection is uniquely characterized by its covariant derivative D_{ω^0} on $\mathrm{T}M$, it is enough to show that the covariant derivative D_{ω^c} induced by ω^c via the isomorphism (2.5.8) coincides with D_{ω^0}. This is done by showing that the extension of ω^c to $O(M)$ is metric and torsionless. By Proposition 1.2.6, we may view any vector field X on M as an H-equivariant mapping $\tilde{X} : G \to \mathfrak{m}$ and, thus,

$$D_{\omega^c}\tilde{X} = \mathrm{d}\tilde{X} + \mathrm{ad}(\omega^c) \circ \tilde{X} = \mathrm{d}\tilde{X} + [\theta_{\mathfrak{h}}, \tilde{X}]\,,$$

cf. Eq. (1.4.2). Let η be the (unique up to a positive factor) $\mathrm{Ad}(H)$-invariant scalar product on $\mathfrak{m}$. By $\mathrm{Ad}(H)$-invariance, we obtain

$$\eta(D_{\omega^c}\tilde{X}, \tilde{Y}) + \eta(\tilde{X}, D_{\omega^c}\tilde{Y}) = \mathrm{d}(\eta(\tilde{X}, \tilde{Y}))\,.$$

This shows that the extension of ω^c to $O(M)$ is metric. It remains to show that this extension is torsionless: restricting the Maurer–Cartan equation to $\mathfrak{m}$ and using (2.5.11) we get

$$D_{\omega^c}\theta_{\mathfrak{m}} = \mathrm{d}\theta_{\mathfrak{m}} + [\theta_{\mathfrak{h}}, \theta_{\mathfrak{m}}] = 0\,.$$

But, by Lemma 2.5.9, $\vartheta^*\theta = \theta_{\mathfrak{m}}$ and, thus, $\vartheta^*\Theta = 0$. By uniqueness of the Levi-Civita connection, the assertion follows.

2. By the Structure Equation, the curvature form of ω^c is given by[30]

$$\Omega^c = -\frac{1}{2}[\theta_{\mathfrak{m}}, \theta_{\mathfrak{m}}]\,.$$

By point 1, $\vartheta^*\Omega^0 = \Omega^c$. These two facts immediately imply (2.5.10).

3. By point 2 and by the Ambrose-Singer Theorem, the Lie algebra of the holonomy group of ω^0 is $[\mathfrak{m}, \mathfrak{m}]$. By point 1 of Remark 2.5.6, $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ and, thus, the Lie algebra of the holonomy group of ω^0 coincides with $\mathfrak{h}$. Since, by construction, M is simply connected, the holonomy group of ω^0 is connected and coincides with the restricted holonomy group. On the other hand, since H is connected, too, we obtain the assertion. It follows that P coincides with the holonomy bundle of ω^0.

4. Since the curvature is constant on P and, thus, H-invariant, the Holonomy Principle 1.7.20 implies the assertion.

5. By Proposition 2.1.22, the geodesics of (M, g) are given by the projections of integral curves of horizontal standard vector fields on $L(M)$. Since they are horizontal, these curves may be chosen to lie in P. The restriction of $B(\mathbf{y})$, $\mathbf{y} \in \mathbb{R}^n$, to P is given by the left-invariant vector field generated by $\mathbf{x} = y^i\mathbf{e}_i \in \mathfrak{m}$, where $\{\mathbf{e}_i\}$ is a basis in $\mathfrak{m}$. Thus, here, the geodesics are given as projections of (global) one-parameter subgroups $t \mapsto \exp(t\mathbf{x})$ and their left translates by arbitrary group elements $g \in G$. ∎

By point 3 of Proposition 2.5.10, the irreducibility of $(\mathfrak{g}, \lambda)$ implies that (M, g) is irreducible. Together with points 4 and 5, this yields the following.

Corollary 2.5.11 (M, g) *is a complete irreducible locally symmetric Riemannian manifold.* ∎

Next, we show that the involutive automorphism λ induces a special symmetry for any point $m \in M$. Since any automorphism of a Lie algebra is the differential of a unique automorphism of the corresponding simply connected Lie group,[31] λ induces a unique automorphism σ of $\tilde{G}$. By (2.5.6), it fulfils $\sigma(\tilde{H}) = \tilde{H}$. Thus, σ descends to an involutive diffeomorphism $s : M \to M$. By construction,

$$s'_{[\mathbb{1}]} : \mathrm{T}_{[\mathbb{1}]}M \to \mathrm{T}_{[\mathbb{1}]}M\,, \quad s'_{[\mathbb{1}]}(X) = -X\,. \tag{2.5.12}$$

Thus, under the identification $\mathrm{T}_{[\mathbb{1}]}M = \mathfrak{m}$, we have $s'_{[\mathbb{1}]} = \lambda_{\restriction\mathfrak{m}}$.

Lemma 2.5.12 *The origin* $[\mathbb{1}]$ *of* M *is an isolated fixed point of* s. *Moreover,* s *is an isometry of the Riemannian metric* g.

---

[30] Since ω^c is a G-invariant connection, this is a special case of point 4 of Remark 1.9.14.

[31] For a proof, see e.g. Theorem 3.27 in [652].

Proof The proof of the first assertion is left to the reader (Exercise 2.5.4). To prove the second statement, we have to show that the mapping

$$s'_m : \mathrm{T}_m M \to \mathrm{T}_m M$$

is isometric. For the point $m = [\mathbb{1}]$, this follows immediately from (2.5.12), because at the origin g coincides with η and the latter is λ-invariant. To prove the invariance for an arbitrary point $m = [g]$, note that for any $g, h \in G$,

$$s(\delta_g[h]) = s([gh]) = [\sigma(g)\sigma(h)] = \delta_{\sigma(g)}[\sigma(h)] = \delta_{\sigma(g)} s([h]) ,$$

that is, $s \circ \delta_g = \delta_{\sigma(g)} \circ s$. Differentiation of this identity yields

$$s'_{[g]} \circ (\delta_g)'_{[\mathbb{1}]} = (\delta_{\sigma(g)})'_{[\mathbb{1}]} \circ s'_{[\mathbb{1}]} .$$

By construction, g is G-invariant and, thus, $(\delta_g)'_{[\mathbb{1}]}$ and $(\delta_{\sigma(g)})'_{[\mathbb{1}]}$ leave g invariant. This yields the assertion. ■

Remark 2.5.13 For every $g \in \tilde{Z}$, we have $\big(\sigma(g)\big)(m) = s \circ g \circ s(m) = s^2(m) = m$. Hence, $\sigma(\tilde{Z}) = \tilde{Z}$ and σ descends to an automorphism of G, denoted by the same symbol. One has $\sigma(H) = H$. ♦

Next, for any $m = [g] \in M$, we define[32]

$$s_m : M \to M , \quad s_m := \delta_g \circ s \circ \delta_{g^{-1}} . \tag{2.5.13}$$

Differentiating (2.5.13), we obtain $s'_m = \delta'_g \circ s'_{[\mathbb{1}]} \circ \delta'_{g^{-1}}$ for any $m = [g] \in M$. Thus, by Lemma 2.5.12, by formula (2.5.12) and by the G-invariance of g, for any $m \in M$, s_m is an involutive isometry of g fulfilling (Exercise 2.5.5)

$$s_m(m) = m , \quad (s_m)'_m = -\,\mathrm{id} . \tag{2.5.14}$$

The following remark yields a geometric interpretation of the symmetry s_m.

Remark 2.5.14 Let $t \to \gamma(t)$ be a geodesic of (M, g) with $\gamma(0) = m$. Since an isometry transforms geodesics to geodesics, $t \mapsto \tau(t) := s_m(\gamma(t))$ is a geodesic, too. By (2.5.14), its tangent vector at $t = 0$ satisfies

$$\dot{\tau}(0) = (s_m)'_m \, \dot{\gamma}(0) = -\dot{\gamma}(0) . \tag{2.5.15}$$

Now, the uniqueness property of geodesics, see Corollary 2.1.23, implies $\tau(t) = \gamma(-t)$. Thus, for any $m \in M$,

[32]Clearly, this definition does not depend on the choice of the representative.

$$s_m(\gamma(t)) = \gamma(-t)\,, \tag{2.5.16}$$

that is, s_m reverses the geodesics through m. ♦

Definition 2.5.15 (*Riemannian globally symmetric space*) A Riemannian manifold (M, g) is called globally symmetric if for each $m \in M$ there exists an involutive isometry $s_m : M \to M$ such that m is an isolated fixed point of s_m. The mapping s_m is called the symmetry of (M, g) at m.

Taking into account that, in the above construction of (M, g), the scalar product on $\mathfrak{m}$ is unique up to a positive constant and that a change of this constant implies a conformal transformation of g, we obtain the following.

Proposition 2.5.16 *To any irreducible*[33] *orthogonal symmetric Lie algebra* $(\mathfrak{g}, \lambda)$ *there corresponds a unique homothetic equivalence class* $(M, [\mathsf{g}])$ *of simply connected irreducible Riemannian globally symmetric spaces.* ■

It should be clear that the locally symmetric Riemannian manifold we started with and the Riemannian globally symmetric space constructed here are deeply related. Indeed, let (M, g) be a locally symmetric space. Let $(\mathfrak{g}, \lambda)$ be its canonical symmetric Lie algebra with canonical decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let $\eta \in S^2(\mathfrak{m}^*)$ be the scalar product on $\mathfrak{m}$ defined by g and let $F \in \mathfrak{K}(\mathfrak{h}) \subset \bigwedge^2 \mathfrak{m}^* \otimes \mathfrak{h}$ be the Riemann curvature of (M, g). Let G/H be the Riemannian globally symmetric space constructed from $(\mathfrak{g}, \lambda)$. Then, for any chosen point $m \in M$, via

$$\mathrm{T}_m M \cong \mathfrak{m} \cong \mathrm{T}_{[\mathbb{1}]} G/H$$

we obtain an isometric isomorphism between $\mathrm{T}_m M$ and $\mathrm{T}_{[\mathbb{1}]} G/H$ and, by point 2 of Proposition 2.5.10, M and G/H have the same Riemann curvature given by the mapping F. By standard arguments,[34] this implies the following.

Corollary 2.5.17 *Every point of a locally symmetric space* (M, g) *admits a neighbourhood isometric to a neighbourhood of the origin of the Riemannian globally symmetric space constructed from the canonical symmetric Lie algebra of* (M, g).

Note, however, that not every locally symmetric space is a Riemannian globally symmetric space. It is even not necessarily homogeneous. As an example,[35] let M be a compact Riemann surface with genus ≥ 2, equipped with a Riemannian metric of constant curvature equal to -1. Then, the isometry group of M is finite and, thus, M is not homogeneous and, consequently, also not globally symmetric.

As an immediate consequence of the existence of the symmetries s_m, we obtain

Proposition 2.5.18 *Any Riemannian globally symmetric space* (M, g) *is complete.*

[33] Remember that irreducibility includes effectiveness, cf. Definition 2.5.3.

[34] See Theorem 7.4 in Chap. VI of [381].

[35] This example is taken from [73].

Proof Consider any geodesic $t \mapsto \gamma(t)$ defined on the interval $[0, t_0[$. Apply the symmetry $s_{\gamma(t_0-\varepsilon)}$ to γ with some ε fulfilling $0 < \varepsilon < \frac{t_0}{2}$. By (2.5.16), this operation extends the domain of γ to $[0, 2t_0 - 2\varepsilon[$. Continuing this procedure, we obtain completeness of (M, g). ■

Next, given a Riemannian globally symmetric space (M, g), for every geodesic $t \mapsto \gamma(t)$ we consider the family of isometries

$$T_t^\gamma := s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}\,, \tag{2.5.17}$$

called the transvections along γ. The following properties are immediate consequences of (2.5.15) and (2.5.16) and are, therefore, left to the reader (Exercise 2.5.3).

Proposition 2.5.19 *Let (M, g) be a Riemannian globally symmetric space and let $t \mapsto \gamma(t)$ be a geodesic. Then,*

1. *T_t^γ acts on γ by translations, that is, $T_t^\gamma(\gamma(s)) = \gamma(t + s)$.*
2. *$(T_t^\gamma)'_{\gamma(s)}$ acts by parallel translation from $\gamma(s)$ to $\gamma(t + s)$ along γ, that is, for any parallel vector field X along γ,*

$$(T_t^\gamma)'_{\gamma(s)}(X(\gamma(s)) = X(\gamma(t + s))\,.$$

3. *$\left\{T_t^\gamma\right\}_{t\in\mathbb{R}}$ is a 1-parameter group of isometries, that is, $T_{t+s}^\gamma = T_t^\gamma \circ T_s^\gamma$.* ■

Recall from Example 2.2.16 that the isometry group $I(M)$ of a Riemannian manifold M is a Lie group. Let us denote its identity component by $I_0(M)$. By point 3 of Proposition 2.5.19, for any geodesic γ, the transvections T_t^γ form a subgroup (called the transvection group) of $I_0(M)$. On the other hand, by a classical theorem of Hopf and Rinow,[36] any two points of a complete Riemannian manifold may be joined by a geodesic. Using these two facts, we obtain the following.

Corollary 2.5.20 *Let (M, g) be a Riemannian globally symmetric space. Then,*

1. *Geodesics in M are images of 1-parameter groups of isometries.*
2. *The identity component $I_0(M)$ acts transitively on M.* ■

Proposition 2.5.21 *Let (M, g) be an irreducible Riemannian globally symmetric space and let G be a Lie group acting transitively and isometrically on M. If G acts effectively, then G coincides with $I_0(M)$.*

Proof Clearly, $I_0(M)$ is the largest connected group of isometries of (M, g). Denote $G' = I_0(M)$ and let $\mathfrak{g}'$ be its Lie algebra. Conjugation by s defines an automorphism σ' of G' which clearly restricts to the automorphism σ of G, cf. Remark 2.5.13. The canonical decompositions $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{g}' = \mathfrak{h}' \oplus \mathfrak{m}'$ necessarily fulfil $\mathfrak{m}' = \mathfrak{m}$. Here $\mathfrak{h}'$ is the Lie algebra of the stabilizer of the chosen point on M under G'. Thus, by Remark 2.5.6,

[36] See, e.g. [352].

$$\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}', \mathfrak{m}'] = \mathfrak{h}' .$$

This implies $\mathfrak{g}' = \mathfrak{g}$ and, thus, $G' = G$. ■

Thus, in the construction leading to Proposition 2.5.16, the Lie group G actually coincides with $I_0(M)$. Now, we are able to prove the converse of Proposition 2.5.16.

Proposition 2.5.22 *To any simply connected irreducible Riemannian globally symmetric space there corresponds a unique irreducible orthogonal symmetric Lie algebra.*

Proof Let (M, g) be a simply connected irreducible Riemannian globally symmetric space. By Corollary 2.5.20, $G = I_0(M)$ acts transitively and effectively on M. Let H be the isotropy group of this Lie group action at a chosen point $o \in M$. By the homotopy sequence of the fibration $H \to G \to G/H$, the simply-connectedness of G/H and the connectedness of G imply that H is connected. Moreover, by Theorem 3.4 in Chap. VI of [381], the isotropy subgroup $I(M)_m$ at any point $m \in M$ is compact. Hence, $H = G \cap I(M)_o$ is compact, too. Thus, $M = G/H$ and, by standard arguments, $\pi : G \to M$ is a submersion. In particular, $\pi' : \mathrm{T}_{\mathbb{1}}G \to \mathrm{T}_o M$ is an H-equivariant surjective linear mapping whose kernel coincides with $\mathrm{T}_{\mathbb{1}}H$.

Let s be the symmetry at o. Since s is an involutive diffeomorphism, the mapping $g \mapsto \sigma(g) := s \circ g \circ s^{-1}$ defines an involutive automorphism of G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. Clearly, $\lambda := \sigma'$ is an involutive automorphism of $\mathfrak{g}$. Let $\mathfrak{m}$ be the eigenspace of λ corresponding to the eigenvalue -1. By (2.5.14), $\pi'(\mathfrak{m}) = \mathrm{T}_o M$. We prove that $\mathfrak{h}$ is the eigenspace of λ corresponding to the eigenvalue $+1$: let

$$G^\sigma := \{g \in G : \sigma(g) = g\}$$

be the fixed point set of σ. By (2.5.14), s'_o commutes with the isotropy representation of H at o and, thus, H is contained in G^σ. Conversely, if $g \in G^\sigma$, then it commutes with s and, thus, for any 1-parameter subgroup $t \mapsto g_t$ of G^σ,

$$s \circ g_t(o) = g_t \circ s(o) = g_t(o) ,$$

that is, the orbit $g_t(o)$ is left invariant pointwise by s. Now, by Lemma 2.5.12, o is an isolated fixed point. Thus, $g_t(o)$ must coincide with o. But, $g_t(o) = o$ implies that the 1-parameter subgroup $t \mapsto g_t$ is contained in H. Since a connected Lie group is generated by its 1-parameter subgroups, we have $(G^\sigma)^0 \subset H$. Thus,

$$(G^\sigma)^0 \subset H \subset G^\sigma .$$

This relation implies that $\mathfrak{h}$ coincides with the $(+1)$-eigenspace of λ, indeed. To summarize, the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is canonical with respect to λ, that is, $(\mathfrak{g}, \lambda)$ is a symmetric Lie algebra. Since H is compact, $\mathrm{ad}(\mathfrak{h})$ is a compactly embedded Lie subalgebra of $\mathfrak{g}$, that is, $(\mathfrak{g}, \lambda)$ is orthogonal. It remains to prove that $(\mathfrak{g}, \lambda)$ is irreducible. Since g is G-invariant, we are in the situation described by Proposition

2.5.10. By this proposition, H coincides with the holonomy group of the Levi-Civita connection of g. Thus, the irreducibility of (M, g), together with the effectiveness of the action of G on M, implies the irreducibility of $(\mathfrak{g}, \lambda)$. ■

Remark 2.5.23 In the course of the above proof, we have found the following structure: a triple (G, H, σ) fulfilling

1. G is a connected Lie group and H is a closed subgroup,
2. σ is an involutive automorphism of G such that $(G^\sigma)^0 \subset H \subset G^\sigma$,
3. $\mathrm{Ad}(H)$ is compact,

is called a Riemannian symmetric pair. This notion clearly constitutes a link between symmetric spaces and symmetric Lie algebras. ♦

Combining Proposition 2.5.16 with Proposition 2.5.22, we obtain the following.

Theorem 2.5.24 *The homothetic equivalence classes of simply connected irreducible Riemannian globally symmetric spaces are in one-to-one correspondence with the irreducible orthogonal symmetric Lie algebras.* ■

This theorem reduces the classification of symmetric spaces of the above type to the classification of irreducible symmetric Lie algebras of compact or of non-compact type. According to a beautiful duality,[37] the problem further reduces to the classification of irreducible symmetric Lie algebras of the non-compact type. The latter can be shown to be in one-to-one correspondence with the real simple Lie algebras of non-compact type. If the complexification of such a Lie algebra is simple as a complex Lie algebra, then M is said to be of type III, otherwise M is said to be of type IV. The corresponding compact irreducible symmetric spaces are obtained by duality and are referred to as of type I and II, respectively. The complete list of simply connected irreducible symmetric spaces with symmetry group being a classical Lie group is given in Tables 2.1 and 2.2.[38] Here, $\mathrm{SO}_0(p, q)$ denotes the identity component of $\mathrm{SO}(p, q)$ and $\mathrm{SO}^*(2n)$ is the subgroup of $\mathrm{SO}(2n, \mathbb{C})$ satisfying

$$g^{\mathrm{T}} \mathsf{J}_0 \overline{g} = \mathsf{J}_0 \,, \quad g^{\mathrm{T}} g = \mathbb{1}_{2n} \,.$$

For the corresponding list with exceptional Lie groups we refer to the textbook of Helgason [293]. As already mentioned, there the reader may find an exhaustive presentation of the whole subject.

Remark 2.5.25 Note that in our considerations, we have excluded the class of symmetric Lie algebras fulfilling $[\mathfrak{m}, \mathfrak{m}] = 0$, cf. case 3 in Proposition 2.5.5. Symmetric Lie algebras with this property are said to be of Euclidean type. By point 2 of Proposition 2.5.10, they are necessarily flat. One can show that if G/H is simply connected,

[37] See Sect. 8 of Chap. XI in [381] or Sect. 2 of Chap. V in [293].

[38] By definition, the rank is the dimension of some maximal Abelian subspace of $\mathfrak{m}$. Any two maximal Abelian subspaces of $\mathfrak{m}$ are $\mathrm{Ad}(H)$-conjugate.

Table 2.1 Classical symmetric spaces of types I and III

Type I	Type III	Dimension	Rank
$\mathrm{SU}(n)/\mathrm{SO}(n)$	$\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$	$(n-1)(n+2)/2$	$n-1$
$\mathrm{SU}(2n)/\mathrm{Sp}(n)$	$\mathrm{SL}(n,\mathbb{H})/\mathrm{Sp}(n)$	$(n-1)(2n+1)$	$n-1$
$\mathrm{SU}(p+q)/S(\mathrm{U}(p)\times\mathrm{U}(q))$	$\mathrm{SU}(p,q)/S(\mathrm{U}(p)\times\mathrm{U}(q))$	$2pq$	$\min(p,q)$
$\mathrm{SO}(p+q)/(\mathrm{SO}(p)\times\mathrm{SO}(q))$	$\mathrm{SO}_0(p,q)/(\mathrm{SO}(p)\times\mathrm{SO}(q))$	pq	$\min(p,q)$
$\mathrm{SO}(2n)/\mathrm{U}(n)$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$n(n-1)$	$[n/2]$
$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathrm{Sp}(n,\mathbb{R})/\mathrm{U}(n)$	$n(n+1)$	n
$\mathrm{Sp}(p+q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$	$\mathrm{Sp}(p,q)/(\mathrm{Sp}(p)\times\mathrm{Sp}(q))$	$4pq$	$\min(p,q)$

Table 2.2 Classical symmetric spaces of types II and IV. For type II, see Proposition X.1.2 and Sect. IV.6 in [293]

Type II	Type IV	Dimension	Rank
$\mathrm{SU}(n+1)$	$\mathrm{SL}(n+1,\mathbb{C})/\mathrm{SU}(n+1)$	$n(n+2)$	n
$\mathrm{Spin}(2n+1)$	$\mathrm{SO}(2n+1,\mathbb{C})/\mathrm{SO}(2n+1)$	$n(2n+1)$	n
$\mathrm{Sp}(n)$	$\mathrm{Sp}(n,\mathbb{C})/\mathrm{Sp}(n)$	$n(2n+1)$	n
$\mathrm{Spin}(2n)$	$\mathrm{SO}(2n,\mathbb{C})/\mathrm{SO}(2n)$	$n(2n-1)$	n

then a symmetric space of this type is isometric to some Euclidean space $\mathbb{R}^n$. Clearly, $\mathbb{R}^n$ itself provides the simplest example, with the symmetry at the origin given by $s : \mathbf{x} \to -\mathbf{x}$. ♦

Next, we show that Riemannian symmetric spaces provide Riemannian manifolds of certain types met before. Recall that if $(\mathfrak{g}, \lambda)$ is irreducible, then $\mathfrak{g}$ is necessarily semisimple and thus, the Killing form k is non-degenerate. As already noted, this implies

$$\eta(\mathbf{x}, \mathbf{z}) = -c\, \mathsf{k}^{\mathfrak{m}}(\mathbf{x}, \mathbf{z})\,, \quad \mathbf{x}, \mathbf{z} \in \mathfrak{m}\,, \tag{2.5.18}$$

for some $c \in \mathbb{R}$, $c \neq 0$, cf. (2.5.7). Recall from point 2 of Proposition 2.5.10 that the curvature mapping $\mathscr{R}$ is given by the mapping F, cf. formula (2.5.10). Substituting $\mathbf{x} = F(\mathbf{u}, \mathbf{v})\mathbf{w}$ into (2.5.18) and using the $\mathrm{ad}(\mathfrak{h})$-invariance of k, we obtain

$$\eta(F(\mathbf{u}, \mathbf{v})\mathbf{w}, \mathbf{z}) = c\, \mathsf{k}^{\mathfrak{m}}([[\mathbf{u}, \mathbf{v}], \mathbf{w}], \mathbf{z}) = c\, \mathsf{k}^{\mathfrak{h}}([\mathbf{u}, \mathbf{v}], [\mathbf{w}, \mathbf{z}])\,. \tag{2.5.19}$$

Setting $\mathbf{x} = \mathbf{u} = \mathbf{z}$ and $\mathbf{y} = \mathbf{v} = \mathbf{w}$ in (2.5.19), we immediately obtain the following formula for the sectional curvature:

$$\eta(F(\mathbf{x}, \mathbf{y})\mathbf{y}, \mathbf{x}) = -c\, \mathsf{k}^{\mathfrak{h}}([\mathbf{x}, \mathbf{y}], [\mathbf{x}, \mathbf{y}])\,. \tag{2.5.20}$$

This yields useful formulae for the Ricci tensor and for the scalar curvature. For any orthonormal basis $\{\mathbf{e}_i\}$ of $\mathfrak{m}$,

$$\mathsf{Ric}(\mathbf{e}_i, \mathbf{e}_j) = -\sum_k \eta([[\mathbf{e}_k, \mathbf{e}_i], \mathbf{e}_j], \mathbf{e}_k)\,, \quad \mathsf{Sc} = -\sum_{k,l} \eta([[\mathbf{e}_k, \mathbf{e}_l], \mathbf{e}_l], \mathbf{e}_k)\,. \tag{2.5.21}$$

Proposition 2.5.26 *Let (M, g) be an irreducible Riemannian globally symmetric space and let $(\mathfrak{g}, \lambda)$ be the corresponding irreducible orthogonal symmetric Lie algebra.*

1. *If $(\mathfrak{g}, \lambda)$ is of compact type, then (M, g) is a compact Einstein manifold with non-negative sectional curvature and positive definite Ricci tensor.*
2. *If $(\mathfrak{g}, \lambda)$ is of non-compact type, then (M, g) is a simply connected Einstein manifold with non-positive sectional curvature and negative definite Ricci tensor. Moreover, M is diffeomorphic to a Euclidean space.*

Proof Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the canonical decomposition. By Theorem 2.5.24, $G = I_0(M)$ acts transitively and effectively on M and g is G-invariant. Since $k^{\mathfrak{h}}$ is negative definite, the statements about the sectional curvature K follow immediately from (2.5.20). Since the Ricci tensor Ric is a symmetric $\mathrm{ad}(\mathfrak{h})$-invariant bilinear form on $\mathfrak{m}$ and since $\mathrm{ad}(\mathfrak{h})$ acts irreducibly on $\mathfrak{m}$, Ric must be proportional to the metric, that is, (M, g) is an Einstein space.

1. Let $(\mathfrak{g}, \lambda)$ be of compact type. Then, K is non-negative and, thus, Ric is semi-positive definite. Since M is Einstein, Ric is either positive definite or zero. But if Ric is zero, then (2.3.27) implies that K must also be zero, which contradicts the non-degeneracy of k and, thus, the irreducibility of $(\mathfrak{g}, \lambda)$. Finally, since $\mathsf{k}^{\mathfrak{m}}$ is negative definite, k is negative definite and, since $\mathfrak{g}$ is semisimple, G is compact. Thus, M is compact.

2. Let $(\mathfrak{g}, \lambda)$ be of non-compact type. Then, by similar arguments, M is Einstein with negative definite Ricci tensor. The remaining statement follows from Theorem 8.3 in Chap. VIII of [381]. ∎

In the remainder of this section, we present the symmetric space structure of a few of the types in Table 2.1 explicitly. By Theorem 2.5.24, it is enough to exhibit the corresponding symmetric Lie algebra structure. For a much more detailed discussion of examples we refer to Chap. XI of [381] and to [692]. We leave it to the reader to check the statements below (Exercise 2.5.6).

Example 2.5.27

1. Consider type I in lines 3, 4, and 7 of Table 2.1. Lines 3 and 7 correspond to the Graßmann manifolds

$$G_{\mathbb{K}}(k, n) \cong \mathrm{U}_{\mathbb{K}}(n)/(\mathrm{U}_{\mathbb{K}}(n-k) \times \mathrm{U}_{\mathbb{K}}(k))\,, \quad \mathbb{K} = \mathbb{C}, \mathbb{H}\,,$$

and line 4 corresponds to the Graßmann manifold of oriented subspaces of $\mathbb{R}^{p+q}$.[39] The corresponding symmetric Lie algebra is given by

$$\mathfrak{u}_{\mathbb{K}}(p+q) = (\mathfrak{u}_{\mathbb{K}}(p) \oplus \mathfrak{u}_{\mathbb{K}}(q)) \oplus \mathfrak{m}\,,$$

where

$$\mathfrak{u}_{\mathbb{K}}(p) \oplus \mathfrak{u}_{\mathbb{K}}(q) = \left\{ \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathfrak{u}_{\mathbb{K}}(p+q) : A \in \mathfrak{u}_{\mathbb{K}}(p)\,,\ B \in \mathfrak{u}_{\mathbb{K}}(q) \right\}\,,$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & -X^\dagger \\ X & 0 \end{bmatrix} \in \mathfrak{u}_{\mathbb{K}}(p+q) : X \in L(\mathbb{K}^p, \mathbb{K}^q) \right\}\,.$$

The action of $\mathrm{Ad}(H)$ on $\mathfrak{m}$ is given by

$$X \mapsto hXk^{-1}\,, \quad h \in \mathrm{U}_{\mathbb{K}}(q)\,,\ k \in \mathrm{U}_{\mathbb{K}}(p)\,,$$

and the involutive automorphism λ acts via

$$\begin{bmatrix} A & -X^\dagger \\ X & B \end{bmatrix} \mapsto \begin{bmatrix} A & X^\dagger \\ -X & B \end{bmatrix}\,.$$

The corresponding involutive automorphism σ is given by conjugation with

$$\mathbb{1}_{p,q} = \begin{bmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{bmatrix}\,. \tag{2.5.22}$$

2. Consider the special case $p = n$ and $q = 1$ for type I in line 4 of Table 2.1:

$$\mathrm{S}^n = S_{\mathbb{R}}(1, n+1) = \mathrm{SO}(n+1)/\mathrm{SO}(n)\,.$$

The underlying symmetric Lie algebra is given by

$$\mathfrak{o}(n+1) = \mathfrak{o}(n) \oplus \mathfrak{m}\,, \tag{2.5.23}$$

where

$$\mathfrak{o}(n) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \in \mathfrak{o}(n+1) : A \in \mathfrak{o}(n) \right\}\,,$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & -\mathbf{x}^{\mathrm{T}} \\ \mathbf{x} & 0 \end{bmatrix} \in \mathfrak{o}(n+1) : \mathbf{x} \in \mathbb{R}^n \right\}\,.$$

Then, $\mathrm{Ad}(\mathrm{SO}(n))$ gets identified with the basic representation of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ and, under the identification $\mathfrak{m} \cong \mathbb{R}^n$, the Euclidean scalar product on $\mathbb{R}^n$ yields

[39] Cf. Example I/7.5.6.

a scalar product on $\mathfrak{m}$ which coincides with the restriction of the Killing form on $\mathfrak{o}(n+1)$ to $\mathfrak{m}$ up to the factor $-2(n-1)$. The involutive automorphisms are read off from the previous point.

3. Consider type I in line 5 of Table 2.1. One easily shows that $\mathrm{SO}(2n)/\mathrm{U}(n)$ is the space of orthogonal complex structures on the $2n$-dimensional Euclidean space.[40] Here we decompose[41]
$$\mathfrak{o}(2n) = \mathfrak{u}(n) \oplus \mathfrak{m}\,,$$
with
$$\mathfrak{u}(n) = \left\{ \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \in \mathfrak{o}(2n) : X, Y \in \mathfrak{gl}(n, \mathbb{R})\,,\ X = -X^{\mathrm{T}}\,,\ Y = Y^{\mathrm{T}} \right\},$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} X & Y \\ Y & -X \end{bmatrix} \in \mathfrak{o}(2n) : X, Y \in \mathfrak{gl}(n, \mathbb{R})\,,\ X = -X^{\mathrm{T}}\,,\ Y = -Y^{\mathrm{T}} \right\}.$$
The involutive automorphism $\lambda : \mathfrak{o}(2n) \to \mathfrak{o}(2n)$ corresponding to this decomposition is given by conjugation with the matrix
$$\mathsf{J}_0 = \begin{bmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}.$$

4. Consider type I in line 1 of Table 2.1. Recall from Sect. 7.6 of Part I that $\mathrm{U}(n)/\mathrm{O}(n)$ is the space of Lagrangian subspaces of $\mathbb{R}^{2n}$ endowed with its canonical symplectic structure. Correspondingly, $\mathrm{SU}(n)/\mathrm{SO}(n)$ is called the space of special Lagrangian subspaces. Here, we decompose
$$\mathfrak{su}(n) = \mathfrak{o}(n) \oplus \mathfrak{m}\,,$$
with
$$\mathfrak{o}(n) = \left\{ \begin{bmatrix} X & 0 \\ 0 & X \end{bmatrix} \in \mathfrak{su}(n) : X \in \mathfrak{gl}(n, \mathbb{R})\,,\ X = -X^{\mathrm{T}}\,,\ \operatorname{tr} X = 0 \right\},$$
$$\mathfrak{m} = \left\{ \begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix} \in \mathfrak{su}(n) : Y \in \mathfrak{gl}(n, \mathbb{R})\,,\ Y = Y^{\mathrm{T}} \right\}.$$
Here, we have used the embedding $\mathfrak{u}(n) \subset \mathfrak{o}(2n)$ from the previous point. Under this embedding, the involutive automorphism $\lambda : \mathfrak{su}(n) \to \mathfrak{su}(n)$ is given by
$$\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \mapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}.$$

[40] Cf. Example I/7.5.5.

[41] Cf. Example 2.2.19.

5. Consider type III in line 4 of Table 2.1 with $p = 1$, that is, $M = \mathrm{SO}_0(1, n)/\mathrm{SO}(n)$. On the level of Lie algebras, we have to consider the pseudo-Euclidean space $(\mathbb{R}^{1,n}, \eta)$ with $\eta = \mathbb{1}_{1,n}$ given by (2.5.22). Then,

$$\mathfrak{o}(n, 1) = \left\{X \in \mathfrak{gl}(n + 1, \mathbb{R}) : X^{\mathrm{T}}\mathbb{1}_{1,n} + \mathbb{1}_{1,n}X = 0\right\} .$$

Embedding $\mathfrak{o}(n) \subset \mathfrak{o}(1, n)$ via $Y \mapsto \begin{bmatrix} 1 & 0 \\ 0 & Y \end{bmatrix}$, we obtain the canonical decomposition

$$\mathfrak{o}(1, n) = \mathfrak{o}(n) \oplus \mathfrak{m} , \quad \mathfrak{m} = \left\{ \begin{bmatrix} 0 & \mathbf{u}^{\mathrm{T}} \\ \mathbf{u} & 0 \end{bmatrix} \in \mathfrak{o}(1, n) : \ \mathbf{u} \in \mathbb{R}^n \right\} .$$

It is obvious that M may be identified with the hypersurface $H_+(1, n) \subset \mathbb{R}^{1,n}$ defined by

$$\eta(\mathbf{u}, \mathbf{u}) = -1 , \quad u^0 \geq 1 .$$

Therefore, M is referred to as the hyperbolic space form of $(\mathbb{R}^{1,n}, \eta)$. ♦

Remark 2.5.28 Consider the example of the n-sphere above. By Example 1.1.18, under the identification $\mathfrak{m} \cong \mathbb{R}^n$, the bundle of orthonormal frames $O(\mathrm{S}^n)$ coincides with the principal $\mathrm{SO}(n)$-bundle $\mathrm{SO}(n + 1) \to \mathrm{SO}(n + 1)/\mathrm{SO}(n)$ and, by Proposition 2.5.10, the Levi-Civita connection on S^n with respect to the natural metric coincides with the $\mathrm{SO}(n + 1)$-invariant canonical connection on this bundle. The curvature (2.5.10) reads $F(\mathbf{x}, \mathbf{y}) = \mathbf{x} \wedge \mathbf{y}$. Comparing with (2.4.2), this shows that S^n has a constant sectional curvature equal to 1. ♦

For applications of the theory of symmetric spaces in this book, see Sects. 6.8 and 7.9.

Exercises

2.5.1 Prove that λ defined by (2.5.1) is an involutive Lie algebra homomorphism.

2.5.2 Prove that the decomposition (2.5.3) is orthogonal with respect to the Killing form.

2.5.3 Prove Proposition 2.5.3.

2.5.4 Prove Lemma 2.5.12.

2.5.5 Prove the following. For an involutive isometry s with isolated fixed point m, one has $s'_m = -\,\mathrm{id}$. *Hint.* Use the eigenspace decomposition of s'_m.

2.5.6 Check the statements in Example 2.5.27.

2.6 Compatible Connections on Vector Bundles

Here, we take up the discussion of Sect. 2.2. We consider real or complex vector bundles endowed with a fibre metric h and an h-compatible connection ∇. Such a structure will be denoted by (E, h, ∇). In the first part, we will collect what we know already for the case of real (pseudo-)Riemannian base manifolds (M, g), and in the second part we will pass to complex base manifolds and Hermitean vector bundles endowed additionally with a holomorphic structure.

First, recall Examples 2.2.19 and 2.2.27.

(a) $\mathrm{O}(k,l)$-structures are in one-to-one correspondence with pseudo-Riemannian manifolds (M, g) of dimension $(k+l)$, where the $\mathrm{O}(k,l)$-structure coincides with the bundle $O(M)$ of frames which are orthonormal with respect to g. A linear connection ω on M is compatible with the $\mathrm{O}(k,l)$-structure iff g is parallel with respect to ω. Such a connection is called metric.

(b) $\mathrm{U}(n)$-structures are in one-to-one correspondence with $2n$-dimensional almost Hermitean manifolds $(M, \mathsf{g}, \mathsf{J})$ or, equivalently, with Hermitean fibre metrics on $\mathrm{T}M$ relative to a given J. A linear connection ω on M is compatible with the $\mathrm{U}(n)$-structure iff both g and J are parallel with respect to ω. Such a connection is called unitary. Equivalently, ω is unitary iff the Hermitean fibre metric h in $\mathrm{T}M$ defined by g and J is parallel with respect to ω.

More generally, as we know from Examples 1.6.6 and 1.6.12, a connection ∇ on a real or complex vector bundle (E, h) is compatible with h iff

$$\nabla \mathsf{h} = 0\,, \tag{2.6.1}$$

which is equivalent to

$$X(\mathsf{h}(s_1, s_2)) = \mathsf{h}(\nabla_X s_1, s_2) + \mathsf{h}(s_1, \nabla_X s_2)\,, \tag{2.6.2}$$

for any $X \in \mathfrak{X}(M)$ and $s_1, s_2 \in \Gamma^\infty(E)$. Since h may be viewed as a section of the associated bundle $L(E) \times_{\mathrm{GL}(n,\mathbb{K})} \mathscr{F}$, where $\mathscr{F}$ denotes the space of fibre metrics, (2.6.1) is equivalent to

$$D_\omega \tilde{\mathsf{h}} = 0\,, \tag{2.6.3}$$

where ω is the connection form on $L(E)$ and $\tilde{\mathsf{h}} : L(E) \to \mathscr{F}$ is the G-homomorphism corresponding to ∇ and h, respectively. The metric h defines a reduction to the subbundle of orthonormal frames

$$O(E) = \left\{ u \in L(E) : \tilde{\mathsf{h}}(u) = \mathsf{h}_0 \right\}\,,$$

where $\mathsf{h}_0 = \mathbb{1}_{p,q}$ in the real and $\mathsf{h}_0 = \mathbb{1}$ in the complex case. By compatibility, ω is reducible to $O(E)$. In the (pseudo-)Riemannian case, the restriction of equation (2.6.3) to $O(E)$ reads

$$\left(\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \omega^{\mathrm{T}}\right)(\mathsf{h}_0) = 0$$

and in the Hermitean case, we obtain

$$\left(\omega^{\mathrm{T}} \otimes \mathbb{1} + \mathbb{1} \otimes \overline{\omega^{\mathrm{T}}}\right)(\mathsf{h}_0) = 0\,.$$

Thus, ∇ is h-compatible iff ω is metric or unitary for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, respectively.

Remark 2.6.1

1. By Proposition 1.3.7, $O(E)$ admits a connection. Thus, every (pseudo-) Riemannian or Hermitean vector bundle admits a compatible connection.
2. Using the isomorphisms given by (1.2.4) and by Proposition 1.6.7, we have

$$E \cong L(E) \times_{\mathrm{GL}(k,\mathbb{K})} \mathbb{K}^k \cong O(E) \times_G \mathbb{K}^k\,,$$

where $G = \mathrm{O}(p, q)$ in the (pseudo-)Riemannian and $G = \mathrm{U}(k)$ in the Hermitean case. Since $\tilde{\mathsf{h}}$ is constant on $O(E)$, without loss of generality, we can limit our attention to the following setting. Let $P(M, G)$ be a principal G-bundle over an oriented (pseudo-)Riemannian manifold (M, g) and let $E = P \times_G F$ be an associated vector bundle such that (F, G, σ) is a finite-dimensional representation space carrying a σ-invariant inner product $\langle\cdot,\cdot\rangle_F$. Then, $\langle\cdot,\cdot\rangle_F$ induces a fibre metric on E via

$$\mathsf{h}(e_1, e_2) := \langle f_1, f_2\rangle_F\,, \tag{2.6.4}$$

with $e_1 = [(p, f_1)]$ and $e_2 = [(p, f_2)]$. By G-invariance of $\langle\cdot,\cdot\rangle_F$, this definition does not depend on the choice of representatives. ♦

For the remainder, let us assume that M is a complex manifold. Recall that a complex manifold of dimension n is a real manifold of dimension $2n$ endowed with an equivalence class of holomorphic atlases.

Definition 2.6.2 A complex vector bundle E over a complex manifold M is called holomorphic if E admits a system of local trivializations whose transition functions are holomorphic.

Note that such a system of trivializations turns E into a complex manifold such that the projection $\pi : E \to M$ is holomorphic. Also note that, since the composition of anti-holomorphic mappings need not be anti-holomorphic, there is no notion of an anti-holomorphic vector bundle.

Remark 2.6.3

1. For a complex manifold of complex dimension n, one can define the principal $\mathrm{GL}(n, \mathbb{C})$-bundle $C(M)$ of complex linear frames in the same way as in the real case, cf. Example 2.2.10. Correspondingly, any holomorphic vector bundle E of rank k over M may be viewed as associated with its complex linear frame bundle $C(E)$, that is, $E \cong C(E) \times_{\mathrm{GL}(k,\mathbb{C})} \mathbb{C}^k$.

2. As in the C^∞-case, any functorial construction in linear algebra gives rise to holomorphic vector bundles. In particular, one can build the dual bundle, direct sums and tensor products, see [336] for details. ♦

The basic example of a holomorphic vector bundle is provided by the holomorphic tangent bundle of a complex manifold M. Let $(U_i, \varphi_i)_{i\in I}$ be a holomorphic atlas of M with transition mappings φ_{ij} and let z^i be the complex coordinates corresponding to φ_i. Consider the Jacobian

$$\mathscr{J}(\varphi_{ij})(\varphi_j(z)) := \frac{\partial \varphi_{ij}^k}{\partial z^l}(\varphi_j(z))$$

of the transition mappings.

Definition 2.6.4 (*Holomorphic tangent bundle*) The holomorphic tangent bundle of a complex manifold M of dimension n is the holomorphic vector bundle $\mathscr{T}M$ over M of rank n given by the transition functions $\psi_{ij}(z) = \mathscr{J}(\varphi_{ij})(\varphi_j(z))$.

The dual $\mathscr{T}^*M$ of $\mathscr{T}M$ is called the holomorphic cotangent bundle. Clearly, $\{\frac{\partial}{\partial z^k}\}$ and $\{\mathrm{d}z^k\}$ provide local frames in $\mathscr{T}M$ and $\mathscr{T}^*M$, respectively.

Let J be the natural almost complex structure of the complex manifold M, cf. Proposition 2.2.11. Consider the decomposition (2.2.17) defined by J. It is easy to see that $\mathrm{T}^{1,0}M$ has the same transition functions as $\mathscr{T}M$ (Exercise 2.6.1). This implies the following.

Proposition 2.6.5 *If M is a complex manifold, then $\mathrm{T}^{1,0}M$ is naturally isomorphic to the holomorphic tangent bundle $\mathscr{T}M$.* ■

Note that the induced tensor bundles $\bigotimes^p \mathrm{T}^{1,0}M$ and $\bigwedge^k \mathrm{T}^{1,0}M$ are holomorphic, whereas $\bigwedge^k \mathrm{T}^{0,1}M$ is not holomorphic.

Next, recall the decomposition (2.2.18). For a complex vector bundle E over a complex manifold M, let $\Omega^{p,q}(M,E)$ be the space of E-valued (p,q)-forms on M.

Proposition 2.6.6 *Let $\pi : E \to M$ be a holomorphic vector bundle. Then, there exists a $\mathbb{C}$-linear differential operator $\bar\partial_E : \Omega^{p,q}(M,E) \to \Omega^{p,q+1}(M,E)$ fulfilling $\bar\partial_E^2 = 0$ and the Leibniz rule*

$$\bar\partial_E(f\alpha) = \bar\partial(f) \wedge \alpha + f\bar\partial_E(\alpha)\,,$$

for any function f on M and any $\alpha \in \Omega^{p,q}(M,E)$.

Proof Let $(e_1, \dots, e_k)$ be a local holomorphic frame[42] in E over $U \subset M$. Then, locally, any $\alpha \in \Omega^{p,q}(M,E)$ may be written as $\alpha = \sum_i \alpha_i \otimes e_i$, with $\alpha_i \in \Omega^{p,q}(M)$. We define

[42] That is, every $e_i : U \to E$ is a holomorphic mapping.

$$\overline{\partial}_E \alpha := \sum_i \overline{\partial}(\alpha_i) \otimes e_i \,.$$

This definition is independent of the choice of frame. Indeed, let $e'_i = g^j{}_i e_j$ be another holomorphic frame. Then, the $g^j{}_i$ are holomorphic functions on M and

$$\overline{\partial}'_E \alpha = \overline{\partial}'_E \big(\sum_i \alpha'_i \otimes g^j{}_i e_j \big) = \sum_i \overline{\partial} \alpha'_i \otimes g^j{}_i e_j = \sum_i \overline{\partial}(g^j{}_i \alpha'_i) \otimes e_j = \sum_i \overline{\partial}(\alpha_i) \otimes e_j \,.$$

Thus, $\overline{\partial}'_E \alpha = \overline{\partial}_E \alpha$. The remaining statements are now obvious. ■

The mapping $\overline{\partial}_E$ is called the Dolbeault operator. It gives rise to a cohomology theory, see Example 5.7.25 and [336] for much more material.[43] Now, let

$$\nabla : \Gamma^\infty(E) \to \Omega^1(M, E)$$

be a connection on E. Taking the complexification of $\mathrm{T}^* M$, we extend it to an operator

$$\nabla : \Gamma^\infty(E) \to \Omega^1_{\mathbb{C}}(M, E) \,.$$

According to (2.2.18), the latter decomposes as follows:

$$\nabla = \nabla^{1,0} + \nabla^{0,1} \,. \tag{2.6.5}$$

Definition 2.6.7 A connection ∇ on a holomorphic vector bundle E is called compatible with the holomorphic structure if $\nabla^{0,1} = \overline{\partial}_E$ on $\Gamma^\infty(E)$.

Note that for a compatible connection, the following are equivalent: for any local section φ of E, $\nabla^{0,1}\varphi = 0$ iff φ is holomorphic.

Proposition 2.6.8 *Let (E, h) be a holomorphic Hermitean vector bundle over the complex manifold M. Then, there exists a unique connection ∇ on E which is compatible both with the holomorphic and with the Hermitean structure.*

Proof Let ∇ be a connection fulfilling the compatibility assumptions and let ω be its connection form. Let $\mathfrak{e} = (e_1, \dots, e_k)$ be a local holomorphic frame, let $\mathscr{A} = \mathfrak{e}^* \omega$ be the local representative of ω and let H be the matrix of h with respect to $\mathfrak{e}$, that is, $H_{ij} = \mathsf{h}(e_i, e_j)$. Taking the pullback of the compatibility condition (2.6.2) under $\mathfrak{e}$, we obtain

$$\mathrm{d}H = \mathscr{A}^{\mathrm{T}} H + H \overline{\mathscr{A}} \,. \tag{2.6.6}$$

To analyze the compatibility of ∇ with the holomorphic structure, we act with ∇ on a local holomorphic section φ. Then,

$$0 = \nabla^{0,1}\varphi = \overline{\partial}\varphi + \mathscr{A}^{0,1}\varphi \,.$$

[43] Note that there is no analogue of the ∂-operator.

Thus, $\mathscr{A}^{0,1} = 0$, that is, $\mathscr{A}$ is of type $(1, 0)$. Now, decomposing both sides of (2.6.6) into their $(1, 0)$ and $(0, 1)$-parts, we read off $\partial H = \mathscr{A}^{\mathrm{T}} H$ and $\overline{\partial} H = H\overline{\mathscr{A}}$ and, thus,

$$\mathscr{A} = \overline{H}^{-1}\partial\overline{H}\,.$$

This formula defines unique compatible connections on each open subset belonging to a system of local trivializations. It is easy to check that, by passing to another local holomorphic frame, these local 1-forms transform properly. Thus, using a partition of unity, they may be glued together to a compatible connection on $C(M)$. ■

Definition 2.6.9 The unique connection given by Proposition 2.6.8 is called the Chern connection, or the canonical connection, of the holomorphic Hermitean vector bundle (E, h).

Corollary 2.6.10 *Let (E, h) be a holomorphic Hermitean vector bundle, let ∇ be its Chern connection and let ω and Ω be the connection and curvature form of ∇, respectively. Let $\mathscr{A} = \mathfrak{e}^*\omega$ and $\mathscr{F} = \mathfrak{e}^*\Omega$ be the local representatives with respect to a local holomorphic frame $\mathfrak{e}$ and let H be the matrix of h with respect to $\mathfrak{e}$. Then,*

$$\mathscr{A} = \overline{H}^{-1}\partial\overline{H}\,,\quad \mathscr{F} = \overline{\partial}\mathscr{A}\,, \tag{2.6.7}$$

that is, $\mathscr{A}$ is of type $(1, 0)$ and $\mathscr{F}$ is of type $(1, 1)$.

Proof The first assertion follows from the proof of Proposition 2.6.8. We show the second one: using the explicit expression for $\mathscr{A}$, together with $\partial^2 = 0$ and $\partial H^{-1} = -H^{-1}\cdot\partial H\cdot H^{-1}$, we obtain $\partial\mathscr{A} = -\mathscr{A}\wedge\mathscr{A}$. Then,

$$\mathscr{F} = \mathrm{d}\mathscr{A} + \mathscr{A}\wedge\mathscr{A} = \overline{\partial}\mathscr{A}\,.$$

Since $\mathscr{A}$ is of type $(1, 0)$, $\mathscr{F}$ is of type $(1, 1)$. ■

Example 2.6.11 In particular, we may consider the holomorphic tangent bundle $\mathscr{T}M$ of a complex manifold M endowed with its Chern connection. According to (2.2.13), $\mathrm{T}M$ viewed as a complex vector bundle is $\mathbb{C}$-linearly isomorphic to $\mathrm{T}^{1,0}M$. On the other hand, by Proposition 2.6.5, $\mathrm{T}^{1,0}M$ is naturally isomorphic to $\mathscr{T}M$. Thus, we have a vector bundle isomorphism $\Phi : \mathrm{T}M \to \mathscr{T}M$ which can be used to transport the Chern connection to $\mathrm{T}M$. The image can be compared with the Levi-Civita connection, see the Appendix to Chap. 4 in [336] for details. In particular, if (M, g) is Kähler, then under Φ, the Chern connection and the Levi-Civita connection coincide. ♦

The following theorem states a converse of Proposition 2.6.8. Our proof is along the lines of [384], cf. Proposition 1.3.7 there.

Theorem 2.6.12 *Let (E, h) be a Hermitean vector bundle over a complex manifold M and let ∇ be a Hermitean connection on E such that its curvature Ω is of type $(1, 1)$, that is, $\Omega \in \Omega^{1,1}(M, \mathrm{End}(E))$. Then, there exists a holomorphic structure on E such that ∇ is the canonical connection with respect to this structure.*

Proof Let $C(E)$ be the principal $\mathrm{GL}(k,\mathbb{C})$-bundle of complex linear frames associated with E, that is, $E \cong C(E) \times_{\mathrm{GL}(k,\mathbb{C})} \mathbb{C}^k$. Clearly, we may view $\mathrm{GL}(k,\mathbb{C})$ as a complex manifold. Let J_M and J_G be the almost complex structures on M and $\mathrm{GL}(k,\mathbb{C})$, respectively, defined by the complex manifold structures. Let ω be the connection form on $C(E)$ corresponding to ∇ and let $\Gamma \subset \mathrm{T}(C(E))$ be its horizontal distribution. Then, we have a unique almost complex structure on $C(E)$ defined by ω, J_M and J_G as follows: Take the splitting $\mathrm{T}(C(E)) = V \oplus \Gamma$, lift J_M from $\mathrm{T}M$ to Γ and define J on $\mathrm{T}(C(E))$ as the direct sum of this lift and of J_G. By construction, J is invariant under the right $\mathrm{GL}(k,\mathbb{C})$-action. Thus, J and the natural almost complex structure of $\mathbb{C}^k$ combine to an almost complex structure on E denoted by the same symbol.

We prove that $\Omega \in \Omega^{1,1}(M, \mathrm{End}(E))$ implies that J is integrable. It is enough to give the proof in a local trivialization of E. For a chosen local trivialization $\pi^{-1}(U) \cong U \times \mathbb{C}^k$, let $(z^1, \ldots, z^n)$ be complex local coordinates on $U \subset M$ and let $(w^1, \ldots, w^k)$ be the complex coordinates on $\mathbb{C}^k$ with respect to the standard basis. Let $\mathscr{A}$ be the local representative of ω on U and let $\mathscr{A}^\alpha{}_\beta$ be its components with respect to the standard basis $\{E^\alpha{}_\beta\}$ of the Lie algebra $\mathfrak{gl}(k,\mathbb{C})$. We decompose $\mathscr{A}$ with respect to J_M,

$$\mathscr{A} = \mathscr{A}^{1,0} + \mathscr{A}^{0,1} .$$

Then, $\left\{\frac{\partial}{\partial \overline{z}^k}\right\}$ locally span $\Gamma^\infty\left(\mathrm{T}^{0,1}M\right)$ and, thus, $\Gamma^\infty\left(\mathrm{T}^{0,1}\Gamma\right)$ is locally spanned by the following vector fields[44]:

$$\left\{\frac{\partial}{\partial \overline{z}^k} - (\mathscr{A}^{0,1})^\alpha{}_\beta \left(\frac{\partial}{\partial \overline{z}^k}\right)(E^\beta{}_\alpha)_*\right\}, \quad k = 1, \ldots, n\,, \;\; \alpha, \beta = 1, \ldots k\,,$$

where $(E^\beta{}_\alpha)_*$ is the Killing vector field generated by $E^\beta{}_\alpha$. Now, the horizontal distribution on E corresponding to Γ is given by (1.3.4). Here, since $\mathbb{C}^k$ is the basic $\mathrm{GL}(k,\mathbb{C})$-module,

$$\iota'_{\mathbf{z}}(A_{*u}) = u(A\mathbf{z})\,, \quad \mathbf{z} \in \mathbb{C}^k\,,\; u \in C(E)\,,\; A \in \mathfrak{gl}(k,\mathbb{C})\,.$$

Thus, $\Gamma^\infty\left(\mathrm{T}^{0,1}E\right)$ is locally spanned by

$$\left\{\frac{\partial}{\partial \overline{z}^k} - \left(\mathscr{A}^{0,1}\right)^\alpha{}_\beta \left(\frac{\partial}{\partial \overline{z}^k}\right) w^\beta \frac{\partial}{\partial w^\alpha}, \frac{\partial}{\partial \overline{w}^\alpha}\right\}.$$

Consequently, its annihilator $\Omega^{1,0}(E)$ is locally spanned by $\{\mathrm{d}z^l, \vartheta^\alpha\}$, where

$$\vartheta^\alpha = \mathrm{d}w^\alpha + \left(\mathscr{A}^{0,1}\right)^\alpha{}_\beta w^\beta .$$

[44] Recall that the horizontal component of a vector field X on a principal G-bundle is given by $X - \Psi'_p(\omega(X))$, cf. formula (1.3.7).

Now, using $\Omega \in \Omega^{1,1}(M, \mathrm{End}(E))$, we calculate

$$\begin{aligned} \mathrm{d}\vartheta^\alpha &= w^\beta \mathrm{d}\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \mathrm{d}w^\beta \\ &= w^\beta \mathrm{d}\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \big(\vartheta^\beta - \big(\mathscr{A}^{0,1}\big)^\beta{}_\gamma w^\gamma\big) \\ &= w^\beta \big(\partial\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta + \big(\Omega^{0,2}\big)^\alpha{}_\beta\big) - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \vartheta^\beta \\ &= w^\beta \partial\big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta - \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \wedge \vartheta^\beta , \end{aligned}$$

that is, $\mathrm{d}\vartheta^\alpha \in \Omega^{1,1}(E)$. By Proposition 2.2.14, this is equivalent to the vanishing of the Nijenhuis tensor and, thus, the Newlander–Nirenberg Theorem 2.2.13 implies that J is integrable.

It remains to prove that, with respect to the holomorphic structure defined by J, ∇ coincides with the Chern connection. That is, we have to prove that a local section $\varphi : U \to E$ fulfilling $\nabla^{0,1}\varphi = 0$ is holomorphic. For that purpose, it is enough to show that any φ fulfilling this condition pulls back every $(1,0)$-form on E to a $(1,0)$-form on M.[45] In the above notation, $\nabla^{0,1}\varphi = 0$ reads

$$\bar{\partial}\varphi^\alpha + \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \varphi^\beta = 0 .$$

Using this, we calculate $\varphi^*(\mathrm{d}z^k) = \mathrm{d}z^k$ and

$$\varphi^*(\vartheta^\alpha) = \mathrm{d}\varphi^\alpha + \big(\mathscr{A}^{0,1}\big)^\alpha{}_\beta \varphi^\beta = \partial\varphi^\alpha .$$

■

For a more general integrability theorem containing Theorem 2.6.12 as a special case, we refer to [35].

Exercises

2.6.1 Prove Proposition 2.6.5.

2.7 Hodge Theory. The Weitzenboeck Formula

Let us recall some basic notions from Sects. 4.4 and 4.5 of Part I. Consider an n-dimensional oriented pseudo-Riemannian manifold (M, g) with signature (r, s). The metric g yields a distinguished volume form v_g, cf. Definition I/4.4.4., and a mapping

$$* : \Omega^k(M) \to \Omega^{n-k}(M) , \quad *\alpha := (-1)^s \mathsf{g}^{-1}(\alpha) \lrcorner \mathsf{v}_\mathsf{g} , \tag{2.7.1}$$

called the Hodge star operator, cf. Definition I/4.5.1. We immediately read off

[45] Recall Exercise 2.2.3.

$$* 1 = (-1)^s \mathsf{v_g}\,, \quad * \mathsf{v_g} = 1\,. \tag{2.7.2}$$

We have the following further basic properties: for any $\alpha, \beta \in \Omega^k(M)$,

$$* * \alpha = (-1)^{k(n-k)+s}\, \alpha\,, \tag{2.7.3}$$

$$\mathsf{g}^{-1}(*\alpha, *\beta) = (-1)^s \mathsf{g}^{-1}(\alpha, \beta)\,, \tag{2.7.4}$$

$$\alpha \wedge *\beta = (-1)^s \mathsf{g}^{-1}(\alpha, \beta)\mathsf{v_g}\,, \tag{2.7.5}$$

cf. Proposition I/4.5.3. Let $\{e_i\}$ be an orthonormal local frame on M and let $\{\vartheta^i\}$ be the dual coframe. Then, locally, we have

$$\mathsf{v_g} = (-1)^s \vartheta^{I_n}\,, \tag{2.7.6}$$

$$* \vartheta^I = \eta^{IJ} e_J \lrcorner \vartheta^{I_n} = \operatorname{sign}\begin{pmatrix} I_n \\ J\ J^c \end{pmatrix} \eta^{IJ}\, \vartheta^{J^c}\,. \tag{2.7.7}$$

Using (2.7.7), for any $\alpha \in \Omega^k(M)$, one easily shows the following:

$$(*\alpha)(X_{k+1}, \ldots, X_n)\mathsf{v_g} = \alpha \wedge \mathsf{g}(X_{k+1}) \wedge \ldots \wedge \mathsf{g}(X_n)\,. \tag{2.7.8}$$

This implies

$$X \lrcorner * \alpha = *(\alpha \wedge \mathsf{g}(X))\,, \tag{2.7.9}$$

$$\mathsf{g}^{-1}(\beta) \lrcorner * \alpha = *(\alpha \wedge \beta)\,, \tag{2.7.10}$$

for any $\alpha \in \Omega^*(M)$, $\beta \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$ (Exercise 2.7.1). The metric induces a natural fibre metric on $E = \bigwedge^k \mathrm{T}^* M$ via

$$\langle \alpha, \beta \rangle := (-1)^s \mathsf{g}^{-1}(\alpha, \beta)\,,$$

which gives rise to an L^2-inner product on the space of square-integrable k-forms:

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \langle \alpha, \beta \rangle \mathsf{v_g} = \int_M \alpha \wedge * \beta\,. \tag{2.7.11}$$

Using this inner product, one defines the Hodge dual $\mathrm{d}^* : \Omega^k(M) \to \Omega^{k-1}(M)$ of the exterior derivative by

$$\langle \mathrm{d}^* \alpha, \beta \rangle_{L^2} := \langle \alpha, \mathrm{d}\beta \rangle_{L^2}\,, \tag{2.7.12}$$

for all $\beta \in \Omega^{k-1}(M)$. For $\alpha \in \Omega^k(M)$, one has

$$\mathrm{d}^* \alpha = (-1)^{n(k-1)+s+1}\, * \mathrm{d} * \alpha\,. \tag{2.7.13}$$

Given the exterior derivative and its Hodge dual, we build the Hodge–Laplace operator of (M, g):

$$\Box : \Omega^k(M) \to \Omega^k(M)\,, \quad \Box := \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}\,. \tag{2.7.14}$$

Clearly,

$$\langle \Box\alpha, \alpha\rangle_{L^2} = \langle \mathrm{d}\alpha, \mathrm{d}\alpha\rangle_{L^2} + \langle \mathrm{d}^*\alpha, \mathrm{d}^*\alpha\rangle_{L^2}\,. \tag{2.7.15}$$

Moreover,

$$\mathrm{d}\,\Box = \Box\,\mathrm{d}\,, \quad \mathrm{d}^*\,\Box = \Box\,\mathrm{d}^*\,, \quad *\Box = \Box\,*\,. \tag{2.7.16}$$

Finally, we note that $\Box$ is symmetric:

$$\langle \Box\alpha, \beta\rangle_{L^2} = \langle \alpha, \Box\beta\rangle_{L^2}\,. \tag{2.7.17}$$

The proof of these elementary properties is left to the reader (Exercise 2.7.2).

Remark 2.7.1 (*Hodge decomposition*) In this Remark, we assume that (M, g) is a compact oriented n-dimensional Riemannian manifold.

Since g is Riemannian, the inner product (2.7.11) is positive definite. Then, (2.7.15) implies that $\Box$ is positive definite and that

$$\Box\alpha = 0 \quad \text{iff} \quad \mathrm{d}\alpha = 0 \ \text{ and } \ \mathrm{d}^*\alpha = 0\,. \tag{2.7.18}$$

Since $\Box = (\mathrm{d} + \mathrm{d}^*)^2$, we also have

$$\ker(\Box) = \ker(\mathrm{d} + \mathrm{d}^*)\,. \tag{2.7.19}$$

A k-form α fulfilling $\Box\alpha = 0$ is called harmonic. We conclude that the only harmonic functions on a compact connected oriented Riemannian manifold are the constant functions. This in turn implies that if, additionally, the first de Rham cohomology of M is trivial, then there does not exist any nontrivial harmonic 1-form on M (Exercise 2.7.3). The space of harmonic k-forms is denoted by

$$\mathscr{H}^k(M) := \left\{\alpha \in \Omega^k(M) : \Box\alpha = 0\right\}\,.$$

In Sect. 5.7 we will see that the Hodge–Laplace operator on a compact oriented Riemannian manifold is elliptic. The theory of elliptic operators implies that, for any k, $\mathscr{H}^k(M)$ is finite-dimensional. Moreover, the following orthogonal direct sum decomposition, called Hodge decomposition, holds.[46]

Theorem 2.7.2 (Hodge Decomposition Theorem)

$$\Omega^k(M) = \mathscr{H}^k(M) \oplus \Box(\Omega^k(M))\,. \tag{2.7.20}$$

[46] Clearly, by the elementary properties of $\Box$ proved above, the second summand can be decomposed further, $\Box(\Omega^k(M)) = \mathrm{d}(\Omega^{k-1}(M)) \oplus \mathrm{d}^*(\Omega^{k+1}(M))$.

The proof will be given in a more general context in Chap. 5, see Theorem 5.7.18. The Hodge decomposition has the following immediate consequences:

1. The natural mapping

$$F : \mathscr{H}^k(M) \to H^k_{\mathrm{dR}}(M)\,, \quad \alpha \mapsto [\alpha]\,,$$

is an isomorphism, that is, every de Rham cohomology class contains a unique harmonic form. To prove injectivity of F, take two harmonic k-forms α and β belonging to the same cohomology class. Then, there exists a $(k-1)$-form τ such that $\alpha - \beta = \mathrm{d}\tau$. Then,

$$\| \alpha - \beta \|^2_{L^2} = \langle \alpha - \beta, \mathrm{d}\tau \rangle_{L^2} = \langle \mathrm{d}^*\alpha - \mathrm{d}^*\beta, \tau \rangle_{L^2} = 0\,,$$

and thus $\alpha = \beta$. To prove surjectivity, take an arbitrary class $[\alpha] \in H^k_{\mathrm{dR}}(M)$ and represent it by some closed form $\alpha \in Z^k(M)$. Then, by the Hodge decomposition (2.7.20), there exists an element $\omega \in \mathscr{H}^k(M)$ and a k-form β such that

$$\alpha = \omega + \Box\beta.$$

Since $\mathrm{d}\omega = 0$, we have $0 = \mathrm{d}\alpha = \mathrm{d}\mathrm{d}^*\mathrm{d}\beta$ and thus

$$\langle \mathrm{d}^*\mathrm{d}\beta, \mathrm{d}^*\mathrm{d}\beta \rangle_{L^2} = \langle \mathrm{d}\beta, \mathrm{d}\mathrm{d}^*\mathrm{d}\beta \rangle_{L^2} = 0\,.$$

This implies $\mathrm{d}^*\mathrm{d}\beta = 0$ and thus $\alpha = \omega + \mathrm{d}\mathrm{d}^*\beta$, showing that $[\omega] = [\alpha]$.
2. The natural pairing

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}\,, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta\,,$$

defines an isomorphism (Poincaré duality) of $H^{n-k}_{\mathrm{dR}}(M)$ with the dual space of $H^k_{\mathrm{dR}}(M)$,

$$H^{n-k}_{\mathrm{dR}}(M) \cong \left(H^k_{\mathrm{dR}}(M)\right)^*\,. \tag{2.7.21}$$

To prove this, given a nonzero cohomology class $[\alpha] \in H^k_{\mathrm{dR}}(M)$, we must find a cohomology class $[\beta] \in H^{n-k}_{\mathrm{dR}}(M)$ such that $\int_M \alpha \wedge \beta \neq 0$. For that purpose, we choose a Riemannian metric g on M. By point 1, we may choose a harmonic representative α of $[\alpha]$ which, of course, cannot vanish identically. Then, by the third identity in (2.7.16), $*\alpha$ is also harmonic and thus, by (2.7.18), it is closed. This means that $*\alpha$ represents a cohomology class in $H^{n-k}_{\mathrm{dR}}(M)$. Pairing this element with $[\alpha]$ yields

$$([\alpha], [*\alpha]) \mapsto \int_M \alpha \wedge *\alpha = \| \alpha \|^2 \neq 0\,.$$

Thus, the above pairing defines an isomorphism of $H^{n-k}_{\mathrm{dR}}(M)$ and $\big(H^k_{\mathrm{dR}}(M)\big)^*$, indeed. ♦

Below, we wish to prove the Weitzenboeck Formula which, combined with the theory of harmonic forms, yields deep insight into the relation between curvature and topology. It compares the Hodge–Laplace operator of (M, g) to the Bochner–Laplace operator built from the Levi-Civita connection ∇ of g. The basic object relating these two quantities is the Weitzenboeck curvature operator built from the curvature endomorphism of ∇. In order to accomplish this goal, we need a unified treatment of these objects in terms of the Koszul calculus. Thus, we consider the vector bundle $E = \bigwedge^k \mathrm{T}^*M$ endowed with its natural fibre metric $\langle\cdot,\cdot\rangle$ defined above and with the natural connection induced from the Levi-Civita connection,[47] which we also denote by ∇. Clearly, ∇ is compatible with $\langle\cdot,\cdot\rangle$. Then, we proceed as follows:

(a) We express the Hodge dual operator d^* in terms of ∇. Recall that d has been already calculated in terms of ∇, cf. formula (2.2.49).
(b) We define the Bochner–Laplace operator and calculate it in terms of ∇. Since this can be done without any modifications for an arbitrary Riemannian (or Hermitean) vector bundle endowed with a compatible connection, we present it for this general case. This will also be useful later on.
(c) We define the Weitzenboeck curvature operator and derive the Weitzenboeck Formula.

(a) Let ω be the connection form of ∇. Let $\mathfrak{e} = \{e_i\}$ be a local frame and let $\{\vartheta^i\}$ be its dual coframe. By (2.1.39), the local representative of ω with respect to $\mathfrak{e}$ is given by $\mathfrak{e}^*\omega^i{}_k = \Gamma^i{}_{jk}\vartheta^j$, where $\Gamma^i{}_{jk}$ are the Christoffel symbols with respect to $\mathfrak{e}$.

Lemma 2.7.3 *For any $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^*(M)$,*

$$\nabla_X \mathsf{v}_{\mathsf{g}} = 0\,, \quad \nabla_X * \alpha = *\nabla_X \alpha\,. \tag{2.7.22}$$

Proof As an immediate consequence of (2.7.6), (2.1.47) and (2.2.44), for any orthonormal frame $\{e_i\}$, we have

$$\nabla_{e_i}\mathsf{v}_{\mathsf{g}} = (-1)^{s+1}\sum_j \Gamma^j{}_{ij}\vartheta^1\wedge\ldots\wedge\vartheta^n = 0\,.$$

This proves the first assertion. To prove the second one, we act with ∇_X on equation (2.7.5). Using $\nabla_X\mathsf{v}_{\mathsf{g}} = 0$, $\nabla_X\mathsf{g} = 0$ and once again (2.7.5), we obtain

$$\begin{aligned}\nabla_X\alpha\wedge*\beta + \alpha\wedge\nabla_X*\beta &= (-1)^s\big(\mathsf{g}^{-1}(\nabla_X\alpha,\beta) + \mathsf{g}^{-1}(\alpha,\nabla_X\beta)\big)\mathsf{v}_{\mathsf{g}}\\ &= \nabla_X\alpha\wedge*\beta + \alpha\wedge*\nabla_X\beta\,,\end{aligned}$$

for arbitrary forms α and β. From this we read off the second assertion. ■

[47] Cf. Exercise 2.1.7.

Lemma 2.7.4 *Let (M, g) be a pseudo-Riemannian manifold and let $\alpha \in \Omega^k(M)$. Let $\{e_i\}$ be a local frame and let $\{\vartheta^i\}$ be its dual coframe. Then,*

$$\mathrm{d}^*\alpha = -\mathsf{g}^{-1}(\vartheta^j) \lrcorner \nabla_{e_j}\alpha\,. \tag{2.7.23}$$

Proof Let $\alpha \in \Omega^k(M)$. Using (2.2.47), Lemma 2.7.3 and (2.7.10), we calculate

$$\begin{aligned} *\mathrm{d} * \alpha &= * \left(\vartheta^j \wedge \nabla_{e_j} * \alpha\right) \\ &= (-1)^{n-k} * \left(*(\nabla_{e_j}\alpha) \wedge \vartheta^j\right) \\ &= (-1)^{n-k} \left(\mathsf{g}^{-1}(\vartheta^j) \lrcorner (*^2 \nabla_{e_j}\alpha)\right) \\ &= (-1)^{(n-k)(k+1)+s} \left(\mathsf{g}^{-1}(\vartheta^j) \lrcorner \nabla_{e_j}\alpha\right)\,. \end{aligned}$$

Comparison with (2.7.13) yields the assertion. ■

Remark 2.7.5 Since the operator d^* is intrinsically defined, formula (2.7.23) does not depend on the choice of the frame. Using $\mathsf{g}^{-1}(\vartheta^j) = \mathsf{g}^{jk} e_k$, it reads

$$(\mathrm{d}^*\alpha)(X_2, \dots, X_k) = -\mathsf{g}^{jl} \left(\nabla_{e_j}\alpha\right)(e_l, X_2, \dots, X_k)\,. \tag{2.7.24}$$

For some purposes, it is useful to rewrite this as

$$(\mathrm{d}^*\alpha)(X_2, \dots, X_k) = -\left(\mathrm{tr}^{\mathsf{g}}_{12}(\nabla\alpha)\right)(X_2, \dots, X_k)\,. \tag{2.7.25}$$

Here, $\nabla\alpha \in \Gamma^\infty(\mathrm{T}^*M \otimes \bigwedge^k \mathrm{T}^*M)$ and $\mathrm{tr}^{\mathsf{g}}_{12}$ means contracting the first two tensor indices of $\nabla\alpha$ with g. The quantity $\mathrm{tr}^{\mathsf{g}}_{12}(\nabla\alpha)$ is called the divergence of α and is denoted by $\mathrm{div}^{\mathsf{g}}\alpha$. In this terminology, we have

$$\mathrm{d}^*\alpha = -\mathrm{div}^{\mathsf{g}}\alpha\,. \tag{2.7.26}$$

In particular, for a 1-form $\alpha \in \Omega^1(M)$, we obtain (Exercise 2.7.4)

$$\mathrm{div}^{\mathsf{g}}(\alpha)\,\mathsf{v}_{\mathsf{g}} = \mathrm{d}(\mathsf{g}^{-1}(\alpha) \lrcorner \mathsf{v}_{\mathsf{g}})\,. \tag{2.7.27}$$

♦

(b) Next, instead of $(\bigwedge^k \mathrm{T}^*M, \langle\cdot,\cdot\rangle, \nabla)$, consider any Riemannian or Hermitean vector bundle E with a fibre metric $\langle\cdot,\cdot\rangle$ and a compatible connection ∇ over a pseudo-Riemannian manifold (M, g). As in the above special case, $\langle\cdot,\cdot\rangle$ and g induce a natural L^2-inner product on $\Gamma^\infty(E)$ via

$$\langle s_1, s_2\rangle_{L^2} := \int_M \langle s_1, s_2\rangle \mathsf{v}_{\mathsf{g}}\,. \tag{2.7.28}$$

If we endow T^*M with the natural fibre metric given by g^{-1}, then we may extend $\langle\cdot,\cdot\rangle_{L^2}$ to an inner product on $\Gamma^\infty(\mathrm{T}^*M \otimes E)$ which we denote by the same symbol.

We define the formal adjoint $\nabla^* : \Gamma^\infty(\mathrm{T}^*M \otimes E) \to \Gamma^\infty(E)$ of ∇ by

$$\langle s, \nabla^* \varphi \rangle_{L^2} = \langle \nabla s, \varphi \rangle_{L^2} ,$$

for any $s \in \Gamma^\infty(E)$ and $\varphi \in \Gamma^\infty(\mathrm{T}^*M \otimes E)$.

Proposition 2.7.6 *For any* $\varphi \in \Gamma^\infty(\mathrm{T}^*M \otimes E)$*,*

$$\nabla^* \varphi = -\operatorname{tr}^{\mathsf{g}}_{12}(\nabla \varphi) .$$

Proof Let $s \in \Gamma^\infty(E)$. For a given local frame $\{e_i\}$ and its dual coframe $\{\vartheta^i\}$, decompose

$$\nabla s = \vartheta^i \otimes \nabla_{e_i} s , \quad \varphi = \vartheta^j \otimes \varphi(e_j) ,$$

and calculate

$$\langle \nabla s, \varphi \rangle = \langle \vartheta^i \otimes \nabla_{e_i} s, \vartheta^j \otimes \varphi(e_j) \rangle = \mathsf{g}^{ij} \langle \nabla_{e_i} s, \varphi(e_j) \rangle .$$

Since ∇ is compatible with the fibre metric, (2.6.2) implies

$$e_i(\langle s, \varphi(e_j) \rangle) = \langle \nabla_{e_i} s, \varphi(e_j) \rangle + \langle s, \nabla_{e_i}(\varphi(e_j)) \rangle ,$$

and, thus,

$$\begin{aligned} \langle \nabla s, \varphi \rangle &= \mathsf{g}^{ij} \left(e_i(\langle s, \varphi(e_j) \rangle) - \langle s, \nabla_{e_i}(\varphi(e_j)) \rangle \right) \\ &= \mathsf{g}^{ij} \left(e_i(\langle s, \varphi(e_j) \rangle) - \langle s, \varphi(\nabla_{e_i} e_j) \rangle - \langle s, (\nabla_{e_i} \varphi)(e_j) \rangle \right) . \end{aligned}$$

Defining a 1-form $\beta \in \Omega^1(M)$ by $\beta(X) := \langle s, \varphi(X) \rangle$, where $X \in \mathfrak{X}(M)$, we obtain

$$\mathsf{g}^{ij} \left(e_i(\langle s, \varphi(e_j) \rangle) - \langle s, \varphi(\nabla_{e_i} e_j) \rangle \right) = \mathsf{g}^{ij} (\nabla_{e_i} \beta)(e_j) = \operatorname{div}^{\mathsf{g}} \beta .$$

Then, (2.7.27) implies

$$\langle \nabla s, \varphi \rangle = \mathrm{d}(\mathsf{g}^{-1}(\beta) \lrcorner \mathsf{v}_{\mathsf{g}}) - \mathsf{g}^{ij} \langle s, (\nabla_{e_i} \varphi)(e_j) \rangle .$$

Integrating this identity with v_{g} and using Stokes' Theorem, we find

$$\langle \nabla s, \varphi \rangle_{L^2} = -\langle s, \mathsf{g}^{ij} (\nabla_{e_i} \varphi)(e_j) \rangle_{L^2} = -\langle s, \operatorname{tr}^{\mathsf{g}}_{12}(\nabla \varphi) \rangle_{L^2} .$$

■

Remark 2.7.7 By Proposition 2.7.6, $\nabla^* \varphi = -\mathsf{g}^{ij} (\nabla_{e_i} \varphi)(e_j)$ for any local frame $\{e_i\}$ and, thus,

$$\nabla^* \varphi = \mathsf{g}^{ij} \left(\varphi(\nabla_{e_i} e_j) - \nabla_{e_i}(\varphi(e_j)) \right) . \tag{2.7.29}$$

♦

Definition 2.7.8 (*Bochner–Laplace operator*) The mapping

$$\nabla^* \nabla : \Gamma^\infty(E) \to \Gamma^\infty(E)$$

is called the Bochner–Laplace operator.[48]

By Proposition 2.7.6, we have

$$\nabla^* \nabla s = -\operatorname{tr}^{\mathsf{g}}_{12}(\nabla \nabla s)\,, \quad s \in \Gamma^\infty(E)\,, \tag{2.7.30}$$

and, by (2.7.29),

$$\nabla^* \nabla s = -\mathsf{g}^{ij} \left(\nabla_{e_i} \nabla_{e_j} s - \nabla_{\nabla_{e_i} e_j} s \right)\,. \tag{2.7.31}$$

Moreover, since $\langle \nabla^* \nabla s_1, s_2 \rangle_{L^2} = \langle \nabla s_1, \nabla s_2 \rangle_{L^2} = \langle s_1, \nabla^* \nabla s_2 \rangle_{L^2}$, the Bochner–Laplace operator is formally self-adjoint.

(c) It is convenient to consider $\bigwedge^k \mathrm{T}^* M$ as associated with the reduced bundle of orthonormal frames $O(M)$. Then, σ is induced from the basic representation of the orthogonal group $\mathrm{O}(r, s)$ of the pseudo-Euclidean metric η on $\mathbb{R}^n$. It acts on $\bigwedge^k (\mathbb{R}^n)^*$ via

$$\sigma(a)(\xi_1 \wedge \ldots \wedge \xi_k) = \left(\left(a^{-1}\right)^{\mathrm{T}} \xi_1 \right) \wedge \xi_2 \wedge \ldots \wedge \xi_k + \ldots + \xi_1 \wedge \ldots \wedge \xi_{k-1} \wedge \left(\left(a^{-1}\right)^{\mathrm{T}} \xi_k \right).$$

Identifying $\bigwedge^k (\mathbb{R}^n)^* \cong \bigwedge^k \mathbb{R}^n$ via the metric, we obtain the representation σ' of the Lie algebra $\mathfrak{o}(r, s)$ on $\bigwedge^k (\mathbb{R}^n)^*$:

$$\sigma'(A)(\xi_1 \wedge \ldots \wedge \xi_k) = (A\xi_1) \wedge \xi_2 \wedge \ldots \wedge \xi_k + \ldots + \xi_1 \wedge \ldots \wedge \xi_{k-1} \wedge (A\xi_k)\,, \tag{2.7.32}$$

that is, $A \in \mathfrak{o}(r, s)$ acts as a derivation on $\bigwedge^k (\mathbb{R}^n)^*$. Accordingly, the curvature endomorphism form

$$\mathsf{R}^{\Lambda}_m(X, Y) = \iota_p \circ \sigma'(\Omega_p(X^h, Y^h)) \circ \iota_p^{-1}$$

of ∇ is a 2-form on M with values in $\operatorname{End}(\bigwedge^k \mathrm{T}^* M)$ acting as a derivation. For the convenience of the reader, we recall the following.

Remark 2.7.9 (*Contraction and exterior multiplication*) Let V be a real vector space endowed with a metric $\eta = \langle \cdot, \cdot \rangle$. The contraction mapping $\iota : V^* \to \operatorname{End}(\bigwedge V)$ is defined by $\iota(\xi) 1 = 0$ and

$$\iota(\xi)(v_1 \wedge \ldots \wedge v_k) = \sum_{i=1}^{k} (-1)^{i-1} \langle \xi, v_i \rangle v_1 \wedge \ldots \hat{v}_i \ldots \wedge v_k\,,$$

where $\xi \in V^*$ and $v_1, \ldots, v_k \in V$. We will also write $\iota(\xi) \equiv \xi \lrcorner$. Since

$$\iota(\xi)\iota(\zeta) + \iota(\zeta)\iota(\xi) = 0$$

[48] Some authors call it the rough Laplacian.

for all $\xi, \zeta \in V^*$, by the universal property of the exterior algebra, ι extends to an algebra morphism $\iota : \bigwedge V^* \to \mathrm{End}(\bigwedge V)$. We denote the operation of exterior multiplication with an element $v \in V$ by

$$\varepsilon(v)(\alpha) := v \wedge \alpha$$

and note the following basic identity (Exercise 2.7.6):

$$\varepsilon(v)\iota(\xi) + \iota(\xi)\varepsilon(v) = \langle \xi, v \rangle \cdot 1\,. \tag{2.7.33}$$

Let $\{\mathbf{e}_j\}$ be an orthonormal basis of V, let $\{\vartheta^j\}$ be the dual basis and denote $\varepsilon_j := \varepsilon(\mathbf{e}_j)$ and $\iota^k := \iota(\vartheta^k)$. In this notation, the natural action $\mathrm{End}(V) \to \mathrm{Der}(\bigwedge V)$ of $\mathrm{End}(V)$ by derivations on the exterior algebra,

$$A^\Lambda(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge v_2 \wedge \cdots \wedge v_k + \cdots + v_1 \wedge \cdots \wedge v_{k-1} \wedge Av_k\,,$$

is given by

$$A^\Lambda = \eta^{jl}\eta(\mathbf{e}_l, A\mathbf{e}_k)\varepsilon_j\iota^k\,. \tag{2.7.34}$$

In terms of the matrix elements $A_{ij} = \eta(\mathbf{e}_i, A\mathbf{e}_j)$, we have

$$A^\Lambda = A^j{}_k\varepsilon_j\iota^k\,. \tag{2.7.35}$$

♦

By (2.7.34), the curvature endomorphism $\mathsf{R}^\Lambda_m(X, Y)$ acts as a derivation on $\bigwedge^k \mathrm{T}^*M$ as follows:

$$\mathsf{R}^\Lambda(e_i, e_j) = \eta^{km}\mathsf{g}(\mathsf{R}(e_i, e_j)e_m, e_l)e^l{}_k\,, \tag{2.7.36}$$

where $e^l{}_k := \varepsilon^l\iota_k$ and where $\{e_j\}$ is any local orthonormal frame.

Definition 2.7.10 The Weitzenboeck curvature operator $\mathfrak{R}^\Lambda : \Omega^k(M) \to \Omega^k(M)$ of ∇ is defined by

$$\mathfrak{R}^\Lambda(\alpha)(X_1, \ldots, X_k) := \sum_i \eta^{jl}\left(\mathsf{R}^\Lambda(e_j, X_i)\alpha\right)(X_1, \ldots, \overset{\overset{i}{\curlyvee}}{e_l}, \ldots, X_k)\,, \tag{2.7.37}$$

where $X_1, \ldots, X_k \in \mathfrak{X}(M)$ and $\{e_j\}$ is an arbitrary orthonormal local frame.[49]

Let us calculate $\mathfrak{R}^\Lambda$ in the frame $\{e^k{}_l\}$. Using (2.7.36), together with the symmetry properties of R, we obtain

[49] We have only made the summation over i explicit. The remaining summations are in accordance with the Einstein summation convention.

$$
\begin{aligned}
\mathfrak{R}^{\Lambda}(\alpha)(X_1,\dots,X_k) &= \sum_i \eta^{jl}\big(\mathsf{R}^{\Lambda}(e_j,X_i)\alpha\big)(X_1,\dots,e_l,\dots,X_k)\\
&= \sum_i \eta^{jl}\eta^{kp}\mathsf{g}\big(\mathsf{R}(e_p,e_m)e_j,X_i\big)(e^m{}_k\alpha)(X_1,\dots,e_l,\dots,X_k)\\
&= \sum_i (e^m{}_k\alpha)(X_1,\dots,\eta^{jl}\eta^{kp}\mathsf{g}\big(\mathsf{R}(e_p,e_m)e_j,X_i\big)e_l,\dots,X_k)\\
&= -\sum_i (e^m{}_k\alpha)(X_1,\dots,\eta^{kl}\mathsf{R}(e_l,e_m)X_i,\dots,X_k)\\
&= \big(\eta^{km}\mathsf{R}^{\Lambda}(e_m,e_l)\circ e^l{}_k\big)(\alpha)(X_1,\dots,X_k)\,.
\end{aligned}
$$

In the last step, we have used that R^{Λ} is a derivation which acts trivially on zero-forms. Using (2.7.36) once again, we obtain

$$
\mathfrak{R}^{\Lambda} = \mathsf{R}_{ijkl}\varepsilon^i\iota^j\varepsilon^k\iota^l\,. \tag{2.7.38}
$$

Now we are able to formulate the main result of the second part of this section.

Theorem 2.7.11 (Weitzenboeck Formula) *Let $\alpha \in \Omega^k(M)$. Then,*

$$
\Box\alpha \;=\; \nabla^*\nabla\alpha + \mathfrak{R}^{\Lambda}(\alpha)\,. \tag{2.7.39}
$$

Proof We choose an orthonormal local frame $\{e_i\}$ and the dual coframe $\{\vartheta^i\}$. Using Lemma 2.7.4, (2.2.47), (2.1.46) and and the first equation in (2.2.44), we calculate

$$
\begin{aligned}
\mathrm{d}\mathrm{d}^*\alpha &= -\mathrm{d}\left(\eta^{jl}e_l\lrcorner\nabla_{e_j}\alpha\right)\\
&= -\vartheta^i\wedge\nabla_{e_i}\left(\eta^{jl}e_l\lrcorner\nabla_{e_j}\alpha\right)\\
&= -\vartheta^i\wedge\left(\nabla_{e_i}e_l\lrcorner\nabla_{e_j}\alpha + e_l\lrcorner\nabla_{e_i}\nabla_{e_j}\alpha\right)\eta^{jl}\\
&= e^i{}_l\left(\nabla_{\nabla_{e_i}e_j}\alpha - \nabla_{e_i}\nabla_{e_j}\alpha\right)\eta^{jl}\,.
\end{aligned}
$$

On the other hand, again by Lemma 2.7.4, together with (2.1.47), we obtain

$$
\begin{aligned}
\mathrm{d}^*\mathrm{d}\alpha &= \mathrm{d}^*\left(\vartheta^i\wedge\nabla_{e_i}\alpha\right)\\
&= -\eta^{jl}e_l\lrcorner\left(\nabla_{e_j}\left(\vartheta^i\wedge\nabla_{e_i}\alpha\right)\right)\\
&= -\eta^{jl}e_l\lrcorner\left(\nabla_{e_j}\vartheta^i\wedge\nabla_{e_i}\alpha + \vartheta^i\wedge\nabla_{e_j}\nabla_{e_i}\alpha\right)\\
&= \eta^{jl}e_l\lrcorner\left(\vartheta^i\wedge\nabla_{\nabla_{e_j}e_i}\alpha - \vartheta^i\wedge\nabla_{e_j}\nabla_{e_i}\alpha\right)\\
&= \eta^{ji}\left(\nabla_{\nabla_{e_j}e_i}\alpha - \nabla_{e_j}\nabla_{e_i}\alpha\right) - \eta^{jl}e^i{}_l\left(\nabla_{\nabla_{e_j}e_i}\alpha - \nabla_{e_j}\nabla_{e_i}\alpha\right)\,.
\end{aligned}
$$

Adding up these two equations and using (2.7.31) yields

$$
\Box\alpha = \nabla^*\nabla\alpha - \eta^{jl}e^i{}_l\left(\nabla_{e_i}\nabla_{e_j}\alpha - \nabla_{e_j}\nabla_{e_i}\alpha - \nabla_{(\nabla_{e_i}e_j-\nabla_{e_j}e_i)}\alpha\right)\,.
$$

Since the Levi-Civita connection is torsionless, we have $\nabla_{e_i} e_j - \nabla_{e_j} e_i = [e_i, e_j]$ and, thus, by point 2 of Remark 1.5.12 and Eqs. (2.1.32) and (2.7.36),

$$\Box\alpha = \nabla^*\nabla\alpha - \eta^{jl} e^i{}_l \big(\mathsf{R}^\Lambda(e_i, e_j)\alpha\big) = \nabla^*\nabla\alpha + \mathsf{R}_{ijkl}\varepsilon^i \iota^j \varepsilon^k \iota^l \alpha\,.$$

Comparing with (2.7.38), we obtain the assertion. ■

Clearly, the second term in the Weitzenboeck Formula may be analyzed in more detail for every form degree k. To do so, recall the presentation of the Ricci tensor in a local frame given by (2.3.27),

$$\mathsf{Ric}(X, Y) = -\eta^{ij} \mathsf{g}\left(\mathsf{R}(X, e_i)Y, e_j\right)\,, \quad X, Y \in \mathfrak{X}(M)\,. \tag{2.7.40}$$

Associated with the Ricci tensor, one has the Ricci mapping

$$\mathsf{Ric} : \mathrm{T}M \to \mathrm{T}M\,, \quad \mathsf{Ric}(X) := \eta^{ij} \mathsf{R}(X, e_i)e_j\,. \tag{2.7.41}$$

Being an endomorphism of $\mathrm{T}M$, the Ricci mapping naturally extends to a derivation Ric^Λ of $\bigwedge \mathrm{T}M$. In degree 2, it is common to denote this derivation by $\mathsf{Ric} \wedge \mathrm{id}$. We have

$$(\mathsf{Ric} \wedge \mathrm{id})(X, Y) := \mathsf{Ric}(X) \wedge Y + X \wedge \mathsf{Ric}(Y)\,.$$

Analogously, associated with the curvature endomorphism form, one has the mapping

$$\mathsf{R} : \textstyle\bigwedge^2 \mathrm{T}M \to \bigwedge^2 \mathrm{T}M\,, \quad X \wedge Y \mapsto \eta^{ij} e_i \wedge \mathsf{R}(X, Y)e_j\,. \tag{2.7.42}$$

In applications, the cases $k = 1$ and $k = 2$ are of special importance.

Corollary 2.7.12

1 *For* $k = 1$, *the Weitzenboeck Formula* (2.7.39) *reads*

$$\Box\alpha \;=\; \nabla^*\nabla\alpha + \alpha \circ \mathsf{Ric}\,. \tag{2.7.43}$$

2 *For* $k = 2$, *the Weitzenboeck Formula may be rewritten as follows:*

$$\Box\alpha \;=\; \nabla^*\nabla\alpha + \alpha \circ (\mathsf{R} + \mathsf{Ric} \wedge \mathrm{id})\,, \tag{2.7.44}$$

where R *is the mapping defined by* (2.7.42).

Proof 1. For $k = 1$, by (2.7.37), (2.7.36) and (2.7.41),

$$\begin{aligned}\mathfrak{R}^{\Lambda}(\alpha)(X) &= \eta^{jl}\big(\mathsf{R}^{\Lambda}(e_j, X)\alpha\big)(e_l)\\ &= \eta^{il}\eta^{km}\mathsf{g}\big(\mathsf{R}(X, e_i)e_l, e_m\big)\alpha(e_k)\\ &= \eta^{km}\mathsf{g}\big(\mathsf{Ric}(X), e_m\big)\alpha(e_k)\\ &= \alpha(\mathsf{Ric}(X))\,.\end{aligned}$$

2. By a similar calculation as under point 1, using additionally the algebraic Bianchi identity (2.3.16), together with (2.1.52) and (2.3.25), one gets:

$$\begin{aligned}\mathfrak{R}^{\Lambda}(\alpha)(e_i, e_j) &= -\mathsf{R}_{kj}\alpha^k{}_i + \mathsf{R}_{ki}\alpha^k{}_j + \mathsf{R}_{ijkl}\alpha^{kl}\\ &= \alpha(\mathsf{Ric}(e_i), e_j) - \alpha(\mathsf{Ric}(e_j), e_i) + \eta^{kl}\alpha(e_k, \mathsf{R}(e_i, e_j)e_l)\\ &= \big(\alpha \circ (\mathsf{Ric} \wedge \mathrm{id}) + \alpha \circ \mathsf{R}\big)(e_i, e_j)\,.\end{aligned}$$

■

The proof of the following example is left to the reader (Exercise 2.7.5).

Example 2.7.13 For S^n, endowed with the canonical Riemannian metric, the mapping (2.7.42) is given by $\mathsf{R} = -\,\mathrm{id}$ and the Ricci mapping reads $\mathsf{Ric}(X) = (n-1)X$. Using (2.7.38), one finds

$$\mathfrak{R}^{\Lambda} = k(n-k)\,\mathrm{id} \tag{2.7.45}$$

on k-forms. ♦

Combining the Weitzenboeck Formula with the theory of harmonic forms, one gets insight into the relation between curvature and topology. Let us discuss a simple application of this type. We will write $\mathsf{Ric} \geq 0$ if $\mathsf{Ric}_m(X, X) \geq 0$ for all $m \in M$ and all $X \in \mathrm{T}_m M$, and $\mathsf{Ric}_m > 0$ if $\mathsf{Ric}_m(X, X) > 0$ for all $0 \neq X \in \mathrm{T}_m M$.

Proposition 2.7.14 (Bochner) *Let (M, g) be an n-dimensional compact connected and oriented Riemannian manifold with $\mathsf{Ric} \geq 0$. Then, the following statements hold.*

1. *Every harmonic 1-form α is parallel and fulfils $\mathsf{Ric}(\mathsf{g}^{-1}(\alpha), \mathsf{g}^{-1}(\alpha)) = 0$.*
2. *If, additionally, $\mathsf{Ric}_m > 0$ for some point $m \in M$, then all harmonic 1-forms are trivial.*

Proof 1. By formula (2.7.43), for any $\alpha \in \Omega^1(M)$, we have

$$\langle \Box\alpha, \alpha\rangle_{L^2} = \| \nabla\alpha \|^2_{L^2} + \int_M \mathsf{Ric}(\mathsf{g}^{-1}(\alpha), \mathsf{g}^{-1}(\alpha))\mathsf{v}_{\mathsf{g}}\,.$$

If α is harmonic, then the left hand side vanishes. Since both terms on the right hand side are non-negative, they must vanish, too.

2. Let $\alpha \in \Omega^1(M)$ be harmonic. Then, it is parallel. Since, for any $X \in \mathfrak{X}(M)$,

$$\nabla_X(\| \alpha \|) = X(\| \alpha \|) = 2\langle \nabla_X \alpha, \alpha \rangle \,,$$

α has locally constant length. Thus, since M is connected, $\alpha_m = 0$ implies $\alpha = 0$ everywhere and, therefore, the evaluation mapping $\alpha \mapsto \alpha_m$ is injective. Also by point 1, $\mathsf{Ric}(\mathsf{g}^{-1}(\alpha), \mathsf{g}^{-1}(\alpha)) = 0$. Since $\mathsf{Ric}_m > 0$ for some point $m \in M$, we conclude $\alpha_m = 0$ and, by the injectivity of the evaluation mapping, $\alpha = 0$. ■

From the above proof it is clear that the vector space of harmonic 1-forms has at most dimension n. Combining this with point 1 of Remark 2.7.1 we get the following.

Corollary 2.7.15 *Under the assumptions of Proposition 2.7.14 on* (M, g)*, we have*

1. *If* $\mathsf{Ric} \geq 0$*, then* $b_1(M) = \dim H^1_{\mathrm{dR}}(M) \leq n$*.*
2. *If, additionally,* $\mathsf{Ric}_m > 0$ *for some point* $m \in M$*, then* $b_1(M) = 0$*.* ■

Example 2.7.16

1. Since for the torus $b_1(\mathrm{T}^n) = n \neq 0$, we conclude that this manifold does not admit a Riemanian metric with positive Ricci curvature.
2. Using (2.7.45), for S^n endowed with the canonical Riemannian metric, we get $\mathfrak{R}^{\Lambda}(\alpha) = k(n-k)\alpha$, and thus the Weitzenboeck Formula implies $\Box > 0$ for $0 < k < n$. Consequently, there are no nontrivial harmonic forms for $0 < k < n$ and the Betti numbers of M vanish for all $k \neq 0, n$. ♦

In the remainder of this section, we show that the Weitzenboeck Formula generalizes in a straightforward way to the case of differential forms on M with values in a Riemannian (or Hermitean) vector bundle E endowed with a fibre metric $\langle \cdot, \cdot \rangle$ and a compatible connection ∇. In this form, it will play a crucial role both for the study of the instanton moduli space and for the investigation of stability of solutions to the Yang-Mills equations.

Recall from point 2 of Remark 2.6.1 that, without loss of generality, we may limit our attention to associated bundles $E = P \times_G F$ with fibre metrics $\langle \cdot, \cdot \rangle$ induced from G-invariant inner products $\langle \cdot, \cdot \rangle_F$ on F. First, note that the fibre metric $\langle \cdot, \cdot \rangle$ induces a pairing $\Omega^k(M, E) \times \Omega^l(M, E) \to \Omega^{k+l}(M)$ as follows. Let $\alpha \in \Omega^k(M, E)$ and $\beta \in \Omega^l(M, E)$. For any $m \in M$, we choose a local frame $s_i : U \to E, i = 1, \ldots, \dim F$, on an open neighbourhood $U \subset M$ of m, decompose $\alpha = \alpha^i \otimes s_i$ and $\beta = \beta^j \otimes s_j$, and define

$$(\alpha \dot{\wedge} \beta)_m := \alpha^i_m \wedge \beta^j_m \, \langle s_i(m), s_j(m) \rangle \,. \tag{2.7.46}$$

Clearly, this definition does not depend on the choice of the local frame.

In particular, using the metric g on M and extending the Hodge-star on M to $\Omega^k(M, E)$ by putting $*\alpha := (*\alpha^i) \otimes s_i$, we obtain a pairing

$$\Omega^k(M, E) \times \Omega^k(M, E) \to \Omega^n(M) \,, \quad (\alpha, \beta) \mapsto \alpha \dot{\wedge} * \beta \,. \tag{2.7.47}$$

The latter can be used to define an L^2-inner product[50] on $\Omega^k(M, E)$,

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \alpha \mathbin{\dot\wedge} * \beta \,. \tag{2.7.48}$$

Decomposing $\alpha = \alpha_I \vartheta^I$ and $\beta = \beta_J \vartheta^J$ with respect to a local orthonormal coframe $\{\vartheta^I\}$ in the bundle of k-forms on M, we have

$$\alpha \mathbin{\dot\wedge} * \beta = \langle \alpha_I, \beta_J \rangle \vartheta^I \wedge * \vartheta^J = \eta^{IJ} \langle \alpha_I, \beta_J \rangle \, \mathsf{v}_{\mathsf{g}} \,. \tag{2.7.49}$$

This shows that to the above pairing, there corresponds a natural inner product on $\Omega^k(M, E)$ given by the tensor product of the fibre metric $\langle \cdot, \cdot \rangle$ with the fibre metric η^{IJ} in $\Omega^k(M)$. If $\langle \cdot, \cdot \rangle$ is positive definite and g is Riemannian, then this inner product is positive definite.

Remark 2.7.17 Let $\tilde\alpha \in \Omega^k_{\sigma,\mathrm{hor}}(P, F)$ and $\tilde\beta \in \Omega^l_{\sigma,\mathrm{hor}}(P, F)$ be the horizontal forms corresponding to $\alpha \in \Omega^k(M, E)$ and $\beta \in \Omega^l(M, E)$ according to Proposition 1.2.12. Then, one easily shows (Exercise 2.7.7)

$$\tilde\alpha \mathbin{\dot\wedge} \tilde\beta = \pi^*(\alpha \mathbin{\dot\wedge} \beta) \,. \tag{2.7.50}$$

♦

Next, recall the covariant exterior derivative $\mathrm{d}_\omega \alpha : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ associated with the connection form ω of ∇, cf. Definition 1.5.1. We define its dual $\mathrm{d}^*_\omega \alpha : \Omega^{k+1}(M, E) \to \Omega^k(M, E)$ in the sense of Hodge by

$$\langle \alpha, \mathrm{d}^*_\omega \beta \rangle_{L^2} = \langle \mathrm{d}_\omega \alpha, \beta \rangle_{L^2} \,, \tag{2.7.51}$$

for $\alpha \in \Omega^k(M, E)$ and $\beta \in \Omega^{k+1}(M, E)$. The operator d^*_ω will be called the covariant exterior coderivative. Note that, given this operator, we have a natural generalization of the Hodge-Laplacian, cf. (2.7.14),

$$\Box_\omega := \mathrm{d}_\omega \circ \mathrm{d}^*_\omega + \mathrm{d}^*_\omega \circ \mathrm{d}_\omega : \;\; \Omega^k(M, E) \to \Omega^k(M, E) \,. \tag{2.7.52}$$

Proposition 2.7.18 *For $\alpha \in \Omega^k(M, E)$,*

$$\mathrm{d}^*_\omega \alpha = (-1)^{n(k-1)+s+1} * \mathrm{d}_\omega * \alpha \,. \tag{2.7.53}$$

Proof Using (2.7.50), (1.5.1) and the G-invariance of $\langle \cdot, \cdot \rangle_F$, for $\beta \in \Omega^{k+1}(M, E)$, we calculate

[50] Again, we must restrict ourselves to square-integrable forms. In particular, we may consider forms with compact support.

$$\begin{aligned}\pi^*(\mathrm{d}_\omega\alpha \,\dot\wedge\, *\beta) &= D_\omega\tilde\alpha \,\dot\wedge\, \widetilde{*\beta}\\ &= (\mathrm{d}\tilde\alpha + \sigma'(\omega)\wedge\tilde\alpha)\,\dot\wedge\,\widetilde{*\beta}\\ &= \mathrm{d}(\tilde\alpha\,\dot\wedge\,\widetilde{*\beta}) - (-1)^k\tilde\alpha\,\dot\wedge\,(\mathrm{d}\widetilde{*\beta} + \sigma'(\omega)\wedge\widetilde{*\beta})\\ &= \mathrm{d}(\tilde\alpha\,\dot\wedge\,\widetilde{*\beta}) - (-1)^k\tilde\alpha\,\dot\wedge\, D_\omega(\widetilde{*\beta})\\ &= \pi^*\big(\mathrm{d}(\alpha\,\dot\wedge\, *\beta)\big) - (-1)^k\pi^*(\alpha\,\dot\wedge\,\mathrm{d}_\omega *\beta)\,.\end{aligned}$$

Thus,

$$\mathrm{d}_\omega\alpha\,\dot\wedge\, *\beta = \mathrm{d}(\alpha\,\dot\wedge\, *\beta) - (-1)^k\alpha\,\dot\wedge\,\mathrm{d}_\omega *\beta\,.$$

Integrating this identity over M, using Stokes' Theorem, we obtain

$$\langle \mathrm{d}_\omega\alpha, \beta\rangle_{L^2} = \langle\alpha, (-1)^{nk+s+1} * \mathrm{d}_\omega * \beta\rangle_{L^2}\,.$$

Comparing with (2.7.51), we read off the assertion. ■

As above, we need a unified description in terms of the Koszul calculus. For that purpose, it will be convenient to view the space $\Omega^k(M, E)$ as follows. Denote

$$T^r_s = \mathbb{R}^n \otimes \overset{r}{\cdots} \otimes\mathbb{R}^n \otimes \mathbb{R}^{n*}\otimes \overset{s}{\cdots} \otimes\mathbb{R}^{n*}\,.$$

Consider the fibre product[51] $O(M)\times_M P$ over M with structure group $\mathrm{O}(k,l)\times G$ and the associated bundle with typical fibre $T^r_s\otimes F$,

$$E_{r,s} = (O(M)\times_M P)\times_{\mathrm{O}(k,l)\times G}(T^r_s\otimes F)\,,$$

which is clearly isomorphic to the tensor product $\mathrm{T}^r_s(M)\otimes E$ of vector bundles. The left actions of $\mathrm{O}(k,l)$ and G on T^r_s and F are denoted by μ and σ, respectively. By Remark 1.3.17, the Levi-Civita connection form ω^o on $O(M)$ and the gauge connection form ω on P induce a connection form $\omega^o+\omega$ on $O(M)\times_M P$, cf. (1.3.16).[52] As usual, we denote the induced covariant exterior derivative acting on $\Omega^k_{(\mu,\sigma),\mathrm{hor}}(O(M)\times_M P, T^r_s\otimes F)$ by $D_{(\omega^o+\omega)}$, its counterpart acting on $\Omega^k(M, E_{r,s})$ by $\mathrm{d}_{(\omega^o+\omega)}$ and the corresponding covariant derivative acting on sections of $E_{r,s}$ by $\nabla^{(\omega^o+\omega)}$. By the general theory,

$$D_{\omega^o+\omega}\tilde\Phi = \mathrm{d}\tilde\Phi + \big(\mu'(\omega^o)\otimes\mathrm{id}_F + \mathrm{id}_{T^r_s}\otimes\sigma'(\omega)\big)\circ\tilde\Phi\,, \tag{2.7.54}$$

cf. (1.4.2). Clearly, $\mu'(\omega^o)\otimes\mathrm{id}_F + \mathrm{id}_{T^r_s}\otimes\sigma'(\omega)$ must be viewed as a 1-form on $Q\times_M P$ with values in $\mathrm{End}(T^r_s\otimes F)$. It is obtained by differentiating the tensor product representation $\mu\otimes\sigma$. Moreover, $\Omega^k_{(\mu,\sigma),\mathrm{hor}}(O(M)\times_M P, T^r_s\otimes F)$ may be viewed as a subspace of

---

[51]Cf. Remark 1.1.9/2.

[52]For simplicity, we omit the canonical projections onto $O(M)$ and P, respectively.

$$\mathrm{Hom}_{\mathrm{O}(k,l)\times G}(O(M)\times_M P, T^r_{s+k}\otimes F)$$

consisting of those elements whose last k covariant tensor indices are anti-symmetric. By Proposition 1.2.12, the latter space in turn may be identified with $\Gamma^\infty(E_{r,s+k})$. Elements of this space may be viewed as tensor fields of type $(r, s+k)$ on M with values in the associated bundle E. In particular, we get the following identification:

$$\Omega^k(M, E) \cong \Omega^k(M, E_{0,0})\,. \tag{2.7.55}$$

Now, the generalization of the Weitzenboeck Formula is straightforward. First, for $(r, s) = (0, 0)$, the action μ is trivial and hence (2.7.54) implies

$$\mathrm{d}_{(\omega^o+\omega)}\alpha = \mathrm{d}_\omega\alpha\,, \quad \mathrm{d}^*_{(\omega^o+\omega)}\alpha = \mathrm{d}^*_\omega\alpha\,,$$

for any $\alpha \in \Omega^k(M, E)$. This implies

$$\Box_\omega = \Box_{(\omega^o+\omega)}\,. \tag{2.7.56}$$

Lemma 2.7.19 *Let $\alpha \in \Omega^k(M, E)$. Then, under the identification (2.7.55),*

$$\mathrm{d}_\omega\alpha(X_0, \dots, X_k) = \sum_j (-1)^j \left(\nabla^{(\omega^o+\omega)}_{X_j}\alpha\right)\left(X_0, \overset{j}{\curlyvee}, X_k\right), \tag{2.7.57}$$

$$(\mathrm{d}^*_\omega\alpha)(X_2, \dots, X_k) = -\sum_{j,l}\eta^{jl}\left(\nabla^{(\omega^o+\omega)}_{e_j}\alpha\right)(e_l, X_2, \dots, X_k)\,, \tag{2.7.58}$$

for $X_0, \dots, X_k \in \mathfrak{X}(M)$ and $\{e_l\}$ being an orthonormal frame on (M, g).

We note the following immediate consequence of (2.7.57):

$$\mathrm{d}_\omega\alpha = \sum_j \vartheta^j \wedge \nabla^{(\omega^o+\omega)}_{e_j}\alpha\,, \tag{2.7.59}$$

where $\{\vartheta^j\}$ is the coframe dual to $\{e_j\}$.

Proof To prove (2.7.57), it is enough to consider elements $\alpha = \phi\otimes\beta$, where $\phi \in \Gamma^\infty(E)$ and $\beta \in \Omega^k(M)$. Then, again using that the action μ is trivial, for the left hand side of (2.7.57) we get

$$\mathrm{d}_\omega\alpha = \mathrm{d}_\omega\phi\wedge\beta + \phi\otimes\mathrm{d}\beta\,.$$

To analyze the right hand side, we use the derivation property of the covariant derivative,

$$\nabla_X^{(\omega^o+\omega)}\alpha = \nabla_X^{\omega}\phi \otimes \beta + \phi \otimes \nabla_X^{\omega^o}\beta\,.$$

This, together with formula (2.2.49), implies the assertion.

The proof of (2.7.58) is analogous to the proof of (2.7.23). We replace d by d_ω and use (2.7.59). ■

Now, by the same calculation as in the proof of Theorem 2.7.11, we obtain the following Generalized Weitzenboeck Formula

$$\Box_\omega\alpha = \left(\nabla^{(\omega^0+\omega)}\right)^*\nabla^{(\omega^0+\omega)}\alpha + \eta^{jl}{e^i}_l\left(\mathsf{R}^{\nabla^{(\omega^0+\omega)}}(e_j, e_i)\alpha\right), \tag{2.7.60}$$

where $\mathsf{R}^{\nabla^{(\omega^0+\omega)}}$ is the curvature endomorphism form of the connection $\omega^0+\omega$ given by (1.5.13). Here, it reads

$$\mathsf{R}_m^{\nabla^{(\omega^0+\omega)}}(X,Y) := \iota_z \circ \left\{\mu'(\Omega_z^o(X^h,Y^h)) \otimes \mathrm{id} + \mathrm{id} \otimes \sigma'(\Omega_z(X^h,Y^h))\right\} \circ \iota_z^{-1}\,,$$

where $m \in M$, $z \in \pi^{-1}(m) \subset \mathrm{O}(M) \times_M P$, $X, Y \in \mathrm{T}_m M$ and X^h and Y^h are the horizontal lifts of X and Y to z, respectively. Clearly, by the additivity of $\mathsf{R}^{\nabla^{(\omega^0+\omega)}}$, the second term on the right hand side of (2.7.60) is the sum of the Weitzenboeck curvature operators for the representations μ and σ, respectively, cf. Definition 2.7.10. This yields the following.

Theorem 2.7.20 (Generalized Weitzenboeck Formula) *For $\alpha \in \Omega^k(M,E)$,*

$$\Box_\omega\alpha = \left(\nabla^{(\omega^0+\omega)}\right)^*\nabla^{(\omega^0+\omega)}\alpha + \mathfrak{R}^{\nabla^{\omega^o}}(\alpha) + \mathfrak{R}^{\nabla^{\omega}}(\alpha)\,. \tag{2.7.61}$$

■

As above, formula (2.7.61) may be analyzed degreewise. Clearly, the terms coming from the Levi-Civita connection are identical with those in Corollary 2.7.12. Thus, we obtain the following.

Corollary 2.7.21

1. *For $\alpha \in \Omega^1(M,E)$, the Weitzenboeck Formula (2.7.61) reads*

$$\Box_\omega\alpha = \left(\nabla^{(\omega^0+\omega)}\right)^*\nabla^{(\omega^0+\omega)}\alpha + \alpha \circ \mathsf{Ric} + \mathfrak{R}^{\nabla^{\omega}}(\alpha)\,. \tag{2.7.62}$$

2. *For $\alpha \in \Omega^2(M,E)$, formula (2.7.61) may be rewritten as follows:*

$$\Box_\omega\alpha = \left(\nabla^{(\omega^0+\omega)}\right)^*\nabla^{(\omega^0+\omega)}\alpha + \alpha \circ (\mathsf{R} + \mathsf{Ric} \wedge \mathrm{id}) + \mathfrak{R}^{\nabla^{\omega}}(\alpha)\,. \tag{2.7.63}$$

■

The Generalized Weitenboeck Formula will be taken up again in Example 5.6.7. There, it will be discussed from the point of view of Dirac operator theory. It will play a basic role in the analysis of the stability of Yang-Mills connections.

Exercises

2.7.1 Prove the formulae (2.7.8)–(2.7.10).

2.7.2 Prove the identities contained in (2.7.15)–(2.7.17).

2.7.3 Prove that on a compact connected oriented Riemannian manifold fulfilling $H^1_{\mathrm{dR}}(M) = 0$ there does not exist any nontrivial harmonic 1-form. Construct a nontrivial harmonic 1-form on the 2-torus $\mathrm{T}^2 \subset \mathbb{R}^4$.

2.7.4 Prove formula (2.7.27).

2.7.5 Prove the statements of Example 2.7.13.

2.7.6 Prove formula (2.7.33).

2.7.7 Prove formula (2.7.50).

2.8 Four-Dimensional Riemannian Geometry. Self-duality

In this section, we deal with 4-dimensional (oriented) Riemannian manifolds. We will show that, in contrast to other dimensions, they admit a rich additional structure. Let us explain the reason for that. Given an oriented Riemannian manifold (M, g), we know from Sect. 2.4 that g yields a reduction of the frame bundle $L(M)$ to the principal SO(4)-bundle $O_+(M)$ of oriented orthonormal frames. Correspondingly, all tensor bundles over M become associated with $O_+(M)$ with their typical fibres carrying representations of SO(4). Now, among all rotation groups, SO(4) is the unique group which is not simple. This has striking consequences, as we will see below. Recall from Example I/5.1.10 the isomorphism

$$\mathrm{Sp}(1) \to \mathrm{SU}(2)\,, \quad a = z + \mathbf{j}\, w \mapsto \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}, \tag{2.8.1}$$

where we have identified $\mathbb{C}$ with $\mathrm{span}\{\mathbf{1}, \mathbf{i}\} \subset \mathbb{H}$ and $\mathbb{H}$ with $\mathbb{C}^2$ by writing quaternions in the form $z + \mathbf{j}w$ with $z\,, w \in \mathbb{C}$. Also recall from Example I/5.1.11 that Sp(1) and $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ are the universal (two-fold) covering groups[53] of SO(3) and SO(4), respectively. Denoting by $\iota : \mathrm{Sp}(1) \to \mathrm{Sp}(1) \times \mathrm{Sp}(1)$ the diagonal embedding, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(1) \arrow[r, "\iota"] \arrow[d] & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \arrow[d, "f"] \\ \mathrm{SO}(3) \arrow[r] & \mathrm{SO}(4) \end{tikzcd} \tag{2.8.2}$$

[53] In Chap. 5, we will see that these are the spin groups in 3 and 4 dimensions, respectively.

This fact reduces the representation theory of SO(4) to that of Sp(1). By the isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, we are led to consider complex representations built from the basic representation of SU(2) on $V \cong \mathbb{C}^2$. By a standard theorem in representation theory [689], up to isomorphisms, the set of irreducible complex SU(2)-modules is

$$\left\{ S^r V : r \geq 0 \right\} ,$$

where $S^r V$ denotes the subspace of $\otimes^r V$ of totally symmetric tensors. Equivalently, this subspace may be identified with the space of homogeneous polynomials of degree r in two variables. Thus, $\dim_{\mathbb{C}}(S^r V) = r + 1$. Moreover,

$$S^p V \otimes S^q V \cong \bigoplus_{r=0}^{min(p,q)} S^{p+q-2r} V . \tag{2.8.3}$$

Note that $S^2 V$ is the (complexified) adjoint representation space.

Now, any complex SO(4)-module (W, σ) may be viewed as an $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$-module via the mapping

$$\sigma \circ f : \quad \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Aut}(W) .$$

Let us denote the basic representation spaces corresponding to the first and the second factor in $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$, respectively, by V_+ and V_-. Then, again, by standard representation theory, the irreducible complex $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))$-modules are given by

$$S^{p,q} = S^p V_+ \otimes S^q V_- , \quad p, q \geq 0 . \tag{2.8.4}$$

Clearly, an irreducible representation $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{Aut}(S^{p,q})$ factors through the covering homomorphism f, giving a representation of SO(4), iff $p + q$ is even. Moreover, in that case, $S^{p,q}$ is the complexification of a real representation which we denote by $S_r^{p,q}$. It is common to call $S_r^{p,q}$ the real representation underlying $S^{p,q}$. Also note that

$$\dim_{\mathbb{C}}(S^{p,q}) = \dim_{\mathbb{R}}(S_r^{p,q}) = (p + 1)(q + 1) .$$

In particular, the basic complex SO(4)-module is $S^{1,1} = V_+ \otimes V_-$. We denote $T := S_r^{1,1}$ and write T^* for the dual (contragredient) representation space. Clearly, we may use the Euclidean metric on T to identify $T \cong T^*$. Now, calculating (Exercise 2.8.1)

$$\bigwedge^2 T_{\mathbb{C}}^* \cong \bigwedge^2 (V_+ \otimes V_-) \cong \left(S^2 V_+ \otimes \bigwedge^2 V_- \right) \oplus \left(\bigwedge^2 V_+ \otimes S^2 V_- \right)$$

and using that $\bigwedge^2 V$ is the trivial Sp(1)-module, we obtain

$$\bigwedge^2 T_{\mathbb{C}}^* \cong S^2 V_+ \oplus S^2 V_- . \tag{2.8.5}$$

Since $S^2 V$ is the adjoint representation of $\mathrm{Sp}(1)$, $\bigwedge^2 T^*_{\mathbb{C}}$ is the (complexified) adjoint representation space of $\mathrm{SO}(4)$ with (2.8.5) corresponding to the Lie algebra decomposition $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$. Thus, we have the underlying isomorphism of real representations

$$\bigwedge^2 T^* \cong S_r^{2,0} \oplus S_r^{0,2} \tag{2.8.6}$$

corresponding to the decomposition $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Next, we will relate the above decompositions to the Hodge star operator. Thus, let $* : \bigwedge^r T^* \to \bigwedge^{4-r} T^*$ be the Hodge star operator with respect to the Euclidean metric on T. By Proposition I/4.5.3,

$$* \circ * = \mathrm{id}_{\bigwedge^2 T^*}\,, \tag{2.8.7}$$

that is, on two-forms, the Hodge star operator is an involution. Thus, we may decompose $\bigwedge^2 T^*$ into an orthogonal direct sum of eigenspaces of $*$ corresponding to the eigenvalues ± 1,

$$\bigwedge^2 T^* = \bigwedge^2{}_+ T^* \oplus \bigwedge^2{}_- T^*\,. \tag{2.8.8}$$

Elements of $\bigwedge^2_+ T^*$ are called self-dual and elements of $\bigwedge^2_- T^*$ are called anti-self-dual. Since the Hodge star operator is invariant under the action of $\mathrm{SO}(4)$, the subspaces $\bigwedge^2_\pm T^*$ are $\mathrm{SO}(4)$-invariant and, thus, they coincide with the direct summands in (2.8.6),

$$\bigwedge^2{}_+ T^* \cong S_r^{2,0}\,, \quad \bigwedge^2{}_- T^* \cong S_r^{0,2}\,. \tag{2.8.9}$$

For the corresponding complexifications, we get

$$\bigwedge^2{}_+ T^*_{\mathbb{C}} \cong S^2 V_+\,, \quad \bigwedge^2{}_- T^*_{\mathbb{C}} \cong S^2 V_-\,. \tag{2.8.10}$$

Remark 2.8.1

1. Let $\vartheta^1, \ldots, \vartheta^4$ be an oriented orthonormal basis in T^*. Then, the irreducible subspaces $\bigwedge^2_\pm T^*$ are spanned by

$$\begin{aligned}
\varphi^1_\pm &= \frac{1}{\sqrt{2}}\left(\vartheta^1 \wedge \vartheta^2 \pm \vartheta^3 \wedge \vartheta^4\right)\,,\\
\varphi^2_\pm &= \frac{1}{\sqrt{2}}\left(\vartheta^1 \wedge \vartheta^3 \mp \vartheta^2 \wedge \vartheta^4\right)\,,\\
\varphi^3_\pm &= \frac{1}{\sqrt{2}}\left(\vartheta^1 \wedge \vartheta^4 \pm \vartheta^2 \wedge \vartheta^3\right)\,.
\end{aligned}$$

2. In the same way as above, we can calculate

$$\begin{aligned} S^2 T^*_{\mathbb{C}} &\cong S^2(V_+ \otimes V_-) \\ &\cong (S^2 V_+ \otimes S^2 V_-) \oplus (\bigwedge^2 V_+ \otimes \bigwedge^2 V_-) \\ &\cong (S^2 V_+ \otimes S^2 V_-) \oplus \mathbb{C}\,. \end{aligned}$$

Thus, using (2.8.10),

$$S^2_0 T^* \cong \bigwedge^2{}_+ T^* \otimes \bigwedge^2{}_- T^*\,, \tag{2.8.11}$$

where the subindex zero refers to tracelessness. ♦

Comparison of the decompositions (2.8.8) with (2.2.16) yields the following deep insight. Let T^* be endowed with the complex structure[54]

$$\mathsf{J} = \begin{bmatrix} \mathsf{J}_1 & 0 \\ 0 & \mathsf{J}_1 \end{bmatrix},$$

where J_1 is the standard complex structure on $\mathbb{R}^2$. With respect to this structure, the decomposition (2.2.16) reads

$$\bigwedge^2 T^*_{\mathbb{C}} = \left(\bigwedge^{2,0} T^*_{\mathbb{C}} \oplus \bigwedge^{0,2} T^*_{\mathbb{C}}\right) \oplus \bigwedge^{1,1} T^*_{\mathbb{C}}\,. \tag{2.8.12}$$

As already noted, the left hand side may be identified with the Lie algebra $\mathfrak{o}(4, \mathbb{C})$. In analogy to (2.2.22), J induces an embedding $\mathrm{U}(2) \subset \mathrm{SO}(4)$ and the summands on the right hand side of (2.8.12) carry representations of $\mathrm{U}(2)$. Observe that the almost symplectic form β defined by (2.2.26) belongs to $\bigwedge^{1,1} T^*_{\mathbb{C}}$ and is $\mathrm{U}(2)$-invariant. Thus, we have an orthogonal decomposition

$$\bigwedge^{1,1} T^*_{\mathbb{C}} = \mathbb{C} \oplus \bigwedge^{1,1}_0 T^*_{\mathbb{C}}$$

into $\mathrm{U}(2)$-irreducible components. By dimension counting, $\bigwedge^{1,1}_0 T^*_{\mathbb{C}} \cong \mathfrak{sl}(2, \mathbb{C})$ (the complexification of $\mathfrak{su}(2)$) and, thus, (2.8.12) corresponds to the complexification of the Lie algebra decomposition $\mathfrak{o}(4) = \mathbb{R} \oplus \mathfrak{su}(2) \oplus \mathfrak{m}$, cf. point 3 of Example 2.5.27.

Lemma 2.8.2 *We have*

$$\bigwedge^2{}_+ T^*_{\mathbb{C}} = \mathbb{C} \oplus \left(\bigwedge^{2,0} T^*_{\mathbb{C}} \oplus \bigwedge^{0,2} T^*_{\mathbb{C}}\right), \quad \bigwedge^2{}_- T^*_{\mathbb{C}} = \bigwedge^{1,1}_0 T^*_{\mathbb{C}}\,. \tag{2.8.13}$$

Proof Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_4\}$ be the standard basis in the basic $\mathrm{SO}(4)$-module $T = \mathbb{R}^4$ and let $\{\vartheta^1, \ldots, \vartheta^4\}$ be the dual basis in T^*. Clearly, $\bigwedge^{1,0} T^*_{\mathbb{C}}$ is spanned by

[54] This choice is made in order to be compatible with standard conventions in gauge theory. It is obtained by combining the standard complex structure J_0 on $\mathbb{R}^4$ with the transformation defined by permuting the standard basis vectors $\mathbf{e}_2$ and $\mathbf{e}_3$. Beware that J and J_0 induce different orientations.

$$\psi^1 = \vartheta^1 + i\vartheta^2\,, \quad \psi^2 = \vartheta^3 + i\vartheta^4\,.$$

Now, using point 1 of Remark 2.8.1, we express the generators of the $\mathrm{U}(n)$-modules on the right hand side of (2.8.12) in terms of the bases $\{\varphi^i_\pm\}$ of $\bigwedge^2_\pm T^*$:

$$\begin{aligned}
\frac{1}{2}i(\psi^1 \wedge \bar{\psi}^1 + \psi^2 \wedge \bar{\psi}^2) &= \varphi^1_+\,,\\
\psi^1 \wedge \psi^2 &= \varphi^2_+ + i\varphi^3_+\,,\\
\bar{\psi}^1 \wedge \bar{\psi}^2 &= \varphi^2_+ - i\varphi^3_+\,,\\
\frac{1}{2}i(\psi^1 \wedge \bar{\psi}^1 - \psi^2 \wedge \bar{\psi}^2) &= \varphi^1_-\,,\\
\psi^1 \wedge \bar{\psi}^2 &= \varphi^2_- - i\varphi^3_-\,,\\
\bar{\psi}^2 \wedge \bar{\psi}^1 &= -\varphi^2_- + i\varphi^3_-\,.
\end{aligned}$$

■

Corollary 2.8.3 *A 2-form on $\mathbb{R}^4$ is anti-self-dual iff it is of type $(1,1)$ for all compatible complex structures.* ■

As we will see, the following lemma is of basic importance in 4-dimensional Riemannian geometry [592].

Lemma 2.8.4 *We have*

$$S^2\left(\bigwedge\nolimits^2 T^*\right) \cong S^{0,0}_r \oplus S^{0,0}_r \oplus S^{2,2}_r \oplus S^{4,0}_r \oplus S^{0,4}_r\,. \tag{2.8.14}$$

Proof Using (2.8.8), we calculate

$$S^2\left(\bigwedge\nolimits^2_+ T^* \oplus \bigwedge\nolimits^2_- T^*\right) \cong S^2\left(\bigwedge\nolimits^2_+ T^*\right) \oplus \left(\bigwedge\nolimits^2_+ T^* \otimes \bigwedge\nolimits^2_- T^*\right) \oplus S^2\left(\bigwedge\nolimits^2_- T^*\right)\,.$$

By (2.8.9), the second term on the right hand side corresponds to $S^{2,2}_r$. The complexification of the first term corresponds via (2.8.10) to the symmetric component of $S^2 V_+ \otimes S^2 V_+$ and thus has complex dimension 6. By (2.8.3),

$$S^2 V_+ \otimes S^2 V_+ = S^4 V_+ \oplus S^2 V_+ \oplus S^0 V_+\,.$$

By counting dimensions, we find that the symmetric component corresponds to $S^4 V_+ \oplus S^0 V_+$. It follows that

$$S^2\left(\bigwedge\nolimits^2_+ T^*\right) = S^{4,0}_r \oplus S^{0,0}_r \tag{2.8.15}$$

and, analogously, $S^2\left(\bigwedge^2_+ T^*\right) = S^{0,4}_r \oplus S^{0,0}_r$. ■

Now, we can apply the above results to the 4-dimensional Riemannian manifold (M, g). By Proposition I/4.5.3, the Hodge star operator is an isometric involution on the bundle of two forms, that is, $* : \bigwedge^2 \mathrm{T}^* M \to \bigwedge^2 \mathrm{T}^* M$ fulfils

$$* \circ * = \mathrm{id}_{\bigwedge^2 \mathrm{T}^* M}\,, \quad \langle *\alpha, *\beta \rangle_{L^2} = \langle \alpha, \beta \rangle_{L^2}\,, \tag{2.8.16}$$

and, corresponding to (2.8.8), we have the splitting

$$\bigwedge^2 \mathrm{T}^* M = \bigwedge^2{}_+ \mathrm{T}^* M \oplus \bigwedge^2{}_- \mathrm{T}^* M\,. \tag{2.8.17}$$

Clearly, the decomposition (2.8.17) implies a decomposition of 2-forms on M,

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M)\,. \tag{2.8.18}$$

Thus, any $\alpha \in \Omega^2(M)$ may be decomposed as follows:

$$\alpha = \alpha^+ + \alpha^-\,, \quad *\alpha^+ = \alpha^+\,, \quad *\alpha^- = -\alpha^-\,, \tag{2.8.19}$$

where $\alpha^\pm = \frac{1}{2}(\alpha \pm *\alpha)$. Elements of $\Omega^2_+(M)$ are called self-dual and elements of $\Omega^2_-(M)$ are called anti-self-dual 2-forms. Finally, for a local oriented orthonormal frame $\vartheta^1, \ldots, \vartheta^4$ in $\bigwedge^1 \mathrm{T}^* M$, the subbundles $\bigwedge^2_\pm \mathrm{T}^* M$ are locally spanned by $\{\varphi^i_\pm\}$ given by the same formulae as in Remark 2.8.1/2.

Next, let us consider the Riemann curvature endomorphism form

$$\mathsf{R} \in \Omega^2(M, \mathrm{End}(\mathrm{T}M))$$

of (M, g). By Remark 2.3.7, pointwise, it may be viewed as a symmetric endomorphism of $\bigwedge^2 \mathrm{T}^*_m M$,

$$\mathsf{R}(m) \in S^2 \left(\bigwedge^2 \mathrm{T}^*_m M \right)\,. \tag{2.8.20}$$

Correspondingly, for every $u \in O(M)$, it may be viewed as an element

$$\mathscr{R}(u) \in \bigwedge^2 \left(\mathbb{R}^4\right)^* \overset{s}{\otimes} \bigwedge^2 \left(\mathbb{R}^4\right)^* \equiv S^2 \left(\bigwedge^2 \left(\mathbb{R}^4\right)^* \right)\,. \tag{2.8.21}$$

We wish to derive the counterpart of the general decomposition formula (2.3.21) for $n = 4$. Here, according to the additional structures at a our disposal, this can be done in two different ways. First, using (2.8.17), we can write

$$\mathsf{R}(m) = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & C \end{bmatrix}\,. \tag{2.8.22}$$

Here, $B \in \mathrm{Hom}(\bigwedge^2_- \mathrm{T}^*_m M, \bigwedge^2_+ \mathrm{T}^*_m M)$, $A \in \mathrm{End}(\bigwedge^2_+ \mathrm{T}^*_m M)$ and $C \in \mathrm{End}(\bigwedge^2_- \mathrm{T}^*_m M)$. Since $\mathsf{R}(m) \in S^2(\bigwedge^2 \mathrm{T}^*_m M)$, both A and C are symmetric endomorphisms. Note that B^{T} is the adjoint of B.

Lemma 2.8.5 *We have*

$$\mathrm{tr}\, A = \mathrm{tr}\, C = -\frac{1}{4}\mathsf{Sc}\,,$$

where Sc *denotes the scalar curvature of* ∇.

Proof This is a simple exercise which we leave to the reader (Exercise 2.8.2). ■

Remark 2.8.6 We show that the decomposition (2.8.22) corresponds to the decomposition of $S^2(\bigwedge^2 T^*)$ into irreducible components of SO(4) given by Lemma 2.8.4, with one of the two $S^{0,0} \cong \mathbb{R}$-summands removed. For that purpose, we choose an orthonormal basis in $\mathrm{T}_m M$ and use it to identify $\mathrm{T}_m M$ with T. Using (2.8.15), we obtain

$$A \in S^2\left(\textstyle\bigwedge^2_+ T^*\right) = S^{4,0}_r \oplus S^{0,0}_r\,, \quad C \in S^{0,4}_r \oplus S^{0,0}_r\,.$$

Moreover,

$$B \in \mathrm{Hom}\left(\textstyle\bigwedge^2_- T^*, \bigwedge^2_+ T^*\right) \cong S^{2,0}_r \otimes S^{0,2}_r \cong S^{2,2}_r\,.$$

Finally, by Lemma 2.8.5, one of the summands $S^{0,0}_r$ is removed and we obtain the following 4-dimensional counterpart of the decomposition (2.3.21) of the space of Riemann curvatures

$$\mathfrak{K}(m) = S^{0,0}_r \oplus S^{2,2}_r \oplus S^{4,0}_r \oplus S^{0,4}_r\,,$$

with

$$\mathsf{R}(m) = (\mathrm{tr}\, A, B, A - \tfrac{1}{3}\,\mathrm{tr}\, A, C - \tfrac{1}{3}\,\mathrm{tr}\, C)\,. \tag{2.8.23}$$

This result belongs to Singer and Thorpe [592]. ♦

We denote

$$\mathsf{W}_+ := A - \tfrac{1}{3}\,\mathrm{tr}\, A\,, \quad \mathsf{W}_- := C - \tfrac{1}{3}\,\mathrm{tr}\, C\,, \tag{2.8.24}$$

and call

$$\mathsf{W} := \begin{bmatrix} \mathsf{W}_+ & 0 \\ 0 & \mathsf{W}_- \end{bmatrix}$$

the Weyl tensor. Note that $\mathsf{W}_\pm : \bigwedge^2_\pm \to \bigwedge^2_\pm$ are symmetric endomorphisms with vanishing trace. Summarizing the above discussion, we obtain the following.

Theorem 2.8.7 (Singer-Thorpe) *The Riemann curvature* R *of an oriented* 4-*dimensional Riemannian manifold defines a symmetric endomorphism of* $\bigwedge^2 \mathrm{T}^* M$ *which decomposes as*

$$\mathsf{R} = -\frac{\mathsf{Sc}}{12}\mathbb{1} + \begin{bmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{bmatrix} + \mathsf{W}\,. \tag{2.8.25}$$

The statements of the following remark are left as an exercise to the reader (Exercise 2.8.3).

Remark 2.8.8 In a local orthonormal frame on M, the decomposition (2.8.25) reads as follows:

$$\mathsf{R}_{ijkl} = \frac{\mathsf{Sc}}{6}\big(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il}\big) + \frac{1}{2}\big(\mathsf{R}_{il}\delta_{jk} + \mathsf{R}_{jk}\delta_{il} - \mathsf{R}_{ik}\delta_{jl} - \mathsf{R}_{jl}\delta_{ik}\big) + \mathsf{W}_{ijkl}\,, \tag{2.8.26}$$

where R_{ij} are the components of the Ricci tensor. Clearly, the Weyl tensor $\mathsf{W}_{ijkl} = \mathsf{g}(\mathsf{W}(e_i, e_j)e_k, e_l)$ inherits the properties (2.3.15) from the curvature tensor. By construction, we have $\sum_i \mathsf{W}_{ijki} = 0$. ♦

Definition 2.8.9 An oriented Riemannian 4-manifold is called self-dual or anti-self-dual if, respectively, $\mathsf{W}_- = 0$ or $\mathsf{W}_+ = 0$.

By direct inspection of (2.8.26), one can check that M is Einstein if $B = 0$.

Example 2.8.10

1. The manifolds S^4, $\mathrm{S}^1 \times \mathrm{S}^3$ and T^4, endowed with their natural metrics, have a vanishing Weyl tensor and are, thus, both self-dual and anti-self-dual (Exercise 2.8.4).
2. $\mathbb{C}\mathrm{P}^2$ with its standard metric and orientation is self-dual. For a detailed proof we refer to [689]. ♦

Exercises

2.8.1 Prove formula (2.8.5). *Hint.* Construct explicit bases for the occuring representation spaces.

2.8.2 Prove Lemma 2.8.5.

2.8.3 Prove the statements of Remark 2.8.8.

2.8.4 Prove the statements of Example 2.8.10/1.

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http://www.springer.com/978-94-024-0958-1

Differential Geometry and Mathematical Physics
Part II. Fibre Bundles, Topology and Gauge Fields
Rudolph, G.; Schmidt, M.
2017, XVI, 830 p. 15 illus., 2 illus. in color., Hardcover
ISBN: 978-94-024-0958-1