Periodic Trajectories of Dynamical Systems Having a One-Parameter Group of Symmetries

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Abstract We study a class of dynamical systems on a compact (semi-)Riemannian manifold endowed with a non trivial 1-parameter (pre-compact) group of symmetries, and we determine the existence of a class of periodic trajectories of these systems.

1 Introduction

The present was originally meant to be the note of an invited lecture given by the second author at the *International Research School "Differential Geometry and Symmetry"*, held at the *Universidad de Murcia*, Spain, in March 2009. During that lecture, emphasis was given mostly to the study of topological and geometrical properties of compact Lorentzian manifolds endowed with a Killing vector field which is time-like somewhere. The main results presented concern some questions of compactness for 1-parameter subgroups of the isometry group of such manifolds, and a proof of existence of non trivial periodic geodesics. The material of the talk is almost entirely contained in references [8, 20].

Actually, some of the techniques employed in [8] to prove the existence of non trivial periodic geodesics in compact Lorentzian manifolds, apply as well in the more general case of periodic solutions of dynamical systems. In this note we will show how to extend the results of [8] to this more general situation using suitable notions of symmetry, thus fitting in the general theme of the *Research School*. We will consider here two types of dynamical systems whose configuration space is a

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(compact) Riemannian or semi-Riemannian manifold (M, g), namely, *conservative* systems, i.e., of the type kinetic energy plus potential, and *exact magnetic* dynamical systems. Trajectories of these systems are curves $x : [0, L] \to M$ that are solutions of a certain second order differential equation of the type $\frac{\mathrm{D}}{\mathrm{d}t}\dot{x} = F(x, \dot{x})$, where $\frac{\mathrm{D}}{\mathrm{d}t}$ is the operator of covariant differentiation for vector fields along x induced by the Levi–Civita connection of g, and $F : TM \to \mathbb{R}$ is a smooth map defined by the potential energy or the magnetic field. When $F \equiv 0$, then solutions of the dynamical system are geodesics of (M, g). A trajectory $x : [0, L] \to M$ of the system is *periodic* if $x(0) = x(L)$ and $\dot{x}(0) = \dot{x}(L)$.

We will define a notion of *symmetry* for such systems (Definitions 2.1, 3.1), which is an isometry of the base manifold that preserves the potential energy or the magnetic field. The first key observation here is that when the dynamical systems admits a non trivial 1-parameter group of isometries, or, equivalently, a Killing vector field whose flow preserves the potential energy or the magnetic field, then some of the flow lines of the group are trajectories of the system. Such special flow lines have a variational characterization, i.e., they are those flow lines passing through the critical point of some natural smooth function on the base manifold (Propositions 2.6, 2.7, 3.6). In particular, being solutions of a *first order* differential equation, such special trajectories do not have self-intersections. It is interesting to observe that infinitesimal symmetries of dynamical systems produce conservation laws for the solutions of such systems (Lemmas 2.4 and 3.4); these are special cases of *Noether's theorem* first theorem on conserved quantities from symmetries, see [18].

When the base manifold (M, g) is compact and Riemannian, then its isometry group is compact. The second important observation is that the compactness of the isometry group implies that when the dynamical system admits a non trivial one-parameter group of symmetries, then it also admits a non trivial one-parameter group of symmetries all of whose flow lines are *closed* (Proposition 4.5). The proof of this fact is based on elementary Lie group techniques; it implies in particular that if there exists a non trivial one-parameter group of symmetries, then the manifold M has the topology of a *generalized Seifert space*, i.e., it admits a smooth circle action without fixed points (Proposition 5.1). Using these two observations, periodic trajectories of dynamical systems on compact Riemannian manifolds are obtained from flow lines of the group of symmetries. Multiplicity of periodic trajectories can be studied using *equivariant Ljusternik–Schnirelmann category* theory, which provides a lower bound for critical orbits of a smooth function on a compact manifold invariant by the action of a compact group of transformations (Sect. 5).

The very same conclusions can be drawn for dynamical systems on arbitrary compact semi-Riemannian manifolds (M, g) having a non trivial one-parameter group of symmetries which is *pre-compact* in the isometry group of (M, g). Also in this situation one has the existence of a non trivial 1-parameter group of symmetries all of whose flows lines are closed. Recall that, unlike the Riemannian case, the (connected component of the identity of the) isometry group of an arbitrary compact semi-Riemannian manifold is in general not compact, and thus the pre-compactness of 1-parameter subgroups has to be explicitly assumed. However, there are important situations where this property is satisfied. For instance, when the compact manifold

(M, g) is Lorentzian, simply connected, and *real analytic*. Namely, in this case a celebrated result of D'Ambra [4] guarantees that the isometry group of (M, g) is compact. A second important case of pre-compact 1-parameter group of symmetries of a Lorentzian manifold is when the infinitesimal generator of the subgroup, which is a Killing field, is timelike at some point.

Most of the paper is dedicated to the study of dynamical systems on Riemannian manifolds, while the semi-Riemannian extension of the results, which basically reduces to the study of the pre-compactness question for the group of symmetries, is discussed briefly in the last section of the paper.

Potential readers of this note are graduate students; virtually everything here should be accessible to anybody with a basic knowledge of calculus in Riemannian manifolds and Lie group theory. Basic references for the background material in Riemannian, Lorentzian and semi-Riemannian geometry are the classical textbooks [1, 5, 19].

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2 Conservative Dynamical Systems with Symmetries on Riemannian Manifolds

Let (M, g) be a Riemannian manifold; we will denote by $\mathfrak{X}(M)$ the Lie algebra of all smooth vector fields on M and by ∇ the Levi–Civita connection of g. Given a smooth function $f : M \to \mathbb{R}$, the *gradient* of f is the vector field ∇f defined by $\mathrm{d}f = g\big(\nabla f, \cdot\big)$, and the *Hessian of* f is the smooth symmetric $(0, 2)$-tensor on M defined by $\mathrm{Hess}^f(v, w) = g\big(\nabla_v(\nabla f), w\big)$ for all $v, w \in TM$.

2.1 Conservative Dynamical Systems and Maupertuis Principle

By a *conservative dymanical system* we will mean a triple (M, g, V), where (M, g) is a Riemannian manifold, and $V : M \to \mathbb{R}$ is a smooth function. The manifold M represents the *configuration space* of the dynamical system, and its dimension is the number of degrees of freedom of the system. The function V is the potential energy of some conservative force. By a *trajectory* of the dynamical system we mean a curve $x : [0, L] \to M$ which is the solution of the second order differential equation:

$$\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{x} = -\nabla V(x), \tag{1}$$

where $\frac{\mathrm{D}}{\mathrm{d}t}$ is the operator of covariant differentiation for vector fields along x relatively to the connection of ∇. If $x : [0, L] \to M$ is a trajectory of the dynamical system, then there exists a constant $E_x \in \mathbb{R}$, called the *(total) energy of* x, such that following identity holds on the interval $[0, L]$:

$$\tfrac{1}{2} g(\dot{x}, \dot{x}) + V(x) \equiv E_x.$$

The classical *Maupertuis principle* (see for instance [10, 24]) states that, up to a reparameterization, trajectories of the dynamical system (M, g, V) having energy E are into one-to-one correspondence with geodesics in the open set:

$$M_E = \big\{p \in M : V(p) < E\big\} \tag{2}$$

endowed with the metric $g_E = \phi_E \cdot g$ which is conformal to g, with conformal factor $\phi_E(p) = E - V(p)$.

2.2 Symmetries of Conservative Systems

Let $\mathrm{Iso}(M, g)$ denote the Lie group of all isometries of (M, g), and let $\mathrm{Iso}_0(M, g)$ be the connected component of the identity of $\mathrm{Iso}(M, g)$.

Definition 2.1 An isometry $\phi \in \mathrm{Iso}(M, g)$ is a symmetry for the dynamical system (M, g, V) if V is ϕ-invariant, i.e., if $V \circ \phi = V$.

An immediate calculation shows that if ϕ is a symmetry for (M, g, V), then

$$\mathrm{d}\phi(p)\nabla V(p) = \nabla V\big(\phi(p)\big),$$

and if $x : [0, L] \to M$ is a trajectory of (M, g, V), then also $\phi \circ x : [0, L] \to M$ is a trajectory of (M, g, V). Let us denote by $\mathrm{Sym}(M, g, V)$ the subset of $\mathrm{Iso}(M, g)$ consisting of symmetries of (M, g, V).

Lemma 2.2 $\mathrm{Sym}(M, g, V)$ *is a closed Lie subgroup of* $\mathrm{Iso}(M, g)$.

Proof $\mathrm{Sym}(M, g, V)$ is clearly a subgroup of $\mathrm{Iso}(M, g)$, and it is closed with respect to the compact-open topology. Thus it is a Lie subgroup of $\mathrm{Iso}(M, g)$. □

Let us now introduce the notion of infinitesimal symmetry for a conservative dynamical system. A vector field $K \in \mathfrak{X}(M)$ is said to be a *Killing vector field of* (M, g) if its flow consists of local isometries of g. Equivalently, K is Killing if the *Lie derivative* $\mathcal{L}_K(g)$ of the metric tensor g in the direction K vanishes identically, i.e., if $g(\nabla_v K, w) + g(v, \nabla_w K) = 0$ for all pairs of tangent vectors $v, w \in TM$.

Definition 2.3 A Killing field K of (M, g) is an infinitesimal symmetry for the dynamical system (M, g, V) if V is invariant by the flow of K, i.e., if $K(V) = g(K, \nabla V) = 0$ on M.

An infinitesimal symmetry K of the dynamical system (M, g, V) gives a conservation law for the trajectories of (M, g, V).

Lemma 2.4 *If K is an infinitesimal symmetry of (M, g, V) and $x : [0, L] \to M$ is a trajectory of (M, g, V), then the quantity $g(\dot{x}, K)$ is constant along x.*

Proof An immediate calculation:

$$\tfrac{\mathrm{d}}{\mathrm{d}t} g(\dot{x}, K) = g\big(\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{x}, K\big) + g\big(\dot{x}, \nabla_{\dot{x}} K\big) = -g(\nabla V, K) = 0.$$

□

Denote by $\mathfrak{Kill}(M, g)$ the Lie algebra of Killing vector field of (M, g), endowed with the Lie bracket $[K_1, K_2] = \nabla_{K_1} K_2 - \nabla_{K_2} K_1$. Moreover, let $\mathfrak{Sym}(M, g, V)$ the space of all infinitesimal symmetries of (M, g, V).

Lemma 2.5 *$\mathfrak{Sym}(M, g, V)$ is a Lie subalgebra of $\mathfrak{Kill}(M, g)$.*

Proof $\mathfrak{Sym}(M, g, V)$ is obviously a vector subspace of $\mathfrak{Kill}(M, g)$. Moreover, given $K_1, K_2 \in \mathfrak{Sym}(M, g, V)$, then $[K_1, K_2]V = K_1(K_2 V) - K_2(K_1 V) = 0$, i.e., $[K_1, K_2] \in \mathfrak{Sym}(M, g, V)$. □

2.3 *Infinitesimal Symmetries and Trajectories of Conservative Systems*

Given an infinitesimal symmetry K of (M, g, V), let us introduce $G_K : M \to \mathbb{R}$ the *characteristic function of K*, which is the smooth map defined by:

$$G_K(p) = \tfrac{1}{2} g(K_p, K_p) - V(p), \qquad p \in M.$$

It is easy to see that G_K is K-invariant, i.e., $K(G_K) = g(\nabla G_K, K) = 0$ everywhere. In particular, if p is a critical point of G_K, then every point of the integral line of K through p consists of critical points of G_K. Critical points of G_K are important because of the following property.

Proposition 2.6 *Let K be an infinitesimal symmetry of (M, g, V). An integral curve of K is a trajectory of (M, g, V) is and only if it passes through a critical point of G_K.*

Proof Let $p \in M$ be a point and let $x : [0, b] \to M$ the integral curve of K such that $x(0) = p$. Assume that x is a trajectory of (M, g, V), then:

$$\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{x}(0) = \nabla_{K_p} K = -\nabla V(p).$$

If $w \in T_p M$ is an arbitrary vector, then:

$$w(G_K) = g(\nabla_w K, K) - g\big(\nabla V(p), w\big) = -g(\nabla_{K_p} K, w) - g\big(\nabla V(p), w\big) = 0,$$

i.e., p is a critical point of G_K. The converse is established by the same argument, using the observation that if x is an integral line of K passing through some critical point of G_K, then every point of x is critical for G_K. □

When $V = 0$, Proposition 2.6 says that the integral lines of a Killing vector field K passing through the critical points of the function $g(K, K)$ are geodesics, see [13, Chap. VI, Proposition 5.7, p. 252].

By Lemma 2.4, the trajectories of (M, g, V) that are obtained as integral curves of K satisfy $g(\dot{x}, \dot{x}) = g(\dot{x}, K) = \text{const.}$; in particular, also V is constant along such solutions.

If one is interested in determining trajectories of (M, g, V) with a prescribed value E of the total energy, there is a more appropriate approach. Given an infinitesimal symmetry K of (M, g, V) and a fixed real number $E > \min\limits_M V$, let us consider the following smooth non negative function $F_{K,E} : M \to \mathbb{R}$:

$$F_{K,E}(p) = \big(E - V(p)\big)\, g(K_p, K_p), \qquad p \in M. \tag{3}$$

As in the case of the characteristic function G_K, also $F_{K,E}$ is K-invariant; moreover, if p is a critical point of $F_{K,E}$, then the integral curve of K through p consists entirely of critical points of $F_{K,E}$.

Proposition 2.7 *Let K be an infinitesimal symmetry of (M, g, V), let $E > \min\limits_M V$ be fixed and let x be an integral curve of K that passes through some critical point of $F_{K,E}$ in M_E. Then, there exists a reparameterization of x which is a trajectory of (M, g, V) having total energy E.*

Proof Using Maupertuis principle, it suffices to show that an integral curve of K passing through some critical point of $F_{K,E}$ in M_E is a geodesic in the Riemannian metric (M_E, g_E). Note that since the flow of K preserves V, then M_E is invariant by the flow of K, i.e., integral curves of K that pass through some point of M_E are entirely contained in M_E. The fact that integral curves of K passing through some critical point of $F_{K,E}$ in M_E are geodesics in the Riemannian metric (M_E, g_E) follow immediately from Proposition 2.6 applied to the conservative dynamical system whose underlying Riemannian manifold is (M_E, g_E) and whose potential function is $V_E \equiv 0$, observing that for this dynamical system the characteristic function G_K coincides with $F_{K,E}$. □

3 Exact Magnetic Equation with Symmetries in Riemannian Manifolds

We will now consider a different dynamical system, that is the mathematical model for trajectories of electric charges under the action of a magnetic field.

3.1 Exact Magnetic Fields

Let (M, g) be a Riemannian manifold, and let ω be a 1-form on M; the vector field $B \in \mathfrak{X}(M)$ defined by $\omega = g(B, \cdot)$ will be called the *dual* vector field to ω. The exterior differential $\mathrm{d}\omega$ is given by:

$$\mathrm{d}\omega(v, w) = g(\nabla_v B, w) - g(v, \nabla_w B),$$

for all $v, w \in TM$. Let $\widehat{F}$ be the g-anti-symmetric $(1, 1)$-tensor on M defined by:

$$\mathrm{d}\omega(v, w) = g(v, \widehat{F}w),$$

for all $v, w \in TM$; an easy calculations shows that $\widehat{F}$ can be written in terms of B as:

$$\widehat{F} = (\nabla B)^\star - \nabla B,$$

where $(\nabla B)^\star$ is the g-adjoint of ∇B, defined by:

$$g\big((\nabla B)^\star v, w\big) = g(\nabla_w B, v)$$

for all $v, w \in TM$. The triple (M, g, ω) will be called an *exact magnetic dynamical system*. Here M represents the configuration space of an electric charge, and the exact 2-form $\mathrm{d}\omega$ represents the magnetic field. By a *trajectory* of the magnetic dynamical system we mean a smooth curve $z : [0, T] \to M$ for which there exists a constant $\varrho \in \mathbb{R}$ such that z is a solution of the second order differential equation:

$$\tfrac{\mathrm{D}}{\mathrm{d}t}\dot{z} = \varrho \cdot \widehat{F}(z)\dot{z}. \tag{4}$$

Solutions of (4) represent trajectories of particles having electric charge equal to (a suitable multiple of) ϱ (see [3, 9, 16]); such constant ϱ will be called the *electric charge* of the trajectory z. If $z : [0, L] \to M$ is a trajectory of (M, g, ω), then the quantity $g(\dot{z}, \dot{z})$ is constant on $[0, L]$.

3.2 Symmetries of Exact Magnetic Fields

Let (M, g, ω) be an exact magnetic dynamical system and let B be the dual vector field to ω.

Definition 3.1 An isometry $\phi \in \mathrm{Iso}(M, g)$ is called a symmetry of (M, g, ω) if ω is ϕ-invariant, i.e., if the pull-back $\phi^*(\omega)$ coincides with ω.

It is easy to show that if ϕ is a symmetry of (M, g, ω) and $z : [0, L] \to M$ is a trajectory of (M, g, ω), then also $\phi \circ z$ is a trajectory of (M, g, ω). Let $\mathrm{Sym}(M, g, \omega)$ denote the subset of $\mathrm{Iso}(M, g)$ consisting of symmetries of (M, g, ω). Clearly, $\mathrm{Sym}(M, g, \omega)$ is a subgroup of $\mathrm{Iso}(M, g)$, and it is closed (relatively to the topology of C^1-convergence on compact subsets).

Thus, in perfect analogy with Lemma 2.2, we have:

Lemma 3.2 $\mathrm{Sym}(M, g, \omega)$ *is a closed Lie subgroup of* $\mathrm{Iso}(M, g)$.

An infinitesimal symmetry of (M, g, ω) will be a Killing vector field $K \in \mathfrak{Kill}(M, g)$ whose flow consists of (local) symmetries for (M, g, ω). More precisely.

Definition 3.3 A Killing field $K \in \mathfrak{Kill}(M, g)$ is an infinitesimal symmetry of (M, g, ω) if the Lie derivative $\mathcal{L}_K(\omega)$ of ω vanishes.

The condition $\mathcal{L}_K(\omega) = 0$ can be written in terms of the vector field B dual to ω, observing that, for all $v \in TM$:

$$\mathcal{L}_K(\omega)v = g\big(\nabla_K B, v\big) + g\big(B, \nabla_v K\big) = g\big(\nabla_K B - \nabla_B K, v\big) = g\big([K, B], v\big).$$

Thus, $K \in \mathfrak{Kill}(M, g)$ is an infinitesimal symmetry of (M, g, ω) if and only if:

$$\mathcal{L}_K(B) = [K, B] = 0. \tag{5}$$

If K is an infinitesimal symmetry of (M, g, ω), then the quantity $g(B, K)$ is constant along the flow lines of K; namely:

$$\begin{aligned} K\big(g(B, K)\big) = g\big(\nabla_K B, K\big) + g\big(B, \nabla_K K\big) = g\big(\nabla_K B, K\big) - g\big(K, \nabla_B K\big) \\ = g\big([K, B], K\big) = 0. \end{aligned}$$

Moreover, an infinitesimal symmetry K of (M, g, ω) gives the following conservation law for the trajectories of (M, g, ω).

Lemma 3.4 *If K is an infinitesimal symmetry of (M, g, ω) and $z : [0, L] \to M$ is a trajectory of (M, g, ω) with electric charge ϱ, then the quantity*

$$g(\dot{z}, K) + \varrho \cdot g(B, K)$$

is constant along z.

Proof An immediate calculation:

$$
\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\big[g(\dot z, K) + \varrho \cdot g(B, K)\big] \\
&\qquad = g\big(\tfrac{\mathrm{D}}{\mathrm{d}t}\dot z, K\big) + \big(\dot z, \nabla_{\dot z} K\big) + \varrho \cdot g\big(\nabla_{\dot z} B, K\big) + \varrho \cdot g\big(B, \nabla_{\dot z} K\big) \\
&\qquad = \varrho\big[g\big(\nabla_K B, \dot z\big) - g\big(\nabla_{\dot z} B, K\big) + g\big(\nabla_{\dot z} B, K\big) - g\big(\nabla_B K, \dot z\big)\big] \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = g\big([K, B], \dot z\big) = 0.
\end{aligned}
$$

□

Let $\mathfrak{Sym}(M, g, \omega)$ be the space of all infinitesimal symmetries of (M, g, ω).

Lemma 3.5 *$\mathfrak{Sym}(M, g, \omega)$ is a Lie subalgebra of $\mathfrak{Kill}(M, g)$.*

Proof Clearly, $\mathfrak{Sym}(M, g, \omega)$ is a vector subspace of $\mathfrak{Kill}(M, g)$.

If $K_1, K_2 \in \mathfrak{Sym}(M, g, \omega)$, then $[K_1, B] = [K_2, B]$ and thus, by Jacobi identity:

$$\big[[K_1, K_2], B\big] = \big[[K_1, B], K_2\big] + \big[K_1, [K_2, B]\big] = 0,$$

i.e., $[K_1, K_2] \in \mathfrak{Sym}(M, g, \omega)$. □

3.3 Infinitesimal Symmetries and Trajectories of Magnetic Dynamical Systems

Given an infinitesimal symmetry K of (M, g, ω), we introduce two functions on M, denoted by $H_K^+, H_K^- : M \to \mathbb{R}$ and defined by:

$$H_K^{\pm}(p) = \tfrac{1}{2}\big[g(K_p, K_p) \pm \omega_p(K_p)^2\big] = \tfrac{1}{2}\big[g(K_p, K_p) \pm g(B_p, K_p)^2\big].$$

An immediate calculation shows that $H_K^{\pm}$ is K-invariant; in particular, if p is a critical point of $H_K^{\pm}$, then the integral curve of K through p consists entirely of critical points of $H_K^{\pm}$.

Proposition 3.6 *Let K be an infinitesimal symmetry of (M, g, ω). An integral curve of K through a critical point p of $H_K^{\pm}$ is a trajectory of (M, g, ω) with electric charge $\varrho = \mp g(K_p, B_p)$.*

Proof Since the integral curve z of K through p consists entirely of critical points of $H_K^{\pm}$, it suffices to show that z satisfies (4) at the point p. Set $\varrho = \mp g(K_p, B_p)$; note that $g(K, B)$ is constant along z, as it follows easily from Lemma 3.4. If w is an arbitrary vector in T_pM, then:

$$
\begin{aligned}
0 = w\big[\tfrac{1}{2}g(K,K) \pm \tfrac{1}{2}g(B,K)^2\big] \\
= g\big(\nabla_w K, K_p) \pm g(K_p, B_p)\big[g\big(\nabla_w B, K_p\big) + g\big(B_p, \nabla_w K\big)\big] \\
= -g\big(\nabla_{K_p} K, w\big) - \varrho\big[g\big(\nabla_w B, K_p\big) - g\big(w, \nabla_{B_p} K\big)\big] \\
\overset{\text{by (5)}}{=} -g\big(\nabla_{K_p} K, w\big) - \varrho\big[g\big(\nabla_w B, K_p\big) - g\big(w, \nabla_{K_p} B\big)\big] \\
= -g\big(\nabla_{K_p} K, w\big) - \varrho\big[g\big(\big[\nabla B - (\nabla B)^\star\big]w, K_p\big),
\end{aligned}
$$

thus:

$$
\nabla_{K_p} K = \varrho\big[(\nabla B)^\star - \nabla B\big]K_p,
$$

which concludes the proof. □

4 Existence of Periodic Trajectories

We will now establish an existence result for periodic trajectories of a conservative dynamical system (M, g, V) and of an exact magnetic dynamical system (M, g, ω). We will assume henceforth that M **is compact**; this implies in particular that the isometry group $\mathrm{Iso}(M, g)$ of (M, g) is compact, see [12]. If K is a Killing vector field for (M, g), then its flow consists of a 1-parameter group of (globally defined) isometries of (M, g).

Lemma 4.1 $\mathfrak{Sym}(M, g, V)$ *is the Lie algebra of the Lie subgroup* $\mathrm{Sym}(M, g, V)$ *and* $\mathfrak{Sym}(M, g, \omega)$ *is the Lie algebra of the Lie subgroup* $\mathrm{Sym}(M, g, \omega)$.

Proof Left to the reader as an exercise. □

Thus, the conservative dynamical system (M, g, V) admits a non trivial symmetry if and only if $\dim\big(\mathrm{Sym}(M, g, V)\big) > 0$; similarly, the magnetic dynamical system $(M, g, \omega))$ admits a non trivial symmetry if and only if $\dim\big(\mathrm{Sym}(M, g, \omega)\big) > 0$.

4.1 Closed Infinitesimal Symmetries

Propositions 2.6 and 3.6 suggest that we can look for *periodic* trajectories of (M, g, V) and (M, g, ω) that are *closed* integral curves of an infinitesimal symmetry K. An infinitesimal symmetry K of (M, g, V) or of (M, g, ω) will be called *closed* if all its integral curves are closed, i.e., homeomorphic to circles or constant. Let us recall the following result from [8]:

Lemma 4.2 *Let* (M, g) *be a compact Riemannian manifold and let* K *be a Killing vector field. Then* K *is closed if and only if* K *generates a closed* 1*-parameter subgroup of isometries of* $\mathrm{Iso}(M, g)$.

A non trivial closed 1-parameter subgroup of $\mathrm{Iso}(M, g)$ is isomorphic to the circle $\mathbb{S}^1$; thus, the existence of a non trivial closed Killing vector field for (M, g) is equivalent to the existence of a smooth isometric action of $\mathbb{S}^1$ without fixed points, see Sect. 5.

4.2 Existence of Periodic Solutions

A preliminary result on the existence of periodic trajectories:

Proposition 4.3 *Assume that the conservative dynamical system (M, g, V) admits a non trivial closed infinitesimal symmetry K. Then, for every $E > \min_M V$ there exists a periodic trajectory x of (M, g, V) having total energy E.*

Similarly, if the exact magnetic dynamical system (M, g, ω) has a non trivial closed infinitesimal symmetry, then there exist a positive constant $\varrho_+ > 0$ and a (non constant) periodic solution $x_+ : [0, L_+] \to M$ of (4) with $\varrho = \varrho_+$.

Proof Using Maupertuis principle, it suffices to show that for all $E > \min_M V$, the conformal metric g_E in the set M_E has non trivial closed geodesics. Since the flow of K preserves the metric g and the function V, then the flow of K preserves the set M_E, and K is a Killing field also for the conformal metric g_E. The function $F_{K,E}$ is equal to $g_E(K, K)$, and applying Proposition 2.7 to the conservative dynamical system with underlying Riemannian manifold (M_E, g_E) and potential function $V_E \equiv 0$, we obtain that an integral curve of K passing through some critical point of $F_{K,E}$ is a geodesic for (M_E, g_E). Since K is closed, any integral curve of K is closed, thus it suffices to show that $F_{K,E}$ has a critical point p in M_E where $K_p \neq 0$. We claim that $F_{K,E}$ has maximum in M_E, which is not zero; the proof of the first statement in the thesis will be concluded once we prove the claim. Namely, since K is Killing for (M, g), then K cannot vanish identically on any non empty open set of M, see [19, Chap. 9, Lemma 27]; thus, since $E > \min V$, then K does not vanish identically on M_E. Since $F_{K,E}$ vanishes on the boundary ∂M_E of M_E, then the (positive) maximum of $F_{K,E}$ on the compact set $\overline{M}_E = M_E \bigcup \partial M_E$ is attained at some point p of M_E. This proves the claim.

As to the second statement of the thesis, if $K \in \mathfrak{Sym}(M, g, \omega)$ is closed and non trivial, then let $\varrho_+ = H_K^+(p_+)$ be the maximum of the function Hk_K on M, attained at the point $p_+ \in M$. It must be $\varrho_+ > 0$, because K does not vanish identically. If x_+ denotes the integral curve of K through p_+, then by Proposition 3.6, x_+ is a solution of (4) with $\varrho = \varrho_+$; x is a periodic trajectory of (M, g, ω) because K is closed. $\square$

Remark 4.4 More precise estimates on the number of periodic trajectories of a conservative dynamical system or of an exact magnetic dynamical system can be done with an analysis of the critical points of the functions $F_{K,E}$ and $H_K^\pm$. This is left as an exercise for the reader. (A hint for the proof of a second non constant periodic trajectory of (M, g, ω): if either $\min_M H_K^-$ or $\max_M H_K^-$ are non zero, then the corresponding integral curve of K is a non constant periodic trajectory. If $\min_M H_K^- = \max_M H_K^- = 0$, then every integral curve of K is a periodic trajectory.)

It is interesting to observe that the periodic solution x whose existence is proved in Proposition 4.3 does not have self-intersection. This follows easily from the fact that it is the integral line of a vector field, i.e., it is the solution of a first order differential equation.

We will now establish an existence result for closed infinitesimal symmetries.

Proposition 4.5 *Given an infinitesimal symmetry K of the conservative system (M, g, V) (resp., of (M, g, ω)), there exists a sequence K_n of* closed *infinitesimal symmetries K_n of (M, g, V) (resp., of (M, g, ω)) that converges uniformly to K as $n \to \infty$. In particular, if K is not identically zero, then there exists a closed infinitesimal symmetry of (M, g, V) (resp., of (M, g, ω)) which is not identically zero.*

Proof Since $\mathrm{Iso}(M, g)$ is compact, then by Lemma 2.2, $\mathrm{Sym}(M, g, V)$ is compact (resp., by Lemma 3.2, $\mathrm{Sym}(M, g, \omega)$ is compact). Using Lemma 4.2, we will show that K is the limit of a sequence of Killing vector fields that generate a closed subgroup of isometries in $\mathrm{Sym}(M, g, V)$ (resp., in $\mathrm{Sym}(M, g, \omega)$). Let $G \subset \mathrm{Sym}(M, g, V)$ (resp., $G \subset \mathrm{Sym}(M, g, \omega)$) be the 1-parameter group of isometries generate by K. If G is not closed, let $\overline{G}$ be its closure, which is a compact abelian subgroup of $\mathrm{Sym}(M, g, V)$ (resp., of $\mathrm{Sym}(M, g, \omega)$). Then, $\overline{G}$ is a torus of dimension greater than or equal to 2. Thus, the tangent vector $\mathfrak{v} \in T_e\overline{G} \subset \mathfrak{Sym}(M, g, V)$ (resp., $\mathfrak{v} \in T_e\overline{G} \subset \mathfrak{Sym}(M, g, \omega)$), where e is the identity, is limit of a sequence $\mathfrak{v}_n \in \mathfrak{Sym}(M, g, V)$ (resp., $\mathfrak{v}_n \in \mathfrak{Sym}(M, g, \omega)$) such that the corresponding Killing field $K^{\mathfrak{v}_n}$ are closed. This concludes the proof. □

Corollary 4.6 *If $\mathfrak{Sym}(M, g, V) \neq \{0\}$, then for every $E > \min_M V$ there exists a periodic trajectory x of (M, g, V) having total energy E.*

Similarly, $\mathfrak{Sym}(M, g, \omega) \neq \{0\}$, then there exists a periodic trajectory x of (M, g, ω) with positive electric charge ϱ.

Proof By Proposition 4.5, if $\mathfrak{Sym}(M, g, V) \neq \{0\}$ (resp., $\mathfrak{Sym}(M, g, \omega) \neq \{0\}$) then there exists at least one non trivial closed infinitesimal symmetry K of (M, g, V) (resp., of (M, g, ω). The conclusion now follows from Proposition 4.3. □

5 Topology of *M* and Multiplicity of Periodic Solutions

The material in this section follows closely [8, Sect. 3].

5.1 Fibration Associated to a Closed Killing Vector Field

A compact manifold M will be called a *generalized Seifert fibered space* if it admits a smooth action of the circle $\mathbb{S}^1$ without fixed points or, equivalently, with finite isotropy. The orbits of a fixed point free action of $\mathbb{S}^1$, that are diffeomorphic to $\mathbb{S}^1$, are called the *fibers* of the fibered space. Low dimensional simply connected generalized Seifert fibered spaces are classified, see [6, 7, 14, 22, 23]. By standard results on group actions, the orbit space of a smooth action of a compact Lie group on a compact manifold having finite isotropy has the structure of a compact orbifold (see the Appendix of E. Salem in [17] for details on orbifolds; the book contains also a more general result on the orbifold structure of orbit spaces in the context of Riemannian foliations).

Proposition 5.1 *A compact manifold M admits a Riemannian metric tensor g whose isometry group* $\mathrm{Iso}(M, g)$ *has positive dimension if and only if it is diffeomorphic to a generalized Seifert fibered space.*

Proof If $\dim\big(\mathrm{Iso}(M, g)\big) > 0$, then by Proposition 4.5 (applied with $V \equiv 0$) there exists a non trivial closed Killing vector field K of (M, g). The one-parameter group of isometries generated by such a Killing field gives a smooth action of $\mathbb{S}^1$ without fixed points; K is tangent to the fibers of this action. Conversely, given a smooth action of $\mathbb{S}^1$ on M without fixed points, by a standard averaging argument one can find a Riemannian metric tensor g which makes such action isometric, i.e., the infinitesimal generator K of this action is Killing (see for instance [11]). $\square$

Remark 5.2 Given a *free* action of $\mathbb{S}^1$ on a compact manifold M, then the orbit space $M/\mathbb{S}^1$ is a smooth manifold (see for instance [2, Theorem 23.4] or [13, Theorem 4.3]). We observe however that in general the quotient space $M_0 = M/\mathbb{S}^1$ is not a manifold. As an example, consider M to be the Klein bottle obtained as the quotient of $\mathbb{R}^2$ endowed with the Euclidean metric $\mathrm{d}x^2 + \mathrm{d}y^2$ by the action of the group generated by the isometries $(x, y) \mapsto (x + 1, y)$ and $(x, y) \mapsto (1 - x, y + 1)$. The vector field $K = \frac{\partial}{\partial y}$ on M is Killing; all its integral lines are periodic. It is easily seen that in this case the $\mathbb{S}^1$-action induced by the flow of K has exactly two exceptional orbits, and that the orbit space $M/\mathbb{S}^1$ is homeomorphic to the closed interval $[0, \frac{1}{2}]$.

As a corollary of Proposition 5.1, we get a somewhat better estimate on the number of periodic solutions of dynamical systems based on the *Ljusternik–Schnirelman category*. Recall that the Ljusternik–Schnirelman category (shortly, LS category) $\mathrm{cat}(\mathcal{X})$ of a topological space $\mathcal{X}$ is the cardinality (possibly infinite) of a minimal family of closed contractible subsets of $\mathcal{X}$ whose union covers $\mathcal{X}$. If $\mathcal{X}$ is G-space, i.e., a topological space on which a compact group G is acting continuously, then one can define the equivariant notion of Ljusternik–Schnirelman G-category $\mathrm{cat}_G(\mathcal{X})$ (see for instance [15]). A homotopy $H : U \times [0, 1] \to \mathcal{X}$ of an open G-invariant set $U \subset \mathcal{X}$ is called *G-equivariant* if $gH(x, t) = H(gx, t)$ for any $g \in G$, $x \in U$ and $t \in [0, 1]$. The set U is *G-categorical* if there is a G-homotopy H with $H(\cdot, 0)$ the identity, and $H(\cdot, 1)$ maps U to a single orbit. The *equivariant category* $\mathrm{cat}_G(\mathcal{X})$ is the cardinality of a minimal family of G-categorical open sets whose union covers $\mathcal{X}$.

If G is a compact Lie group, $\mathcal{X}$ is a smooth G-manifold, and $h : \mathcal{X} \to \mathbb{R}$ is a smooth function which is G-invariant, then h has at least $\mathrm{cat}_G(\mathcal{X})$ distinct critical G-orbits (see [15, Theorem 3.2]).

Corollary 5.3 *Let (M, g, V) be a conservative dynamical system, with M compact, having a non trivial closed infinitesimal symmetry K. Consider the $\mathbb{S}^1$-action on M determined by K. Then, for all $E > \max_M V$, there are at least $\mathrm{cat}_{\mathbb{S}^1}(M)$ distinct periodic non self-intersecting periodic solutions of (M, g, V) having energy E.*

Proof If $E > \max_M V$, then the open set M_E in (2) coincides with M. The function $F_{K,E} : M \to \mathbb{R}$ (recall (3)) is constant on the orbits of $G = \mathbb{S}^1$, thus it has at least $\mathrm{cat}_G(M)$ critical orbits. Hence, the proof follows by observing that distinct critical G-orbits of $F_{K,E}$ in M correspond to distinct non self-intersecting periodic solutions of (M, g, V) having total energy equal to E. □

We have an analogous result for periodic trajectories of exact magnetic dynamical systems:

Corollary 5.4 *Let (M, g, ω) be an exact magnetic dynamical system, with M compact, having a non trivial closed infinitesimal symmetry K. Consider the $\mathbb{S}^1$-action on M determined by K. Then, there are at least $\mathrm{cat}_{\mathbb{S}^1}(M)$ distinct periodic non self-intersecting periodic solutions of (M, g, ω).*

Proof Such solutions are integral curves of K that are critical orbits of the function H_K^+ on M. The estimate on the number of periodic trajectories of (M, g, ω) can be improved considering critical orbits of the function H_K^-. This is left as an exercise for the reader. □

In Corollaries 5.3 and 5.4, note that the integer $\mathrm{cat}_{\mathbb{S}^1}(M)$ is greater than or equal to 2. Namely, if it were $\mathrm{cat}_{\mathbb{S}^1}(M) = 1$, then M would be (equivariantly) homotopic to an orbit of $\mathbb{S}^1$, which is diffeomorphic to $\mathbb{S}^1$. But, no compact manifold of dimension greater than or equal to 2 is homotopic to $\mathbb{S}^1$. Observe also that in general the equivariant LS category $\mathrm{cat}_{\mathbb{S}^1}(M)$ is greater than or equal to the LS category $\mathrm{cat}(M/\mathbb{S}^1)$ of the quotient space $M/\mathbb{S}^1$. The Klein bottle in Remark 5.2 provides an example where such inequality is strict: here the quotient space $M/\mathbb{S}^1$ is contractible, and thus $\mathrm{cat}(M/\mathbb{S}^1) = 1$, while it is easily computed $\mathrm{cat}_{\mathbb{S}^1}(M) = 2$.

6 Dynamical Systems in Semi-Riemannian Manifolds

It is desirable to extend the results of existence of periodic solutions to the case of dynamical systems whose underlying geometry is given by a compact *semi-Riemannian manifold* (M, g). Recall that g is a semi-Riemannian metric tensor if it is everywhere nondegenerate, but not necessarily positive definite; a semi-Riemannian metric tensor g is *Lorentzian* if it has index 1.

The notions of conservative dynamical system and exact magnetic system extend naturally by the very same definition to the case of semi-Riemannian manifolds. Several results discussed here for dynamical systems in Riemannian manifolds, namely those in Sects. 2 and 3 as well as Proposition 4.3 (where the positive definite character of the metric tensor has not been used) extend by the same arguments to general semi-Riemannian dynamical systems. However, the main difference is that for an arbitrary semi-Riemannian manifold the (connected component of the identity of the) isometry group is not compact. Thus, the central result of Proposition 4.5 does not hold, unless one assumes that the 1-parameter subgroup of isometries generated by the infinitesimal symmetry K is *precompact*. Recall that, by the Arzelà–Ascoli theorem, precompactness is equivalent to equicontinuity.

Proposition 6.1 *The result of Proposition 4.5 holds for conservative or exact magnetic dynamical systems on a compact semi-Riemannian manifold* (M, g) *under the assumption that the infinitesimal symmetry K generates a precompact one-parameter subgroup of the isometry group* $\mathrm{Iso}(M, g)$.

The question of precompactness of one-parameter subgroups of the isometry group of a compact Lorentzian manifold has been studied recently in [8] and especially in [20]. Let us recall the following situation where Proposition 6.1 can be used.

- If (M, g) is a compact real-analytic Lorentzian manifold, then $\mathrm{Iso}(M, g)$ is compact; this is proved in [4].
- if (M, g) is a compact Lorentzian manifold and K is a Killing vector field which is timelike at some point $p \in M$, i.e., $g(K_p, K_p) < 0$, then K generates a precompact one-parameter subgroup of isometries of (M, g); this is proved in [8] and in [20], see also [21].
- Let (M, g) be a compact Lorentz manifold that admits a Killing vector field which is timelike at some point. Then, the identity component of its isometry group is compact, unless it contains a group locally isomorphic to $\mathrm{SL}(2, \mathbb{R})$ or to an *oscillator group* (in particular, the one-parameter subgroup generated by any Killing vector field is precompact). This is proved in [20].

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