

Chapter 2

Extensions of the Linear Theory

Abstract For more than a century since it was first formulated, the stable population model has been used as a standard theory in demography. In fact, after World War II, the population explosions in developing countries became an important challenge to demographers working with prediction and control problems of Malthusian populations. Since the latter half of the 1970s, the onset of a *second demographic transition* (namely the persistence of below-replacement fertility) has been observed in developed countries, and the subsequent population problems require much more sophisticated tools to deal with the heterogeneities of individuals and environments. In this chapter, we introduce some linear extensions of the stable population model. These extensions are essentially multidimensional stable population models, so it is not surprising that the strong ergodicity theorem again holds. In contrast, we also examine non-autonomous systems to address the population dynamics in heterogeneous environments. In this case, we cannot expect a stable age distribution to exist, and thus introduce the idea of *weak ergodicity*. Using these extended models, we establish more elaborate demographic indices and characterizations for heterogeneous populations, and demonstrate their importance for applications in demography, epidemiology, and population biology.

2.1 Multistate Stable Population Model

In the previous chapter, we considered a population structured only by age and sex. Here, we deal with a model for a *multistate population*. This is an age-structured population divided according to some discrete state variables that indicate individual heterogeneity, such as the region of residence, parity status, labor status, and marital status. During the 1970s, Andrei Rogers and Herve Le Bras independently extended the stable population model to deal with multiregional populations and their migration between regions [70, 89]. In particular, Rogers and his collaborators developed numerical methods for parameter estimation from real data and computer simulations, and their results were successfully applied to both regional populations and other multistate populations [66, 98]. It is clear that extending the stable population model to the multistate model was an important first step in accounting for

the heterogeneity of populations. Because many quantitative problems in the social sciences, biology, and epidemiology can be formulated using multistate populations; the idea of multistate demography has a number of potential applications.

As we shall show in this chapter, the asymptotic behavior of the linear multistate population model is analogous to that of the stable population model. However, it is far more difficult to calculate the parameters of the multistate model from available data, so multistate demography has a greater focus on the parameter estimation problem. In this chapter, we present only a sketch of the mathematical properties of linear multistate population models.

Suppose that a one-sex population is divided into n states, and denote each different state by the index $i = 1, 2, \dots, n$. Each state of a population refers to any individual trait that could affect the life cycle parameters, for example, the place of residence, labor force status, marital status, or parity status (the number of children a woman has had). Let $p_i(t, a)$ be the age-density function of the i -th state, and let

$$p(t, a) = (p_1(t, a), \dots, p_n(t, a))^T$$

be the population vector, where T denotes the transpose of the vector. For simplicity, we assume that $\omega = \infty$ and the state space of a population density $p_i(t, \cdot)$ is $L_+^1(\mathbb{R}_+)$.

Let $q_{ij}(a)$, $i \neq j$, be the *force of transition* from state j to state i , and let $\mu_j(a)$ be the force of mortality at state j . Moreover, we define

$$q_{jj}(a) = -\mu_j(a) - \sum_{i \neq j} q_{ij}(a), \quad 1 \leq j \leq n. \quad (2.1)$$

Let $Q(a)$ be an $n \times n$ matrix whose (i, j) -th entry is given by $q_{ij}(a)$, $1 \leq i, j \leq n$. Let $m_{ij}(a)$ be the age-specific fertility rate that gives rise to i -th state newborns from j -th state individuals. Let $M(a)$ be an $n \times n$ matrix whose (i, j) -th element is given by $m_{ij}(a)$, $1 \leq i, j \leq n$.

The *multistate stable population model* is then formulated by the following initial-boundary value problem of the vector-type McKendrick equation:

$$\begin{aligned} \frac{\partial p(t, a)}{\partial t} + \frac{\partial p(t, a)}{\partial a} &= Q(a)p(t, a), \quad t > 0, \quad a > 0, \\ p(t, 0) &= \int_0^\infty M(a)p(t, a)da, \quad t > 0, \\ p(0, a) &= p_0(a), \end{aligned} \quad (2.2)$$

where $p_0(a)$ is the initial population vector.

To derive the renewal equation for the multistate model, let us consider the evolutionary system (transition matrix) $\{L(a, \sigma) : a \geq \sigma \geq 0\}$ satisfying the forward equation

$$\frac{\partial}{\partial a} L(a, \sigma) = Q(a)L(a, \sigma), \quad L(\sigma, \sigma) = I, \quad (2.3)$$

where I is the $n \times n$ identity matrix. Then, $L(a, \sigma)$ is the fundamental solution matrix for the master equation

$$\frac{d\ell(a)}{da} = Q(a)\ell(a),$$

where $\ell(a) := (\ell_1(a), \dots, \ell_n(a))^T$; $\ell_j(a)$ denotes the probability that an individual is in state j at age a , and $\ell(0)$ gives the state distribution at age zero, so $\sum_{j=1}^N \ell_j(0) = 1$. The evolutionary property

$$L(a, \tau)L(\tau, \sigma) = L(a, \sigma), \quad a \geq \tau \geq \sigma$$

holds, and the (i, j) -th entry $L(a, \sigma)$ denotes the probability that an individual of age σ and state j will survive to age a and state i . For simplicity, $L(a, 0)$ is often written as $L(a)$. This is a multidimensional extension of the survival probability $\ell(a)$ in the scalar model, because its (i, j) -th entry $\ell_{ij}(a)$ gives the probability that an individual born in state j will survive to age a and state i . From (2.1), $Q(a)$ is an *essentially non-negative matrix*¹; that is, all off-diagonal elements are non-negative, and the fundamental solution matrix $L(a, \sigma)$ is a non-negative, non-singular matrix.²

Using the transition matrix, the McKendrick equation can be integrated along the characteristic line to obtain the following expression:

$$p(t, a) = \begin{cases} L(a, 0)B(t - a), & t - a > 0, \\ L(a, a - t)p_0(a - t), & a - t > 0, \end{cases} \quad (2.4)$$

where $B(t) := p(t, 0)$. Inserting expression (2.4) into the boundary condition in (2.2), we obtain the renewal integral equation

$$B(t) = G(t) + \int_0^t \Psi(a)B(t - a)da, \quad (2.5)$$

where $\Psi(a) := M(a)L(a, 0)$ and

$$G(t) := \int_t^\infty M(a)L(a, a - t)p_0(a - t)da.$$

For our population problems, we can assume that $G(\cdot)$ and $\Psi(\cdot)$ are bounded and compactly supported integrable (vector-valued and matrix-valued) functions. In such a case, it is well known that the vector-type renewal integral equation (2.5) has

¹An essentially non-negative matrix is also called *quasi-positive* if it is not zero [104].

²If a square matrix A is irreducible and essentially non-negative, it is called *essentially positive*. A is essentially positive if and only if $A + sI_d$ is a non-negative, irreducible, and primitive matrix for all sufficiently large $s > 0$. Moreover, a square matrix A is essentially non-negative if and only if $e^{At} \geq 0$ for all $t \geq 0$, and A is essentially positive if and only if $e^{At} > 0$ for all $t > 0$ (see Chap. 10, [17, 107]).

a unique locally integrable solution for $t \in \mathbb{R}_+$. If we know the behavior of $B(t)$, $p(t, a)$ is completely determined by (2.4).

Starting from the renewal equation (2.5), we can again determine the asymptotic behavior of the multistate model (2.2) as in Chap. 1. However, instead of repeating the same kind of argument, here we sketch another intuitive approach. A rigorous justification of the following is given by the semigroup approach [49, 50].

In the previous chapter, we took the Laplace transform of $B(t)$. Here, we take the Laplace transform of $p(t, a)$ directly:

$$\hat{p}(\lambda, a) := \int_0^\infty e^{-\lambda t} p(t, a) dt, \quad \lambda \in \mathbb{C}.$$

Because the norm of $p(t, a)$ grows at most exponentially, the Laplace transform of $p(t, a)$ exists for λ with a sufficiently large real part. By taking the Laplace transform of both sides of (2.2), we have

$$\begin{aligned} \frac{d\hat{p}(\lambda, a)}{da} &= p_0(a) + (-\lambda + Q(a))\hat{p}(\lambda, a), \\ \hat{p}(\lambda, 0) &= \int_0^\infty M(a)\hat{p}(\lambda, a)da. \end{aligned} \tag{2.6}$$

Solving the differential equation in (2.6), it follows that

$$\hat{p}(\lambda, a) = e^{-\lambda a} L(a) \hat{p}(\lambda, 0) + \int_0^a e^{-\lambda(a-s)} L(a, s) p_0(s) ds.$$

Inserting the above expression into the right-hand side of the boundary condition of (2.6) to solve for $\hat{p}(\lambda, 0)$, we obtain

$$\hat{p}(\lambda, 0) = (I - \hat{\Psi}(\lambda))^{-1} \int_0^\infty \int_s^\infty e^{-\lambda(a-s)} \Psi(a) da L^{-1}(s) p_0(s) ds.$$

Therefore, $\hat{p}(\lambda, a)$ is expressed as

$$\begin{aligned} \hat{p}(\lambda, a) &= \int_0^a e^{-\lambda(a-s)} L(a, s) p_0(s) ds \\ &+ e^{-\lambda a} L(a) (I - \hat{\Psi}(\lambda))^{-1} \int_0^\infty \int_s^\infty e^{-\lambda(a-s)} \Psi(a) da L^{-1}(s) p_0(s) ds. \end{aligned}$$

Taking the inverse transformation, we obtain the formal solution as

$$p(t, a) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\lambda t} \hat{p}(\lambda, a) d\lambda, \tag{2.7}$$

where the real number x is sufficiently large that $\hat{p}(x + i\mathbb{R})$ converges absolutely. Let us define a set of complex numbers denoted by Ω such that

$$\Omega := \{\lambda \in \mathbb{C} : \det(I - \hat{\Psi}(\lambda)) = 0\} = \{\lambda \in \mathbb{C} : 1 \in \sigma(\hat{\Psi}(\lambda))\},$$

where $\sigma(A)$ denotes the set of spectra of matrix A . Then, as $\hat{p}(\lambda, a)$ is a meromorphic function with poles at Ω , we can obtain its asymptotic expansion by shifting the integral path to the left, just as for the single-state stable population model in Chap. 1.

Let us again examine the structure of the set of characteristic roots Ω . First, define $\underline{\mu} := \inf \mu_j(a)$, which leads to the estimate:

Lemma 2.1

$$|L(b, a)| \leq e^{-\underline{\mu}(b-a)}, \quad (2.8)$$

where the norm of a matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ and of a vector $x = (x_1, \dots, x_n)$ are given by:

$$|x| = \sum_{i=1}^n |x_i|, \quad |A| = \max_j \sum_{i=1}^n |a_{ij}|.$$

Proof Let $\ell_{ij}(b, a)$ be the (i, j) -th element of $L(b, a)$. It follows that

$$\frac{\partial}{\partial b} \ell_{ij}(b, a) = \sum_{k=1}^n q_{ik}(b) \ell_{kj}(b, a), \quad \ell_{ij}(a, a) = \delta_{ij},$$

and

$$\begin{aligned} \frac{\partial}{\partial b} \sum_{i=1}^n \ell_{ij}(b, a) &= \sum_{i=1}^n \sum_{k=1}^n q_{ik}(b) \ell_{kj}(b, a) = \sum_{k=1}^n \sum_{i=1}^n q_{ik}(b) \ell_{kj}(b, a) \\ &= \sum_{k=1}^n (-\mu_k(b)) \ell_{kj}(b, a) \leq (-\underline{\mu}) \sum_{i=1}^n \ell_{ij}(b, a). \end{aligned}$$

Therefore, we can conclude that

$$\sum_{i=1}^n \ell_{ij}(b, a) \leq e^{-\underline{\mu}(b-a)},$$

which implies that (2.8) holds. \square

From the above preparation, we can state the following proposition [49, 50]:

Proposition 2.1 *Let $\bar{m} := \sup_{0 \leq a \leq \omega} |M(a)|$. Then, the following holds:*

- (1) Ω is included in the half-plane $\Re \lambda \leq \bar{m} - \underline{\mu}$.
- (2) If $\lambda \in \Omega$, then $\bar{\lambda} \in \Omega$.
- (3) For any $x \in \mathbb{R}$, the half-plane $\Re \lambda > x$ contains at most finitely many $\lambda \in \Omega$.

Proof We omit the proofs for (2) and (3), because they are the same as those for Proposition 1.7. Let us show (1). Because the matrix $\hat{\Psi}(\lambda)$ is non-negative for real λ , it has the Frobenius root $F(\lambda)$, $\lambda \in \mathbb{R}$. Let $\hat{\Psi}^*(\lambda)$, $\lambda \in \mathbb{C}$ be the matrix whose (i, j) -th element is given by the absolute value $|\hat{\psi}_{ij}(\lambda)|$ of the (i, j) -th element $\hat{\psi}_{ij}(\lambda)$ of $\hat{\Psi}(\lambda)$, and denote its Frobenius root as $F^*(\lambda)$. If we denote $r(A)$ as the *spectral radius*³ of a matrix A , it can be proved that $r(\hat{\Psi}(\lambda)) \leq F^*(\lambda)$. Moreover, it follows from $\hat{\Psi}^*(\lambda) \leq \hat{\Psi}(\Re \lambda)$ that $F^*(\lambda) \leq F(\Re \lambda)$. Hence, we have $r(\hat{\Psi}(\lambda)) \leq F(\Re \lambda)$. Observe that for $\Re \lambda > -\underline{\mu}$,

$$|\hat{\Psi}(\Re \lambda)| \leq \int_0^\infty e^{-\Re \lambda a} |\Psi(a)| da \leq \int_0^\infty e^{-\Re \lambda a} |M(a)| |L(a)| da \leq \frac{\bar{m}}{\Re \lambda + \underline{\mu}}.$$

Thus, it follows that

$$F(\Re \lambda) \leq \max_j \sum_{i=1}^n \hat{\psi}_{ij}(\Re \lambda) = |\hat{\Psi}(\Re \lambda)| \leq \frac{\bar{m}}{\Re \lambda + \underline{\mu}}.$$

Therefore, if $\Re \lambda > \bar{m} - \underline{\mu}$, we have $r(\hat{\Psi}(\lambda)) \leq F(\Re \lambda) < 1$. However, if $\lambda \in \Omega$, we have $r(\hat{\Psi}(\lambda)) \geq 1$. We can conclude that there is no element of Ω in the half-plane $\Re \lambda > \bar{m} - \underline{\mu}$. \square

For the multistate stable population model, we define the *net reproduction matrix* (*next-generation matrix* in epidemic terminology, see Chap. 9) by

$$K := \hat{\Psi}(0) = \int_0^\infty \Psi(a) da, \quad (2.9)$$

and the *net reproduction rate* (basic reproduction number)⁴ R_0 by its spectral radius

$$R_0 = r(\hat{\Psi}(0)). \quad (2.10)$$

Let $\hat{\psi}_{ij}(0)$ be the (i, j) -th entry of the net reproduction matrix:

$$\hat{\psi}_{ij}(0) = \sum_{k=1}^n \int_0^\infty m_{ik}(a) \ell_{kj}(a) da.$$

³ $r(A) = \max_{\lambda \in \sigma(A)} |\lambda|$, where $\sigma(A)$ denotes the set of eigenvalues of A , and it follows that $r(A) = \lim_{n \rightarrow \infty} |A^n|^{1/n}$.

⁴Readers are referred to [58] for a historical review of the multistate net reproduction rate.

We then define the native-dependent reproduction number by

$$R_{0j} := \sum_{i=1}^n \hat{\psi}_{ij}(0),$$

which is the expected number of newborns produced by an individual born in state j during her entire lifetime. As it is known that

$$\min \sum_{i=1}^n \hat{\psi}_{ij}(x) \leq F(x) \leq \max \sum_{i=1}^n \hat{\psi}_{ij}(x), \quad (2.11)$$

we can conclude that $R_0 < 1$ if $\max_{1 \leq j \leq n} R_{0j} < 1$ and $R_0 > 1$ if $\min_{1 \leq j \leq n} R_{0j} > 1$, whereas $R_0 = 1$ if $R_{0j} = 1$ for all of state j . An $n \times n$ matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ is called *decomposable* (*reducible*) if the set of numbers $S = \{1, 2, \dots, n\}$ is the sum of two disjoint subsets S_1 and S_2 such that $a_{ij} = 0$ if $i \in S_1$ and $j \in S_2$. If a matrix is not decomposable, it is called *indecomposable* (or *irreducible*). If A is irreducible and $A^n = (a_{ij}^{(n)})_{1 \leq i, j \leq n}$, for any i and j there exists an integer n_0 such that $a_{ij}^{(n_0)} > 0$. Thus, any one state is accessible from any other state by multiple transitions [80].

If $\hat{\Psi}(0)$ is decomposable, it follows that for almost all $a \in \mathbb{R}_+$,

$$\sum_{k=1}^n m_{ik}(a) \ell_{kj}(a) = 0, \quad i \in S_1, \quad j \in S_2,$$

which implies that individuals born in state S_2 cannot produce children in state S_1 . Thus, if we divide the population into two subpopulations, one composed of people born in state S_1 and the other composed of people born in state S_2 , then the S_1 subpopulation can be reproduced only by the S_1 subpopulation and has become a closed subpopulation with respect to reproduction. In such a case, we cannot generally expect there to be a common Malthusian parameter for both subpopulations. Conversely, the indecomposability (irreducibility) of the net reproduction matrix implies the existence of the dominant Malthusian solution [49, 50]:

Proposition 2.2 *If the net reproduction matrix $\hat{\Psi}(0)$ is indecomposable, the following holds:*

- (1) *There exists $\lambda_0 \in \Omega \cap \mathbb{R}$ such that $\lambda_0 > \sup\{\Re \lambda : \lambda \in \Omega \setminus \{\lambda_0\}\}$ and $F(\lambda_0) = 1$. Moreover, it follows that*

$$\text{sign}(\lambda_0) = \text{sign}(R_0 - 1). \quad (2.12)$$

- (2) *Let $R(\lambda) := (I - \hat{\Psi}(\lambda))^{-1}$. Then, $R(\lambda)$ has a pole of order one at $\lambda = \lambda_0$ and its residue R_{-1} is a one-dimensional projection given by:*

$$R_{-1}\phi = \frac{v_0(0)^T \phi}{v_0(0)^T \Psi_1 u_0(0)} u_0(0), \quad (2.13)$$

where $v_0(0)$ and $u_0(0)$ are the left and right positive eigenvectors of $\hat{\Psi}(\lambda_0)$ corresponding to the Frobenius root $F(\lambda_0) = 1$ and Ψ_1 is given by

$$\Psi_1 = \int_0^\infty a\Psi(a)e^{-\lambda_0 a} da.$$

Proof Under the assumption, $\hat{\Psi}(x)$ ($x \in \mathbb{R}$) is a non-negative indecomposable matrix. Its Frobenius root $F(x) > 0$ therefore exists, is a strictly decreasing function of real x , and satisfies (2.11). Therefore, we have $\lim_{x \rightarrow -\infty} F(x) = +\infty$ and $\lim_{x \rightarrow \infty} F(x) = 0$. Thus, there exists a unique real root λ_0 of $F(x) = 1$ such that $\lambda_0 > 0$ if $F(0) > 1$, $\lambda_0 = 0$ if $F(0) = 1$ and $\lambda_0 < 0$ if $F(0) < 1$. From the definition, it is clear that $\lambda_0 \in \Omega$. For any $\lambda \in \Omega$, it follows from $1 \in \sigma(\hat{\Psi}(\lambda))$ and $\hat{\Psi}(\Re \lambda) \geq \hat{\Psi}^*(\lambda)$ that

$$r(\hat{\Psi}(\Re \lambda)) = F(\Re \lambda) \geq F^*(\lambda) \geq r(\hat{\Psi}(\lambda)) \geq 1.$$

As $F(x)$ is monotone decreasing, we have $\Re \lambda \leq \lambda_0$. Moreover, if $\Im \lambda \neq 0$, the equality of $\hat{\Psi}^*(\lambda) \leq \hat{\Psi}(\Re \lambda)$ does not hold, so we obtain

$$1 \leq r(\hat{\Psi}(\lambda)) \leq F^*(\lambda) < F(\Re \lambda).$$

Therefore, we have $\Re \lambda < \lambda_0$. We then know that λ_0 is a unique real root in Ω , and $\lambda_0 > \sup\{\Re \lambda : \lambda \in \Omega \setminus \{\lambda_0\}\}$. Next we show (2). By differentiating the identity

$$(I - \hat{\Psi}(\lambda))\text{adj}(I - \hat{\Psi}(\lambda)) = \det(I - \hat{\Psi}(\lambda)) \cdot I,$$

we have

$$\begin{aligned} & \left. \frac{d}{d\lambda}(I - \hat{\Psi}(\lambda)) \right|_{\lambda=\lambda_0} \text{adj}(I - \hat{\Psi}(\lambda)) + (I - \hat{\Psi}(\lambda)) \left. \frac{d}{d\lambda} \text{adj}(I - \hat{\Psi}(\lambda)) \right|_{\lambda=\lambda_0} \\ &= \left. \frac{d}{d\lambda} \det(I - \hat{\Psi}(\lambda)) \right|_{\lambda=\lambda_0} \cdot I. \end{aligned} \quad (2.14)$$

Observe that for any vector ϕ , the following holds:

$$(I - \hat{\Psi}(\lambda_0))\text{adj}(I - \hat{\Psi}(\lambda_0))\phi = 0.$$

As the eigenspace of $\hat{\Psi}(\lambda_0)$ associated with its Frobenius root of unity is one-dimensional, there exists a scalar $c(\phi)$ such that

$$\text{adj}(I - \hat{\Psi}(\lambda_0))\phi = c(\phi)u_0(0).$$

Using $v_0(0)^T(I - \hat{\Psi}(\lambda_0)) = 0$, we multiply (2.14) by $v_0(0)$ and ϕ from the left-hand side and right-hand side, respectively, and let $\lambda = \lambda_0$. Thus, we arrive at

$$c(\phi)v_0(0)^T\Psi_1u_0(0) = v_0(0)^T\phi \left. \frac{d}{d\lambda} \det(I - \hat{\Psi}(\lambda)) \right|_{\lambda=\lambda_0}.$$

Therefore, we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)(I - \hat{\Psi}(\lambda))^{-1}\phi &= \left[\left. \frac{d}{d\lambda} \det(I - \hat{\Psi}(\lambda)) \right|_{\lambda=\lambda_0} \right]^{-1} \text{adj}(I - \hat{\Psi}(\lambda_0))\phi \\ &= \frac{v_0(0)^T\phi}{v_0(0)^T\Psi_1u_0(0)}u_0(0), \end{aligned}$$

which shows that λ_0 is a simple pole of $(I - \hat{\Psi}(\lambda))^{-1}$ and its residue at $\lambda = \lambda_0$ is given by (2.13). We conclude that (2) holds. \square

From Proposition 2.2, for the indecomposable multistate stable population system, we can shift the integral path in (2.7) to the left so that $\lambda_0 > x > \sup\{\Re\lambda : \lambda \in \Omega \setminus \{\lambda_0\}\}$. It then follows from the residue theorem that

$$p(t, a) = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)e^{\lambda t} \hat{p}(\lambda, a) + \frac{1}{2\pi i} \int_{y-i\infty}^{y+i\infty} e^{\lambda t} \hat{p}(\lambda, a) d\lambda, \quad (2.15)$$

where we can calculate the first term as

$$\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)e^{\lambda t} \hat{p}(\lambda, a) = e^{\lambda_0 t} \frac{\langle v_0, p_0 \rangle}{\langle v_0, u_0 \rangle} u_0(a).$$

Although we omit the proof, it can be shown that the growth bound of the second term of (2.15) is less than λ_0 . Therefore, we have the following proposition [49, 50]:

Proposition 2.3 *For the multistate stable population model with an irreducible net reproduction matrix, there exists some $\varepsilon > 0$ such that*

$$p(t, a) = e^{\lambda_0 t} \frac{\langle v_0, p_0 \rangle}{\langle v_0, u_0 \rangle} u_0(a) + O(e^{(\lambda_0 - \varepsilon)t}), \quad (2.16)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of functions given by

$$\langle f, g \rangle := \sum_{i=1}^n \int_0^\infty f_i(a) g_i(a) da$$

for $f = (f_1, \dots, f_n)^T$, $g = (g_1, \dots, g_n)^T$ and $u_0(a)$ and $v_0(a)$ are positive vectors given by

$$u_0(a) := e^{-\lambda_0 a} L(a) u_0(0), \quad v_0(a)^T := v_0(0)^T \int_a^\infty e^{-\lambda_0(s-a)} M(s) L(s, a) ds.$$

The normalized stable age distribution is given by $u_0(a)/\|u_0\|_{L^1}$. The vector $v_0(a)$ is called the *multistate reproductive value*, which is a multidimensional extension of the reproductive value in the scalar stable population theory.

Corollary 2.1 *For the birth rate vector of (2.2), it follows that*

$$\lim_{t \rightarrow \infty} e^{-\lambda_0 t} B(t) = \frac{\langle v_0, p_0 \rangle}{\langle v_0, u_0 \rangle} u_0(0) = \frac{v_0(0)^T \hat{G}(\lambda_0)}{v_0(0)^T \Psi_1 u_0(0)} u_0(0). \quad (2.17)$$

From the above argument, and under the assumption of the irreducibility (indecomposability) of the net reproduction matrix, we know that there exists a dominant exponential solution for (2.2), so the sign of the intrinsic growth rate (Malthusian parameter) determines the stability of the zero solution of (2.2).

Remark 2.1 In the multistate stable population model, we assume that the state transition process satisfies the *Markovian assumption*, that is, age-dependent forces of transition $q_{ij}(a)$, $1 \leq i \leq n$ are independent of the past history of individuals and depend only on their age and the present state. However, the Markovian assumption would not necessarily be satisfied for real human migration. In fact, the human migration pattern depends on individual migration histories, and especially the birth places of the individuals. Moreover, the duration in a region of residence could affect the force of migration. Non-Markovian models that can recognize these realistic aspects have been proposed by several authors [4, 72, 83], but it is very difficult to obtain real data for many related parameters. For Markovian models, parameter estimation methods are well developed, so there has been greater progress in their application and utilization [89–91].

Remark 2.2 If the timescale of the migration dynamics is much faster than that of the birth and death process, the multistate system can be aggregated to a scalar stable population model, where the fast process has attained an equilibrium and the global variable is the total population size [5, 6, 19, 73].

2.2 Inhomogeneous Linear Problems

2.2.1 Stable Population Model with Immigration

We now extend the stable population model to allow immigration. For simplicity, the birth and death rates of immigrants are the same as those of the native population. This type of model has been investigated by several authors to consider the demographic effect of immigration [8, 20, 36, 81, 82].

If we let $f(t, \cdot) \in L^1_+(0, \omega; \mathbb{R}^n)$ be the age-density function of immigrants, we obtain an inhomogeneous problem of the McKendrick equation:

$$\begin{aligned}
\frac{\partial p(t, a)}{\partial t} + \frac{\partial p(t, a)}{\partial a} &= Q(a)p(t, a) + f(t, a), \quad t > 0, \quad 0 < a < \omega, \\
p(t, 0) &= \int_0^\omega M(a)p(t, a)da, \quad t > 0, \\
p(0, a) &= p_0(a).
\end{aligned} \tag{2.18}$$

To address this inhomogeneous problem, a functional analytics approach is most effective, as shown in Chap. 10 of this text and by Inaba [51] under relaxed conditions. Here, however, we present a classical elementary calculation under restrictive conditions. For simplicity, we only deal with the autonomous case, that is, we assume that the migration term $f(t, a)$ is time-independent and given by a known function $f(a)$.

By integrating the McKendrick equation in (2.18) along characteristic lines, we have

$$p(t, a) = \begin{cases} L(a)B(t-a) + \int_0^a L(a, \rho)f(\rho)d\rho, & t-a > 0, \\ L(a, a-t)p_0(a-t) \\ \quad + \int_0^t L(a, a-t+\rho)f(a-t+\rho)d\rho, & a-t > 0, \end{cases} \tag{2.19}$$

where $B(t) := p(t, 0)$. Inserting this expression into the boundary condition in (2.18), we arrive at the renewal integral equation

$$B(t) = G(t) + H_1(t) + H_2(t) + \int_0^t \Psi(a)B(t-a)da, \tag{2.20}$$

where $\Psi(a) := M(a)L(a, 0)$ and

$$\begin{aligned}
G(t) &:= \int_t^\infty M(a)L(a, a-t)p_0(a-t)da, \\
H_1(t) &:= \int_0^t M(a) \int_0^a L(a, \rho)f(\rho)d\rho da, \\
H_2(t) &:= \int_t^\infty M(a) \int_0^t L(a, a-t+\rho)f(a-t+\rho)d\rho da.
\end{aligned}$$

Note that $G(t) = H_2(t) = 0$ and $H_1(t) = H_1(\beta_2)$ for $t > \beta_2$, where β_2 is the upper bound of the reproductive age. We can divide (2.20) into two renewal equations:

$$\begin{aligned}
B_1(t) &= H_1(t) + \int_0^t \Psi(a)B_1(t-a)da, \\
B_2(t) &= G(t) + H_2(t) + \int_0^t \Psi(a)B_2(t-a)da.
\end{aligned} \tag{2.21}$$

It is then clear that the solution $B(t)$ of (2.20) is given as the sum of the two solutions to the renewal equation (2.21). Because the initial data $G(t) + H_2(t)$ of the second equation in (2.21) have a finite support, its asymptotic behavior is the same as the multistate Lotka integral equation (2.5), and is given by (2.17).

However, the initial data of the first equation in (2.21) are not integrable on \mathbb{R}_+ , so we cannot directly apply result (2.17). If we assume $H_1 \in W^{1,\infty}(\mathbb{R}_+)$, then $B_1 \in W_{\text{loc}}^{1,\infty}(\mathbb{R}_+)$, and it follows from $B_1(0) = 0$ that

$$B_1'(t) = H_1'(t) + \int_0^t \Psi(a) B_1'(t-a) da. \quad (2.22)$$

It then follows that $H_1'(t) \in L_+^1(\mathbb{R}_+)$ and $\hat{H}_1'(0) = H_1(\infty)$, and we can apply Proposition 10.38 to (2.22). First, suppose that $\lambda_0 < 0$. Then, we have

$$B_1(\infty) = \int_0^\infty B_1'(t) dt = (I - \hat{\Psi}(0))^{-1} H_1(\infty).$$

It follows from $\lim_{t \rightarrow \infty} B_2(t) = 0$ that

$$\lim_{t \rightarrow \infty} B(t) = \lim_{t \rightarrow \infty} B_1(t) = (I - \hat{\Psi}(0))^{-1} H_1(\infty). \quad (2.23)$$

Next, suppose that $\lambda_0 = 0$. Then, we have

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t B_1'(t) dt = \lim_{t \rightarrow \infty} \frac{B_1(t)}{t} = \frac{v_0(0)^T H_1(\infty)}{v_0(0)^T \Psi_1 u_0(0)} u_0(0),$$

where $v_0(0)$ and $u_0(0)$ are the left and right eigenvectors of $\hat{\Psi}(\lambda_0)$ associated with the eigenvalue unity. However, it follows from the renewal theorem that

$$\lim_{t \rightarrow \infty} B_2(t) = \frac{v_0(0)^T (\hat{G}(0) + \hat{H}_2(0))}{v_0(0)^T \Psi_1 u_0(0)} u_0(0).$$

Then, $\lim_{t \rightarrow \infty} B_2(t)/t = 0$ and we obtain

$$\lim_{t \rightarrow \infty} \frac{B(t)}{t} = \lim_{t \rightarrow \infty} \frac{B_1(t)}{t} = \frac{v_0(0)^T H_1(\infty)}{v_0(0)^T \Psi_1 u_0(0)} u_0(0). \quad (2.24)$$

Finally, suppose that $\lambda_0 > 0$. In this case, the initial data of (2.20) are bounded above for $t \geq 0$. Hence, for any positive number $\sigma > 0$, we have $e^{-\sigma t}(G(t) + H_1(t) + H_2(t)) \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. We can then repeat the same argument as for Proposition 1.9 for the vector-type renewal equation (2.20) to prove the following:

$$\lim_{t \rightarrow \infty} e^{-\lambda_0 t} B(t) = \frac{v_0(0)^T (\hat{G}(\lambda_0) + \hat{H}_1(\lambda_0) + \hat{H}_2(\lambda_0))}{v_0(0)^T \Psi_1 u_0(0)} u_0(0). \quad (2.25)$$

From (2.19), (2.23), (2.24) and (2.25), we can conclude the following [49]:

Proposition 2.4 *Suppose that Ψ is differentiable and $\hat{\Psi}(0)$ is indecomposable. For the stable population model of (2.18) with time-independent immigration term f , the following holds:*

(1) *If $\lambda_0 < 0$, then*

$$\lim_{t \rightarrow \infty} p(t, a) = L(a)(I - \hat{\Psi}(0))^{-1} H_1(\infty) + \int_0^a L(a, \rho) f(\rho) d\rho,$$

where

$$H_1(\infty) = \int_0^\infty d\rho \int_\rho^\infty M(a) L(a, \rho) da f(\rho) d\rho da.$$

(2) *If $\lambda_0 = 0$, then*

$$\lim_{t \rightarrow \infty} \frac{p(t, a)}{t} = \frac{\langle v_0, f \rangle}{\langle v_0, u_0 \rangle} u_0(a),$$

where $u_0(a)$ and $v_0(a)$ are given in Proposition 2.3 with $\lambda_0 = 0$.

(3) *If $\lambda_0 > 0$, then*

$$\lim_{t \rightarrow \infty} e^{-\lambda_0 t} p(t, a) = \frac{\langle v_0, p_0 \rangle + \lambda_0^{-1} \langle v_0, f \rangle}{\langle v_0, u_0 \rangle} u_0(a),$$

where $u_0(a)$ and $v_0(a)$ are given in Proposition 2.3.

From Proposition 2.4, we know that if $\lambda_0 > 0$, the asymptotic Malthusian parameter is not affected by the existence of immigration and the asymptotic age structure is the stable age distribution associated with the net reproduction matrix $\Psi(a) = M(a)L(a)$. If $\lambda_0 = 0$, the size of the population is asymptotically growing in a linear manner, and the final age structure is the same as the stationary population structure $L(a)u_0(0)$ of a closed population. If $\lambda_0 < 0$, the population converges to a stationary population. Therefore, if the population has below-replacement fertility, a stationary population with an appropriate size can be achieved by a constant stream of immigration, but the final age structure will be more aged in comparison with the stationary state of a closed population (see Chap. 10).

2.2.2 Population Dynamics of Marine Invertebrates

We now introduce an ecological application of the stable population theory as an example of the inhomogeneous boundary value problem of the McKendrick equation. The McKendrick equation with inhomogeneous boundary conditions is investigated in [111].

The population dynamics of marine invertebrates such as *barnacles*, in which sessile adults and pelagic larvae are contained in a local area, are very much different from the population dynamics of vertebrates. Although the sessile adults can be viewed as living in a limited area, their larvae can freely move from one area to another, because each area (patch) is connected by the pelagic pool containing the larvae. That is, such a population system in a local area is essentially “open,” because newly settled larvae can be carried from outside the region, while the whole multipatch system can be “closed” if the larvae are produced by the sessile adults in each patch [63]. Moreover, it has been observed in sessile marine populations that the space to be settled by the larvae is a principal limiting resource, and the number of settlements is approximately proportional to the free space available to larvae.

Under the observations mentioned above, Roughgarden et al. [92] proposed an age-structured population model for sessile invertebrates living in a local area. They derived a sufficient condition for the local stability of the uniquely existing steady state and used numerical examples to suggest that this steady state could be destabilized if the settlement rate were sufficiently high, leading to a limit cycle oscillation. The open marine population model has been further developed by Roughgarden and Iwasa [93, 94] and mathematically investigated by several authors [57, 63, 112, 113].

Originally, Roughgarden et al. recognized the absurd drawback of their model, whereby the population density may become negative for some initial conditions. The main reason for this shortcoming is that the demographic parameters of the size growth rate and mortality are assumed to be independent of the population density. In reality, these parameters will depend on the density of the population or the available free space. However, we overlook this limitation and simply sketch the mathematical formulation of the original linear model.

Let $p(t, a)$ denote the density of adults of age a at time t , A be the total area of available substrate, $F(t)$ be the size of the free space available to the larvae at time t , k be the instantaneous settling rate per unit of free space, $\beta(a)$ be the size of individuals of age a , $\mu(a)$ be the age-specific death rate, and ω be the upper bound of the age of individuals. The Roughgarden–Iwasa–Baxter age-structured population model for sessile invertebrates living in a local area can then be formulated as

$$\begin{aligned} \frac{\partial p(t, a)}{\partial t} + \frac{\partial p(t, a)}{\partial a} &= -\mu(a)p(t, a), \\ p(t, 0) &= kF(t), \\ F(t) &= A - \int_0^\omega \beta(a)p(t, a)da, \end{aligned} \tag{2.26}$$

where we assume that the maximum attainable age of individuals is finite, although $\omega = \infty$ in the original model.

We assume that $\beta \in L_+^\infty(0, \omega)$, $\mu(a)$ is positive for all $a \in [0, \omega]$, locally integrable on $[0, \omega)$ and $\int_0^\omega \mu(\sigma)d\sigma = \infty$. The survival probability (the proportion of newly settled larvae who can survive to age a) is given by $\ell(a) := \exp(-\int_0^a \mu(\sigma)d\sigma)$. The state space Ω of the age-density function is

$$\Omega = \left\{ p \in L_+^1(0, \omega) : \int_0^\omega \mu(a)p(a)da < \infty, \int_0^\omega \beta(a)p(a)da \leq A \right\}.$$

Let us factor out the natural death rate in the basic model (2.26). If we define a new function $q(t, a)$ by $p(t, a) = \ell(a)q(t, a)$, then system (2.26) reduces to a simpler system for q as follows:

$$\begin{aligned} \frac{\partial q(t, a)}{\partial t} + \frac{\partial q(t, a)}{\partial a} &= 0, \\ q(t, 0) &= k \left(A - \int_0^\omega \phi(a)q(t, a)da \right), \\ q(0, a) &= q_0(a) := \frac{p_0(a)}{\ell(a)}, \end{aligned} \tag{2.27}$$

where $\phi(a) := \beta(a)\ell(a)$ is the expected space size occupied by the population at age a and we assume that $p_0/\ell \in L_+^1(0, \omega)$. If $\mu\ell \in L^\infty(0, \omega)$, we have $p = q\ell \in \Omega$ when $\int_0^\omega q(a)\phi(a)da \leq A$.

For model (2.27), there always exists the unique positive steady state

$$q^*(a) = \frac{kA}{1 + k \int_0^\omega \phi(a)da}. \tag{2.28}$$

To rewrite the basic model, so as to have a homogeneous boundary condition, let us introduce a new variable $u(t, a)$ as $u(t, a) := q(t, a) - q^*(a)$. System (2.27) can then be rewritten as the following homogeneous system:

$$\begin{aligned} \frac{\partial u(t, a)}{\partial t} + \frac{\partial u(t, a)}{\partial a} &= 0, \\ u(t, 0) &= -k \int_0^\omega \phi(a)u(t, a)da, \\ u(0, a) &= q_0(a) - q^*(a). \end{aligned} \tag{2.29}$$

Therefore, the open marine population model (2.26) can be reduced to a linear homogeneous age-dependent population system in L^1 , and the method for the stable population model can be applied to this linear system.

Integrating (2.29) along the characteristic line, we obtain the following expression:

$$u(t, a) = \begin{cases} b(t - a), & t - a > 0, \\ u_0(a - t), & t - a < 0, \end{cases} \tag{2.30}$$

where $b(t) := u(t, 0)$. Inserting (2.30) into the boundary condition of (2.29), we have the renewal integral equation

$$b(t) = -g(t) - k \int_0^t \phi(a)b(t-a)da, \quad (2.31)$$

where $g(t)$ is defined by

$$g(t) := k \int_t^\infty \phi(a)u_0(a-t)da.$$

Here, we extend the domain of $\phi(a)$ to $\phi(a) = 0$ for $a > \omega$. Let Λ be the set of characteristic roots given by:

$$\Lambda := \{\lambda \in \mathbb{C} : 1 + k\hat{\phi}(\lambda) = 0\}, \quad (2.32)$$

where $\hat{\phi}$ denotes the Laplace transform of ϕ . We can then state the following:

Proposition 2.5 *$\Lambda \cap \mathbb{R} = \emptyset$ and Λ is composed of a countably infinite number of complex conjugate pairs. For any real number α , there are at most finitely many roots in the right half-plane $\Re \lambda > \alpha$, and there is a dominant pair whose real part is greater than the real part of any other characteristic root.*

Using the same kind of arguments as in Sect. 1.2, we obtain an asymptotic expansion of $b(t)$:

$$\begin{aligned} b(t) &= b_0 e^{\lambda_0 t} + \bar{b}_0 e^{\bar{\lambda}_0 t} + O(e^{(\Re \lambda_0 - \varepsilon)t}) \\ &= e^{\Re \lambda_0 t} [\Re b_0 \cos(\Im \lambda_0 t) - \Im b_0 \sin(\Im \lambda_0 t)] + O(e^{(\Re \lambda_0 - \varepsilon)t}), \end{aligned} \quad (2.33)$$

where λ_0 and $\bar{\lambda}_0$ are the dominant pair of characteristic roots, $\varepsilon > 0$ is a small number such that $\{\lambda : \lambda \in \Lambda \setminus \{\lambda_0, \bar{\lambda}_0\}\} \subset \{\lambda : \Re \lambda \leq \Re \lambda_0 - \varepsilon\}$ and b_0 is given by:

$$b_0 := \frac{\int_0^\omega e^{-\lambda_0 t} g(t) dt}{k \int_0^\omega a e^{-\lambda_0 a} \phi(a) da}. \quad (2.34)$$

Hence, the asymptotically dominant part of the solution of the basic model (2.26) is given as

$$\ell(a)q^*(a) + |b_0|e^{\Re \lambda_0(t-a)} \ell(a) \cos(\Im \lambda_0(t-a) + \theta), \quad (2.35)$$

where $\theta := \arctan(\Im b_0 / \Re b_0)$. Thus, there is no Malthusian solution, and the steady state is globally asymptotically stable if $\Re \lambda_0 < 0$, whereas it is unstable if $\Re \lambda_0 > 0$. In particular, if $\Re \lambda_0 > 0$, the positivity of the population density will be destroyed as time evolves.

An interesting point, however, is that the following *50% free space rule* for stability holds [92]:

Proposition 2.6 *Let π be the proportion of free space in the steady state:*

$$\pi := \frac{F^*}{A} = \frac{1}{1 + k \int_0^\omega \phi(a) da}. \quad (2.36)$$

If $\pi > 1/2$, the steady state is globally asymptotically stable.

Proof Suppose that $\pi > 1/2$ and there exists a characteristic root $\lambda = x + iy$ with $x \geq 0$. Then, it follows from (2.32) that

$$1 = k \left| \int_0^\omega e^{-\lambda a} \phi(a) da \right| \leq k \int_0^\omega e^{-x a} \phi(a) da \leq k \int_0^\omega \phi(a) da = \frac{1}{\pi} - 1.$$

This contradicts our assumption.

We will again encounter the *fifty percent stability rule* in the Pease influenza model (Chap. 8). Note that the condition $\pi > 1/2$ can be rewritten as

$$R := k \int_0^\omega \phi(a) da < 1, \quad (2.37)$$

where R denotes the cumulative area occupied by settled larvae per unit area. As shown in the next chapter, there is a possibility that the characteristic Eq. (2.32) has a pair of characteristic roots with positive real parts if k is sufficiently large, destabilizing the linear system. If there exists a characteristic root with a positive real part, the amplitude of the solution of (2.26) will grow without bound. In this case, the physical condition $0 \leq F(t) \leq A$ will be lost and the linear model breaks down.

As pointed out by Roughgarden et al., the above shortcoming can be overcome if we make the more realistic assumption that the mortality of the adult population increases as the free space decreases. Using numerical simulations, Roughgarden et al. found that the destabilization of the steady state in the density dependent model can lead to a limit cycle as the free space becomes exhausted.

Remark 2.3 Note that in a formal sense, the open marine population dynamics are formulated by the Gurtin and MacCamy nonlinear model, which we investigate in Chap. 3. In fact, if we define the size-dependent fertility rate as

$$m(a, S(t)) := \frac{k(A - S(t))}{S(t)} \beta(a),$$

where $S(t) = \int_0^\omega \beta(a) p(t, a) da$, the open marine population model can be written as the Gurtin and MacCamy model [41], although it has a singularity at $S = 0$.

2.3 Linear Marriage Models

2.3.1 *First-Marriage Model*

In modern societies, fertility change is a most important factor in understanding population changes. From a demographic point of view, when considering the causal analysis of human reproduction, it is important to divide the population according to life stages that could affect the reproduction process. Thus, we need to model the birth process so as to clarify the relationship between the macrodynamics of populations and life cycle variables such as maturity, marriage, family building, and divorce. In this section, we consider a multistate stable population model that takes into account the marital status of individuals.

We consider a society in which there is a strong tradition of monogamous marriage, the divorce rate is still low and there is very little childbearing among the unmarried. That is, almost all newborns are produced by the legitimate first marriage, and the contribution of ex-nuptial phenomena to the total number of births is negligible. The reproductivity of such a population is determined by two factors: how many people will get married and how many children will be produced by married couples. Hence, changes in fertility should be factored into changes in nuptiality and marital fertility. Such a situation has been observed in Japan as well as in Mediterranean countries and other traditional societies. Understanding the fertility trends of those countries is one of the motivating factors behind the following theoretical framework [16, 56].

For simplicity, we assume that children are produced only by first-marriage couples. To formulate a dynamic population model based on nuptiality and marital fertility, we use the duration-specific marital fertility rate by age at marriage, instead of the age-specific fertility rate. This is because, for populations in which childbearing occurs predominantly within marriage, the duration of marriage is a more appropriate and influential variable than chronological age for describing controlled fertility [87, 96]. Of course, it should be noted that the duration-specific marital fertility by age at marriage implicitly accounts for the effect of chronological age, because the chronological age is given as the sum of the age at marriage and the duration of marriage. In addition, we assume that the force of marriage is a given function of age, that is, we disregard the nonlinear interaction between both sexes. The pair formation phenomena will be studied in Chap. 4.

Let us divide the population into three groups: p_0 (never married), p_1 (within first marriage), and p_2 (widowed or divorced population including remarried). Let $p_0(t, a)$ be the age density of the never-married population at time t and age a , let $p_1(t, \tau; \zeta)$ be the density of married individuals at time t and marital duration τ whose age at first marriage is ζ (that is, their chronological age is $\tau + \zeta$) and let $p_2(t, a)$ be the age density of widowed/divorced individuals at time t and age a . Let $\lambda(a)$ be the *force of first marriage* at age a , $\mu(a)$ be the force of mortality (for simplicity, we do not assume differential mortality according to individuals' status), $m(\tau; \zeta)$ be the *marital fertility rate* of duration τ by age at first marriage ζ , $\delta(\tau; \zeta)$

be the force of dissolution of couples with duration τ by age at marriage ζ and γ be the proportion of female newborns. In the real reproduction process, the *parity* (the number of children that a woman has had) is also an important variable. However, we only use the marital fertility function aggregated with respect to the parity. A parity-structured model will be examined in the next section.

We can formulate the following one-sex model for the population reproduced by first marriage [56]:

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p_0(t, a) &= -(\mu(a) + \lambda(a)) p_0(t, a), \\
 \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau} \right) p_1(t, \tau; \zeta) &= -\mu(\tau + \zeta) p_1(t, \tau; \zeta) - \delta(\tau; \zeta) p_1(t, \tau; \zeta), \\
 \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p_2(t, a) &= -\mu(a) p_2(t, a) + \int_0^a \delta(\tau; a - \tau) p_1(t, \tau; a - \tau) d\tau, \\
 p_0(t, 0) &= \gamma \int_0^\infty \int_0^\infty m(\tau; \zeta) p_1(t, \tau; \zeta) d\tau d\zeta, \\
 p_1(t, 0; \zeta) &= \lambda(\zeta) p_0(t, \zeta), \\
 p_2(t, 0) &= 0,
 \end{aligned} \tag{2.38}$$

where the initial data are given by:

$$p_0(0, a) = k_0(a), \quad p_1(0, \tau; \zeta) = k_1(\tau; \zeta), \quad p_2(0, a) = k_2(a).$$

If we define $n(t, a)$ as the age-density function of the total female population, it follows that

$$n(t, a) = p_0(t, a) + \int_0^a p_1(t, a - \zeta; \zeta) d\zeta + p_2(t, a),$$

and it is easy to see that $n(t, a)$ satisfies the McKendrick equation

$$\begin{aligned}
 \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) n(t, a) &= -\mu(a) n(t, a), \\
 n(t, 0) &= p_0(t, 0), \\
 n(0, a) &= k_0(a) + \int_0^a k_1(a - \zeta; \zeta) d\zeta + k_2(a).
 \end{aligned}$$

Let $B(t) := p_0(t, 0)$ be the birth rate at time t . By integrating the McKendrick equation of (2.38) along the characteristic line, we obtain the following expression:

$$\begin{aligned}
p_0(t, a) &= \begin{cases} \Lambda(a)\ell(a)B(t-a), & t-a > 0 \\ \frac{\Lambda(a)\ell(a)}{\Lambda(a-t)\ell(a-t)}k_0(a-t), & a-t > 0 \end{cases} \\
p_1(t, \tau; \zeta) &= \begin{cases} k_1(\tau-t; \zeta) \frac{\ell(\tau+\zeta)}{\ell(\tau+\zeta-t)} \exp\left(-\int_{\tau-t}^{\tau} \delta(s; \zeta) ds\right), & t < \tau \\ k_0(\tau+\zeta-t) \frac{\ell(\tau+\zeta)\phi(\zeta)}{\ell(\tau+\zeta-t)\Lambda(\tau+\zeta-t)} \exp\left(-\int_0^{\tau} \delta(s; \zeta) ds\right), & \tau < t < \tau+\zeta \\ B(t-\tau-\zeta)\phi(\zeta)\ell(\tau+\zeta) \exp\left(-\int_0^{\tau} \delta(s; \zeta) ds\right), & \tau+\zeta < t, \end{cases} \\
p_2(t, a) &= n(t, a) - p_0(t, a) - \int_0^a p_1(t, a-\zeta; \zeta) d\zeta,
\end{aligned} \tag{2.39}$$

where $\ell(a)$ is the survival probability, $\Lambda(a)$ is the proportion of unmarried females at age a , and $\phi(a) := \lambda(a)\Lambda(a)$ is the (incomplete) probability density of first marriage:

$$\ell(a) := \exp\left(-\int_0^a \mu(\rho) d\rho\right), \quad \Lambda(a) := \exp\left(-\int_0^a \lambda(\sigma) d\sigma\right).$$

Inserting (2.39) into the boundary condition in (2.38) and changing the order of integrals, we obtain the renewal equation for $B(t)$:

$$B(t) = G(t) + \int_0^t \psi(a)B(t-a)da, \tag{2.40}$$

where $\beta(a)$ is the age-specific birth rate and $\psi(a)$ is the net reproduction function given by

$$\psi(a) = \gamma\beta(a)\ell(a), \quad \beta(a) = \int_0^a m(a-\zeta; \zeta) e^{-\int_0^{a-\zeta} \delta(s; \zeta) ds} \phi(\zeta) d\zeta$$

and

$$\begin{aligned}
G(t) &:= \int_t^\infty d\tau \int_0^\infty m(\tau; \zeta) k_1(\tau-t; \zeta) \frac{\ell(\tau+\zeta)}{\ell(\tau+\zeta-t)} e^{-\int_{\tau-t}^{\tau} \delta(s; \zeta) ds} d\zeta \\
&\quad + \int_0^t d\tau \int_{t-\tau}^\infty m(\tau; \zeta) \frac{k_0(\tau+\zeta-t)\ell(\tau+\zeta)\phi(\zeta)}{\ell(\tau+\zeta-t)\Lambda(\tau+\zeta-t)} e^{-\int_0^{\tau} \delta(s; \zeta) ds} d\zeta.
\end{aligned}$$

If we let β_2 be the upper bound of reproductive age, we have $m(\tau; \zeta) = 0$ for $\tau+\zeta > \beta_2$ and $\psi(a) = 0$, $G(t) = 0$ for $t > \beta_2$. Thus, the strong ergodicity theorem holds for the renewal equation, and there is an intrinsic growth rate λ_0 such that

$$\lim_{t \rightarrow \infty} e^{-\lambda_0 t} B(t) = \frac{\int_0^\infty e^{-\lambda_0 s} G(s) ds}{\int_0^\infty a e^{-\lambda_0 a} \psi(a) da},$$

where λ_0 is the unique real root of the characteristic equation

$$\int_0^\infty e^{-\lambda_0 a} \psi(a) da = 1.$$

It is easy to calculate the stable age distributions by marital status as follows:

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{p_0(t, a)}{\int_0^\infty n(t, a) da} &= w(a) \Lambda(a), \\ \lim_{t \rightarrow \infty} \frac{\int_0^a p_1(t, \tau; a - \tau) d\tau}{\int_0^\infty n(t, a) da} &= w(a) \Gamma(a), \\ \lim_{t \rightarrow \infty} \frac{p_2(t, a)}{\int_0^\infty n(t, a) da} &= w(a) (1 - \Lambda(a) - \Gamma(a)), \end{aligned}$$

where $w(a)$ is the age profile of the stable population given by

$$w(a) := \frac{e^{-r_0 a} \ell(a)}{\int_0^\infty e^{-r_0 a} \ell(a) da},$$

and $\Gamma(a)$ is the proportion of married population at age a in the stable distribution defined by

$$\Gamma(a) := \int_0^a \phi(\zeta) e^{-\int_0^{a-\zeta} \delta(\sigma; \zeta) d\sigma} d\zeta.$$

From the renewal equation (2.40), we can calculate the basic reproduction number (net reproduction rate) R_0 and the total fertility rate as follows:

$$\begin{aligned} R_0 &= \int_0^\infty \psi(a) da = \gamma \int_0^\infty \int_0^\infty m(\tau; \zeta) \ell(\tau + \zeta) e^{-\int_0^\tau \delta(s; \zeta) ds} d\tau \phi(\zeta) d\zeta, \\ \text{TFR} &= \int_0^\infty \beta(a) da = \int_0^\infty \int_0^\infty m(\tau; \zeta) e^{-\int_0^\tau \delta(s; \zeta) ds} d\tau \phi(\zeta) d\zeta. \end{aligned} \quad (2.41)$$

The period total fertility rate (TFR) observed at time t is calculated as

$$\text{period TFR} = \int_0^\infty \frac{B(t, a)}{n(t, a)} da,$$

where $B(t, a)$ denotes the number of newborns produced by mothers aged a at time t . For our first-marriage model, we have

$$B(t, a) = \int_0^a m(a - \zeta; \zeta) p_1(t, a - \zeta; \zeta) d\zeta.$$

Then, the period TFR can change as a function of time t and coincides with the (time-independent) cohort TFR if and only if the population structure by marital status is consistent with the given marriage schedule, that is, p_1 is given as

$$p_1(t, a - \zeta; \zeta) = n(t, a)\phi(\zeta)e^{-\int_0^{a-\zeta} \delta(\sigma; \zeta) d\sigma}.$$

The above condition will be satisfied as the influence of the initial population disappears over time.

Let $T(\zeta)$ be the expected total number of children produced by a woman married at age ζ in the case that her reproduction process is not terminated by death. We then have

$$T(\zeta) = \int_0^\infty m(\tau; \zeta) e^{-\int_0^\tau \delta(s; \zeta) ds} d\tau.$$

Using $T(\zeta)$, TFR can be expressed as

$$\text{TFR} = \int_0^\infty T(\zeta)\phi(\zeta)d\zeta.$$

Let $\Phi(a)$ be the probability density of age at first marriage:

$$\Phi(a) = \frac{\phi(a)}{\int_0^\infty \phi(z)dz} = \frac{\phi(a)}{1 - \Lambda(\infty)}.$$

We can then rewrite TFR as

$$\text{TFR} = (1 - \Lambda(\infty)) \int_0^\infty T(\zeta)\Phi(\zeta)d\zeta, \quad (2.42)$$

where $1 - \Lambda(\infty)$ is the *proportion ever marrying* (PEM), so we know that TFR can be factored into the PEM and the average number of children produced per marriage in the case that the marital birth process is not terminated by death of female partner. Moreover, let us define $S(\zeta)$ by

$$S(\zeta) = \int_0^\infty m(\tau; \zeta) \frac{\ell(\tau + \zeta)}{\ell(\zeta)} e^{-\int_0^\tau \delta(s; \zeta) ds} d\tau.$$

Then, $S(\zeta)$ is the expected total number of children produced per marriage by age at marriage ζ . Using $S(\zeta)$, the basic reproduction number is given by:

$$R_0 = \gamma \int_0^\infty S(\zeta)\ell(\zeta)\phi(\zeta)d\zeta. \quad (2.43)$$

As Henry [47] observed for *natural fertility*, if conception occurs randomly for each marital status, the *completed fertility* by age at first marriage given by

$$U(\zeta) := \int_0^\infty m(\tau; \zeta) d\tau$$

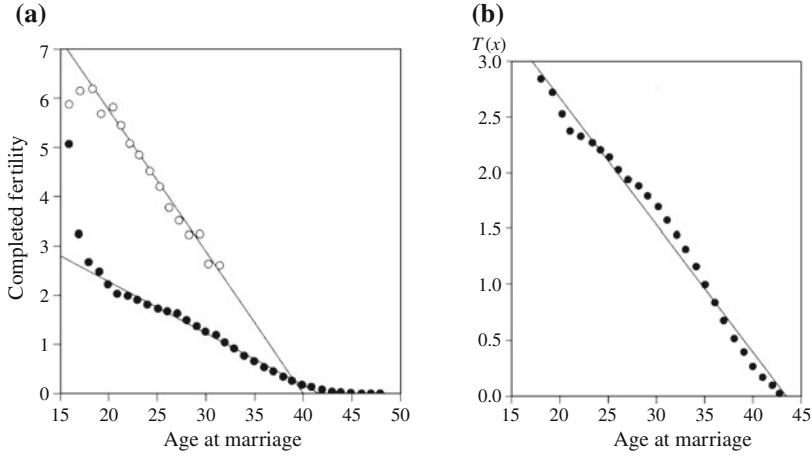


Fig. 2.1 **a** Observed data for completed fertility rates in Japan (1940, \circ) and England (1939, \bullet); **b** Observed data of $T(x)$ in Japan (1985) and the regression line [56, 62]

would become an almost linear function of the age at marriage. Even for the weakly controlled fertility historically observed in England and Japan before World War II, $U(\zeta)$ was approximately linear. More surprisingly, we can see that $T(\zeta)$ and $S(\zeta)$ for Japanese women in the 1980s, who had strong control over when they conceived, are also well approximated by linear functions (see Fig. 2.1 and [56]). Similar phenomena have been observed among modern Mediterranean countries such as Italy [16].

According to the above observations, we can write $T(\zeta)$ as

$$T(\zeta) = u + v\zeta + \xi(\zeta).$$

The coefficients of the linear part $u + v\zeta$ can be determined from a regression equation using real data. Inserting the above expression into (2.42), we then have

$$\text{TFR} = (1 - \Lambda(\infty))(u + va_0) + \int_0^\infty \xi(\zeta)\phi(\zeta)d\zeta, \quad (2.44)$$

where $\Lambda(\infty)$ is the proportion of lifelong unmarried, and a_0 is the mean age at first marriage. Though $T(\zeta)$ would not exhibit a linear pattern in the neighborhood of the upper and lower bounds of reproductive age, the residual term in (2.44) is very small because $\phi(\zeta)$ takes a very small value around the limits of reproductive age. As a result, the linear part of (2.44) is a sufficient approximation for TFR.

From the Japanese marital fertility table in 1985 (see [62] and Fig. 2.1), if we neglect the effect of remarriage, $T(\zeta)$ can be estimated as

$$T(\zeta) \approx 4.927 - 0.1136\zeta, \quad 18 \leq \zeta \leq 43,$$

where the coefficient of determination is 0.986. If we neglect the residual term in (2.44), we have

$$\text{TFR} \approx (1 - \Lambda(\infty))(4.927 - 0.1136a_0).$$

For Japanese women born between 1945 and 1950 (the Japanese baby boom cohort after World War II), $\Lambda(\infty)$ is less than 5% and the average age at marriage is 24 years. The TFR of this generation calculated by the above first approximation formula is 2.09, which is almost equal to the completed fertility rate observed from sample surveys and near to the population replacement level (critical fertility rate).

Remark 2.4 Japanese demographers were concerned about the effect of nuptiality on fertility under pronatalist policy during World War II. After the war, however, the issue received less attention in Japan, because the first Japanese *demographic transition* occurred over the decade following the war, mainly because of a rapid decline in marital fertility, whereas the effect of nuptiality was relatively faint. Itoh [61] revived concern for the nuptiality–fertility relation to explain the *second demographic transition* that began in 1974 in Japan and derived “Itoh’s formula” given in (2.41). In 1974, the period TFR of Japanese women was 2.05, which is below the replacement level ($R_0 = 0.97$). From 1974 to 2005, the period TFR in Japan monotonically decreased (except for a temporal increase from 1982 to 1984), falling to a postwar low of 1.26 in 2005. After 2005, TFR began to increase, reaching 1.41 in 2012. National fertility surveys, however, show that marital fertility is relatively stable. In fact, the completed marital fertility rate is 2.01 for cohorts born from 1960 to 1965, and the social norm of two children per couple is still accepted, even among younger generations. These observations indicate that a decline in period TFR since 1974 in Japan could be attributable to a change in nuptiality. In particular, delayed marriage and the increased proportion of lifelong unmarried are seen as major causes of fewer children being born and the declining population of Japan.

2.3.2 Reproduction by Non-persistent Unions

The assumption of reproduction only by first marriage gives a good approximation of real human reproduction if childbearing outside of marriage is exceptional and the union formation is persistent, a situation that has often been observed in traditional societies. However, this model is not so good when the pair formation process is unstable and temporary, as in post-modern societies.

We now consider the case in which each woman repeatedly (re)marries and divorces. This divides the female population into three states: never-married single (p_0), married (p_1), and divorced single (p_2). Alternatively, we can interpret p_0 as the unmatured state, p_1 as the susceptible state and p_2 as the non-susceptible state. Using the same notation as for the first-marriage model, the basic system is formulated as follows [55]:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) p_0(t, a) &= -(\lambda(a) + \mu(a)) p_0(t, a), \\
\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial a} + \frac{\partial}{\partial s}\right) p_1(t, a, s) &= -(\mu(a) + \sigma(a, s)) p_1(t, a, s), \\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} + \frac{\partial}{\partial s}\right) p_2(t, a, s) &= -(\mu(a) + \delta(a, s)) p_2(t, a, s),
\end{aligned} \tag{2.45}$$

$$\begin{aligned}
p_0(t, 0) &= \int_0^\infty \int_0^a m(a, s) p_1(t, a, s) ds da, \\
p_1(t, a, 0) &= \lambda(a) p_0(t, a) + \int_0^a \delta(a, s) p_2(t, a, s) ds, \\
p_2(t, a, 0) &= \int_0^a \sigma(a, s) p_1(t, a, s) ds, \\
p_1(t, a, s) &= p_2(t, a, s) = 0, \quad a \leq s,
\end{aligned}$$

where $\lambda(a)$ is the force of first marriage, $\sigma(a, s)$ is the force of dissolution at age a and marriage duration s (by divorce or death of spouse), $\delta(a, s)$ is the force of remarriage at age a and duration (since the last divorce) s , and $m(a, s)$ is the fertility rate at age a for duration s . For simplicity, we assume that there is no differential mortality among the three states, state transitions and natural death are mutually independent phenomena, and newborns are only produced by married individuals.

To solve the basic system, let us consider the age–duration-dependent state transition process in a female cohort using survival probabilities. Let $q_0(a)$ be the survival probability at age a in the never-married single state, $q_1(a, s)$ be the survival probability at age a and duration (since the last marriage) s in the marriage state, and $q_2(a, s)$ be the survival probability at age a and duration (since the last dissolution) s in the divorced single state. Neglecting the natural death rate, we obtain the following system:

$$\begin{aligned}
\frac{dq_0(a)}{da} &= -\lambda(a) q_0(a), \\
\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial s}\right) q_1(a, s) &= -\sigma(a, s) q_1(a, s), \\
\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial s}\right) q_2(a, s) &= -\delta(a, s) q_2(a, s),
\end{aligned} \tag{2.46}$$

where the boundary conditions are given by:

$$\begin{aligned}
q_0(0) &= 1, \\
q_1(a, 0) &= \lambda(a) q_0(a) + \int_0^a \delta(a, s) q_2(a, s) ds, \\
q_2(a, 0) &= \int_0^a \sigma(a, s) q_1(a, s) ds,
\end{aligned}$$

and we adopt the convention that $q_1(a, s) = q_2(a, s) = 0$ for $a \leq s$.

If we can determine the survival probabilities q_j and assume that the number of newborns per unit time at time t , denoted by $B(t)$, is given for $t \in \mathbb{R}$, the age-density functions of each state can be expressed as follows:

$$\begin{aligned} p_0(t, a) &= \ell(a)q_0(a)B(t - a), \\ p_1(t, a, s) &= \ell(a)q_1(a, s)B(t - a), \\ p_2(t, a, s) &= \ell(a)q_2(a, s)B(t - a), \end{aligned} \quad (2.47)$$

where $\ell(a) := \exp\left(-\int_0^a \mu(\sigma)d\sigma\right)$ is the survival probability with respect to natural death.

By integrating (2.46) along characteristic lines, we have the following expression:

$$\begin{aligned} q_0(a) &= e^{-\int_0^a \lambda(z)dz}, \\ q_1(a, s) &= q_1(a - s, 0)e^{-\int_0^s \sigma(a-s+z, z)dz}, \quad a > s, \\ q_2(a, s) &= q_2(a - s, 0)e^{-\int_0^s \delta(a-s+z, z)dz}, \quad a > s. \end{aligned} \quad (2.48)$$

The boundary values $q_1(a, 0)$ and $q_2(a, 0)$ are the incidences of marriage and divorce at age a , respectively.

Let $\phi(a) := q_1(a, 0)$ and $\psi(a) := q_2(a, 0)$. Inserting expression (2.48) into the boundary condition in (2.46), we obtain a Volterra integral equation system for the unknown vector of boundary values $X(a) := (\phi(a), \psi(a))^T$ as follows:

$$X(a) = G(a) + \int_0^a H(a, s)X(a - s)ds, \quad (2.49)$$

where

$$\begin{aligned} G(a) &:= \begin{pmatrix} \lambda(a)q_0(a) \\ 0 \end{pmatrix}, \\ H(a, s) &:= \begin{pmatrix} 0 & \delta(a, s)e^{-\int_0^s \delta(a-s+z, z)dz} \\ \sigma(a, s)e^{-\int_0^s \sigma(a-s+z, z)dz} & 0 \end{pmatrix} \end{aligned}$$

If we define $K(a, \rho) := H(a, a - \rho)$, then (2.49) can be rewritten as

$$X(a) = G(a) + \int_0^a K(a, \rho)X(\rho)d\rho. \quad (2.50)$$

Define the resolvent $R(a, \rho)$ as the solution of

$$R(a, \rho) = K(a, \rho) + \int_0^a K(a, \sigma)R(\sigma, \rho)d\sigma,$$

in which case $X(a)$ can be expressed as

$$X(a) = G(a) + \int_0^a R(a, \rho)G(\rho)d\rho.$$

Using this solution, we have ϕ and ψ , and hence the age–duration–dependent survival probabilities by state can be calculated from (2.48).

From (2.47), omitting the initial data, we obtain a homogeneous renewal equation

$$B(t) = \int_0^\infty \left[\int_0^a m(a, s)q_1(a, s)ds \right] \ell(a)B(t-a)da. \quad (2.51)$$

The basic reproduction number and TFR are then given by:

$$\begin{aligned} R_0 &= \int_0^\infty \int_0^a m(a, s)q_1(a, s)ds\ell(a)da = \int_0^\infty S(a)\phi(a)\ell(a)da, \\ \text{TFR} &= \int_0^\infty \int_0^a m(a, s)q_1(a, s)dsda = \int_0^\infty T(a)\phi(a)da, \end{aligned} \quad (2.52)$$

where

$$\begin{aligned} S(a) &:= \int_0^\infty m(a+s, s) \frac{\ell(a+s)}{\ell(a)} e^{-\int_0^s \sigma(a+z, z)dz} ds, \\ T(a) &:= \int_0^\infty m(a+s, s) e^{-\int_0^s \sigma(a+z, z)dz} ds, \end{aligned}$$

$S(a)$ denotes the average number of newborns produced per marriage and $T(a)$ is the average number of newborns produced per marriage, provided that the marriage is not terminated by death. From (2.51), we can determine $B(t)$ for $t > 0$ and all age-density functions by (2.47), provided that the initial data $B(t)$, $t < 0$, are given.

Finally, let us consider the special case in which the forces of marriage and divorce do not depend on age, but only on duration. In such a case, $K(a, \sigma)$ depends only on $a - \sigma$, and (2.50) becomes a convolution integral equation with a kernel $K(a, \sigma) = K(a - \sigma)$. Integrating both sides, we have

$$\int_0^\infty X(a)da = \int_0^\infty G(a)da + \int_0^\infty K(a)da \int_0^\infty X(a)da,$$

from which it follows that

$$\int_0^\infty X(a)da = \left(I - \int_0^\infty K(a)da \right)^{-1} \int_0^\infty G(a)da,$$

where I denotes the 2×2 identity matrix. By calculating the matrices of the right-hand side of the above equation, we obtain

$$\int_0^\infty \phi(a) da = \frac{1 - \Lambda(\infty)}{1 - (1 - \Delta(\infty))(1 - \Sigma(\infty))},$$

$$\int_0^\infty \psi(a) da = \frac{(1 - \Delta(\infty))(1 - \Sigma(\infty))}{1 - (1 - \Delta(\infty))(1 - \Sigma(\infty))},$$

where $\Lambda(a) := \exp(-\int_0^a \lambda(z) dz)$ is the survival probability in the never-married state, $\Sigma(a) := \exp(-\int_0^a \sigma(z) dz)$ is the survival probability in the married state, and $\Delta(a) := \exp(-\int_0^a \delta(z) dz)$ is the survival probability in the divorced or widowed single state. The integrals of ϕ and ψ give the average number of marriages and the average number of dissolutions per person, respectively, during their lifetime, provided that there is no termination by death.

In fact, the force of a third marriage is different from the force of a second marriage, the marital fertility of remarried couples differs from that of first-marriage couples, and so on. Thus, our model may need some extensions to capture the reality. If we introduce more fine-grained physiological aspects, model (2.46) could be extended as a model for the conception cycle. Moreover, if we take into account that the force of union formation λ depends on the supply of the other sex, the basic model will become a nonlinear two-sex model. We will encounter such a nonlinear version as an epidemiological reinfection model in Sect. 8.4.

2.4 Parity Progression Model

In demography, *parity* refers to the number of children a woman has had. It is very important to divide the female population into subclasses according to their parity status, as women's childbearing decisions are strongly dependent on their parity status and the period since the last birth. In this section, we extend the stable population model to recognize the parity status and the duration of each parity status.

Let $p_i(t, a, s)$, ($i = 1, 2, \dots, N$) be the density of the female population whose parity is i at age a , time t , and duration (since the last live birth) s . N denotes the maximum parity, which is estimated as $N = 10 \sim 15$ for humans. For simplicity, we assume that p_i are defined as zero for $i \geq N + 1$. Let $p_0(t, a)$ be the age density of women who have parity zero at age a and time t , $\lambda_0(a)$ be the force of the first birth at parity zero and age a , and $\lambda_i(a, s)$ be the force of the $i + 1$ -th birth at parity i , age a , and duration s . If β_2 is the upper bound of reproductive age, then $\lambda_0(a) = \lambda_i(a, s) = 0$ for $a > \beta_2$ or $i \geq N$.

Biologically, $\lambda_i(a, s)$ is defined only for $a \geq s$. However, we adopt the convention that $\lambda_i(a, s) = 0$ for $a \leq s$. $\lambda_i(a, s)$ and $\lambda_0(a)$ are the age-duration-dependent forces of parity progression. Let $\mu(a)$ be the natural force of mortality, which is, for simplicity, assumed to be independent of parity status. Under the above assumption, the *parity progression model* is formulated as follows:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right) p_0(t, a) &= -(\mu(a) + \lambda_0(a))p_0(t, a), \\
p_0(t, 0) &= \gamma \sum_{i=1}^{\infty} \int_0^{\infty} p_i(t, a, 0) da, \\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} + \frac{\partial}{\partial s}\right) p_i(t, a, s) &= -(\mu(a) + \lambda_i(a, s))p_i(t, a, s), \quad (i \geq 1), \\
p_1(t, a, 0) &= \lambda_0(a)p_0(t, a), \\
p_i(t, a, 0) &= \int_0^a \lambda_{i-1}(a, s)p_{i-1}(t, a, s)ds, \quad (i \geq 2), \\
p_0(0, a) &= \phi_0(a), \\
p_i(0, a, s) &= \phi_i(a, s), \quad (i \geq 1),
\end{aligned} \tag{2.53}$$

where $\phi_0(a)$ and $\phi_i(a, s)$ denote the initial data and γ is the proportion of female children among newborns. For simplicity, we assume that a woman produces at most one child per unit time, and we neglect stillbirths. The number of live births is then equal to the number of women whose parity increases.

The above model was first formulated by Inaba [53], although many authors have independently developed similar models [4, 21, 22, 30, 65]. This kind of stage structure is also observed in cell populations [42, 43, 84].

We define survival functions for the parity progression model as follows:

$$\begin{aligned}
\Lambda_0(a) &:= \exp\left(-\int_0^a \lambda_0(\rho)d\rho\right), \\
\Lambda_i(h; a, s) &:= \exp\left(-\int_0^h \lambda_i(a + \rho, s + \rho)d\rho\right), \quad (i \geq 1).
\end{aligned}$$

Let $\ell(a) := \exp(-\int_0^a \mu(\sigma)d\sigma)$ be the survival probability with respect to natural death. We assume that $\ell(\infty) = \Lambda_0(\infty) = \Lambda(\infty; a, s) = 0$. Let $\zeta_i(a)$ be the probability that the i -th birth occurs at age a . Then, $\zeta_i(a)$ satisfies the iterative relation

$$\begin{aligned}
\zeta_1(a) &= \lambda_0(a)\Lambda_0(a), \\
\zeta_{i+1}(a) &= \int_0^a \lambda_i(a, s)\Lambda_i(s; a - s, 0)\zeta_i(a - s)ds, \quad (i \geq 1).
\end{aligned}$$

Let $\Psi_i(a)$, $i \geq 0$ be the survival probability with respect to the parity progression rate at age a and parity i , defined as

$$\Psi_0(a) = \Lambda_0(a), \quad \Psi_i(a) = \int_0^a \Lambda_i(s; a - s, 0)\zeta_{i-1}(a - s)ds, \quad (i \geq 1).$$

It is then easy to see that the following holds for $0 \leq i$:

$$0 \leq \Psi_i(a) \leq 1, \quad \sum_{i=0}^{\infty} \Psi_i(a) = 1,$$

$$\Psi_0(a) = 1 - \int_0^a \zeta(\rho) d\rho, \quad \Psi_i(a) = \int_0^a [\zeta_{i-1}(\rho) - \zeta_i(\rho)] d\rho.$$

Using the above functions, we can induce useful demographic indices. For example, $\zeta_i(a)\ell(a)$ is the probability that the i -th birth occurs at age a and

$$A_i := \int_0^{\infty} \zeta_i(a)\ell(a) da$$

is the probability that at least i births occur. In the demographic literature,

$$\alpha_i := \frac{A_{i+1}}{A_i}$$

is called the *parity progression ratio*. If we let $A_0 = 1$, α_i ($i \geq 0$) denotes the ratio of individuals who bear an $(i + 1)$ -th child among the individuals who have produced i children. In this case, the basic reproduction number is given by:

$$R_0 = \gamma \sum_{i \geq 1} \int_0^{\infty} \zeta_i(a)\ell(a) da = \gamma \sum_{i \geq 1} i(A_i - A_{i+1}). \quad (2.54)$$

Let $T_i(a)$ be the expected sojourn time for individuals who have entered parity state i ($i \geq 1$) at age a . It follows that

$$\begin{aligned} T_i(a) &= \int_0^{\infty} s(\mu(a+s) + \lambda_i(a+s, s)) \Lambda(s; a, 0) \frac{\ell(a+s)}{\ell(a)} ds \\ &= \int_0^{\infty} \Lambda(s; a, 0) \frac{\ell(a+s)}{\ell(a)} ds. \end{aligned}$$

Let us define the expected sojourn time at parity i by

$$T_i := \int_0^{\infty} T_i(a) \zeta_i(a) \ell(a) da,$$

from which it is easy to see that

$$T_i = \int_0^{\infty} \Psi_i(a) \ell(a) da, \quad (i \geq 0).$$

Of course, $\sum_{i \geq 0} T_i = \int_0^{\infty} \ell(a) da$ gives the life expectancy at birth.

Using the survival functions, age–duration–density functions of each state can be calculated from the boundary values as

$$\begin{aligned}
 p_0(t, a) &= \begin{cases} \Lambda_0(a)\ell(a)B(t-a), & t-a > 0, \\ \frac{\Lambda_0(a)\ell(a)}{\Lambda_0(a-t)\ell(a-t)}\phi_0(a-t), & a-t > 0, \end{cases} \\
 p_i(t, a, s) &= \begin{cases} \Lambda_i(s; a-s, 0)\frac{\ell(a)}{\ell(a-s)}p_i(t-s, a-s, 0), & t-s > 0, \\ \Lambda_i(t; a-t, s-t)\frac{\ell(a)}{\ell(a-t)}\phi_i(a-t, s-t), & a \geq s \geq t, \end{cases}
 \end{aligned} \tag{2.55}$$

where $B(t) := p_0(t, 0)$. For the boundary value $p_i(t, a, 0)$, it is easy to see that

$$p_i(t, a, 0) = \zeta_i(a)\ell(a)B(t-a), \quad t-a > 0.$$

To calculate $p_i(t, a, 0)$ in the domain $a-t > 0$, we need $g_i^j(t, a)$, which is the age density of women who have parity i at the initial time and bear their $i+j$ -th ($j \geq 1$) child at time t . $g_i^j(t, a)$ ($a-t > 0$) is calculated iteratively as follows:

$$\begin{aligned}
 g_0^j(t, a) &= \frac{\zeta_j(a)}{\Lambda_0(a-t)}\phi_0(a-t), \quad j \geq 1, \\
 g_i^1(t, a) &= \int_t^a \lambda_i(a, s)\Lambda_i(t; a-t, s-t)\phi_i(a-t, s-t)ds, \\
 g_i^{j+1}(t, a) &= \int_0^t \lambda_{i+j}(a, s)\Lambda_{i+j}(s; a-s, 0)g_i^j(t-s, a-s)ds.
 \end{aligned}$$

By mathematical induction, we can conclude that:

Lemma 2.2 *The boundary value $p_i(t, a, 0)$ with parity $i \geq 1$ is given as*

$$p_i(t, a, 0) = \begin{cases} \zeta_i(a)\ell(a)B(t-a), & t-a > 0, \\ \left[\sum_{j=0}^{i-1} g_j^{i-j}(t, a) \right] \frac{\ell(a)}{\ell(a-t)}, & a-t > 0. \end{cases} \tag{2.56}$$

Applying (2.56) to the relation

$$B(t) = \gamma \sum_{i=1}^{\infty} \int_0^{\infty} p_i(t, a, 0)da,$$

we arrive at the renewal equation for the parity progression model:

$$B(t) = G(t) + \int_0^t \Phi(a)B(t-a)da, \tag{2.57}$$

where

$$\Phi(a) := \gamma \sum_{i \geq 1} \zeta_i(a) \ell(a), \quad G(t) := \gamma \sum_{j=0}^{i-1} \int_t^{\infty} g_j^{i-j}(t, a) \frac{\ell(a)}{\ell(a-t)} da.$$

Therefore, as in the stable population model, the strong ergodicity theorem also holds for the parity progression model. From the above consideration, we obtain the following result:

Proposition 2.7 ([53]) *The solution of the parity progression model (2.53) is given as follows:*

$$p_0(t, a) = \begin{cases} \Lambda_0(a) \ell(a) B(t-a), & t-a > 0 \\ \frac{\Lambda_0(a) \ell(a)}{\Lambda_0(a-t) \ell(a-t)} \phi_0(a-t), & a-t > 0 \end{cases}$$

$$p_i(t, a, s) = \begin{cases} \Lambda_i(s; a-s, 0) \zeta_i(a-s) \ell(a) B(t-a), & t > a > s, \\ \Lambda_i(s; a-s, 0) \left[\sum_{j=0}^{i-1} g_j^{i-j}(t-s, a-s) \right] \frac{\ell(a)}{\ell(a-t)}, & a \geq t \geq s, \\ \Lambda_i(t; a-t, s-t) \frac{\ell(a)}{\ell(a-t)} \phi_i(a-t, s-t), & a \geq s \geq t, \end{cases}$$

where $B(t)$ is the solution of the renewal equation (2.57).

Proposition 2.8 ([53]) *Let λ_0 be the intrinsic growth rate determined by the net reproduction function $\Phi(a)$. Then, the following holds uniformly on any finite age interval:*

$$\lim_{t \rightarrow \infty} e^{-\lambda_0 t} p_0(t, a) = q_0 e^{-\lambda_0 a} \ell(a) \Lambda_0(a),$$

$$\lim_{t \rightarrow \infty} e^{-\lambda_0 t} p_i(t, a, s) = q_0 e^{-\lambda_0 a} \ell(a) \Lambda_i(s; a-s, 0) \zeta_i(a-s),$$

where q_0 is given by

$$q_0 = \frac{\int_0^{\infty} e^{-\lambda_0 t} G(t) dt}{\int_0^{\infty} a e^{-\lambda_0 a} \Phi(a) da}.$$

Finally, consider the special case in which the parity progression rate $\lambda_i(a, s)$ is age-independent—that is, it depends only on duration and parity status [37]. This type of parity progression model has been widely used in demography, because it depends on fewer parameters, and is therefore easier to apply to real data. We write $\lambda_i(a, s) = \lambda_i(s)$. Moreover, we neglect the age dependency of mortality or simply neglect the natural death rate. Even if we neglect the death rate, the model is still realistic for describing the populations of developed countries, which have very small death rates in the reproductive age classes.

In the following, we neglect the death rate. The survival probability at each parity status is then given by

$$\Lambda_i(s) = \exp\left(-\int_0^s \lambda_i(\rho) d\rho\right).$$

The probability of bearing an $i + 1$ -th child is given by $f_i(s) := \lambda_i(s)\Lambda_i(s)$, where s denotes the duration since the i -th birth. Let us introduce the iterative functions $\zeta_i(a)$ as

$$\zeta_1(a) := f_0(a), \quad \zeta_{i+1}(a) := \int_0^a f_i(s)\zeta_i(a-s)ds, \quad (i \geq 1).$$

$\zeta_i(a)$ gives the probability that the i -th birth occurs at age a . Then,

$$A_i = \int_0^\infty \zeta_i(a)da,$$

and the parity progression ratio is given by

$$\frac{A_{i+1}}{A_i} = \int_0^\infty f_i(s)ds = 1 - \Lambda_i(\infty).$$

In this case, the basic reproduction number is calculated as

$$R_0 = \gamma \sum_{i \geq 1} \int_0^\infty \zeta_i(a)da = \gamma \sum_{i \geq 1} A_i.$$

Let $B_i(t) := \gamma \int_0^\infty p_i(t, a, 0)da$ be the number of i -th births per unit time. From the boundary condition in (2.53) and changing the order of integrals, we have

$$\begin{aligned} B_i(t) &= \gamma \int_0^\infty p_i(t, a, 0)da \\ &= \gamma \int_0^t \int_0^a \lambda_{i-1}(s)p_{i-1}(t, a, s)dsda + \gamma \int_t^\infty \int_0^a \lambda_{i-1}(s)p_{i-1}(t, a, s)dsda \\ &= \gamma \left[\int_0^t ds \int_s^t da + \int_0^t ds \int_t^\infty da + \int_t^\infty ds \int_s^\infty da \right] \lambda_{i-1}(s)p_{i-1}(t, a, s) \\ &= \gamma \int_0^t \lambda_{i-1}(s)\Lambda_{i-1}(s) \int_s^\infty p_{i-1}(t-s, a-s, 0)dads + G_{i-1}(t) \\ &= \int_0^t f_{i-1}(s)B_{i-1}(t-s)ds + G_{i-1}(t), \end{aligned}$$

where $B_0(t) := B(t)$ and

$$G_i(t) := \gamma \int_t^\infty \int_s^\infty f_i(s)\phi_i(a-t, s-t)dads.$$

$G_i(t)$ denotes the number of i -th births produced by the initial population at time t . Therefore, we arrive at the following renewal equation system:

$$B(t) = \sum_{i \geq 1} B_i(t),$$

$$B_i(t) = \int_0^t f_{i-1}(s) B_{i-1}(t-s) ds + G_{i-1}(t), \quad (i \geq 1).$$

Instead of giving the age-duration-density of the population at time $t = 0$ as initial data, if the past data for birth $B_i(t)$, $t < 0$ are given, we obtain the impressively simple system of renewal equations

$$B_i(t) = \int_0^\infty f_{i-1}(s) B_{i-1}(t-s) ds, \quad (i \geq 1).$$

To determine $B_i(t)$ for $t > 0$, we only need the (duration-dependent) parity progression rate $f_i(s)$. The parity progression model plays an important role in fertility analysis because it provides a theoretical framework from which to examine the effects of a delay or advance in the timing of reproduction and parity-dependent fertility decline in modern societies.

2.5 Growth and Diffusion in Continuous State Spaces

In the multistate population models considered so far, each state (trait) of the individuals is described by a finite set of natural numbers. We now consider the case in which individual states are indicated by a continuous (vector) parameter ξ , and the domain of ξ (trait space) is a subset of \mathbb{R}^n , denoted by Ω . The continuous state space may correspond to geographical position, social status, genetic traits, or any kind of individual heterogeneity. We present a formal exposition of how to induce the basic equation.

2.5.1 McKendrick Equation with an Additional Structure

First, we consider the survival process of individuals in the trait space Ω . Let $\ell(a, x)$ be the survival probability of an individual at age a and state $x \in \Omega$ under an initial distribution $\ell(0, \cdot)$ such that

$$\int_{\Omega} \ell(0, x) dx = 1.$$

Let $q(a; x, y)$ be the force of transition⁵ from state $y \in \Omega$ to state $x \in \Omega$ at age a . Then, we obtain the *master equation*

$$\frac{\partial \ell(a, x)}{\partial a} = -\mu(a, x)\ell(a, x) + \int_{\Omega} [q(a; x, \eta)\ell(a, \eta) - q(a; \eta, x)\ell(a, x)]d\eta, \quad (2.58)$$

where $\mu(a, x)$ denotes the force of mortality at age a and state x . Observe that

$$\frac{d}{da} \int_{\Omega} \ell(a, x)dx = - \int_{\Omega} \mu(a, x)\ell(a, x)dx.$$

Thus, we have

$$0 \leq \int_{\Omega} \ell(a, x)dx \leq 1$$

for all $a \geq 0$ if $\mu \in L^{\infty}(\mathbb{R}_+ \times \Omega)$.

The solution of the master equation can be expressed using a Green's function (transition operator) W as

$$\ell(a, x) = \int_{\Omega} W(a, x|\sigma, \zeta)\ell(\sigma, \zeta)d\zeta, \quad (2.59)$$

where $W(a, x|\sigma, y)$ satisfies

$$\begin{aligned} \frac{\partial W(a, x|\sigma, \zeta)}{\partial a} &= -\mu(a, x)W(a, x|\sigma, \zeta) \\ &+ \int_{\Omega} [q(a; x, \eta)W(a, \eta|\sigma, \zeta) - q(a; \eta, x)W(a, x|\sigma, \zeta)]d\eta, \end{aligned} \quad (2.60)$$

$$\lim_{a \downarrow \sigma} W(a, x|\sigma, \zeta) = \delta(x - \zeta).$$

From (2.59), if $\ell(0, x) = \delta(x - \zeta)$, we have $\ell(a, x) = W(a, x|0, \zeta)$. Biologically speaking, $W(a, x|0, \zeta)$ is the survival probability for an individual born at ζ reaching age a and state x .

Define a two-parameter family of operators $\{L(a, \sigma) : a \geq \sigma\}$ on $L^1(\Omega)$ as a solution operator of (2.58) such that $\ell(a, \cdot) = L(a, \sigma)\ell(\sigma, \cdot)$, that is,

$$(L(a, \sigma)f)(x) := \int_{\Omega} W(a, x|\sigma, \xi)f(\xi)d\xi. \quad (2.61)$$

Then, $L(a, \sigma)$, $a \geq \sigma$, forms an evolutionary system and $L(a, \sigma)$ maps the state-specific density function at age σ to the density at age a , that is, $L(a, \sigma)$ describes the aging and state transition process in a birth cohort, so it is an infinite-dimensional

⁵That is, we only deal with the case in which the transition measure is absolutely continuous with density q . A more general formulation can be found in [33].

analog of the survival probability (transition) matrix introduced in Sect. 2.1. If an individual's traits do not change throughout their lifetime, there is no migration process in Ω , and the survival probability at each state x is simply given by

$$\ell(a, x) = e^{-\int_0^a \mu(\sigma, x) d\sigma} \ell(0, x).$$

In this case, the trait-specific survival probability is given by $e^{-\int_0^a \mu(\sigma, x) d\sigma}$, which is an analog of the survival probability $\ell(a)$ in the scalar model.

Under appropriate conditions, the master equation can be reduced to a reaction–diffusion equation. For example, let us consider a one-dimensional case such that $\Omega = \mathbb{R}$, and assume that the transition intensity can be decomposed as

$$q(a; x, y) = v(a, y)\phi(x - y),$$

where ϕ is the probability density function

$$\int_{\Omega} \phi(x) dx = 1.$$

Then, we have

$$\begin{aligned} \int_{\Omega} q(a; x, \eta) \ell(a, \eta) d\eta &= \int_{\Omega} v(a, x - \xi) \ell(a, x - \xi) \phi(\xi) d\xi, \\ \int_{\Omega} q(a; \eta, x) \ell(a, x) d\eta &= v(a, x) \ell(a, x). \end{aligned}$$

A Taylor expansion gives

$$\begin{aligned} v(a, x - \xi) \ell(a, x - \xi) &= v(a, x) \ell(a, x) - \xi \frac{\partial}{\partial x} v(a, x) \ell(a, x) \\ &\quad + \frac{\xi^2}{2} \frac{\partial^2}{\partial x^2} v(a, x) \ell(a, x) + O(\xi^3). \end{aligned}$$

Neglecting the higher-order terms, we obtain

$$\begin{aligned} \int_{\Omega} q(a; x, \eta) \ell(a, \eta) d\eta &\sim v(a, x) \ell(a, x) - \langle \xi \rangle \frac{\partial}{\partial x} v(a, x) \ell(a, x) \\ &\quad + \frac{\langle \xi^2 \rangle}{2} \frac{\partial^2}{\partial x^2} v(a, x) \ell(a, x), \end{aligned}$$

where

$$\langle \xi^n \rangle := \int_{\Omega} \xi^n \phi(\xi) d\xi.$$

Inserting the above expression into (2.58), we have

$$\frac{\partial \ell(a, x)}{\partial a} = -\mu(a, x)\ell(a, x) - \langle \xi \rangle \frac{\partial}{\partial x} v(a, x)\ell(a, x) + \frac{\langle \xi^2 \rangle}{2} \frac{\partial^2}{\partial x^2} v(a, x)\ell(a, x), \quad (2.62)$$

which is a *Fokker–Planck equation* or a *forward Kolmogorov equation* [86].

The Fokker–Planck equation can be written as a reaction–diffusion equation:

$$\frac{\partial \ell(a, x)}{\partial a} = -\mu(a, x)\ell(a, x) + \mathcal{L}(a)\ell(a, x), \quad (2.63)$$

where $\mathcal{L}(a)$ is a differential operator given by

$$\begin{aligned} \mathcal{L}(a) &:= -\frac{\partial}{\partial x} C(a, x) + \frac{\partial}{\partial x} \left(D(a, x) \frac{\partial}{\partial x} \right), \\ C(a, x) &:= \langle \xi \rangle v(a, x) - \frac{\langle \xi^2 \rangle}{2} \frac{\partial v(a, x)}{\partial x}, \quad D(a, x) := \frac{\langle \xi^2 \rangle}{2} v(a, x). \end{aligned}$$

Conversely, if we start from a reaction–diffusion equation describing the survival process, $W(a, x|\sigma, \zeta)$ will be obtained as its fundamental solution such that

$$\begin{aligned} \frac{\partial W(a, x|\sigma, \zeta)}{\partial a} &= (-\mu(a, x) + \mathcal{L}(a))W(a, x|\sigma, \zeta), \\ \lim_{a \downarrow \sigma} W(a, x|\sigma, \zeta) &= \delta(x - \zeta). \end{aligned} \quad (2.64)$$

Next, let $\beta(a, x, \xi)$ be the fertility function describing whether an individual at age a and state $\xi \in \Omega$ produces a newborn of state $x \in \Omega$, and let

$$\psi(a, x, \xi) := \int_{\Omega} \beta(a, x, \eta) W(a, \eta|0, \xi) d\eta \quad (2.65)$$

be the *net reproduction function* depending on the birth state $\xi \in \Omega$.

We define a one-parameter operator $\Psi(a)$ on $L^1(\Omega)$, called the *net reproduction operator*, by

$$\begin{aligned} (\Psi(a)f)(x) &:= \int_{\Omega} \psi(a, x, \xi) f(\xi) d\xi \\ &= \int_{\Omega} \int_{\Omega} \beta(a, x, \eta) W(a, \eta|0, \xi) d\eta f(\xi) d\xi \\ &= \int_{\Omega} \beta(a, x, \eta) (L(a, 0)f)(\eta) d\eta. \end{aligned} \quad (2.66)$$

$\Psi(a)$ maps the state vector f of newborns to the state vector of their own newborns at age a .

Let $b(t, x)$ be the density of newborns of state x at time t . It follows that

$$b(t, x) = \int_0^\infty \int_\Omega \beta(a, x, \xi) p(t, a, \xi) d\xi da, \quad (2.67)$$

where p is the population density at age a and state ξ . The population density is given by:

$$p(t, a, \xi) = \begin{cases} (L(a, 0)b(t-a, \cdot))(\xi), & t-a > 0, \\ (L(a, a-t)p(0, a-t, \cdot))(\xi), & a-t > 0. \end{cases} \quad (2.68)$$

Inserting the above expression into (2.67), we obtain the renewal equation

$$b(t, x) = g(t, x) + \int_0^t \int_\Omega \psi(a, x, \xi) b(t-a, \xi) d\xi da, \quad (2.69)$$

where

$$g(t, x) := \int_t^\infty da \int_\Omega d\eta \int_\Omega d\xi \beta(a, x, \xi) W(a, \xi | a-t, \eta) p(0, a-t, \eta).$$

If we consider $b(t) := b(t, \cdot)$ as a vector-valued function in the state space $L^1(\Omega)$, (2.69) can be written as an abstract renewal integral equation in $L^1(\Omega)$:

$$b(t) = g(t) + \int_0^t \Psi(a) b(t-a) da. \quad (2.70)$$

Let us consider $W(a, x | \sigma, \zeta)$ to be given by the fundamental solution of the reaction–diffusion equation (2.63). For $t > a$, we have

$$p(t, a, x) = \int_\Omega W(a, x | 0, \zeta) b(t-a, \zeta) d\zeta.$$

Observe that

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p(t, a, x) &= \int_\Omega \frac{\partial W(a, x | 0, \zeta)}{\partial a} b(t-a, \zeta) d\zeta \\ &= \int_\Omega (-\mu(a, x) + \mathcal{L}(a)) W(a, x | 0, \zeta) b(t-a, \zeta) d\zeta \\ &= -\mu(a, x) p(t, a, x) + (\mathcal{L}(a) p(t, \cdot, \cdot))(a, x). \end{aligned}$$

Therefore, we obtain an age-dependent population problem as

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) p(t, a, x) &= -\mu(a, x) p(t, a, x) + \mathcal{L}(a) p(t, a, x), \\ p(t, 0, x) &= \int_0^\infty \int_\Omega \beta(a, x, \xi) p(t, a, \xi) d\xi da, \end{aligned} \quad (2.71)$$

which is the partial differential equation (PDE) formulation corresponding to the renewal equation (2.69). This type of PDE model (age-structured population models with an additional structure) has been studied by several authors, such as Tucker and Zimmerman [105] and Thieme [101, 102].

Remark 2.5 Usually the basic Eq. (2.71) can be simply induced as follows: Let Ω_0 be a subset with a smooth boundary $\partial\Omega_0$ in Ω . Let $J(a, x)$ be the population flux vector and let \mathbf{n} be the outward normal at (a, x) on Ω_0 . Then, the outward flux at (a, x) is given by $\mathbf{n} \cdot J(a, x)$, and the population balance on a cohort is expressed by

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \int_{\Omega_0} p(t, a, x) dx + \int_{\partial\Omega_0} \mathbf{n} \cdot J d\sigma = - \int_{\Omega_0} \mu(a, x) p(t, a, x) dx,$$

where σ denotes the Lebesgue measure on $\partial\Omega_0$. From the Gauss divergence theorem, we have

$$\int_{\partial\Omega_0} \mathbf{n} \cdot J d\sigma = \int_{\Omega_0} \nabla J dx.$$

Therefore, it follows that

$$\int_{\Omega_0} \left[\frac{\partial p(t, a, x)}{\partial t} + \frac{\partial p(t, a, x)}{\partial a} + \nabla J(a, x) + \mu(a, x) p(t, a, x) \right] dx = 0.$$

Because Ω_0 is arbitrary, we obtain

$$\frac{\partial p(t, a, x)}{\partial t} + \frac{\partial p(t, a, x)}{\partial a} + \nabla J(a, x) + \mu(a, x) p(t, a, x) = 0.$$

The population flux is given by the product of density and velocity as $J = \mathbf{v} \cdot p$, so we arrive at (2.71).

As shown in Chap. 10, the asymptotic analysis in Chap. 1 for a scalar-type Volterra integral equation can be extended to the infinite-dimensional case of (2.70) in the same manner as long as the Laplace transformation $\hat{\Psi}(\lambda)$ is a non-supporting compact operator for $\lambda \in \mathbb{R}$.⁶ That is, there exists a dominant characteristic root (Malthusian parameter) λ_0 and a number $\eta > 0$ such that

$$b(t) = g(t) + e^{\lambda_0 t} \left[\frac{\langle v_0, \hat{g}(\lambda_0) \rangle}{\langle v_0, \Psi_1 u_0 \rangle} u_0 + O(e^{-\eta t}) \right], \quad (2.72)$$

where v_0 and u_0 are the positive eigenfunctional and the positive eigenfunction of the positive linear operator $\hat{\Psi}(\lambda_0)$ associated with the eigenvalue unity and

⁶Readers can find early attempts to analyze the abstract renewal equation (2.70) in [46, 77].

$$\psi_1 = - \left. \frac{d}{d\lambda} \hat{\psi}(\lambda) \right|_{\lambda=\lambda_0}.$$

Moreover, if we define

$$K := \int_0^\infty \Psi(a) da, \quad (2.73)$$

it follows that

$$\text{sign}(\lambda_0) = \text{sign}(r(K) - 1), \quad (2.74)$$

where $r(K)$ denotes the spectral radius of K . Therefore, it is reasonable to define K as the *next-generation operator* (NGO), and its spectral radius $r(K)$ is the *basic reproduction number* (see Chap. 9).

Note that the NGO is expressed as an integral operator on $L^1(\Omega)$. In fact, it follows from (2.66) that

$$(Kf)(x) = \int_0^\infty da \int_\Omega \psi(a, x, \xi) f(\xi) d\xi = \int_\Omega \left[\int_0^\infty \psi(a, x, \xi) da \right] f(\xi) d\xi.$$

Therefore, we obtain

$$(Kf)(x) = \int_\Omega k(x, \xi) f(\xi) d\xi, \quad (2.75)$$

where

$$k(x, \xi) := \int_0^\infty \psi(a, x, \xi) da.$$

Biologically speaking, $k(x, \xi)$ is the total number of newborns at location x produced by an individual born at ξ .

If the vital parameters are age-independent (non-age-structured population models), the survival rate $L(a, 0)$ is given by a strongly continuous semigroup $T(a) = e^{Qa}$ with a generator Q whose growth bound is negative, that is, $\omega(Q) < 0$ and β is a function of only x , and we have

$$K = \int_0^\infty MT(a) da = M(-Q)^{-1}, \quad (2.76)$$

where M is a multiplication operator on $L^1(\Omega)$ defined by $(Mf)(x) := \beta(x)f(x)$ and $(Qf)(x) = -\mu(x)f(x) + (\mathcal{L}f)(x)$. In fact, if we use the well-known fact that $0 \in \rho(Q)$ and the resolvent operator is given by a Laplace transform of $T(a)$,

$$(\lambda - Q)^{-1} = \int_0^\infty e^{-\lambda a} T(a) da, \quad \lambda \in \rho(Q),$$

where $\rho(Q)$ denotes the resolvent set of Q , we have (2.76). The expression in (2.76) is an infinite-dimensional analog of the familiar framework for calculating the NGO (see Chaps. 9 and 10).

Note that the operator $(-Q)^{-1}$ is a positive integral operator on $L^1(\Omega)$ that gives the expected sojourn time at state x for a typical individual. In fact, it follows from (2.61) that

$$\begin{aligned} \int_0^\infty (L(a, 0)f)(x)da &= \int_0^\infty da \int_\Omega W(a, x|0, \xi) f(\xi) d\xi \\ &= \int_\Omega \left[\int_0^\infty W(a, x|0, \xi) da \right] f(\xi) d\xi, \end{aligned}$$

so we obtain the expression

$$\int_0^\infty (L(a, 0)f)(x)da = ((-Q)^{-1}f)(x) = \int_\Omega E(x, \xi) f(\xi) d\xi, \quad (2.77)$$

where

$$E(x, \xi) := \int_0^\infty W(a, x|0, \xi) da,$$

which gives the sojourn time at state x for an individual born in state ξ .

Integrating the reaction–diffusion equation (2.64) with age-independent parameters from zero to ∞ with respect to a , we have

$$-\delta(x - \zeta) = (-\mu(x) + \mathcal{L})E(x, \zeta), \quad (2.78)$$

where

$$\mathcal{L} = -\frac{\partial}{\partial x} C(x) + \frac{\partial}{\partial x} \left(D(x) \frac{\partial}{\partial x} \right),$$

which is a differential equation satisfied by the integral kernel $E(x, \zeta)$. Using $E(a, \zeta)$, the NGO is given by:

$$(Kf)(x) = \beta(x) \int_\Omega E(x, \zeta) f(\zeta) d\zeta. \quad (2.79)$$

Because (2.78) is an ordinary differential equation, we can obtain the NGO without solving the master equation if the vital parameters are age-independent [79].

The age-dependent diffusion model was proposed by Gurtin [40] and subsequently investigated by many authors from the late 1970s to the 1990s [67, 77, 110]. Readers should note that the precise definition of R_0 for a structured population was first formulated by Diekmann et al. in 1990 [32]. Therefore, early studies on the age-dependent diffusion dynamics could not use the idea of R_0 and its threshold property.

Recently, some authors have incorporated R_0 into the study of reaction–diffusion models [1, 79, 88, 109].

2.5.2 Traveling Wave Solutions

An interesting phenomenon other than the asymptotic behavior in the renewal system (2.69) is the existence of *traveling wave solutions*. In fact, if the NGO K is not a compact operator, our argument in the previous subsection does not necessarily hold, and we do not always expect the exponential invasion mode.

For simplicity, we assume that individuals do not change their traits during their lifetime, but newborns can exhibit different traits from their mothers, and the transmission rate from mother ξ to daughter x depends only on their distance $x - \xi$. Moreover, the survival process is independent of traits. That is, we assume that the net reproduction function is given as

$$\psi(a, x, \xi) = \beta_0(a)V(x - \xi)\ell_0(a) = \phi(a)V(x - \xi),$$

where ℓ_0 is the survival rate, $\phi(a) := \beta_0(a)\ell_0(a)$ is a trait-independent net reproduction function, and $V(x)$ is a trait transmission probability from mother to daughter satisfying

$$\int_{\Omega} V(x)dx = 1, \quad V(x) = V(-x).$$

The basic reproduction number is then given by

$$R_0 = \int_0^{\infty} \phi(a)da.$$

The renewal equation (2.69) becomes as follows:

$$b(t, x) = \int_0^{\infty} \phi(a) \int_{\Omega} V(x - \xi)b(t - a, \xi)d\xi da, \quad (2.80)$$

where we assume that the initial population has died out.⁷

Inserting a trial solution $b(t, x) = w(x + ct)$ into (2.80), we have

$$w(x) = \int_{-\infty}^{\infty} V_c(x - \xi)w(\xi)d\xi, \quad (2.81)$$

where

$$V_c(z) := \int_0^{\infty} \phi(a)V(z - ca)da.$$

⁷A more general case is studied in [103].

From the symmetry of V , if $w(x)$ is a solution of (2.81), then $w(-x)$ becomes a solution of (2.81) by replacing V_c with V_{-c} . Because our argument can proceed symmetrically, we assume, without loss of generality, that $c > 0$, that is, we only consider the traveling wave to the left-hand side.

Note that the characteristic equation of (2.81) is given by:

$$\hat{V}_c(\lambda) := \int_{-\infty}^{\infty} e^{-\lambda z} V_c(z) dz = 1, \quad (2.82)$$

where $\hat{V}_c(0) = R_0$. If the characteristic equation has a real root λ_0 with multiplicity $k \geq 1$, then $z^n e^{\lambda_0 z}$ ($n = 0, 1, \dots, k-1$) become real solutions of (2.81). Therefore, there exists a traveling wave solution if there is some $c > 0$ such that the characteristic equation has a real root. Suppose that $\hat{V}_c(\lambda)$ is defined in the right neighborhood of $\lambda = 0$, and $\hat{V}_c(0) = R_0 > 1$. Because

$$\hat{V}_c(\lambda) = \int_0^{\infty} e^{-\lambda c a} \phi(a) da \int_{-\infty}^{\infty} e^{-\lambda z} V(z) dz,$$

we obtain

$$\frac{d\hat{V}_c(0+)}{d\lambda} = -c \int_0^{\infty} a \phi(a) da < 0,$$

$$\frac{d^2 \hat{V}_c(\lambda)}{d\lambda^2} = \int_{-\infty}^{\infty} z^2 e^{-\lambda z} V_c(z) dz > 0.$$

As the graph of $\hat{V}_c(\lambda)$ is downwardly convex for $\lambda > 0$ and $\hat{V}_c(\lambda)$ is monotone decreasing for $c > 0$, for a sufficiently large $c > 0$, there must exist some $\lambda > 0$ such that $\hat{V}_c(\lambda) < 1$. Because $\hat{V}_c(0) > 1$, the characteristic equation (2.82) has a positive root. Consider the set defined by $\{c > 0 \mid \text{there exists } \lambda > 0 \text{ such that } \hat{V}_c(\lambda) < 1\}$. This set becomes a half line (c_0, ∞) . Then, if $R_0 > 1$, there exists a traveling wave solution for $c > c_0$ [31].

2.6 Ergodicity Theorems for Non-autonomous Systems

Although we have so far assumed time-independent vital rates for the stable population model and its linear extensions, this assumption is at best only approximately satisfied during a limited time interval in real-world applications. In fact, the essence of the autonomous models is to reveal the *potential power* of vital rates—they are not descriptive but normative, especially for modern populations. The vital rates of biological populations are always varying under socioeconomic and environmental changes, or as a result of interventions or controls.

We now examine the characteristics of the age profile dynamics for time-dependent vital rates. However, we do not consider any nonlinear interaction among the population parameters, but generally consider a linear one-sex population model with time-dependent vital rates. For simplicity, we consider only scalar models, although the main results can easily be extended to multistate or vector-valued models under appropriate additional assumptions.

A fundamental principle of non-autonomous positive linear evolutionary systems is *weak ergodicity*, which implies the asymptotic independence of the system state from its initial state. Although the idea of weak ergodicity can be traced back to the work of Norton [23, 85], our formulation essentially depends on the general theory of linear multiplicative processes that was mainly developed by Birkhoff in the context of lattice theory [18]. For further details, readers are referred to Chap. 10, [52, 69, 95].

For non-autonomous weakly ergodic population systems, we cannot generally expect a time-independent asymptotic age structure to exist. However, there are at least two exceptional cases, namely the asymptotically autonomous case and the time periodic case. We consider these two special cases later in this chapter.

2.6.1 Primary System and Ergodicity

A closed one-sex population with time-dependent vital rates is described by a boundary value problem of the non-autonomous McKendrick equation:

$$\begin{aligned} \frac{\partial p(t, a)}{\partial t} + \frac{\partial p(t, a)}{\partial a} &= -\mu(t, a)p(t, a), \quad t \in \mathbb{R}, \quad 0 < a < \omega, \\ p(t, 0) &= \int_0^\omega \beta(t, a)p(t, a)da, \quad t \in \mathbb{R}. \end{aligned} \quad (2.83)$$

We consider the above system for $t \in \mathbb{R}$ and assume that $\omega < \infty$ is not the maximum attainable age, but the (time-independent) least upper bound of reproductive age. Thus, we can assume that for any given $t \in \mathbb{R}$, $\mu(t, \cdot), \beta(t, \cdot) \in L_+^\infty(0, \omega)$.

We define the cohort survival probability by

$$\ell(h; t, a) := \exp \left(- \int_0^h \mu(t + \sigma, a + \sigma) d\sigma \right), \quad h \geq 0.$$

Hence, $\ell(h; t, a)$ is the probability that individuals at time t and age a survive to age $a + h$ and time $t + h$. If we consider an age distribution $p(s, a) = \phi(a)$ at time s as the initial data for system (2.83) and integrate the non-autonomous McKendrick equation (2.83) along characteristic lines for $t > s$, we obtain

$$p(t, a) = \begin{cases} \ell(a; t - a, 0)p(t - a, 0), & t - s > a, \\ \ell(t - s; s, a - t + s)\phi(a - t + s), & a > t - s. \end{cases} \quad (2.84)$$

Inserting expression (2.84) into the boundary condition in (2.83), we have

$$p(t, 0) = \begin{cases} \int_0^{t-s} \beta(t, a)\ell(a; t - a, 0)p(t - a, 0)da \\ + \int_{t-s}^{\omega} \beta(t, a)\ell(t - s; s, a - t + s)\phi(a - t + s)da, & t - s < \omega, \\ \int_0^{\omega} \beta(t, a)\ell(a; t - a, 0)p(t - a, 0)da, & t - s > \omega. \end{cases} \quad (2.85)$$

We now introduce the following functions for $h = t - s > 0$:

$$\begin{aligned} B(h; \phi, s) &:= p(s + h, 0), \\ \Psi(s + h, a) &:= \begin{cases} \beta(s + h, a)\ell(a; s + h - a, 0), & 0 < a < \omega, \\ 0, & a > \omega, \end{cases} \\ G(h; \phi, s) &:= \int_h^{\omega \vee h} \beta(s + h, a)\ell(h; s, a - h)\phi(a - h)da, \end{aligned}$$

where $\Psi(t, a) = \beta(t, a)\ell(a; t - a, 0)$ is the net reproduction function for a cohort born at time $t - a$.

From (2.85), we know that $B(h; \phi, s)$ satisfies the following Volterra integral equation:

$$B(h; \phi, s) = G(h; \phi, s) + \int_0^h \Psi(s + h, a)B(h - a; \phi, s)da, \quad h > 0. \quad (2.86)$$

Let $R(h, x; s)$ be the resolvent kernel corresponding to the integral kernel $\Psi(h + s, h - x)$. That is, $R(h, x; s)$ is the solution of the resolvent equation

$$R(h, x; s) = \Psi(h + s, h - x) + \int_x^h \Psi(h + s, h - z)R(z, x; s)dz. \quad (2.87)$$

Then, (2.86) can be solved as follows:

$$B(h; \phi, s) = G(h; \phi, s) + \int_0^h R(h, \rho; s)G(\rho; \phi, s)d\rho. \quad (2.88)$$

Using the solution $B(h; \phi, s)$, the age distribution for $t > s$ is expressed as follows:

$$p(t, a) = \begin{cases} \ell(a; t - a, 0)B(t - s - a; \phi, s), & t - s > a, \\ \ell(t - s; s, a - t + s)\phi(a - t + s), & a > t - s, \end{cases} \quad (2.89)$$

where $p(s, a) = \phi(a)$.

To consider the time evolution problem of the non-autonomous system (2.83), let us introduce the *population evolution operator*, which is a time evolution operator acting on the state space of the age distributions $L^1(0, \omega)$ as follows:

$$(U(t, s)\phi)(a) := \begin{cases} \ell(a; t - a, 0)B(t - s - a; \phi, s), & t - s > a, \\ \ell(t - s; s, a - t + s)\phi(a - t + s), & a > t - s, \end{cases} \quad (2.90)$$

where $\phi \in L^1(0, \omega)$ and $B(h; \phi, s)$ are given by (2.88). Thus, $U(t, s)$, $t \geq s$, forms a positive exponentially bounded evolutionary family (multiplicative process) on $L_+^1(0, \omega)$:

Definition 2.1 ([24]) On a Banach space E , a family of bounded linear operators $\{U(a, s)\}_{a \geq s}$ with $a, s \in \mathbb{R}$ or $a, s \in \mathbb{R}_+$ is called an exponentially bounded evolutionary family if

1. $U(a, s)U(s, \tau) = U(a, \tau)$ and $U(a, a) = I$ for all $a \geq s \geq \tau$,
2. for each $\phi \in E$, the function $(a, s) \rightarrow U(a, s)\phi$ is continuous for $a \geq s$,
3. there exist constants $M \geq 1$ and $\omega_0 > 0$ such that $\|U(a, s)\| \leq Me^{\omega_0(a-s)}$ for all $a \geq s$.

Readers are referred to [25, 54, 101] for generation theorems regarding the evolutionary system based on the differential operator on $L_+^1(0, \omega)$. We can also apply the Z-space method [29] to obtain $U(t, s)$ as a continuous solution of the extended variation-of-constants formula. As shown in Chap. 1, for a given $\phi \in L^1(0, \omega)$, $U(t, s)\phi$ does not necessarily give a classical solution of the PDE system (2.83) with initial data ϕ at time s . However, if we replace $\frac{\partial}{\partial t} + \frac{\partial}{\partial a}$ by the directional derivative D along the characteristic line defined by (1.37), $p(t) = U(t, s)\phi$ becomes a solution in the weak sense that p satisfies $Dp(t, a) = -\mu(t, a)p(t, a)$ for almost all $(t, a) \in \mathbb{R} \times [0, \omega]$, $p(t, 0) = \int_0^\omega \beta(t, a)p(t, a)da$ for $t > s$, $p(s, \cdot) = \phi(\cdot)$ and $p \in C_+([s, \infty); L^1(0, \omega))$.

Let $f : \mathbb{R} \rightarrow L^1$ be a function that takes values in the state space of age-density functions. The time series $\{f(t)\}_{t \in \mathbb{R}}$ is said to be *consistent* with the evolution system $\{U(t, s)\}_{s \leq t}$ in $\Omega \subset \mathbb{R}$ if $f(t) = U(t, s)f(s)$ for all $s, t \in \Omega$ such that $s \leq t$. The two-parameter family of non-negative linear operators $\{U(t, s)\}_{s \leq t}$ has been extensively studied by Birkhoff in the theory of *multiplicative processes*. Here, we apply Birkhoff's theory to the population evolution operators. The key idea is a classical result given by Norton [85]. To formulate Norton's result, we adopt the following additional assumption for the vital rates:

Assumption 2.9 (1) The vital rates μ and β are uniformly bounded above, that is,

$$\bar{\mu} := \sup_{(t,a) \in \mathbb{R} \times [0, \omega]} \mu(t, a) < \infty, \quad \bar{\beta} := \sup_{(t,a) \in \mathbb{R} \times [0, \omega]} \beta(t, a) < \infty.$$

(2) There exist positive numbers $\varepsilon > 0$, $0 < \gamma_1 < \gamma_2 < \omega$ such that

$$\inf_{(t,a) \in \mathbb{R} \times [\gamma_1, \gamma_2]} \beta(t, a) > \varepsilon.$$

(3) For almost all $(s, x) \in \mathbb{R} \times [0, \omega]$, it follows that

$$\int_0^\omega \beta(s + z, x + z) \ell(x + z; s - x, 0) dz > 0.$$

Assumption (2) implies that there exists a “core” time-independent reproductive age period, which is not unnatural for real populations. Assumption (3) is satisfied if we assume that ω is the time-independent least upper bound of reproductive age, so the fertility function $\beta(t, a)$ is not zero in the left neighborhood of $a = \omega$ for any time t . In such a case, considering the age-density function on $[0, \omega]$ is sufficient to determine the whole population dynamics, because the post-reproductive age period is determined by the population on the age interval $[0, \omega]$. If condition (3) is satisfied, any non-zero population distribution on $[0, \omega]$ is *non-trivial*,⁸ so the population does not become extinct within a finite time.

Lemma 2.3 ([85]) *Under Assumption 2.9, for any δ such that $0 < \delta < (\gamma_2 - \gamma_1)/2$ and for any non-negative integers $n = 0, 1, 2, \dots$, it follows that*

$$R(h, x; s) \geq \alpha^{n+1} \delta^n, \quad h - x \in [\gamma_1 + n(\gamma_1 + \delta), \gamma_2 + n(\gamma_2 - \delta)], \quad (2.91)$$

where $\alpha := \varepsilon \exp(-\bar{\mu} \gamma_2)$.

Proof From the resolvent equation (2.87), if $h - x \in [\gamma_1, \gamma_2]$, we have

$$R(h, x; s) \geq \Psi(h + s, h - x) \geq \varepsilon e^{-\bar{\mu} \gamma_2} = \alpha.$$

Thus, (2.91) holds for $n = 0$. Suppose that (2.91) holds for some non-negative integer n_0 , that is,

$$R(h, x; s) \geq \alpha^{n_0+1} \delta^{n_0}, \quad h - x \in [\gamma_1 + n_0(\gamma_1 + \delta), \gamma_2 + n_0(\gamma_2 - \delta)].$$

⁸A population age distribution is called *trivial* if it cannot produce any offspring.

First, we assume that $x \leq h - \gamma_2$. Then,

$$\begin{aligned} R(h, x; s) &\geq \int_x^h \Psi(h + s, h - z) R(z, x; s) dz \\ &\geq \int_{h-\gamma_2}^{h-\gamma_1} \Psi(h + s, h - z) R(z, x; s) dz. \end{aligned} \quad (2.92)$$

Let J be an interval given by

$$J := [h - \gamma_2, h - \gamma_1] \cap [x + \gamma_1 + n_0(\gamma_1 + \delta), x + \gamma_2 + n_0(\gamma_2 - \delta)].$$

From our assumption, the intervals $[h - \gamma_2, h - \gamma_1]$ and $[x + \gamma_1 + n_0(\gamma_1 + \delta), x + \gamma_2 + n_0(\gamma_2 - \delta)]$ are longer than 2δ . Therefore, if

$$h - x \in [\gamma_1 + (n_0 + 1)(\gamma_1 + \delta), \gamma_2 + (n_0 + 1)(\gamma_2 - \delta)], \quad (2.93)$$

it follows that

$$h - \gamma_1 - (x + \gamma_1 + n_0(\gamma_1 + \delta)) \geq \delta, \quad (x + \gamma_2 + n_0(\gamma_2 - \delta)) - (h - \gamma_2) \geq \delta.$$

Thus, the interval J is not empty and its length is greater than δ . Therefore, if (2.93) holds, it follows from (2.87) and the inductive assumption that

$$\begin{aligned} R(h, x; s) &\geq \int_{h-\gamma_2}^{h-\gamma_1} \Psi(h + s, h - z) R(z, x; s) dz \\ &\geq \int_J \Psi(h + s, h - z) R(z, x; s) dz \\ &\geq \alpha \alpha^{n_0+1} \delta^{n_0} \delta = \alpha^{n_0+2} \delta^{n_0+1}. \end{aligned}$$

Next, suppose that $h - \gamma_2 < x$. If we let

$$K := [x, h - \gamma_1] \cap [x + \gamma_1 + n_0(\gamma_1 + \delta), x + \gamma_2 + n_0(\gamma_2 - \delta)],$$

then the interval K is longer than δ and

$$R(h, x; s) \geq \int_K \Psi(h + s, h - z) R(z, x; s) dz \geq \alpha^{n_0+2} \delta^{n_0+1}.$$

Therefore, (2.91) holds for $n = n_0 + 1$. Mathematical induction then implies that (2.91) holds for any natural number. \square

Lemma 2.4 *Under Assumption 2.9, there exists a positive number $\zeta > 0$ such that the population evolution operator $U(t, s)$ is uniformly positive for $t - s \geq \zeta$ in the sense of Birkhoff.*

Proof From the resolvent equation (2.87), we have

$$R(h, x; s) \leq \bar{\beta} + \bar{\beta} \int_x^h R(z, x; s) dz.$$

Therefore, it follows that

$$R(h, x; s) \leq \bar{\beta} e^{\bar{\beta}(h-x)}.$$

From (2.88) and (2.90), we can observe that for $t - s > 2\omega$,

$$\begin{aligned} (U(t, s)\phi)(a) &= \ell(a; t - a, 0) \int_0^\omega R(t - s - a, \rho; s) G(\rho; \phi, s) d\rho \\ &\leq \ell(a; t - a, 0) \bar{\beta} e^{\bar{\beta}(t-s)} \int_0^\omega G(\rho; \phi, s) d\rho. \end{aligned} \quad (2.94)$$

If we choose a natural number n such that $n > (2\omega - \gamma_2 + \gamma_1)/(\gamma_2 - \gamma_1 - 2\delta)$, then there exists a positive number ζ such that $2\omega + \gamma_1 + n(\gamma_1 + \delta) < \zeta < \gamma_2 + n(\gamma_2 - \delta)$. If we let $t - s = \zeta$, for $a, \rho \in [0, \omega]$ we have $\zeta - a - \rho \in [\gamma_1 + n(\gamma_1 + \delta), \gamma_2 + n(\gamma_2 - \delta)]$. From (2.88), (2.90) and (2.91), we obtain

$$(U(s + \zeta, s)\phi)(a) \geq \ell(a; s + \zeta - a, 0) \alpha^{n+1} \delta^n \int_0^\omega G(\rho; \phi, s) d\rho. \quad (2.95)$$

Define a functional $\lambda_s(\phi)$, $\phi \in L^1(0, \omega)$ by

$$\lambda_s(\phi) := \int_0^\omega G(\rho; \phi, s) d\rho.$$

Changing the order of integrals, it is easy to see that

$$\lambda_s(\phi) = \int_0^\omega S(s, x) \phi(x) dx,$$

where

$$S(s, x) := \int_0^\omega \Psi(s + z, x + z) dz$$

is the total expected number of children who will be born in the remaining life of a mother at age x and time s . From Assumption 2.9, λ_s is a strictly positive functional, that is, $\lambda_s(\phi) > 0$ if $\phi \in L^1_+(0, \omega) \setminus \{0\}$. From (2.94) and (2.95), we have

$$\ell(a; s + \zeta - a, 0) \alpha^{n+1} \delta^n \lambda_s(\phi) \leq U(s + \zeta, s)\phi \leq \ell(a; s + \zeta - a, 0) \bar{\beta} e^{\bar{\beta}\zeta} \lambda_s(\phi).$$

Thus, the projective diameter of $U(s + \zeta, s)$ is estimated to be

$$\Delta(U(s + \zeta, s)) \leq 2 \log \left[\frac{\bar{\beta} e^{\bar{\beta} \zeta}}{\alpha^{n+1} \delta^n} \right].$$

That is, $U(s + \zeta, s)$ is uniformly positive, so $U(t, s)$ is uniformly positive for $t - s \geq \zeta$ in the sense of Birkhoff.⁹ \square

From Lemma 2.4, we obtain the following result (see Chap. 10):

Proposition 2.10 ([52]) *Under Assumption 2.9, the population evolution system $\{U(t, s)\}_{s \leq t}$ is a uniformly primitive multiplicative process for negative and positive time in the sense of Birkhoff, and is exponentially weakly ergodic.*

Corollary 2.2 (The weak ergodicity theorem) *Suppose that $p_j(t, a)$, $t > 0$, ($j = 1, 2$) is a solution of system (2.83) corresponding to the initial data $p_i(0, a) = \phi_i(a) \in L_+^1 \setminus \{0\}$. Under Assumption 2.9, it follows that*

$$\lim_{t \rightarrow \infty} |w_1(t, \cdot) - w_2(t, \cdot)|_{L^1} = 0, \quad (2.96)$$

where $w_j(t, a) := p_j(t, a)/|p(t, \cdot)|_{L^1}$ is the age profile of population p_j .

The result in (2.96) shows the *weak ergodicity* or the *asymptotic proportionality* of the non-autonomous age-structured population dynamics (2.83), whereby any two age distributions that evolve according to the same time-dependent vital rates become proportional to each other as time evolves, so the influences of the initial data disappear. In other words, for a uniformly primitive population evolution process, the long run-time variations in age profiles are uniquely determined by the vital rates.

In the weakly ergodic population evolution process, if the infinite time series of vital rates (birth and death rates) for $(-\infty, t_1]$ (where t_1 is a given time) are given, the time series of age profiles consistent with the given vital rates are uniquely determined. Even if the past vital rate data are given in a finite time interval $[t_0, t_1]$, the age profile at time t_1 is almost completely determined by the vital rate data if $t_1 - t_0$ is sufficiently large, because the convergence speed of the age profile is exponential. As Cohen pointed out [28], it is not necessary to know the prior age structures indefinitely far into the past to explain the current age structure of a population, because the weak ergodic theorem ensures that regardless of the age structure of a population some years previously, the vital rates that follow almost completely determine the current age structure.

If the vital rates are time-independent, the population evolution operator $U(t, s)$ depends only on the time interval $t - s$, and we can write $U(t, s) = T(t - s)$, where $T(t)$ is a population semigroup. In such a case, there exists an exponential solution $e^{\lambda_0(t-a)} \ell(a)$, and its age profile is time-independent. Therefore, for any solution $p(t, a) = (T(t)p_0)(a)$ corresponding to the positive initial data point $p_0 \in L^1 \setminus \{0\}$, it follows from (2.96) that

⁹The definition of uniform positivity is given in Chap. 10.

$$\lim_{t \rightarrow \infty} \left| \frac{p(t, \cdot)}{|p(t, \cdot)|_{L^1}} - w^* \right|_{L^1} = 0,$$

where

$$w^*(a) := \frac{e^{-\lambda_0 a} \ell(a)}{\int_0^\omega e^{-\lambda_0 \sigma} \ell(\sigma) d\sigma}$$

is the stable age profile. This result is simply the strong ergodicity theorem, which was derived in Chap. 1 using an integral equation approach.

Suppose that a population evolves according to a uniformly positive population semigroup for the time interval $t \in (-\infty, t_0]$, that is, $p(t, a) = (T(t-s)p(s, \cdot))(a)$, $t > s > -\infty$. Because the Malthusian solution $e^{\lambda_0(t-a)}\ell(a)$ is always consistent with $T(t)$ for this infinite time interval, it follows from the uniqueness of consistent time series of age profiles on an infinite time interval that for any $t \in (-\infty, t_0]$, $p(t, a)$ must have a stable age profile. Therefore, if we know that the birth and death rates have not changed for a sufficiently long time, we can expect the age distribution to have attained a stable age profile to within practical accuracy. This observation was first noticed by Lotka ([76, 97], and see Chap. 10).

Remark 2.6 In demography, the weak ergodicity of the population evolution process was first pointed out by Coale [26], and a weak ergodicity theorem for the Leslie matrix model was proved by Lopez [74] (Coale–Lopez theorem). Readers can find early demographic discussions on the weak ergodicity theorem in [38, 71, 75]. Stochastic extensions were developed by Cohen [28].

2.6.2 Dual System and Ergodicity

The dual system of the non-autonomous problem in (2.83) is given as follows:

$$\begin{aligned} \frac{\partial v(t, a)}{\partial t} + \frac{\partial v(t, a)}{\partial a} &= \mu(t, a)v(t, a) - \beta(t, a)v(t, 0), \\ v(t, \omega) &= 0. \end{aligned} \quad (2.97)$$

In an abstract setting, the backward problem in the dual space $X^* = L^\infty(0, \omega)$ is

$$\frac{dv(s)}{ds} = -A^*(s)v(s), \quad 0 \leq s < t, \quad v(t) = v^* \in X^*, \quad (2.98)$$

where

$$\begin{aligned} (A^*(s)v)(a) &= \frac{dv(a)}{da} - \mu(s, a)v(a) + \beta(s, a)v(0), \\ v \in \mathcal{D}(A^*(s)) &= \{v \in L^\infty(0, \omega) : A^*(s)v \in L^\infty, v(\omega) = 0\}. \end{aligned}$$

The weak* solution of the backward problem in (2.98) is then given by $U^*(s, t)v^*$ [25], where $U^*(s, t) = U(t, s)^*$ ($t > s > 0$) is the dual evolutionary system of the forward system $U(t, s)$, $t > s$. As shown in Chap. 10 (Proposition 10.21), the dual system has an essentially unique solution (importance functional) $v^*(s) = U^*(s, t)v^*(t)$ for $0 < s < t < \infty$, which may be called the *demographic potential* [34].

Integrating (2.97) along the characteristic line, we have

$$v(t, a) = \int_a^\omega \beta(t - a + \sigma, \sigma) \ell(\sigma - a; t, a) v(t - a + \sigma, 0) d\sigma, \quad (2.99)$$

which again implies that the demographic potential of a woman at age a and time t is equal to the total sum of the demographic potential of newborns produced in her remaining life. Therefore, the boundary value $v(t, 0)$ must satisfy the *backward renewal equation*

$$v(t, 0) = \int_0^\omega \Psi(t + \sigma, \sigma) v(t + \sigma, 0) d\sigma, \quad (2.100)$$

where $\Psi(t, a) := \beta(t, a) \ell(a; t - a, 0)$.

Let $p(t, a)$ be a positive solution of the primal system (2.83) and let $v(t, a)$ be a positive solution of the dual system. Again, it is easy to see that the total demographic potential $\langle v(t, \cdot), p(t, \cdot) \rangle$ is constant, because

$$\begin{aligned} \frac{d}{dt} \langle v(t, \cdot), p(t, \cdot) \rangle &= \left\langle \frac{\partial v(t, \cdot)}{\partial t}, p(t, \cdot) \right\rangle + \langle v(t, \cdot), \frac{\partial p(t, \cdot)}{\partial t} \rangle \\ &= -\langle A^*(t)v(t, \cdot), p(t, \cdot) \rangle + \langle v(t, \cdot), A(t)p(t, \cdot) \rangle = 0. \end{aligned}$$

Assume that $\langle v(0, \cdot), p(0, \cdot) \rangle > 0$. Then, the weighted distribution

$$\phi(t, a) := \frac{v(t, a)p(t, a)}{\langle v(t, \cdot), p(t, \cdot) \rangle} = \frac{v(t, a)p(t, a)}{\langle v(0, \cdot), p(0, \cdot) \rangle}$$

satisfies the following non-autonomous McKendrick equation:

$$\begin{aligned} \frac{\partial \phi(t, a)}{\partial t} + \frac{\partial \phi(t, a)}{\partial a} &= -\frac{v(t, 0)\beta(t, a)}{v(t, a)} \phi(t, a), \\ \phi(t, 0) &= \int_0^\omega \frac{v(t, 0)\beta(t, a)}{v(t, a)} \phi(t, a) da, \\ \phi(0, a) &= \frac{v(0, a)p(0, a)}{\langle v(0, \cdot), p(0, \cdot) \rangle}, \end{aligned} \quad (2.101)$$

which conserves the total size

$$\int_0^\omega \phi(t, a) da = 1,$$

because the total demographic potential is constant. The distribution $\phi(t, \cdot)$ then evolves as a Markov evolutionary system.

Let us fix the importance functional $v^*(t)$ for $0 < t < \infty$. Consider the two distributions

$$\phi_1(t, a) = \frac{v^*(t, a)p_1(t, a)}{\langle v^*(0, \cdot), p_1(0, \cdot) \rangle}, \quad \phi_2(t, a) = \frac{v^*(t, a)p_2(t, a)}{\langle v^*(0, \cdot), p_2(0, \cdot) \rangle},$$

satisfying (2.101), where p_1 and p_2 are two solutions of the forward system. As shown in Lemma 1.7, its Kullback information

$$K(\phi_1, \phi_2) = \int_0^\omega \phi_1(t, a) \log \left(\frac{\phi_1(t, a)}{\phi_2(t, a)} \right) da$$

is monotone decreasing as time evolves. Because this entropy-increasing (information gain decreasing) process does not stop as long as $\phi_1 \neq \phi_2$, the Kullback distance tends to zero as $t \rightarrow \infty$ ([68], Chap. 9). Therefore, in the long term, the weighted distribution (probability density function) ϕ becomes independent of the initial data, and the two age-density functions become proportional to each other:

$$p_1(t, a) \sim \frac{\langle v^*(0, \cdot), p_1(0, \cdot) \rangle}{\langle v^*(0, \cdot), p_2(0, \cdot) \rangle} p_2(t, a), \quad t \rightarrow \infty, \quad (2.102)$$

so the asymptotic proportional coefficient is the ratio of the two total demographic values at the initial time.

2.6.3 Generalized Stable Populations

We now consider the asymptotically autonomous case. If the population vital rates converge to time-independent values as time evolves, we could expect the age structure to converge to a stable age distribution calculated from the limiting vital rates. This type of result was first shown for a matrix population model by Artzrouni [7], who called such an asymptotically stable population the *generalized stable population*. To formulate this idea for the continuous time model, we introduce the following definition [54]:

Definition 2.2 Let $p(t, \cdot) \in L_+^1(0, \omega)$ be the age distribution governed by (2.83). If there exist a real number $\lambda \in \mathbb{R}$, a positive functional $F : L_+^1(0, \omega) \rightarrow \mathbb{R}_+$ and an age distribution $q \in L_+^1(0, \omega)$ such that $\phi(a) = p(0, a)$ and

$$\lim_{t \rightarrow \infty} \int_0^\omega |e^{-\lambda t} p(t, a) - \langle F, \phi \rangle q(a)| da = 0, \quad (2.103)$$

then $p(t, a)$ is called the *generalized stable population* and the population evolution process is said to be strongly ergodic.

Even when there is a limiting system, it generally depends on the speed of convergence of the vital rates whether there exists an asymptotic time-independent age structure. We now introduce a sufficient condition:

Assumption 2.11 In addition to Assumption 2.9, the following holds:

- (1) There exists a time-independent survival probability $\ell(\cdot)$ such that

$$\lim_{t \rightarrow \infty} \int_0^\omega |\ell(a; t - a, 0) - \ell(a)| da = 0. \quad (2.104)$$

- (2) There exists a real number λ_0 such that

$$\int_0^\infty \left| \int_0^\omega e^{-\lambda_0 a} \Psi(t, a) da - 1 \right| dt < \infty, \quad (2.105)$$

where $\Psi(t, a) := \beta(t, a)\ell(a; t - a, 0)$.

- (3) $\Psi(t, a)$ is uniformly Lipschitz continuous with respect to t for all a .

Assumption (2) implies that $t \rightarrow \int_0^\omega e^{-\lambda_0 a} \Psi(t, a) da$ rapidly converges to unity as $t \rightarrow +\infty$. An important case in which (2) is satisfied is when the net reproduction kernel $\Psi(t, a)$ itself rapidly converges to a time-independent net reproduction kernel $\Psi(a)$ (that is, there exists a limiting fertility rate $\beta(a)$) as

$$\int_0^\infty \int_0^\omega |\Psi(t, a) - \Psi(a)| da dt < \infty.$$

In fact, in this case it is easy to see that there exists a unique real value λ_0 such that $\int_0^\omega e^{-\lambda_0 a} \Psi(a) da = 1$, and then (2.105) follows. Thus, the asymptotic growth rate λ_0 is the dominant characteristic root of the limiting net reproduction kernel. The following strong ergodicity theorem then holds [54]:

Proposition 2.12 Under Assumption 2.11, system (2.83) is strongly ergodic in the sense of Definition 2.2. That is, there exists a positive functional F such that

$$\lim_{t \rightarrow \infty} \int_0^\omega |e^{-\lambda_0 t} p(t, a) - \langle F, p_0 \rangle e^{-\lambda_0 a} \ell(a)| da = 0.$$

The proof of the above statement is omitted, as it involves the theory of multiplicative processes and a generalized variation-of-constants formula for evolution operators, and these are beyond the scope of this chapter. However, we do provide a sketch of the proof. Under Assumption 2.11, we can decompose the net reproduction function $\Psi(t, a)$ to $\Psi(t, a) = \Psi_0(t, a) + (h(t) - 1)\Psi_0(t, a)$, where

$$\Psi_0(t, a) = \frac{\Psi(t, a)}{h(t)}, \quad h(t) := \int_0^\omega e^{-\lambda_0 a} \Psi(t, a) da.$$

It then follows that

$$\int_0^\omega e^{-\lambda_0 a} \Psi_0(t, a) da = 1.$$

If we consider a non-autonomous Lotka–McKendrick system with fertility rate $\beta(t, a)/h(t)$ and survival probability $\ell(a; t - a, 0)$, we obtain the exponential solution $e^{\lambda_0(t-s)}\ell(a; t - a, 0)$, and all positive solutions are asymptotically proportional to the exponential solution because of the weak ergodicity. Hence, the evolutionary system produced by the fertility rate $\beta(t, a)$ and the survival probability $\ell(a; t - a, 0)$ is a perturbed system of a strongly ergodic evolutionary system, and we can prove that the original system itself also becomes strongly ergodic because $h(t)$ rapidly converges to unity by $\int_0^\infty |h(t) - 1| dt < \infty$ [54, Proposition 5.6].

Finally, note that Iannelli [48] used a classical calculation to show that a renewal theorem holds for the non-autonomous Volterra integral equation with a rapidly convergent integral kernel.

2.6.4 Periodic Stable Populations

We now consider a weakly ergodic non-autonomous population system (2.83) whose vital rates μ and β have a period $\theta > 0$ with respect to t :

$$\beta(t + \theta, a) = \beta(t, a), \quad \mu(t + \theta, a) = \mu(t, a), \quad \forall (t, a) \in \mathbb{R} \times [0, \omega].$$

In the periodic environment, the population evolution operator satisfies

$$U(t + \theta, s + \theta) = U(t, s). \quad (2.106)$$

In fact, $p(t) = U(t + \theta, s + \theta)p_0$ is the solution of (2.83) with the initial value $p(s) = p_0$, so it can be expressed as $p(t) = U(t, s)p_0$. Because $U(t + \theta, s + \theta)p_0 = U(t, s)p_0$ holds for any $p_0 \in L^1$, (2.106) holds.

In the following, we seek a positive *exponential solution* with exponent λ for (2.83) such that $p(t, a) = e^{\lambda t}\phi(t, a)$, where $\lambda \in \mathbb{R}$ and $\phi(t, \cdot)$ is a θ -periodic L^1_+ -valued positive function. In fact, as shown in Chap. 10, if the population evolution process given by $U(t, s)$, $t \geq s$, is weakly ergodic, then all positive solutions are asymptotically proportional to the positive exponential solution (if it exists), so its exponent can be seen as the intrinsic growth rate of the periodic system, and we can prove that the age profile converges to a periodic age profile. Moreover, as shown below, we can also define the basic reproduction number R_0 for the periodic stable population system. The exponential solution $e^{\lambda t}\phi(t, a)$ can then be called the *periodic stable population*.

If we insert $e^{\lambda t} \phi(t, a)$ into (2.83), we obtain the eigenvalue problem

$$\begin{aligned} (-D - \mu(t, a))\phi(t, a) &= \lambda\phi(t, a), \\ \phi(t, 0) &= \int_0^\omega \beta(t, a)\phi(t, a)da, \end{aligned} \quad (2.107)$$

where D denotes the directional derivative along the characteristic line defined in (1.37).

Remark 2.7 From the perspective of functional analysis, the Malthusian parameter λ is given as a real eigenvalue of the infinitesimal generator of the *evolution semigroup* $S(\sigma)$, $\sigma \geq 0$, associated with $U(t, s)$, which is defined as

$$(S(\sigma)f)(t) = U(t, t - \sigma)f(t - \sigma), \quad f \in \mathbb{F}_\theta,$$

where $\mathbb{F}_\theta = C_\theta(\mathbb{R}; L^1(0, \omega))$ or $\mathbb{F}_\theta = L^1_\theta(\mathbb{R}; L^1(0, \omega))$ [104]. In fact, the infinitesimal generator of the evolution semigroup $S(\sigma)$ is given by $-d/dt + A(t)$, where $A(t)$ is a generator of $U(t, s)$ such that $A(t) = -d/da - \mu(t, \cdot)$ with the domain $\mathcal{D}(A(t)) = \{\phi \in L^1(0, \omega) : A(t)\phi \in L^1, \phi(0) = \int_0^\omega \beta(t, a)\phi(a)da\}$.

Let $B(t)$ be a θ -periodic function. It is then easy to see that $e^{-\lambda a} \ell(a; t - a, 0)B(t - a)$ is the eigenfunction associated with the eigenvalue λ for problem (2.107) if $B(t)$ satisfies the boundary condition

$$B(t) = \int_0^\infty e^{-\lambda a} \Psi(t, a)B(t - a)da, \quad (2.108)$$

where $\Psi(t, a) := \beta(t, a)\ell(a; t - a, 0)$ is the net reproduction function. Here, we adopt the convention that $\Psi = 0$ for $a > \omega$ and apply a calculation method in [10, 59]. From the periodicity of the vital rates, we have $\Psi(t + \theta, a) = \Psi(t, a)$. Using the periodicity of Ψ , we can rewrite (2.108) as

$$B(u) = \int_0^\theta \Theta_\lambda(u, \sigma)B(\sigma)d\sigma, \quad u \in [0, \theta], \quad (2.109)$$

where

$$\Theta_\lambda(u, \sigma) := \begin{cases} \sum_{n=0}^{[\omega/\theta]+1} e^{-\lambda(u-\sigma+n\theta)} \Psi(u, u - \sigma + n\theta), & u - \sigma > 0, \\ \sum_{n=1}^{[\omega/\theta]+1} e^{-\lambda(u-\sigma+n\theta)} \Psi(u, u - \sigma + n\theta), & u - \sigma < 0. \end{cases}$$

In fact, for any t , there exists an integer n such that $t = u + n\theta$, $u \in [0, \theta]$. It follows from the periodicity that (2.108) can be rewritten as

$$\begin{aligned}
B(u) &= \int_0^\infty e^{-\lambda a} \Psi(u, a) B(u - a) da = \int_{-\infty}^u e^{-\lambda(u-\sigma)} \Psi(u, u - \sigma) B(\sigma) d\sigma, \\
&= \int_0^u e^{-\lambda(u-\sigma)} \Psi(u, u - \sigma) B(\sigma) d\sigma \\
&\quad + \sum_{n=1}^\infty \left\{ \int_0^u d\sigma + \int_u^\theta d\sigma \right\} e^{-\lambda(u-\sigma+n\theta)} \Psi(u, u - \sigma + n\theta) B(\sigma).
\end{aligned}$$

Thus, we have expression (2.109), because $\Psi(u, u - \sigma + n\theta) = 0$ for $n \geq [\omega/\theta] + 2$.

For real λ , the right-hand side of (2.109) defines a positive linear operator on $L^1(0, \theta)$, denoted by $J(\lambda)$:

$$(J(\lambda)f)(u) := \int_0^\theta \Theta_\lambda(u, \sigma) f(\sigma) d\sigma, \quad u \in [0, \theta].$$

If we assume that $J(\lambda)$ is compact and non-supporting, then its spectral radius $r(J(\lambda))$ is a positive eigenvalue of $J(\lambda)$ and is strictly decreasing from $+\infty$ to zero with respect to λ . Therefore, there exists a unique λ_0 such that $r(J(\lambda_0)) = 1$ and a positive eigenfunction $B_0 \in L_+^1(0, \theta)$ such that $B_0 = J(\lambda)B_0$. The periodic extension $B(t) = B_0(t - [t/\theta]\theta)$ for $t \in \mathbb{R}$ then satisfies (2.109) with $\lambda = \lambda_0$, and $e^{\lambda_0(t-a)}\ell(a; t - a, 0)B(t - a)$ becomes a positive exponential solution. Moreover, it follows that

$$\text{sign}(\lambda_0) = \text{sign}(r(J(0)) - 1), \quad (2.110)$$

and λ_0 can be seen as the asymptotic growth rate, because any positive solution of (2.83) is asymptotically proportional to the exponential solution $e^{\lambda_0(t-a)}\ell(a; t - a, 0)B(t - a)$.

The above arguments suggest that the positive linear operator acting on the θ -periodic locally integrable functions f defined by

$$(K_\theta f)(t) := \int_0^\infty \Psi(t, a) f(t - a) da \quad (2.111)$$

can be seen as the NGO whose spectral radius gives the basic reproduction number R_0 for the periodic stable population system, because $r(K_\theta) = r(J(0))$. In fact, we can show that the spectral radius $r(K_\theta)$ gives the asymptotic per-generation growth factor of the periodic stable population model (see Chap. 9 and [60]). An early attempt to use $r(K_\theta)$ as a threshold value of population growth appeared in [2, 3], but the first precise recognition that $r(K_\theta)$ is the basic reproduction number was given in [9]. It is clear that the spectral radius of the monodromy operator $U(t + \theta, t)$ is a surrogate index for the basic reproduction number [44, 45].

Finally, to see the asymptotic behavior of the periodic stable population system, let us consider the *dual* eigenvalue problem

$$\begin{aligned} \frac{\partial \phi(t, a)}{\partial t} + \frac{\partial \phi(t, a)}{\partial a} &= (\lambda_0 + \mu(t, a))\phi(t, a) - \beta(t, a)\phi(t, 0), \\ \phi(t, \omega) &= 0, \end{aligned} \quad (2.112)$$

where ϕ is a θ -periodic L^∞ -valued positive function.

Let $p^\dagger(t, a) = e^{\lambda_0 t} u(t, a)$ be the exponential solution of (2.83), and assume that its population evolutionary system is uniformly primitive. From Proposition 10.30, the dual problem in (2.112) has a positive solution $\phi(t, a)$ corresponding to the eigenvalue λ_0 , and $v^\dagger(t, a) := e^{-\lambda_0 t} \phi(t, a)$ is an exponential solution (importance functional) of the dual system (2.97) such that

$$\langle v^\dagger(t, \cdot), p^\dagger(t, \cdot) \rangle = \langle \phi(t, \cdot), u(t, \cdot) \rangle = 1.$$

It then follows from Propositions 10.19 and 10.30 that we can choose $p^\dagger(t, \cdot)$ as a reference distribution such that

$$\lim_{t \rightarrow \infty} \int_0^\omega |e^{-\lambda_0 t} p(t, a) - \langle v^\dagger(0, \cdot), p(0, \cdot) \rangle u(t, a)| da = 0, \quad (2.113)$$

where

$$\langle v^\dagger(0, \cdot), p(0, \cdot) \rangle = \langle \phi(0, \cdot), p(0, \cdot) \rangle$$

is the total demographic potential of the initial data $p(0, \cdot)$.

Bacaër and Abdurahman [12] defined $\phi(t, a)$ as the *reproductive value* at time t and age a in the periodic environment. With the total reproductive value defined by $V(t) := \langle \phi(t, \cdot), p(t, \cdot) \rangle$, it follows from Proposition 10.18 that $\langle v^\dagger(t, \cdot), p(t, \cdot) \rangle$ is constant, and so

$$V(t) = \langle \phi(t, \cdot), p(t, \cdot) \rangle = e^{\lambda_0 t} \langle v^\dagger(t, \cdot), p(t, \cdot) \rangle = e^{\lambda_0 t} \langle v^\dagger(0, \cdot), p(0, \cdot) \rangle = e^{\lambda_0 t} V(0),$$

which can be thought of as an extension of Fisher's theorem (Proposition 1.12) for the reproductive value in a constant environment. To date, although there is no consensus on how to define a reproductive value in a time-heterogeneous environment [35, 106], Bacaër and Abdurahman's definition is a natural extension of Fisher's reproductive value, because it mathematically describes a positive eigenfunction (corresponding to the Malthusian parameter as an eigenvalue) of the generator of the evolution semigroup associated with the population evolution system $U(t, s)$.

Using $V(t)$, it follows from (2.113) that

$$p(t, a) \sim V(t) u(t, a) = V(0) p^\dagger(t, a), \quad t \rightarrow \infty.$$

This shows that the periodic stable population model does not have a time-independent stable age distribution, but is strongly ergodic in the sense that its asymptotic age profile is independent of the initial data and has a Malthusian parameter λ_0 and basic reproduction number R_0 , which satisfies the sign relation.

In a demographic sense, the periodic stable population was first observed by Coale [27], whereas the Volterra integral equation with periodic kernel and its renewal theorem was first mathematically established by Thieme [99]. Two decades later, based on Thieme's results, Bacaër and Guernaoui [9] defined R_0 for an age-structured periodic epidemic system using the spectral radius of an integral operator. For the essential progress of periodic population systems, readers should refer to a series of works by Bacaër and co-workers [9–15].

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