Chapter 2
Mathematical Preliminaries

In this chapter, some basic mathematical tools, which will be used in the other chapters of this book, are introduced.

Throughout this chapter, I_r denotes the r-dimensional identity matrix, and I denotes an identity matrix with appropriate dimensions. The field of scalars denoted by $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. The notation $|\cdot|$ denotes the absolute value of a real number, or the modulus of a complex number. In addition, the symbol i is used to denote the imaginary unit, that is, $\mathrm{i} = \sqrt{-1}$. For two integers $m \leq n$, $\mathbb{I}[m, n]$ is used to denote the set $\{m, m+1, \ldots, n\}$.

For matrices, the following symbols are used. $A = \left[a_{ij}\right]_{m \times n}$ represents an $m \times n$ matrix A whose entry in the i-th row j-th column position is a_{ij}. For a square matrix A, $\mathrm{tr}\,(A)$ is used to denote its trace; in addition, it is denoted that

$$f_A\,(s) = \det(sI - A), \text{ and } g_A\,(s) = \det(I - sA),$$

where $\det(\cdot)$ represents the determinant of a matrix. For a general matrix A, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ denote the real and imaginary parts of A, respectively.

For partitioned matrices the similar symbol is also used. That is, $A = \left[A_{ij}\right]_{m \times n}$ represents a matrix A with m row partitions and n column partitions, whose block in i-th row j-th column position is A_{ij}.

It should be pointed out that all the aforementioned symbols are also valid throughout this monograph.

2.1 Kronecker Products

In this section, we introduce the matrix operation of Kronecker products, a useful notation that has several important applications. A direct application of Kronecker products is to analyze/solve linear matrix equations, which will be seen at the end of this section.

A.-G. Wu and Y. Zhang, *Complex Conjugate Matrix Equations for Systems and Control*, Communications and Control Engineering,
DOI 10.1007/978-981-10-0637-1_2

Definition 2.1 Given $A = \left[a_{ij}\right]_{m_1 \times n_1} \in \mathbb{F}^{m_1 \times n_1}$ and $B = \left[b_{ij}\right]_{m_2 \times n_2} \in \mathbb{F}^{m_2 \times n_2}$, the *Kronecker product* of A and B is defined to be the partitioned matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2n_1}B \\ \cdots & \cdots & & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{m_1 n_1}B \end{bmatrix} = \left[a_{ij}B\right]_{m_1 \times n_1} \in \mathbb{F}^{m_1 m_2 \times n_1 n_2}. \tag{2.1}$$

The Kronecker product is also called the direct product, or tensor product.

Similarly, the *left Kronecker product* of $A = \left[a_{ij}\right]_{m_1 \times n_1}$ and $B = \left[b_{ij}\right]_{m_2 \times n_2}$ can be defined as

$$C = \begin{bmatrix} Ab_{11} & Ab_{12} & \cdots & Ab_{1n_2} \\ Ab_{21} & Ab_{22} & \cdots & Ab_{2n_2} \\ \cdots & \cdots & & \cdots \\ Ab_{m_2 1} & Ab_{m_2 2} & \cdots & Ab_{m_2 n_2} \end{bmatrix}.$$

Obviously, the above defined matrix C satisfies $C = B \otimes A$. This fact implies that it is sufficient to only define the Kronecker product expressed in (2.1).

According to Definition 2.1, the following result can be readily obtained.

Lemma 2.1 *For two matrices A and B with A in the partitioned form $A = [A_{ij}]_{m \times n}$, there holds*

$$A \otimes B = \left[A_{ij} \otimes B\right]_{m \times n}.$$

The next proposition provides some properties of the Kronecker product. The results can follow immediately from the definition, and thus the proof is omitted.

Proposition 2.1 *If the dimensions of the involved matrices A, B, and C, are compatible for the defined operations, then the following conclusions hold.*

(1) *For $\mu \in \mathbb{F}$, $\mu \otimes A = A \otimes \mu = \mu A$;*
(2) *For $a, b \in \mathbb{F}$, $(aA) \otimes (bB) = (ab)(A \otimes B)$;*
(3) $(A + B) \otimes C = A \otimes C + B \otimes C$, $A \otimes (B + C) = A \otimes B + A \otimes C$;
(4) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$, $(A \otimes B)^{\mathrm{H}} = A^{\mathrm{H}} \otimes B^{\mathrm{H}}$;
(5) $\mathrm{tr}(A \otimes B) = (\mathrm{tr}A)(\mathrm{tr}B)$;
(6) *For two column vectors u and v, $u^{\mathrm{T}} \otimes v = vu^{\mathrm{T}} = v \otimes u^{\mathrm{T}}$.*

The following proposition provides the associative property and distributive property for the Kronecker product.

Proposition 2.2 *For the matrices with appropriate dimensions, the following results hold.*

(1) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$;
(2) $(A \otimes B)(C \otimes D) = AC \otimes BD$.

Proof (1) Let $A = \left[a_{ij}\right]_{m\times n} \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{p\times q}$, and $C \in \mathbb{F}^{s\times t}$. By using Lemma 2.1 and the 2nd item of Proposition 2.1, one has

$$
\begin{aligned}
&(A \otimes B) \otimes C \\
&= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \otimes C \\
&= \begin{bmatrix} (a_{11}B) \otimes C & (a_{12}B) \otimes C & \cdots & (a_{1n}B) \otimes C \\ (a_{21}B) \otimes C & (a_{22}B) \otimes C & \cdots & (a_{2n}B) \otimes C \\ \cdots & \cdots & & \cdots \\ (a_{m1}B) \otimes C & (a_{m2}B) \otimes C & \cdots & (a_{mn}B) \otimes C \end{bmatrix} \\
&= \begin{bmatrix} a_{11}\left(B \otimes C\right) & a_{12}\left(B \otimes C\right) & \cdots & a_{1n}\left(B \otimes C\right) \\ a_{21}\left(B \otimes C\right) & a_{22}\left(B \otimes C\right) & \cdots & a_{2n}\left(B \otimes C\right) \\ \cdots & \cdots & & \cdots \\ a_{m1}\left(B \otimes C\right) & a_{m2}\left(B \otimes C\right) & \cdots & a_{mn}\left(B \otimes C\right) \end{bmatrix} \\
&= A \otimes \left(B \otimes C\right).
\end{aligned}
$$

This is the first conclusion.

(2) Let $A = \left[a_{ij}\right]_{m\times n} \in \mathbb{F}^{m\times n}$, $C = \left[c_{ij}\right]_{n\times p} \in \mathbb{F}^{n\times p}$. Then, one has

$$
\begin{aligned}
&(A \otimes B)(C \otimes D) \\
&= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & & \cdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix} \begin{bmatrix} c_{11}D & c_{12}D & \cdots & c_{1p}D \\ c_{21}D & c_{22}D & \cdots & c_{2p}D \\ \cdots & \cdots & & \cdots \\ c_{n1}D & c_{n2}D & \cdots & c_{np}D \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{j=1}^{n} \left(a_{1j}B\right)\left(c_{j1}D\right) & \sum\limits_{j=1}^{n} \left(a_{1j}B\right)\left(c_{j2}D\right) & \cdots & \sum\limits_{j=1}^{n} \left(a_{1j}B\right)\left(c_{jp}D\right) \\ \sum\limits_{j=1}^{n} \left(a_{2j}B\right)\left(c_{j1}D\right) & \sum\limits_{j=1}^{n} \left(a_{2j}B\right)\left(c_{j2}D\right) & \cdots & \sum\limits_{j=1}^{n} \left(a_{2j}B\right)\left(c_{jp}D\right) \\ \cdots & \cdots & & \cdots \\ \sum\limits_{j=1}^{n} \left(a_{mj}B\right)\left(c_{j1}D\right) & \sum\limits_{j=1}^{n} \left(a_{mj}B\right)\left(c_{j2}D\right) & \cdots & \sum\limits_{j=1}^{n} \left(a_{mj}B\right)\left(c_{jp}D\right) \end{bmatrix}
\end{aligned}
$$

$$
= \begin{bmatrix} \left(\sum_{j=1}^{n} a_{1j}c_{j1}\right)BD & \left(\sum_{j=1}^{n} a_{1j}c_{j2}\right)BD & \cdots & \left(\sum_{j=1}^{n} a_{1j}c_{jp}\right)BD \\ \left(\sum_{j=1}^{n} a_{2j}c_{j1}\right)BD & \left(\sum_{j=1}^{n} a_{2j}c_{j2}\right)BD & \cdots & \left(\sum_{j=1}^{n} a_{2j}c_{jp}\right)BD \\ \cdots & \cdots & & \cdots \\ \left(\sum_{j=1}^{n} a_{mj}c_{j1}\right)BD & \left(\sum_{j=1}^{n} a_{mj}c_{j2}\right)BD & \cdots & \left(\sum_{j=1}^{n} a_{mj}c_{jp}\right)BD \end{bmatrix}
$$
$$
= (AC) \otimes (BD) .
$$

The proof is thus completed. ■

By using Item (2) of Proposition 2.2, the following two conclusions can be obtained.

Proposition 2.3 *If U and V are two square unitary matrices, then $U \otimes V$ is also a unitary matrix.*

Proof Since U and V are unitary, then $UU^{\mathrm{H}} = I$, and $VV^{\mathrm{H}} = I$. By using Item (4) of Proposition 2.1 and Item (2) of Proposition 2.2, one has

$$
\begin{aligned}
(U \otimes V)(U \otimes V)^{\mathrm{H}} &= (U \otimes V)\left(U^{\mathrm{H}} \otimes V^{\mathrm{H}}\right) = \left(UU^{\mathrm{H}}\right) \otimes \left(VV^{\mathrm{H}}\right) \\
&= I \otimes I = I.
\end{aligned}
$$

This implies that $U \otimes V$ is a unitary matrix. The proof is thus completed. ■

Proposition 2.4 *For two nonsingular matrices A_1 and A_2, there holds*

$$
(A_1 \otimes A_2)^{-1} = A_1^{-1} \otimes A_2^{-1}.
$$

Proof Since A_1 and A_2 are nonsingular, then $A_1A_1^{-1} = I$, $A_2A_2^{-1} = I$. By using Item (2) of Proposition 2.2, one has

$$
\begin{aligned}
&(A_1 \otimes A_2)\left(A_1^{-1} \otimes A_2^{-1}\right) \\
&= \left(A_1A_1^{-1}\right) \otimes \left(A_2A_2^{-1}\right) \\
&= I \otimes I \\
&= I.
\end{aligned}
$$

This implies that the conclusion is true. ■

According to Proposition 2.4, the Kronecker product of two nonsingular matrices is also nonsingular. With this fact, by further using Item (2) of Proposition 2.2 the following result can be readily obtained.

Proposition 2.5 *Given two matrices $A_1 \in \mathbb{F}^{m\times n}$ and $A_2 \in \mathbb{F}^{s\times t}$, there holds*

$$\text{rank}(A_1 \otimes A_2) = (\text{rank}A_1)\,(\text{rank}A_2)\,.$$

Proof Suppose that $\text{rank}A_1 = r_1$, and $\text{rank}A_2 = r_2$. Denote

$$\Lambda_1 = \text{diag}(I_{r_1}, 0_{(m-r_1)\times(n-r_1)}),\ \Lambda_2 = \text{diag}(I_{r_2}, 0_{(s-r_2)\times(t-r_2)}).$$

According to basic matrix theory, there exist nonsingular matrices P_i, Q_i, $i = 1, 2$, with appropriate dimensions such that $A_i = P_i\Lambda_i Q_i$, $i = 1, 2$. By using Item (2) of Proposition 2.2, one has

$$\begin{aligned} A_1 \otimes A_2 &= (P_1\Lambda_1 Q_1) \otimes (P_2\Lambda_2 Q_2) \\ &= (P_1 \otimes P_2)\,(\Lambda_1 \otimes \Lambda_2)\,(Q_1 \otimes Q_2)\,. \end{aligned} \tag{2.2}$$

Since P_i, Q_i, $i = 1, 2$, are all nonsingular, both $(P_1 \otimes P_2)$ and $(Q_1 \otimes Q_2)$ are nonsingular according to Proposition 2.4. In addition, it is obvious that rank $(\Lambda_1 \otimes \Lambda_2) = r_1 r_2$ by using the definition of Kronecker products. Combining these facts with (2.2), gives

$$\text{rank}(A_1 \otimes A_2) = \text{rank}(\Lambda_1 \otimes \Lambda_2) = r_1 r_2 = (\text{rank}A_1)\,(\text{rank}A_2)\,.$$

The proof is thus completed. ■

At the end of this section, the operation of vectorization is considered for a matrix. It will be seen that this operation is very closely related to the Kronecker product. For a matrix $A \in \mathbb{F}^{m\times n}$, if it is written as $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$ with $a_j \in \mathbb{F}^m$, $j \in \mathbb{I}[1, n]$, then its vectorization is defined as

$$\text{vec}(A) = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix} \in \mathbb{F}^{mn}.$$

That is to say, vec (A) is the vector formed by "stacking" the columns of A into a long vector. It is very obvious that the operation "vec" is linear:

$$\text{vec}(\alpha A + \beta B) = \alpha\text{vec}(A) + \beta\text{vec}(B),$$

for any $A,\ B \in \mathbb{F}^{m\times n}$ and $\alpha,\ \beta \in \mathbb{F}$. The next result indicates a close relationship between the operation of "vec" and the Kronecker product.

Proposition 2.6 *If $A \in \mathbb{F}^{m\times n}$, $B \in \mathbb{F}^{s\times t}$, and $X \in \mathbb{F}^{n\times s}$, then*

$$\text{vec}(AXB) = (B^{\mathrm{T}} \otimes A)\text{vec}(X).$$

Proof For any matrix $D \in \mathbb{F}^{s\times t}$, it is written as

$$D = \begin{bmatrix} d_1 & d_2 & \cdots & d_t \end{bmatrix},$$

with $d_j \in \mathbb{F}^s$, $j \in \mathbb{I}[1, t]$. In addition, it is denoted that

$$d_j = \begin{bmatrix} d_{1j} & d_{2j} & \cdots & d_{sj} \end{bmatrix}^{\mathrm{T}}.$$

With these notations, one has

$$\begin{aligned}
\mathrm{vec}(AXB) &= \mathrm{vec}\left(\begin{bmatrix} AXb_1 & AXb_2 & \cdots & AXb_t \end{bmatrix}\right) \\
&= \begin{bmatrix} AXb_1 \\ AXb_2 \\ \cdots \\ AXb_t \end{bmatrix} \\
&= \begin{bmatrix} b_{11}Ax_1 + b_{21}Ax_2 + \cdots + b_{s1}Ax_s \\ b_{12}Ax_1 + b_{22}Ax_2 + \cdots + b_{s2}Ax_s \\ \cdots \\ b_{1t}Ax_1 + b_{2t}Ax_2 + \cdots + b_{st}Ax_s \end{bmatrix} \\
&= \begin{bmatrix} b_{11}A & b_{21}A & \cdots & b_{s1}A \\ b_{12}A & b_{22}A & \cdots & b_{s2}A \\ \cdots & \cdots & & \cdots \\ b_{1t}A & b_{2t}A & \cdots & b_{st}A \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_s \end{bmatrix} \\
&= (B^{\mathrm{T}} \otimes A)\mathrm{vec}(X).
\end{aligned}$$

The conclusion is thus proven. ■

The following result establishes a relation between vec $\left(X^{\mathrm{T}}\right)$ and vec (X) for an arbitrarily given matrix X.

Proposition 2.7 *Let m, n be given positive integers. There is a unique matrix $P(m, n) \in \mathbb{F}^{mn\times mn}$ such that*

$$\mathrm{vec}\left(X^{\mathrm{T}}\right) = P\,(m, n)\,\mathrm{vec}(X)\ \textit{for all}\ X \in \mathbb{F}^{m\times n}. \tag{2.3}$$

This matrix $P\,(m, n)$ depends only on the dimensions m and n, and is given by

$$P(m, n) = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij} \otimes E_{ij}^{\mathrm{T}} = \left[E_{ij}^{\mathrm{T}}\right]_{m\times n}, \tag{2.4}$$

where each $E_{ij} \in \mathbb{F}^{m\times n}$ has entry 1 in the i-th row j-th column position and all other entries are zero. Moreover,

$$P\,(m, n) = [P\,(n, m)]^{\mathrm{T}} = [P\,(n, m)]^{-1}.$$

Proof Let $X = \left[x_{ij}\right] \in \mathbb{F}^{m\times n}$, and $E_{ij} \in \mathbb{F}^{m\times n}$ be the unit matrix described in the statement of the proposition. Notice that $E_{ij}^{\mathrm{T}} X E_{ij}^{\mathrm{T}} = x_{ij} E_{ij}^{\mathrm{T}}$ for all $i \in \mathbb{I}[1, m]$ and $j \in \mathbb{I}[1, n]$. Thus,

$$X^{\mathrm{T}} = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} E_{ij}^{\mathrm{T}} = \sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}^{\mathrm{T}} X E_{ij}^{\mathrm{T}}.$$

Now use the identity for the vec of a threefold matrix product given in Proposition 2.6 to write

$$\operatorname{vec}\left(X^{\mathrm{T}}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} \operatorname{vec}\left(E_{ij}^{\mathrm{T}} X E_{ij}^{\mathrm{T}}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} \left(E_{ij} \otimes E_{ij}^{\mathrm{T}}\right) \operatorname{vec}(X),$$

which verifies (2.3). Since $\left(X^{\mathrm{T}}\right)^{\mathrm{T}} = X$ and $X^{\mathrm{T}} \in \mathbb{F}^{n\times m}$, one has

$$\operatorname{vec}(X) = P\,(n, m) \operatorname{vec}\left(X^{\mathrm{T}}\right) = P\,(n, m)\, P\,(m, n) \operatorname{vec}(X),$$

so $P\,(n, m) = [P\,(m, n)]^{-1}$. Finally, let ε_{ij} denote the unit matrices in $\mathbb{F}^{n\times m}$, notice that $\varepsilon_{ij} = E_{ji}^{\mathrm{T}}$, and compute

$$\begin{aligned} P\,(n, m) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \varepsilon_{ij} \otimes \varepsilon_{ij}^{\mathrm{T}} = \sum_{i=1}^{n}\sum_{j=1}^{m} E_{ji}^{\mathrm{T}} \otimes E_{ji} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} \left(E_{ij} \otimes E_{ij}^{\mathrm{T}}\right)^{\mathrm{T}} = [P\,(m, n)]^{\mathrm{T}}. \end{aligned}$$

The proof is thus completed. ■

The matrix $P(m, n)$ in the above proposition is known as a permutation matrix, which is very useful in the field of algebra.

A direct application of Kronecker products is to investigate the following general linear matrix equation

$$\sum_{i=1}^{p} A_i X B_i = C, \tag{2.5}$$

where $A_i \in \mathbb{C}^{m\times s}$, $B_i \in \mathbb{C}^{t\times n}$, $i \in \mathbb{I}[1, p]$, and $C \in \mathbb{C}^{m\times n}$ are known matrices, and $X \in \mathbb{C}^{s\times t}$ is the unknown matrix. By using Proposition 2.6, the matrix equation (2.5) can be reduced to a matrix-vector equation of the form $Gx = c$ where $G \in \mathbb{C}^{mn\times st}$, and $x \in \mathbb{C}^{st}$, $c \in \mathbb{C}^{mn}$.

Theorem 2.1 *Given $A_i \in \mathbb{C}^{m\times s}$, $B_i \in \mathbb{C}^{t\times n}$, $i \in \mathbb{I}[1, p]$, and $C \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{s\times t}$ is a solution of the matrix equation (2.5) if and only if the vector $x = \mathrm{vec}\,(X)$ is a solution of the equation*

$$Gx = c,$$

with

$$G = \sum_{i=1}^{p} \left(B_i^{\mathrm{T}} \otimes A_i\right), c = \mathrm{vec}(C).$$

The conclusion of this theorem can be easily proven by using Proposition 2.6 and the linearity of the operation of vec.

2.2 Leverrier Algorithms

For a matrix $A \in \mathbb{C}^{n\times n}$, the coefficients of its characteristic polynomial play an important role in many applications, such as control theory and matrix algebra. It is well known that the so-called Leverrier algorithm can be used to obtain these coefficients in a successive manner. This is the following theorem.

Theorem 2.2 *Given $A \in \mathbb{C}^{n\times n}$, denote*

$$\begin{aligned} f_A(s) &= \det(sI - A) = a_0 + a_1 s + \cdots + a_n s^n,\ a_n = 1, \\ \mathrm{adj}(sI - A) &= R_0 + R_1 s + \cdots + R_{n-1} s^{n-1}. \end{aligned} \tag{2.6}$$

Then, the coefficients a_i, $i \in \mathbb{I}[0, n-1]$, and the coefficient matrices R_i, $i \in \mathbb{I}[0, n-1]$, can be obtained by the following iteration

$$\begin{cases} a_{n-i} = -\frac{\mathrm{tr}(R_{n-i}A)}{i} \\ R_{n-i} = R_{n-i+1}A + a_{n-i+1}I \end{cases}, \tag{2.7}$$

with the initial value $R_{n-1} = I$.

In the rest of this section, the aim is to give a proof of Theorem 2.2. To this end, some preliminaries are needed.

For the polynomial $f_A(s)$ in (2.6), the companion matrix is defined as

$$C_A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ & 1 & \cdots & 0 & -a_2 \\ & & \ddots & & \cdots \\ & & & 1 & -a_{n-1} \end{bmatrix}, \tag{2.8}$$

whose characteristic polynomial is $f_A(s)$ in (2.6). For the companion matrix C_A in (2.8), one has the following two lemmas.

Lemma 2.2 *For the companion matrix C_A in (2.8) and an integer m less than n, denote*

$$\left[x_1 \; x_2 \; \cdots \; x_n \right] = a_m C_A^m + a_{m-1} C_A^{m-1} + \cdots + a_0 C_A^0.$$

Then,

$$\begin{aligned} x_1 &= \left[a_0 \; a_1 \; \cdots \; a_m \; 0_{1\times(n-m-1)} \right]^{\mathrm{T}}, \\ x_{i+1} &= C_A x_i, \; i \in \mathbb{I}[1, n-1]. \end{aligned}$$

Proof Let e_i, be the i-th column of the identity matrix I_n. By simple calculations, it is easily found that

$$\begin{aligned} C_A e_i &= e_{i+1}, \; i \in \mathbb{I}[1, n-1], \\ C_A e_n &= \left[-a_0 \; -a_1 \; \cdots \; -a_{n-1} \right]^{\mathrm{T}}. \end{aligned}$$

With the above relations, one has

$$\begin{aligned} C_A^k e_i &= e_{i+k}, \; k \in \mathbb{I}[1, n-i], \\ C_A^{n-i+1} e_i &= \left[-a_0 \; -a_1 \; \cdots \; -a_{n-1} \right]^{\mathrm{T}}. \end{aligned} \tag{2.9}$$

Thus, it can be derived from (2.9) that

$$\begin{aligned} x_1 &= \left(a_m C_A^m + a_{m-1} C_A^{m-1} + \cdots + a_0 C_A^0 \right) e_1 \\ &= a_m e_{m+1} + a_{m-1} e_m + \cdots + a_0 e_1 \\ &= \left[a_0 \; a_1 \; \cdots \; a_m \; 0_{1\times(n-m-1)} \right]^{\mathrm{T}}, \end{aligned}$$

and for $i \in \mathbb{I}[1, n-1]$,

$$\begin{aligned} x_{i+1} &= \left(a_m C_A^m + a_{m-1} C_A^{m-1} + \cdots + a_0 C_A^0 \right) e_{i+1} \\ &= a_m C_A^m e_{i+1} + a_{m-1} C_A^{m-1} e_{i+1} + \cdots + a_0 C_A^0 e_{i+1} \\ &= a_m C_A^{m+1} e_i + a_{m-1} C_A^m e_i + \cdots + a_0 C_A^1 e_i \\ &= C_A \left(a_m C_A^m + a_{n-1} C_A^{m-1} + \cdots + a_0 C_A^0 \right) e_i \\ &= C_A x_i. \end{aligned}$$

The proof is thus completed. ∎

Lemma 2.3 *For the companion matrix C_A in (2.8), the first $(n-k)$ elements on the principal diagonal of*

$$E_k = C_A^k + a_{n-1}C_A^{k-1} + \cdots + a_{n-k+1}C_A, k \in \mathbb{I}[1, n-1] \tag{2.10}$$

are 0, *and the other k elements are $-a_{n-k}$.*

Proof Let e_i, $i \in \mathbb{I}[1, n]$, be the i-th column of the identity matrix I_n. For $i \in \mathbb{I}[1, n-k]$, the i-th element on the principal diagonal of E_k is $e_i^{\mathrm{T}} E_k e_i$. Then, by using (2.9) one has

$$\begin{aligned} e_i^{\mathrm{T}} E_k e_i &= e_i^{\mathrm{T}} \left(C_A^k e_i + a_{n-1}C_A^{k-1} e_i + \cdots + a_{n-k+1}C_A e_i \right) \\ &= e_i^{\mathrm{T}} (e_{i+k} + a_{n-1}e_{i+k-1} + \cdots + a_{n-k+1}e_{i+1}) \\ &= 0. \end{aligned} \tag{2.11}$$

To determine the remaining diagonal elements $e_i^{\mathrm{T}} E_k e_i$ of E_k, $i \in \mathbb{I}[n-k+1, n]$, the following Cayley-Hamilton identity for the companion matrix C_A in (2.8) is considered:

$$C_A^n + a_{n-1}C_A^{n-1} + \cdots + a_0 I_n = 0,$$

which can be written as

$$E_k C_A^{n-k} = -(a_{n-k}C_A^{n-k} + \cdots + a_1 C_A + a_0 I_n), \tag{2.12}$$

where E_k is defined in (2.10). Let the matrix within parentheses on the right-hand side in (2.12) have columns $y_1, y_2, \ldots, y_n$. Then by Lemma 2.2 one has

$$y_1 = \left[a_0 \; a_1 \; \cdots \; a_{n-k} \; 0_{1\times(k-1)} \right]^{\mathrm{T}}, \tag{2.13}$$

$$y_{i+1} = C_A y_i, \text{ for } i \in \mathbb{I}[1, n-1]. \tag{2.14}$$

In addition, by using (2.9) it follows from (2.13) that for $i \in \mathbb{I}[1, n]$,

$$\begin{aligned} y_i &= C_A^{i-1} y_1 = C_A^{i-1} (a_0 e_1 + a_1 e_2 + \cdots + a_{n-k} e_{n-k+1}) \\ &= a_0 e_i + a_1 e_{i+1} + \cdots + a_{n-k} e_{n-k+i}. \end{aligned} \tag{2.15}$$

Now denote the columns of E_k in (2.10) by $z_1, z_2, \ldots, z_n$. By a further application of (2.9) it follows that the i-th column on the left-hand side of (2.12) is

$$E_k C_A^{n-k} e_i = E_k e_{n-k+i} = z_{n-k+i}, \quad i \in \mathbb{I}[1, k].$$

Equating the first k columns on either side of (2.12) therefore produces

$$z_{n-k+i} = -y_i, i \in \mathbb{I}[1, k]. \tag{2.16}$$

It follows from (2.16) and (2.15) that for $i \in \mathbb{I}[1, k]$,

$$\begin{aligned} & e_{n-k+i}^{\mathrm{T}} E_k e_{n-k+i} \\ &= e_{n-k+i}^{\mathrm{T}} z_{n-k+i} = -e_{n-k+i}^{\mathrm{T}} y_i \\ &= -e_{n-k+i}^{\mathrm{T}} \left(a_0 e_i + a_1 e_{i+1} + \cdots + a_{n-k} e_{n-k+i}\right) \\ &= -a_{n-k}. \end{aligned} \tag{2.17}$$

With (2.11) and (2.17), the proof is thus completed. ∎

With the previous lemmas as preliminary, now the proof of Theorem 2.2 can be given.

The proof of Theorem 2.2: Obviously, the following identity holds

$$(sI - A)\,\mathrm{adj}(sI - A) = I\det(sI - A).$$

With the notation in Theorem 2.2, the preceding expression gives

$$\begin{aligned} &(sI - A)\left(R_0 + R_1 s + \cdots + R_{n-1} s^{n-1}\right) \\ &= a_0 I + a_1 I s + \cdots + a_n I s^n. \end{aligned}$$

Equating coefficients of $s^{n-1}, s^{n-2}, \ldots, s^0$, in this relation, one immediately obtains the second expression in (2.7). In addition, by the iteration in the second expression of (2.7) it can be derived that for $k \in \mathbb{I}[1, n]$,

$$R_{n-k} = A^{k-1} + a_{n-1} A^{k-2} + \cdots + a_{n-k+1} I. \tag{2.18}$$

On the other hand, let C_A be in (2.8). Then, it is well-known that $f_A(s) = f_{C_A}(s)$. Thus, if

$$f_A(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n),$$

then for an integer $k \geq 0$ there holds

$$\mathrm{tr} A^k = \mathrm{tr} C^k = \sum_{i=1}^{n} \lambda_i^k.$$

With this relation, it follows from (2.18) and Lemma 2.3 that

$$\begin{aligned} -k a_{n-k} &= \mathrm{tr} E_k = \mathrm{tr}\left(C_A^k + a_{n-1} C_A^{k-1} + \cdots + a_{n-k+1} C_A\right) \\ &= \mathrm{tr}\left(A^k + a_{n-1} A^{k-1} + \cdots + a_{n-k+1} A\right) \\ &= \mathrm{tr}(R_{n-k} A). \end{aligned}$$

This gives the first expression in (2.7). The proof is thus completed.

Remark 2.1 It is easily known from the proof that the Leverrier algorithm in (2.7) can also be written as

$$\begin{cases} a_{n-i} = -\frac{\operatorname{tr}(AR_{n-i})}{i} \\ R_{n-i} = AR_{n-i+1} + a_{n-i+1}I \end{cases},$$

with the initial value $R_{n-1} = I$.

2.3 Generalized Leverrier Algorithms

In Sect. 2.2, the celebrated Leverrier algorithm is given. By using this algorithm, one can obtain the determinant and the adjoint matrix of the polynomial matrix $(sI - A)$, whose leading coefficient matrix is the identity matrix. In this section, the aim is to give an algorithm for computing the determinant and the adjoint matrix of the general polynomial matrix $(sM - A)$ for $M, A \in \mathbb{C}^{n\times n}$.

The polynomial matrix $(sM - A)$ can be written as

$$(sM - A) = zI - (zI - sM + A) = zI - \hat{A},$$

where

$$\hat{A} = zI - sM + A \tag{2.19}$$

and z is a new pseudo variable, which does not affect $(sM - A)$, since it can be eliminated. Now it is clearly seen that the Leverrier algorithm in Sect. 2.2 can be applied to compute the inverse of the matrix $zI - \hat{A}$. Denote

$$\operatorname{adj}\left(zI - \hat{A}\right) = R(s) = z^{n-1}R_0(s) + z^{n-2}R_1(s) + \cdots + zR_{n-2}(s) + R_{n-1}(s), \tag{2.20}$$

and

$$\det\left(zI - \hat{A}\right) = q(s) = z^n + q_1(s)z^{n-1} + q_2(s)z^{n-2} + \cdots + q_n(s). \tag{2.21}$$

By using the Leverrier algorithm, one has

$$\begin{array}{ll} R_0(s) = I_n, & q_1(s) = -\operatorname{tr}\left[\hat{A}\right], \\ R_1(s) = \hat{A}R_0(s) + q_1 I_n, & q_2(s) = -\frac{1}{2}\operatorname{tr}\left[\hat{A}R_1(s)\right], \\ R_2(s) = \hat{A}R_1(s) + q_2 I_n, & q_3(s) = -\frac{1}{3}\operatorname{tr}\left[\hat{A}R_2(s)\right], \\ \cdots & \cdots \\ R_{n-1}(s) = \hat{A}R_{n-2}(s) + q_{n-1}I_n, & q_n(s) = -\frac{1}{n}\operatorname{tr}\left[\hat{A}R_{n-1}(s)\right]. \end{array} \tag{2.22}$$

The matrices $R_i(s)$, $i \in \mathbb{I}[1, n-1]$, can also be computed by the following expression:

$$R_i(s) = \hat{A}^i + q_1(s)\hat{A}^{i-1} + q_2(s)\hat{A}^{i-2} + \cdots + q_i(s)I. \tag{2.23}$$

The matrices $R_i(s)$, $i \in \mathbb{I}[1, n-1]$, are no longer the coefficient matrices of the powers of s, but depend on the variable s itself. This can be seen from (2.22), since the matrix $\hat{A}$ depends on s. As (2.20) is independent of z, in the following, for the sake of simplicity, one can take $z = 0$. Therefore, the relations (2.19)–(2.21) can be written as

$$\hat{A} = -Ms + A, \tag{2.24}$$

$$\operatorname{adj}\left(-\hat{A}\right) = R(s) = R_{n-1}(s), \tag{2.25}$$

$$q(s) = q_n(s). \tag{2.26}$$

Note that there are an infinite number of forms of $\hat{A}$, and $q(s)$, depending on the specific value of the pseudo variable z. It is seen from (2.22) that the degree of the polynomial matrix $R_i(s)$, $i \in \mathbb{I}[0, n-1]$, and of the polynomial $q_i(s)$, $i \in \mathbb{I}[1, n]$, is at most equal to i. Hence, $R_i(s)$ and $q_i(s)$ can be written as

$$R_i(s) = \sum_{k=0}^{i} R_{i,k}s^k, \tag{2.27}$$

$$q_i(s) = \sum_{k=0}^{i} q_{i,k}s^k, \tag{2.28}$$

where $R_{i,k}$ and $q_{i,k}$ are the constant coefficient matrices and scalars of the power s, respectively. It follows from (2.27) that, in order to obtain $R(s)$ and $q(s)$, it is sufficient to iteratively compute the coefficient matrices $R_{n-1,k}$, and the coefficients $q_{n,k}$ given by

$$R(s) = -\operatorname{adj}\left(\hat{A}\right) = R_{n-1}(s) = \sum_{k=0}^{n-1} R_{n-1,k}s^k, \tag{2.29}$$

$$q(s) = q_n(s) = \sum_{k=0}^{n} q_{n,k}s^k, \tag{2.30}$$

in terms of $R_{i,k}$, $q_{i,k}$.

Substituting (2.24) and (2.27) into the iterative relations $R_i(s)$, $i \in \mathbb{I}[0, n-1]$, in (2.22), one can obtain the following general iterative relations by equating the coefficients of the powers of s in the two sides of each equation:

$$R_{i+1,k} = \begin{cases} -MR_{i,k-1} + q_{i+1,k}I_n, & \text{if } k = i+1, \\ AR_{i,k} - MR_{i,k-1} + q_{i+1,k}I_n, & \text{if } k \in \mathbb{I}[1,i], \\ AR_{i,0} + q_{i+1,0}I_n, & \text{if } k = 0, \end{cases} \tag{2.31}$$

and

$$q_{i+1,k} = \begin{cases} \frac{1}{i+1}\mathrm{tr}\big[MR_{i,k-1}\big], & \text{if } k = i+1, \\ -\frac{1}{i+1}\mathrm{tr}[AR_{i,k} - MR_{i,k-1}], & \text{if } k \in \mathbb{I}[1,i], \\ -\frac{1}{i+1}\mathrm{tr}[AR_{i,0}], & \text{if } k = 0. \end{cases} \tag{2.32}$$

The previous result can be summarized as the following theorem.

Theorem 2.3 *Given matrices $M, A \in \mathbb{C}^{n\times n}$, denote*

$$\det(sM - A) = q(s) = q_{n,0} + q_{n,1}s + \cdots + q_{n,n}s^n,$$
$$\mathrm{adj}(sM - A) = R(s) = R_{n-1,0} + R_{n-1,1}s + \cdots + R_{n-1,n-1}s^{n-1}.$$

Then, the coefficients $q_{n,i}$, $i \in \mathbb{I}[0,n]$, and the coefficient matrices $R_{n-1,i}$, $i \in \mathbb{I}[0, n-1]$, can be obtained by the iteration (2.31)–(2.32).

The formulas (2.31) and (2.32) are readily reduced to the Leverrier algorithm in the previous section if it is assumed that $M = I$. In this case $R_{i,k}$ and $q_{i,k}$ for $k \neq 0$ are reduced to zero.

In some applications, one may need the determinant and the adjoint matrix of $(I - sM)$ for a given matrix M. Obviously, they can be obtained by using the algorithm in Theorem 2.3. Such a method is a bit complicated since the iteration is 2-dimensional. In fact, they can be obtained by the following result.

Theorem 2.4 *Given a matrix $M \in \mathbb{C}^{n\times n}$, denote*

$$\det(I - sM) = q_n s^n + q_{n-1}s^{n-1} + \cdots + q_0,\ q_0 = 1, \tag{2.33}$$
$$\mathrm{adj}(I - sM) = R_{n-1}s^{n-1} + R_{n-2}s^{n-2} + \cdots + R_0. \tag{2.34}$$

Then, the coefficients q_i, and the coefficient matrices R_i, $i \in \mathbb{I}[1, n-1]$, can be obtained by the iteration

$$\begin{cases} R_i = R_{i-1}M + q_i I \\ q_i = -\frac{\mathrm{tr}(R_{i-1}M)}{i} \end{cases}, \tag{2.35}$$

with the initial value $R_0 = I$.

Proof If (2.33) and (2.34) hold, it is easily known that

$$\det(sI - M) = q_0 s^n + q_1 s^{n-1} + \cdots + q_n,$$
$$\mathrm{adj}(sI - M) = R_0 s^{n-1} + R_1 s^{n-2} + \cdots + R_{n-1}.$$

By using the Leverrier algorithm in Theorem 2.2, the result is immediately obtained. ■

The algorithms in Theorems 2.3 and 2.4 are called the generalized Leverrier algorithms in this book.

2.4 Singular Value Decompositions

The singular value decomposition (SVD) is a very useful tool in matrix analysis and numerical computation. Since it involves only unitary or orthogonal matrices, its solution is regarded to be numerically very simple and reliable. Because of this nice feature, the SVD has found many useful applications in signal processing and statistics. In this book, the SVD will also be repeatedly used. This section is started with the concept of singular values of matrices.

Definition 2.2 For a matrix $M \in \mathbb{C}^{m\times n}$, a positive real number σ is called a singular value of M if there exist unit-length vectors $u \in \mathbb{C}^m$ and $v \in \mathbb{C}^n$ such that

$$Mv = \sigma u, M^{\mathrm{H}}u = \sigma v. \tag{2.36}$$

The vectors u and v are called left-singular and right-singular vectors for σ, respectively.

In this book, $\sigma_{\max}(\cdot)$ is used to denote the maximal singular value of a matrix. In Definition 2.2, the length for a vector is its Euclidean length. For a vector $a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^n$, its Euclidean length is denoted by $\|a\|_2$. That is,

$$\|a\|_2 = \sqrt{\sum_{i=1}^{n} |a_i|^2}.$$

For a matrix $M \in \mathbb{C}^{m\times n}$ of rank r, if (2.36) holds, then one has

$$M^{\mathrm{H}}Mv = M^{\mathrm{H}}\sigma u = \sigma M^{\mathrm{H}}u = \sigma\sigma v = \sigma^2 v.$$

This implies that the square of a singular value of M is an eigenvalue of $M^{\mathrm{H}}M$. On the other hand, it is obvious that $M^{\mathrm{H}}M$ is Hermitian. Thus, $M^{\mathrm{H}}M$ has r positive eigenvalues, and admits an orthonormal basis of eigenvectors. If λ is a positive eigenvalue of $M^{\mathrm{H}}M$, and η is the corresponding unit-length eigenvector, then one has

$$M^{\mathrm{H}}M\eta = \lambda\eta, \eta^{\mathrm{H}}\eta = 1. \tag{2.37}$$

Pre-multiplying by η^{H} both sides of the first expression in (2.37), gives

$$\eta^{\mathrm{H}}M^{\mathrm{H}}M\eta = \lambda\eta^{\mathrm{H}}\eta = \lambda,$$

which implies that

$$\|M\eta\|_2 = \sqrt{\lambda},$$

and thus $\frac{M\eta}{\sqrt{\lambda}}$ is of unit-length. It follows from (2.37) that

$$M^{\mathrm{H}} \frac{M\eta}{\sqrt{\lambda}} = \sqrt{\lambda}\eta. \tag{2.38}$$

In addition, it is obvious that

$$M\eta = \sqrt{\lambda} \left(\frac{M\eta}{\sqrt{\lambda}} \right). \tag{2.39}$$

The expressions in (2.38) and (2.39) reveal that $\sqrt{\lambda}$ is a singular value of the matrix M, and $\frac{M\eta}{\sqrt{\lambda}}$ and η are left-singular and right-singular vectors for $\sqrt{\lambda}$, respectively. The preceding result can be summarized as the following lemma.

Lemma 2.4 *For a matrix $M \in \mathbb{C}^{m \times n}$, a positive real number σ is a singular value of M if and only if σ^2 is an eigenvalue of $M^{\mathrm{H}}M$. Moreover, if η is a unit-length eigenvector of $M^{\mathrm{H}}M$ for σ^2, then $\frac{M\eta}{\sigma}$ and η are left-singular and right-singular vectors of M for σ, respectively.*

Similarly, the following result can also be easily obtained.

Lemma 2.5 *For a matrix $M \in \mathbb{C}^{m \times n}$, a positive real number σ is a singular value of M if and only if σ^2 is an eigenvalue of MM^{H}. Moreover, if π is a unit-length eigenvector of MM^{H} for σ^2, then π and $\frac{M^{\mathrm{H}}\pi}{\sigma}$ are left-singular and right-singular vectors of M for σ, respectively.*

Before proceeding, the following well-known result on Hermitian matrices is needed.

Lemma 2.6 *Given $Q = Q^{\mathrm{H}} \in \mathbb{C}^{n \times n}$, there is a unitary matrix $W \in \mathbb{C}^{n \times n}$ such that WQW^{H} is a diagonal matrix.*

With the preceding preliminary, one can obtain the following theorem on singular value decompositions.

Theorem 2.5 *For a matrix $M \in \mathbb{C}^{m \times n}$ with rank r, there exist two unitary matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ such that*

$$M = U \operatorname{diag}\left(\sigma_1, \sigma_2, \cdots, \sigma_r, 0_{(m-r)\times(n-r)}\right) V^{\mathrm{H}} \tag{2.40}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

Proof It is obvious that $M^{\mathrm{H}}M$ is nonnegative definite. In addition, $\operatorname{rank}M^{\mathrm{H}}M = r$ since $\operatorname{rank}M = r$. With these two facts, it can be known by Lemma 2.6 that there exists a unitary matrix $V \in \mathbb{C}^{n\times n}$ satisfying

$$V^{\mathrm{H}}M^{\mathrm{H}}MV = \operatorname{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_r, 0_{(n-r)\times(n-r)}\right) \tag{2.41}$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$. Denote

$$V = \left[\, \eta_1 \ \eta_2 \ \cdots \ \eta_n \,\right]. \tag{2.42}$$

It is obvious that λ_i, $i \in \mathbb{I}[1, r]$, are the eigenvalues of $M^{\mathrm{H}}M$, and η_i, $i \in \mathbb{I}[1, r]$, are the corresponding unit-length eigenvectors. Denote $\sigma_i = \sqrt{\lambda_i}$, $i \in \mathbb{I}[1, r]$. Then, by using Lemma 2.4 it can be obtained that $u_i = \frac{M\eta_i}{\sigma_i}$ and η_i are left-singular and right-singular vectors of M for σ_i, $i \in \mathbb{I}[1, r]$, respectively. Thus, one has

$$M\eta_i = \sigma_i u_i,\ i \in \mathbb{I}[1, r], \tag{2.43}$$

with $u_i = \frac{M\eta_i}{\sigma_i}$, $i \in \mathbb{I}[1, r]$. The expression (2.43) can be compactly written as the following matrix form:

$$M\left[\, \eta_1 \ \eta_2 \ \cdots \ \eta_r \,\right] = \left[\, u_1 \ u_2 \ \cdots \ u_r \,\right] \operatorname{diag}\left(\sigma_1, \sigma_2, \ \ldots, \sigma_r\right). \tag{2.44}$$

In addition, it is easily derived from (2.41) and (2.42) that

$$\eta_i^{\mathrm{H}} M^{\mathrm{H}} M \eta_i = 0, \text{ for } i \in \mathbb{I}[r+1, n],$$

which implies that

$$M\eta_i = 0, \text{ for } i \in \mathbb{I}[r+1, n]. \tag{2.45}$$

Moreover, since V in (2.42) is unitary, then $\eta_i^{\mathrm{H}}\eta_j = 0$, for $i \neq j$, $i, j \in \mathbb{I}[1, r]$. With this, it follows from (2.41) that for $i \neq j$, $i, j \in \mathbb{I}[1, r]$,

$$\begin{aligned} u_i^{\mathrm{H}} u_j &= \left(\frac{M\eta_i}{\sigma_i}\right)^{\mathrm{H}} \frac{M\eta_j}{\sigma_j} = \frac{\eta_i^{\mathrm{H}} M^{\mathrm{H}} M \eta_j}{\sigma_i \sigma_j} \\ &= \frac{\eta_i^{\mathrm{H}} \lambda_j \eta_j}{\sigma_i \sigma_j} = 0. \end{aligned}$$

That is, $u_i = \frac{M\eta_i}{\sigma_i}$, $i \in \mathbb{I}[1, r]$, are a group of orthogonal vectors. Thus, one can find a group of vectors u_i, $i \in \mathbb{I}[1, m]$, such that

$$U = \left[\, u_1 \ u_2 \ \cdots \ u_m \,\right]$$

is a unitary matrix. With this matrix U, it follows from (2.44) and (2.45) that

$$\begin{aligned}
&U \operatorname{diag}\left(\sigma_1, \sigma_2, \ldots, \sigma_r, 0_{(m-r)\times(n-r)}\right) \\
&= \left[M \left[\eta_1 \ \eta_2 \ \cdots \ \eta_r \right] 0_{m\times(n-r)} \right] \\
&= \left[M \left[\eta_1 \ \eta_2 \ \cdots \ \eta_r \right] M \left[\eta_{r+1} \ \eta_{r+2} \ \cdots \ \eta_n \right] \right] \\
&= M \left[\eta_1 \ \eta_2 \ \cdots \ \eta_n \right] \\
&= MV.
\end{aligned}$$

This can be equivalently written as (2.40). ■

The expression in (2.40) is called the singular value decomposition of the matrix M.

Remark 2.2 The proof of Theorem 2.5 on the SVD can also be done by treating the matrix MM^{H}. The proof procedure is very similar.

At the end of this section, a property of left-singular and right-singular vectors is given in the following lemma. The result can be easily obtained by using Lemma 2.4, and thus the proof is omitted.

Lemma 2.7 *For a matrix $M \in \mathbb{C}^{m\times n}$, if u and v are left-singular and right-singular vectors of M for the singular value σ, respectively, then*

$$u^{\mathrm{H}} M v = \sigma.$$

2.5 Vector Norms and Operator Norms

In this section, some basic concepts and properties on norms are introduced.

2.5.1 Vector Norms

A vector norm is a measure of the size of a vector in a vector space. Norms may be thought of as generalizations of the Euclidean length.

Definition 2.3 Given a vector space V over a field $\mathbb{F}$, a norm on V is a function $\|\cdot\| : V \to \mathbb{R}$ with the following properties:

(1) Absolute homogeneity: $\|ax\| = |a| \, \|x\|$, for all $a \in \mathbb{F}$ and all $x \in V$;
(2) Triangle inequality: $\|x + y\| \le \|x\| + \|y\|$, for all $x, y \in V$;
(3) If $\|x\| = 0$, then $x = 0$.

A vector space V with a norm $\|\cdot\|$ is called a normed space, and denoted by $(V, \ \|\cdot\|)$, or briefly V.

By the first axiom, one has $\|0\| = 0$ and $\|x\| = \|-x\|$ for any $x \in V$. Thus, by triangle inequality one has

$$\|0\| = \|x - x\| \leq \|x\| + \|-x\| = 2\,\|x\|\,.$$

This implies that $\|x\| \geq 0$ for any $x \in V$.

For an n-dimensional Euclidean space $\mathbb{C}^n$, the intuitive notation of length of the vector $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$ is captured by

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2},$$

which is the Euclidean norm on $\mathbb{C}^n$. This is perhaps the best known vector norm since $\|x - y\|_2$ measures the standard Euclidean distance between two points $x, y \in \mathbb{C}^n$.

Lemma 2.8 *For the n-dimensional complex vector space $\mathbb{C}^n$, a norm of $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^n$ can be defined as*

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \tag{2.46}$$

where $p \geq 1$. The norm defined in (2.46) is called p-norm.

Proof For the case of $p = 1$, the conclusion can be easily proven. Next, let us show that the properties of norms are valid for all p-norms with $p > 1$.

(1) For $a \in \mathbb{C}$, $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^n$, one has

$$\begin{aligned} \|ax\|_p &= \left(\sum_{i=1}^{n} |ax_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |a|^p\,|x_i|^p \right)^{\frac{1}{p}} \\ &= \left(|a|^p \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} = |a| \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \\ &= |a|\,\|x\|_p\,. \end{aligned}$$

This implies that the function $\|\cdot\|_p$ in (2.46) satisfies the absolute homogeneity.

(2) It is easily checked that for fixed $0 < \alpha < 1$, there holds

$$\xi^{\alpha} - \alpha\xi + \alpha - 1 \leq 0 \tag{2.47}$$

for all $\xi \geq 0$. Let

$$\xi = \frac{|a|}{|b|},\ \alpha = \frac{1}{p},\ 1 - \alpha = \frac{1}{q}.$$

Thus, by using (2.47) one has

$$\frac{|a|^{\frac{1}{p}}}{|b|^{\frac{1}{p}}} - \frac{1}{p}\frac{|a|}{|b|} \leq \frac{1}{q},$$

which can be equivalently written as

$$|a|^{\frac{1}{p}} |b|^{\frac{1}{q}} \leq \frac{|a|}{p} + \frac{|b|}{q}. \tag{2.48}$$

The expression in (2.48) is the celebrated Young's Inequality.

For two vectors $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$, $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^n$, and two scalars $p > 1$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, by using Young's Inequality one has

$$\frac{(|x_i|^p)^{\frac{1}{p}}}{\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}}} \frac{(|y_i|^q)^{\frac{1}{q}}}{\left(\sum_{j=1}^{n} |y_j|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p}\frac{|x_i|^p}{\sum_{j=1}^{n} |x_j|^p} + \frac{1}{q}\frac{|y_i|^q}{\sum_{j=1}^{n} |y_j|^q},$$

for $i \in \mathbb{I}[1, n]$. Summing both sides of the preceding relations, it can be obtained that

$$\frac{\sum_{i=1}^{n} (|x_i|^p)^{\frac{1}{p}} (|y_i|^q)^{\frac{1}{q}}}{\left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |y_j|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p} + \frac{1}{q} = 1,$$

which is equivalent to

$$\sum_{i=1}^{n} |x_i| \, |y_i| \leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |y_j|^q\right)^{\frac{1}{q}}. \tag{2.49}$$

The expression in (2.49) is the celebrated Hölder's Inequality.

By using Hölder's Inequality, for $p > 1$ one has

$$\begin{aligned} &\sum_{i=1}^{n} |x_i + y_i|^p \\ &= \sum_{i=1}^{n} |x_i + y_i| \, |x_i + y_i|^{p-1} \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1} \\
&\le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right)^{\frac{1}{q}} \\
&= \left[\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}\right] \left(\sum_{i=1}^{n} |x_i + y_i|^{q(p-1)}\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, then $q\,(p-1) = p$. Thus, from the preceding relation one has

$$
\begin{aligned}
&\sum_{i=1}^{n} |x_i + y_i|^p \\
&\le \left[\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}\right] \left(\sum_{i=1}^{n} |x_i + y_i|^{p}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.50}
$$

In view that $1 - \frac{1}{q} = \frac{1}{p}$, it is easily obtained from (2.50) that

$$
\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}. \tag{2.51}
$$

This implies that the function $\|\cdot\|_p$ in (2.46) satisfies the triangle inequality.

(3) It is obvious that $\|0\|_p = 0$.

With the preceding facts, it can be concluded that the function $\|\cdot\|_p$ in (2.46) is a norm of $\mathbb{C}^n$. ■

Remark 2.3 The expression in (2.51) is called Minkowski's Inequality.

In the preceding lemma, if p tends to infinity, one can obtain the following infinity norm of $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^n$

$$
\|x\|_\infty = \max\left\{|x_i|\,,\, i \in \mathbb{I}[1, n]\right\}.
$$

For the n-dimensional complex vector space $\mathbb{C}^n$, the following norm of $x \in \mathbb{C}^n$ is often used:

$$
\|x\|_M = x^{\mathrm{H}} M x,
$$

where $M \in \mathbb{C}^{n\times n}$ is a given positive definite matrix.

In addition, for the vector space $\mathbb{C}^{m\times n}$ in which an element is an $m \times n$ dimensional matrix, the often used vector norm of $X = \left[x_{ij}\right]_{m\times n} \in \mathbb{C}^{m\times n}$ is the so-called Frobenius norm:

$$\|X\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\left|x_{ij}\right|^2} = \mathrm{tr}\left(X^{\mathrm{H}}X\right).$$

2.5.2 Operator Norms

For two normed vector spaces $(V,\ \|\cdot\|_\alpha)$ and $\left(W,\ \|\cdot\|_\beta\right)$, let $\mathscr{A}$ be a linear operator from V to W. The norm of the operator $\mathscr{A}$ is defined as

$$\|\mathscr{A}\|_{\alpha\to\beta} = \max_{x\neq 0}\left\{\frac{\|\mathscr{A}x\|_\beta}{\|x\|_\alpha}\right\}.$$

If $V = \mathbb{C}^n$ and $W = \mathbb{C}^m$, then the operator $\mathscr{A}$ can be represented by a matrix $A \in \mathbb{C}^{m\times n}$. In this case, the preceding defined norm is called an operator norm of the matrix A. That is, an operator norm of A is defined as

$$\|A\|_{\alpha\to\beta} = \max_{x\neq 0}\left\{\frac{\|Ax\|_\beta}{\|x\|_\alpha}\right\}. \tag{2.52}$$

The norm $\|\cdot\|_{\alpha\to\beta}$ in (2.52) is called subordinate to the vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$. In addition, the operator norm defined in (2.52) is referred to as $\alpha \to \beta$ induced norm; if $\alpha = \beta$, it is referred to as α-induced norm, and is denoted by $\|\cdot\|_\alpha$. In (2.52), if α, β are chosen to be real numbers greater than or equal to 1, the vector norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ in (2.52) represent the p-norms defined in (2.46).

Theorem 2.6 *Let $\|\cdot\|_\alpha$ be a norm in $\mathbb{C}^n$, and $\|\cdot\|_\beta$ be a norm in $\mathbb{C}^m$. In addition, let $\|\cdot\|_{\alpha\to\beta}$ be the operator norm defined in (2.52) for all matrices in $\mathbb{C}^{m\times n}$. Then, for the matrices $A, B \in \mathbb{C}^{m\times n}$, and a vector $x_0 \in \mathbb{C}^n$, the following properties hold.*

(1) If $\|A\|_{\alpha\to\beta} = 0$, then $A = 0$;
(2) $\|cA\|_{\alpha\to\beta} = |c|\ \|A\|_{\alpha\to\beta}$, for any $c \in \mathbb{C}$;
(3) $\|A + B\|_{\alpha\to\beta} \le \|A\|_{\alpha\to\beta} + \|B\|_{\alpha\to\beta}$;
(4) $\|Ax_0\|_\beta \le \|A\|_{\alpha\to\beta}\ \|x_0\|_\alpha$.

Proof (1) It is assumed that $\|A\|_{\alpha\to\beta} = 0$, but $A \neq 0$.

Since $A \neq 0$, then there exists a nonzero vector x_0 such that $Ax_0 \neq 0$. Thus, $\|x_0\|_\alpha > 0$ and $\|Ax_0\|_\beta > 0$. In this case, it can be obtained that

$$\|A\|_{\alpha\to\beta} = \max_{x\neq 0}\left\{\frac{\|Ax\|_\beta}{\|x\|_\alpha}\right\} \ge \frac{\|Ax_0\|_\beta}{\|x_0\|_\alpha} > 0.$$

This contradicts $\|A\|_{\alpha\to\beta} = 0$. Therefore, $A = 0$.

(2) By using definitions and properties of vector norms, one has

$$\begin{aligned}\|cA\|_{\alpha\to\beta} &= \max_{x\neq 0}\left\{\frac{\|cAx\|_\beta}{\|x\|_\alpha}\right\} = \max_{x\neq 0}\left\{\frac{|c|\ \|Ax\|_\beta}{\|x\|_\alpha}\right\}\\ &= |c| \max_{x\neq 0}\left\{\frac{\|Ax\|_\beta}{\|x\|_\alpha}\right\} = |c|\ \|A\|_{\alpha\to\beta}\,.\end{aligned}$$

(3) By using definitions and properties of vector norms, one has

$$\begin{aligned}\|A+B\|_{\alpha\to\beta} &= \max_{x\neq 0}\left\{\frac{\|(A+B)\,x\|_\beta}{\|x\|_\alpha}\right\} \leq \max_{x\neq 0}\left\{\frac{\|Ax\|_\beta+\|Bx\|_\beta}{\|x\|_\alpha}\right\}\\ &\leq \max_{x\neq 0}\left\{\frac{\|Ax\|_\beta}{\|x\|_\alpha}\right\} + \max_{x\neq 0}\left\{\frac{\|Bx\|_\beta}{\|x\|_\alpha}\right\}\\ &= \|A\|_{\alpha\to\beta} + \|B\|_{\alpha\to\beta}\,.\end{aligned}$$

(4) This conclusion can be easily derived from the definition of operator norms. ■

Let $(\mathbb{C}^n,\ \|\cdot\|_\alpha)$, $\left(\mathbb{C}^m,\ \|\cdot\|_\beta\right)$, and $\left(\mathbb{C}^l,\ \|\cdot\|_\gamma\right)$ be three normed spaces. Then for any $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{l\times m}$, there holds

$$\|BA\|_{\alpha\to\gamma} \leq \|B\|_{\beta\to\gamma}\ \|A\|_{\alpha\to\beta}\,.$$

This conclusion can be easily obtained from the definition in (2.52). In fact, according to (2.52) one has

$$\begin{aligned}\|BA\|_{\alpha\to\gamma} &= \max_{x\neq 0}\left\{\frac{\|BAx\|_\gamma}{\|x\|_\alpha}\right\} = \max_{x\neq 0}\left\{\frac{\|BAx\|_\gamma}{\|Ax\|_\beta}\frac{\|Ax\|_\beta}{\|x\|_\alpha}\right\}\\ &\leq \max_{x\neq 0}\left\{\frac{\|BAx\|_\gamma}{\|Ax\|_\beta}\right\}\max_{x\neq 0}\left\{\frac{\|Ax\|_\beta}{\|x\|_\alpha}\right\}\\ &= \max_{y\neq 0}\left\{\frac{\|By\|_\gamma}{\|y\|_\beta}\right\}\|A\|_{\alpha\to\beta}\\ &= \|B\|_{\beta\to\gamma}\ \|A\|_{\alpha\to\beta}\,.\end{aligned}$$

Next, several often used operator norms are introduced. These operator norms are induced by p-norms for vectors, and can be calculated independent of the definition (2.52). In each case, the matrix A is denoted by $A = \left[a_{ij}\right]_{m\times n} \in \mathbb{C}^{m\times n}$. During the interpretation, for a vector the notation $\|\cdot\|_p$ represents the p-norm defined in (2.46).

- 1-norm, or column sum norm:

$$\|A\|_1 = \max_{j\in\mathbb{I}[1,n]}\left\{\sum_{i=1}^{m}\left|a_{ij}\right|\right\}.$$

Write A in terms of its column as $A = \begin{bmatrix} a_1 \ a_2 \ \cdots \ a_n \end{bmatrix}$, then

$$\|A\|_1 = \max_{j\in\mathbb{I}[1,n]} \|a_j\|_1 .$$

Denote $x = \begin{bmatrix} x_1 \ x_2 \ \cdots \ x_n \end{bmatrix}^{\mathrm{T}}$. Then, one has

$$\begin{aligned}
\|Ax\|_1 &= \|x_1a_1 + x_2a_2 + \cdots + x_na_n\|_1 \\
&\leq \sum_{j=1}^{n} \|x_ja_j\|_1 = \sum_{j=1}^{n} |x_j| \, \|a_j\|_1 \\
&\leq \left(\sum_{j=1}^{n} |x_j|\right) \max_{k\in\mathbb{I}[1,n]} \{\|a_k\|_1\} = \|x\|_1 \, \|A\|_1 .
\end{aligned}$$

It follows from this expression that

$$\max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} \leq \|A\|_1 . \tag{2.53}$$

If the vector x is chosen to be $x = e_k$ (the k-th unit basic vector), then for any $k \in \mathbb{I}[1,n]$ one has

$$\max_{x\neq 0} \left\{\frac{\|Ax\|_1}{\|x\|_1}\right\} \geq \|1a_k\|_1 = \|a_k\|_1 ,$$

and hence

$$\max_{x\neq 0} \left\{\frac{\|Ax\|_1}{\|x\|_1}\right\} \geq \max_{k\in\mathbb{I}[1,n]} \|a_k\|_1 = \|A\|_1 . \tag{2.54}$$

The relations (2.53) and (2.54) imply that the column norm is the 1-induced norm $\|\cdot\|_1$.

- 2-norm, or maximal singular value norm:

$$\|A\|_2 = \sigma_{\max}(A) . \tag{2.55}$$

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be n eigenvalues of the matrix $A^{\mathrm{H}}A$. According to Lemma 2.4, $\sigma_{\max}(A) = \sqrt{\lambda_1}$. Let ξ_i be the unit-length eigenvector of $A^{\mathrm{H}}A$ corresponding to eigenvalue λ_i, $i \in \mathbb{I}[1,n]$. Since $A^{\mathrm{H}}A$ is Hermitian, then ξ_i, $i \in \mathbb{I}[1,n]$, form an orthonormal basis of $\mathbb{C}^n$. Let

$$x = \sum_{i=1}^{n} \alpha_i \xi_i .$$

Then, one has

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |\alpha_i|^2},$$

and

$$\begin{aligned}\|Ax\|_2 &= \sqrt{x^{\mathrm{H}} A^{\mathrm{H}} A x} = \sqrt{\sum_{i=1}^{n} \lambda_i \, |\alpha_i|^2} \\ &\leq \sqrt{\sum_{i=1}^{n} \lambda_1 \, |\alpha_i|^2} = \sqrt{\lambda_1} \sqrt{\sum_{i=1}^{n} |\alpha_i|^2} = \sigma_{\max}(A) \, \|x\|_2 \, .\end{aligned}$$

This relation implies that

$$\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \leq \sigma_{\max}(A) \, . \tag{2.56}$$

On the other hand, if one chooses $x = \xi_1$, then

$$\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \geq \frac{\|A\xi_1\|_2}{\|\xi_1\|_2} = \sqrt{\xi_1^{\mathrm{H}} A^{\mathrm{H}} A \xi_1} = \sqrt{\lambda_1 \xi_1^{\mathrm{H}} \xi_1} = \sqrt{\lambda_1}. \tag{2.57}$$

The relations (2.56) and (2.57) imply that the maximal singular value norm is the 2-induced norm $\|\cdot\|_2$.

- ∞-norm, or row sum norm:

$$\|A\|_\infty = \max_{i \in \mathbb{I}[1,m]} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} .$$

Denote $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}}$. One has

$$\begin{aligned}\|Ax\|_\infty &= \max_{i \in \mathbb{I}[1,m]} \left\{ \left| \sum_{k=1}^{n} a_{ik} x_k \right| \right\} \\ &\leq \max_{i \in \mathbb{I}[1,m]} \left\{ \sum_{k=1}^{n} |a_{ik}| \, |x_k| \right\} \\ &\leq \max_{i \in \mathbb{I}[1,m]} \left\{ \left(\sum_{k=1}^{n} |a_{ik}| \right) \max_{k \in \mathbb{I}[1,n]} \{|x_k|\} \right\} \\ &= \left[\max_{i \in \mathbb{I}[1,m]} \left\{ \sum_{k=1}^{n} |a_{ik}| \right\} \right] \cdot \left[\max_{k \in \mathbb{I}[1,n]} \{|x_k|\} \right]\end{aligned}$$

$$= \|A\|_\infty \|x\|_\infty \ .$$

It follows from this expression that

$$\max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \leq \|A\|_\infty . \tag{2.58}$$

In addition, it is assumed that

$$\max_{i\in\mathbb{I}[1,m]} \left\{ \sum_{j=1}^{n} \left|a_{ij}\right| \right\} = \sum_{j=1}^{n} \left|a_{sj}\right|$$

for some $s \in \mathbb{I}[1, m]$, and denote $a_{sj} = \left|a_{sj}\right| e^{\mathrm{i}\,\theta_j}, j \in \mathbb{I}[1, n]$. Now, choose

$$z = \left[e^{-\mathrm{i}\,\theta_1} \ e^{-\mathrm{i}\,\theta_2} \ \cdots \ e^{-\mathrm{i}\,\theta_n} \right]^{\mathrm{T}},$$

then $\|z\|_\infty = 1$. With such a choice, one has

$$\begin{aligned} &\max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} \\ &\geq \frac{\|Az\|_\infty}{\|z\|_\infty} = \frac{\|Az\|_\infty}{1} \\ &= \sum_{j=1}^{n} \left|a_{sj}\right| = \max_{i\in\mathbb{I}[1,m]} \left\{ \sum_{j=1}^{n} \left|a_{ij}\right| \right\} = \|A\|_\infty . \end{aligned} \tag{2.59}$$

The relations (2.58) and (2.59) imply that the row norm is the operator norm $\|\cdot\|_\infty$.

Besides the preceding three operator norms, the norm $\|\cdot\|_{1\to 2}$ is provided in the next lemma.

Lemma 2.9 *Given a matrix* $A = \left[a_1 \ a_2 \ \cdots \ a_n \right] \in \mathbb{C}^{m\times n}$*, there holds*

$$\|A\|_{1\to 2} = \max_{x\neq 0} \left\{ \frac{\|Ax\|_2}{\|x\|_1} \right\} = \max_{j\in\mathbb{I}[1,n]} \left\{ \left\|a_j\right\|_2 \right\},$$

where $\|\cdot\|_p$ *is the* p*-norm of vectors defined in (2.46),* $p = 1, 2$.

Proof Denote $x = \left[x_1 \ x_2 \ \cdots \ x_n \right]^{\mathrm{T}} \in \mathbb{C}^n$. Then, by using properties of vector norms one has

$$\begin{aligned} \|Ax\|_2 &= \|x_1 a_1 + x_2 a_2 + \cdots + x_n a_n\|_2 \\ &\leq \sum_{j=1}^{n} \left\|x_j a_j\right\|_2 = \sum_{j=1}^{n} \left|x_j\right| \left\|a_j\right\|_2 \end{aligned}$$

$$\leq \left(\sum_{j=1}^{n} |x_j| \right) \max_{k \in \mathbb{I}[1,n]} \{\|a_k\|_2\}$$
$$= \|x\|_1 \max_{k \in \mathbb{I}[1,n]} \{\|a_k\|_2\}.$$

It follows from this expression that

$$\max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_1} \leq \max_{k \in \mathbb{I}[1,n]} \{\|a_k\|_2\}. \tag{2.60}$$

If the vector x is chosen to be $x = e_k$ (the k-th unit basic vector), then for any $k \in \mathbb{I}[1, n]$ one has

$$\max_{x \neq 0} \left\{ \frac{\|Ax\|_2}{\|x\|_1} \right\} \geq \frac{\|1a_k\|_2}{\|e_k\|_1} = \|a_k\|_2,$$

and hence

$$\max_{x \neq 0} \left\{ \frac{\|Ax\|_2}{\|x\|_1} \right\} \geq \max_{k \in \mathbb{I}[1,n]} \|a_k\|_2. \tag{2.61}$$

The relations (2.60) and (2.61) imply the conclusion of this lemma. ■

The following lemma is a result on the $p \to q$ induced norm of partitioned matrices.

Lemma 2.10 *Let A be a block partitioned matrix with $A = [A_{ij}]_{m \times n}$ where the dimensionality of A_{ij}, $i \in \mathbb{I}[1, m]$, $j \in \mathbb{I}[1, n]$, are compatible. Then for any $p \to q$ induced norm with $p, q \geq 1$ there holds:*

$$\|A\|_{p \to q} \leq \left\| \left[\|A_{ij}\|_{p \to q} \right]_{m \times n} \right\|_{p \to q}.$$

Proof Let a vector x be partitioned consistently with A as

$$x = \left[x_1^{\mathrm{T}} \; x_2^{\mathrm{T}} \; \cdots \; x_n^{\mathrm{T}} \right]^{\mathrm{T}},$$

and note that

$$\|x\|_p = \left\| \left[\|x_1\|_p \; \|x_2\|_p \; \cdots \; \|x_n\|_p \right]^{\mathrm{T}} \right\|_p.$$

Then, by using the preceding relation and Theorem 2.6 it can be obtained that

$$
\begin{aligned}
&\left\| \left[A_{ij}\right]_{m\times n} \right\|_{p\to q} \\
&= \max_{x\neq 0} \left\{ \frac{\left\| \left[A_{ij}\right]_{m\times n} x \right\|_q}{\|x\|_p} \right\} = \max_{x\neq 0} \left\{ \frac{1}{\|x\|_p} \left\| \begin{bmatrix} \sum_{j=1}^n A_{1j}x_j \\ \sum_{j=1}^n A_{2j}x_j \\ \vdots \\ \sum_{j=1}^n A_{mj}x_j \end{bmatrix} \right\|_q \right\} \\
&= \max_{x\neq 0} \left\{ \frac{1}{\|x\|_p} \left\| \begin{bmatrix} \| \sum_{j=1}^n A_{1j}x_j \|_q \\ \| \sum_{j=1}^n A_{2j}x_j \|_q \\ \cdots \\ \| \sum_{j=1}^n A_{mj}x_j \|_q \end{bmatrix} \right\|_q \right\} \\
&\leq \max_{x\neq 0} \left\{ \frac{1}{\|x\|_p} \left\| \begin{bmatrix} \sum_{j=1}^n \|A_{1j}\|_{p\to q} \|x_j\|_p \\ \sum_{j=1}^n \|A_{2j}\|_{p\to q} \|x_j\|_p \\ \cdots \\ \sum_{j=1}^n \|A_{mj}\|_{p\to q} \|x_j\|_p \end{bmatrix} \right\|_q \right\} \\
&= \max_{x\neq 0} \left\{ \frac{1}{\|x\|_p} \left\| \begin{bmatrix} \|A_{11}\|_{p\to q} & \|A_{12}\|_{p\to q} & \cdots & \|A_{1n}\|_{p\to q} \\ \|A_{21}\|_{p\to q} & \|A_{22}\|_{p\to q} & \cdots & \|A_{2n}\|_{p\to q} \\ \vdots & \vdots & & \vdots \\ \|A_{m1}\|_{p\to q} & \|A_{m2}\|_{p\to q} & \cdots & \|A_{mn}\|_{p\to q} \end{bmatrix} \begin{bmatrix} \|x_1\|_p \\ \|x_2\|_p \\ \vdots \\ \|x_n\|_p \end{bmatrix} \right\|_q \right\} \\
&\leq \left\| \left[\|A_{ij}\|_{p\to q} \right]_{m\times n} \right\|_{p\to q}.
\end{aligned}
$$

The proof is thus completed. ■

In the preceding lemma, if $q = p$, the following corollary is readily obtained.

Corollary 2.1 *[316] Let A be a block partitioned matrix with $A = \left[A_{ij}\right]_{m\times n}$ where the dimensionality of $A_{ij}, i \in \mathbb{I}[1, m]$, $j \in \mathbb{I}[1, n]$, are compatible. Then for any p-induced norm with $p \geq 1$ there holds:*

$$
\|A\|_p \leq \left\| \left[\|A_{ij}\|_p \right]_{m\times n} \right\|_p.
$$

The following lemma is on the 2-norm of the Kronecker product of two matrices.

Lemma 2.11 *Given two matrices A and B, there holds*

$$
\|A \otimes B\|_2 = \|A\|_2 \, \|B\|_2 \, .
$$

Proof Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{p\times q}$ with $\operatorname{rank} A = r_1$ and $\operatorname{rank} B = r_2$. By carrying out singular value decompositions for A and B, there exist unitary matrices U_i, V_i, $i = 1, 2$, with appropriate dimensions such that

$$
A = U_1 \Sigma_1 V_1, B = U_2 \Sigma_2 V_2,
$$

with

$$\Sigma_1 = \operatorname{diag}\left(\delta_1, \delta_2, \ldots, \delta_{r_1}, 0_{(m-r_1)\times(n-r_1)}\right), \delta_1 \geq \delta_2 \geq \cdots \geq \delta_{r_1} > 0,$$
$$\Sigma_2 = \operatorname{diag}\left(\sigma_1, \sigma_2, \ldots, \sigma_{r_2}, 0_{(p-r_2)\times(q-r_2)}\right), \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{r_2} > 0.$$

By Item (2) of Proposition 2.2, one has

$$A \otimes B = (U_1 \otimes U_2)(\Sigma_1 \otimes \Sigma_2)(V_1 \otimes V_2). \tag{2.62}$$

By Proposition 2.3, it is known that both $U_1 \otimes U_2$ and $V_1 \otimes V_2$ are unitary. Therefore, it follows from (2.62) that the maximal singular value of $A \otimes B$ is $\delta_1\sigma_1$. The conclusion is immediately obtained from this fact and the expression (2.55). ■

At the end of this subsection, a relation is given for 2-norms and Frobenius norms.

Lemma 2.12 *For a matrix A, there holds*

$$\|A\|_2 \leq \|A\|_{\mathrm{F}}.$$

Proof Let $A \in \mathbb{C}^{m\times n}$ with rank $A = r$. By carrying out singular value decompositions for A, there exist unitary matrices U and V with appropriate dimensions such that

$$A = U\Sigma V,$$

with

$$\Sigma = \operatorname{diag}\left(\sigma_1, \sigma_2, \ldots, \sigma_r, 0_{(m-r)\times(n-r)}\right), \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$

Thus, one has

$$\begin{aligned}
\|A\|_{\mathrm{F}} &= \sqrt{\operatorname{tr}\left(AA^{\mathrm{H}}\right)} = \sqrt{\operatorname{tr}\left(U\Sigma VV^{\mathrm{H}}\Sigma^{\mathrm{H}}U^{\mathrm{H}}\right)} \\
&= \sqrt{\operatorname{tr}\left(U\Sigma\Sigma^{\mathrm{H}}U^{\mathrm{H}}\right)} = \sqrt{\operatorname{tr}\left(U^{\mathrm{H}}U\Sigma\Sigma^{\mathrm{H}}\right)} \\
&= \sqrt{\operatorname{tr}\left(\Sigma\Sigma^{\mathrm{H}}\right)} = \sqrt{\sum_{i=1}^{r}\sigma_i^2} \geq \sigma_1 = \|A\|_2.
\end{aligned}$$

The proof is thus completed. ■

2.6 A Real Representation of a Complex Matrix

Let $A \in \mathbb{C}^{m\times n}$, then A can be uniquely written as $A = A_1 + A_2\,\mathrm{i}$ with $A_1, A_2 \in \mathbb{R}^{m\times n}$. The real representation A_σ of the matrix A is defined as

$$A_\sigma = \begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix} \in \mathbb{R}^{2m\times 2n}. \tag{2.63}$$

In general, one often uses the following real representation for a complex matrix $A = A_1 + A_2\,\mathrm{i}$ with $A_1, A_2 \in \mathbb{R}^{m\times n}$

$$\begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix},$$

which is different from the real representation defined in (2.63).

In the first subsection, some basic properties of the real representation defined in (2.63) are given, and the proof of the result on the characteristic polynomial of the real representation is provided in the second subsection.

2.6.1 Basic Properties

For an $n \times n$ complex matrix A, define $A_\sigma^i = (A_\sigma)^i$, and

$$P_j = \begin{bmatrix} I_j & 0 \\ 0 & -I_j \end{bmatrix},\ Q_j = \begin{bmatrix} 0 & I_j \\ -I_j & 0 \end{bmatrix}, \tag{2.64}$$

where I_j is the $j \times j$ identity matrix. The following lemma gives some basic properties of the real representation defined in (2.63).

Lemma 2.13 (The properties of the real representation) *In the following statements, if the power indices k and l are negative, it is required that the involved matrices A and B are nonsingular.*

(1) If A, $B \in \mathbb{C}^{m\times n}$, $a \in \mathbb{R}$, then

$$\begin{cases} (A+B)_\sigma = A_\sigma + B_\sigma \\ (aA)_\sigma = aA_\sigma \\ P_m A_\sigma P_n = \left(\overline{A}\right)_\sigma \end{cases}.$$

(2) If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times r}$, then

$$(AB)_\sigma = A_\sigma P_n B_\sigma = A_\sigma (\overline{B})_\sigma P_r.$$

(3) If $A \in \mathbb{C}^{n\times n}$, then A is nonsingular if and only if A_σ is nonsingular.
(4) If $A \in \mathbb{C}^{n\times n}$, and k is an integer, then $A_\sigma^{2k} = ((A\overline{A})^k)_\sigma P_n$.
(5) If $A \in \mathbb{C}^{m\times n}$, then

$$Q_m A_\sigma Q_n = A_\sigma,\ \left(\left(A^{\mathrm{T}}\right)_\sigma\right)^{\mathrm{T}} = A_\sigma.$$

(6) If $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{m\times n}$, k and l are integers, then

$$A_\sigma^k C_\sigma B_\sigma^l = \begin{cases} \left(\left(A\overline{A}\right)^s \left(A\overline{C}B\right) \left(\overline{B}B\right)^t\right)_\sigma, & k = 2s+1, l = 2t+1 \\ \left(\left(A\overline{A}\right)^s C \left(\overline{B}B\right)^t\right)_\sigma, & k = 2s, l = 2t \end{cases}.$$

(7) Given three matrices A, B, and C with appropriate dimensions, there holds

$$(ABC)_\sigma = A_\sigma \left(\overline{B}\right)_\sigma C_\sigma.$$

Proof By direct calculation, it is easily known that Items (1), (2) and (5) hold. Also Item (3) follows from (2) directly.

Now, let us show Item (4) by induction. The conclusion is obvious for $k = 0$. When $k = 1$, by using Item (2) it is easily known that the conclusion holds.

It is assumed that the conclusion holds for $k = i$. That is, $A_\sigma^{2i} = ((A\overline{A})^i)_\sigma P_n$. With this assumption, by Item (2) one has

$$\begin{aligned} A_\sigma^{2(i+1)} &= (A_\sigma)^{2(i+1)} = (A_\sigma)^{2i} (A_\sigma)^2 = (A_\sigma)^{2i} \left(A\overline{A}\right)_\sigma P_n \\ &= ((A\overline{A})^i)_\sigma P_n \left(A\overline{A}\right)_\sigma P_n = \left((A\overline{A})^i A\overline{A}\right)_\sigma P_n \\ &= ((A\overline{A})^{i+1})_\sigma P_n. \end{aligned}$$

This implies that the conclusion holds for $k = i + 1$. By induction, it is known that the conclusion holds for integer $k \geq 0$.

Next, the case with $k < 0$ is considered for Item (4). First, by using Item (2) it is obtained that

$$\mathscr{A}_\sigma = P_n \left[\left(\mathscr{A}^{-1}\right)_\sigma\right]^{-1} P_n$$

for $\mathscr{A} \in \mathbb{C}^{n\times n}$. With this relation and Item (4) with $k \geq 0$, for integer $k < 0$ one has

$$\begin{aligned} A_\sigma^{2k} &= \left(A_\sigma^{-2k}\right)^{-1} = \left[((A\overline{A})^{-k})_\sigma P_n\right]^{-1} \\ &= (P_n)^{-1} \left[((A\overline{A})^{-k})_\sigma\right]^{-1} \\ &= P_n \left[\left[\left[(A\overline{A})^k\right]^{-1}\right]_\sigma\right]^{-1} P_n P_n \\ &= \left[\left((A\overline{A})^k\right)_\sigma\right] P_n. \end{aligned}$$

This is the conclusion. The preceding facts imply that Item (4) holds for all integer k.

Item (6) can be easily checked by using Item (4).

Finally, let us show Item (7). Let $B \in \mathbb{C}^{n\times m}$. By Item (2), one has

$$(ABC)_\sigma = (AB)_\sigma P_m C_\sigma = (A)_\sigma \left(\overline{B}\right)_\sigma P_m P_m C_\sigma = (A)_\sigma \left(\overline{B}\right)_\sigma C_\sigma.$$

The lemma is thus proven. ∎

Lemma 2.14 *Given a matrix $A \in \mathbb{C}^{n \times n}$, if $\gamma \in \lambda(A_\sigma)$, then $\{\pm\gamma, \pm\overline{\gamma}\} \subset \lambda(A_\sigma)$.*

Proof Let $\left[\alpha_1^{\mathrm{T}} \ \alpha_2^{\mathrm{T}}\right]^{\mathrm{T}}$ with $\alpha_i \in \mathbb{C}^n$, $i = 1, 2$, be an eigenvector of A_σ corresponding to the eigenvalue γ. That is,

$$A_\sigma \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \gamma \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}. \tag{2.65}$$

By the definition of A_σ, the following relations can be easily obtained from (2.65)

$$A_\sigma \begin{bmatrix} \alpha_2 \\ -\alpha_1 \end{bmatrix} = (-\gamma) \begin{bmatrix} \alpha_2 \\ -\alpha_1 \end{bmatrix},$$

$$A_\sigma \begin{bmatrix} \overline{\alpha_1} \\ \overline{\alpha_2} \end{bmatrix} = \overline{\gamma} \begin{bmatrix} \overline{\alpha_1} \\ \overline{\alpha_2} \end{bmatrix},$$

$$A_\sigma \begin{bmatrix} \overline{\alpha_2} \\ -\overline{\alpha_1} \end{bmatrix} = (-\overline{\gamma}) \begin{bmatrix} \overline{\alpha_2} \\ -\overline{\alpha_1} \end{bmatrix}.$$

Thus, the proof is completed. ■

The following lemma gives some results on the 2-norm and Frobenius norm of the real representation defined in (2.63). From now on, $\|\cdot\|$ and $\|\cdot\|_2$ are used to represent the Frobenius norm and 2-norm for a given matrix, respectively.

Lemma 2.15 *Given a complex matrix A, the following relations hold.*

(1) $\|A_\sigma\|^2 = 2\,\|A\|^2$;
(2) $\|A_\sigma\|_2 = \|A\|_2$.

Proof The conclusion of Item (1) can be easily obtained by the definitions of the real representations and Frobenius norms.

Now, let us show the conclusion of Item (2).

Given a matrix $A \in \mathbb{C}^{m \times n}$ with $\operatorname{rank} A = r$, $r \leq \min(m, n)$. Performing the singular value decomposition for $A \in \mathbb{C}^{m \times n}$, gives

$$A = U \Sigma V, \tag{2.66}$$

where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are two unitary matrices, and

$$\Sigma = \begin{bmatrix} \Pi & 0 \\ 0 & 0_{(m-r)\times(n-r)} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

is a real matrix with $\Pi = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$, $\sigma_i > 0$, $i \in \mathbb{I}[1, r]$. Let

$$\Upsilon_1 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & I_{m-r} & 0 \\ 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & I_{m-r} \end{bmatrix}, \Upsilon_2 = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ 0 & I_{n-r} & 0 & 0 \\ 0 & 0 & 0 & I_{n-r} \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} \Pi & & \\ & \Pi & \\ & & 0_{2(m-r)\times 2(n-r)} \end{bmatrix}.$$

Then, by Lemma 2.13 one has from (2.66)

$$\begin{aligned} A_\sigma &= (U_\sigma P_m)\, \Sigma_\sigma\, (P_n V_\sigma) \\ &= (U_\sigma P_m)\, \Upsilon_1 \Sigma_1 \Upsilon_2\, (P_n V_\sigma)\,. \end{aligned} \tag{2.67}$$

In addition, it is easily derived that

$$\begin{aligned} & (U_\sigma P_m \Upsilon_1)\, (U_\sigma P_m \Upsilon_1)^{\mathrm{H}} \\ = {} & (U_\sigma P_m \Upsilon_1)\, \Upsilon_1^{\mathrm{H}} P_m^{\mathrm{H}}\, (U_\sigma)^{\mathrm{H}} \\ = {} & U_\sigma\, (U_\sigma)^{\mathrm{H}}\,. \end{aligned} \tag{2.68}$$

Now let $U = U_1 + \mathrm{i}U_2$ with $U_1, U_2 \in \mathbb{R}^{m\times m}$. Since U is a unitary matrix, then

$$\begin{aligned} UU^{\mathrm{H}} &= (U_1 + \mathrm{i}U_2)\left(U_1^{\mathrm{T}} - \mathrm{i}U_2^{\mathrm{T}}\right) \\ &= \left(U_1U_1^{\mathrm{T}} + U_2U_2^{\mathrm{T}}\right) + \mathrm{i}\left(U_2U_1^{\mathrm{T}} - U_1U_2^{\mathrm{T}}\right) \\ &= I, \end{aligned}$$

which implies that

$$U_1U_1^{\mathrm{T}} + U_2U_2^{\mathrm{T}} = I,\ \ U_2U_1^{\mathrm{T}} - U_1U_2^{\mathrm{T}} = 0.$$

In view of the definition of the real representation, it follows from the preceding relations that

$$\begin{aligned} & U_\sigma\, (U_\sigma)^{\mathrm{H}} \\ = {} & \begin{bmatrix} U_1 & U_2 \\ U_2 & -U_1 \end{bmatrix} \begin{bmatrix} U_1^{\mathrm{T}} & U_2^{\mathrm{T}} \\ U_2^{\mathrm{T}} & -U_1^{\mathrm{T}} \end{bmatrix} \\ = {} & \begin{bmatrix} U_1U_1^{\mathrm{T}} + U_2U_2^{\mathrm{T}} & U_1U_2^{\mathrm{T}} - U_2U_1^{\mathrm{T}} \\ U_2U_1^{\mathrm{T}} - U_1U_2^{\mathrm{T}} & U_2U_2^{\mathrm{T}} + U_1U_1^{\mathrm{T}} \end{bmatrix} \\ = {} & I. \end{aligned}$$

It is known from this relation and (2.68) that $U_\sigma P_m \Upsilon_1$ is a unitary matrix. Similarly, it is easily shown that $\Upsilon_2 P_n V_\sigma$ is also a unitary matrix. Therefore, the expression in (2.67) gives a singular value decomposition for the real representation matrix A_σ. In view of the definitions of Σ and 2-norm, the conclusion is immediately obtained. ■

The following theorem is concerned with the characteristic polynomial of the real representation of a complex matrix.

Theorem 2.7 *Let $A \in \mathbb{C}^{n\times n}$, then*

$$f_{A_\sigma}(s) = f_{A\overline{A}}(s^2) = f_{\overline{A}A}(s^2) \in \mathbb{R}[s].$$

Proof The proof is provided in Subsection 2.6.2. ■

For a matrix $A \in \mathbb{C}^{n\times n}$, let

$$f_A(s) = \det(sI - A) = \sum_{i=0}^{n} a_i s^i.$$

It is easily known that

$$g_A(s) = \det(I - sA) = \sum_{i=0}^{n} a_{n-i} s^i.$$

With this simple fact, the following result can be immediately obtained from Theorem 2.7.

Theorem 2.8 *Let $A \in \mathbb{C}^{n\times n}$, then*

$$g_{A_\sigma}(s) = g_{A\overline{A}}(s^2) = g_{\overline{A}A}(s^2) \in \mathbb{R}[s].$$

With the above two theorems, the following results can be readily derived by using Item (4) of Lemma 2.13.

Lemma 2.16 *Let $A \in \mathbb{C}^{n\times n}$ and $F \in \mathbb{C}^{p\times p}$. Then*

$$f_{A_\sigma}(F_\sigma) = (f_{A\overline{A}}(F\overline{F}))_\sigma P_p.$$

Lemma 2.17 *Let $A \in \mathbb{C}^{n\times n}$ and $F \in \mathbb{C}^{p\times p}$. Then*

$$g_{A_\sigma}(F_\sigma) = (g_{A\overline{A}}(F\overline{F}))_\sigma P_p.$$

2.6.2 Proof of Theorem 2.7

This subsection is devoted to give a proof of Theorem 2.7.

Lemma 2.18 *[143] Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times m}$ with $m \le n$. Then, BA has the same eigenvalues as AB, counting multiplicity, together with an additional $n - m$ eigenvalues equal to* 0*. That is,*

$$f_{BA}(s) = s^{n-m} f_{AB}(s).$$

Proof Consider the following two identities involving block matrices in $\mathbb{C}^{(m+n)\times(m+n)}$:

$$\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} AB & ABA \\ B & BA \end{bmatrix},$$
$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = \begin{bmatrix} AB & ABA \\ B & BA \end{bmatrix}.$$

Since the block matrix

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}$$

is nonsingular (all its eigenvalues are 1), then one has

$$\begin{bmatrix} I & A \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}\begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}. \tag{2.69}$$

That is, the two $(m+n) \times (m+n)$ matrices

$$C_1 = \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \text{ and } C_2 = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$$

are similar. The eigenvalues of C_1 are the eigenvalues of AB together with n zeros. The eigenvalues of C_2 are the eigenvalues of BA together with m zeros. Since the eigenvalues of C_1 and C_2 are the same by (2.69), including multiplicities, the assertion of the theorem follows. ■

The result in the preceding lemma is standard. According to this lemma, the following result is readily obtained.

Lemma 2.19 *Let $A \in \mathbb{C}^{n\times n}$, then $f_{A\overline{A}}(s) = f_{\overline{A}A}(s) \in \mathbb{R}[s]$.*

Proof It is obvious from the preceding lemma that $f_{A\overline{A}}(s) = f_{\overline{A}A}(s)$. Thus one has $\lambda(A\overline{A}) = \lambda(\overline{A}A)$. On the other hand, if $\gamma \in \lambda(A\overline{A})$, and the corresponding eigenvector chain is $\left\{v_{ij},\ j \in \mathbb{I}[1, p_i],\ i \in \mathbb{I}[1, q]\right\}$, then

$$A\overline{A}v_{ij} = \gamma v_{ij} + v_{i,j-1},\ v_{i0} = 0.$$

By this relation, one has

$$\overline{A}A\overline{v_{ij}} = \overline{\gamma}\ \overline{v_{ij}} + \overline{v_{i,j-1}},$$

which implies that $\overline{\gamma} \in \lambda(\overline{A}A)$, and $\left\{\overline{v_{ij}},\ j \in \mathbb{I}[1, p_i], i \in \mathbb{I}[1, q]\right\}$ is the eigenvector chain corresponding to $\overline{\gamma}$. Therefore, $\overline{\gamma}$ and γ has the same algebraic multiplicity.

With the preceding two aspects, it follows that $f_{A\overline{A}}(s) = f_{\overline{A}A}(s) \in \mathbb{R}[s]$. ■

Now let us give another real representation of a complex matrix. In order to distinguish it from the real representation given in (2.63), it is called the first real representation. The definition is given as follows.

Definition 2.4 For a complex matrix $A = A_1 + \mathrm{i}A_2$, with $A_i \in \mathbb{R}^{m\times n}$, $i = 1, 2$, the first real representation $\bar{\sigma}$ is defined as

$$A_{\bar{\sigma}} = \begin{bmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{bmatrix}.$$

It should be pointed out that the so-called first real representation is the often used one in the area of complex matrices. For $A \in \mathbb{C}^{n\times n}$, let

$$P = \begin{bmatrix} -\mathrm{i}I & \mathrm{i}I \\ -I & -I \end{bmatrix},$$

then one has

$$P \operatorname{diag}\left(A, \overline{A}\right) P^{-1} = A_{\bar{\sigma}}.$$

Based on this relation, the following conclusion on the first real representation can be obtained from basic matrix theory.

Lemma 2.20 *Let $A \in \mathbb{C}^{n\times n}$. Then*

$$f_{A_{\bar{\sigma}}}(s) = f_{\overline{A}_{\bar{\sigma}}}(s) = f_A(s) f_{\overline{A}}(s).$$

Lemma 2.21 *Let $A \in \mathbb{C}^{n\times n}$. Then, there holds*

$$\left(A\overline{A}\right)_{\bar{\sigma}} = A_{\sigma}^2.$$

Proof Let $A = A_1 + \mathrm{i}\,A_2$, then

$$\left(A\overline{A}\right)_{\bar{\sigma}} = \begin{bmatrix} A_1^2 + A_2^2 & A_1A_2 - A_2A_1 \\ A_2A_1 - A_1A_2 & A_1^2 + A_2^2 \end{bmatrix} = (A_{\sigma})^2. \tag{2.70}$$

This implies that the conclusion is true. ■

Before giving the next lemmas, the following symbols are defined:

$$J_1(\lambda, n) = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \in \mathbb{C}^{n\times n},$$

$$J_2(\lambda, n) = \begin{bmatrix} \lambda & -1 & & \\ & \lambda & \ddots & \\ & & \ddots & -1 \\ & & & \lambda \end{bmatrix} \in \mathbb{C}^{n\times n},$$

$$\Theta(n) = \operatorname{diag}((-1)^0, (-1)^1, \ldots, (-1)^{n-1}).$$

For these three matrices, the following two lemmas can be obtained by simple computations.

Lemma 2.22 *Given $\lambda \in \mathbb{C}$, the following relation holds:*

$$\Theta(n)J_1(\lambda, n) = J_2(\lambda, n)\Theta(n).$$

Lemma 2.23 *(1) For $0 \neq \lambda \in \mathbb{C}$, $J_1^2(\lambda, n)$ is similar to $J_1(\lambda^2, n)$.*
(2) When n is odd, $J_1^2(0, n)$ is similar to $\operatorname{diag}\left(J_1(0, \frac{n+1}{2}), J_1(0, \frac{n-1}{2})\right)$. *When n is even, $J_1^2(0, n)$ is similar to* $\operatorname{diag}\left(J_1(0, \frac{n}{2}), J_1(0, \frac{n}{2})\right)$.

Based on the above lemmas, the following conclusion on the characteristic polynomial of the square of a matrix can be obtained.

Lemma 2.24 *For a given matrix $A \in \mathbb{C}^{n\times n}$, if*

$$f_A(s) = \prod_{i=1}^{n}(s - \lambda_i),$$

then

$$f_{A^2}(s) = \prod_{i=1}^{n}(s - \lambda_i^2).$$

Lemma 2.25 *Given a matrix $A \in \mathbb{C}^{n\times n}$, if $0 \neq \gamma \in \lambda(A_\sigma)$, then $-\gamma \in \lambda(A_\sigma)$, and γ and $-\gamma$ have the same Jordan structure. In details, it is assumed that the eigenvalue γ has p Jordan blocks with the orders q_i, $i \in \mathbb{I}[1, p]$, the eigenvalue $-\gamma$ has also p Jordan blocks with the orders of q_i, $i \in \mathbb{I}[1, p]$.*

Proof Let $\left[\alpha_{1ij}^{\mathrm{T}}\ \alpha_{2ij}^{\mathrm{T}}\right]^{\mathrm{T}}$, $j \in \mathbb{I}[1, q_i]$, $i \in \mathbb{I}[1, p]$, be a group of eigenvector chains of A_σ corresponding to the eigenvalue γ. Thus, one has

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix} \begin{bmatrix} \alpha_{1ij} \\ \alpha_{2ij} \end{bmatrix} = \gamma \begin{bmatrix} \alpha_{1ij} \\ \alpha_{2ij} \end{bmatrix} + \begin{bmatrix} \alpha_{1i,j-1} \\ \alpha_{2i,j-1} \end{bmatrix}, \begin{bmatrix} \alpha_{1i,0} \\ \alpha_{2i,0} \end{bmatrix} = 0, \quad (2.71)$$
$$j \in \mathbb{I}[1, q_i], i \in \mathbb{I}[1, p],$$

that is,

$$\begin{cases} A_1\alpha_{1ij} + A_2\alpha_{2ij} = \gamma\alpha_{1ij} + \alpha_{1i,j-1} \\ A_2\alpha_{1ij} - A_1\alpha_{2ij} = \gamma\alpha_{2ij} + \alpha_{2i,j-1} \end{cases}.$$

Rearranging the above relations, one can obtain

$$\begin{cases} A_1(-\alpha_{2ij}) + A_2\alpha_{1ij} = (-\gamma)(-\alpha_{2ij}) + \alpha_{2i,j-1} \\ A_2(-\alpha_{2ij}) - A_1\alpha_{1ij} = (-\gamma)\alpha_{1ij} - \alpha_{1i,j-1} \end{cases},$$

that is,

$$\begin{bmatrix} A_1 & A_2 \\ A_2 & -A_1 \end{bmatrix} \begin{bmatrix} -\alpha_{2ij} \\ \alpha_{1ij} \end{bmatrix} = (-\gamma) \begin{bmatrix} -\alpha_{2ij} \\ \alpha_{1ij} \end{bmatrix} - \begin{bmatrix} -\alpha_{2i,j-1} \\ \alpha_{1i,j-1} \end{bmatrix}. \quad (2.72)$$

Let

$$V_i = \begin{bmatrix} -\alpha_{2i1} & -\alpha_{2i2} & \cdots & -\alpha_{2iq_i} \\ \alpha_{1i1} & \alpha_{1i2} & \cdots & \alpha_{1iq_i} \end{bmatrix}, i \in \mathbb{I}[1, p].$$

Then it follows from (2.72) that

$$A_\sigma V_i = V_i J_2(-\gamma, q_i), i \in \mathbb{I}[1, p].$$

In view of Lemma 2.22, one has

$$A_\sigma \left(V_i \Theta(q_i)\right) = V_i J_2(-\gamma, q_i)\Theta(q_i) = \left(V_i\Theta(q_i)\right) J_1(-\gamma, q_i), i \in \mathbb{I}[1, p].$$

This implies the conclusion. ■

With the above preliminaries, now the proof of Theorem 2.7 can be presented.
Proof of Theorem 2.7: According to Lemma 2.25, one can assume that

$$f_{A_\sigma}(s) = \prod_{i=1}^{n} \left[s - (-\lambda_i)\right](s - \lambda_i) = \prod_{i=1}^{n} (s^2 - \lambda_i^2). \quad (2.73)$$

By using Lemma 2.24, it is obtained that

$$f_{A_\sigma^2}(s) = \prod_{i=1}^{n} \left[s - (-\lambda_i)^2\right]\left(s - \lambda_i^2\right) = \prod_{i=1}^{n} (s - \lambda_i^2)^2. \quad (2.74)$$

It follows from Lemmas 2.19, 2.20, and 2.21 that

$$f_{A_{\sigma}^{2}}(s) = f_{(A\overline{A})_{\tilde{\sigma}}}(s) = f_{A\overline{A}}(s) f_{\overline{A}A}(s) = f_{A\overline{A}}^{2}(s) = f_{\overline{A}A}^{2}(s). \tag{2.75}$$

Combining (2.75) with (2.74), in view of Lemma 2.19 one has

$$f_{A\overline{A}}(s) = f_{\overline{A}A}(s) = \prod_{i=1}^{n}(s - \lambda_i^2) \in \mathbb{R}[s]. \tag{2.76}$$

The conclusion of Theorem 2.7 can be immediately obtained from (2.73) and (2.76).

2.7 Consimilarity

For complex matrices, besides similarity there is another equivalence relation, consimilarity.

Definition 2.5 Two matrices $A,\ B \in \mathbb{C}^{n\times n}$ are said to be consimilar if there exists a nonsingular matrix $S \in \mathbb{C}^{n\times n}$ such that $A = SB\overline{S}^{-1}$. If the matrix S can be taken to be unitary, A and B are said to be unitarily consimilar.

If $A = SB\overline{S}^{-1}$ and $S = U$ is unitary, then $A = SB\overline{S}^{-1} = UBU^{\mathrm{T}}$; if $S = R$ is a real nonsingular matrix, then $A = SB\overline{S}^{-1} = RBR^{-1}$. Thus, special cases of consimilarity include congruence and ordinary similarity.

Analogously to the case of similarity, one may be concerned with the equivalence classes containing triangular or diagonal representatives under consimilarity.

Definition 2.6 A matrix $A \in \mathbb{C}^{n\times n}$ is said to be contriangularizable if there exists a nonsingular $S \in \mathbb{C}^{n\times n}$ such that $S^{-1}A\overline{S}$ is upper triangular; it is said to be condiagonalizable if S can be chosen so that $S^{-1}A\overline{S}$ is diagonal. It is said to be unitarily contriangularizable or unitarily condiagonalizable if it can be reduced by consimilarity to the required form via a unitary matrix.

If $A \in \mathbb{C}^{n\times n}$ is condiagonalizable and

$$S^{-1}A\overline{S} = \Lambda = \mathrm{diag}(\lambda_1,\ \lambda_2, \ldots,\ \lambda_n),$$

then $A\overline{S} = S\Lambda$. If $S = \begin{bmatrix} s_1 & s_2 & \cdots & s_n \end{bmatrix}$ with each $s_i \in \mathbb{C}^n$, this identity says that $A\overline{s_i} = \lambda_i s_i$ for $i \in \mathbb{I}[1, n]$. This equation is similar to, but crucially different from, the usual eigenvector-eigenvalue equation.

Definition 2.7 Let $A \in \mathbb{C}^{n\times n}$ be given. A nonzero vector $x \in \mathbb{C}^n$ such that $A\overline{x} = \lambda x$ for some $\lambda \in \mathbb{C}$ is said to be a coneigenvector of A; the scalar λ is a coneigenvalue of A.

The identity $A\overline{S} = S\Lambda$ with Λ diagonal says that every nonzero column of S is a coneigenvector of A. Since the columns of S are independent if and only if S is nonsingular, one can see that a matrix $A \in \mathbb{C}^{n\times n}$ is condiagonalizable if and only if it has n independent coneigenvectors. To this extent, the theory of condiagonalization is entirely analogous to the theory of ordinary diagonalization.

But every matrix has at least one eigenvalue, and it has only finitely many distinct eigenvalues; in this regard, the theory of coneigenvalues is rather different. If $A\overline{x} = \lambda x$, then for all $\theta \in \mathbb{R}$ there holds

$$e^{-\mathrm{i}\theta} A\overline{x} = A\left(\overline{e^{\mathrm{i}\theta} x}\right) = e^{-\mathrm{i}\theta}\lambda x = \left(e^{-2\mathrm{i}\theta}\lambda\right)\left(e^{\mathrm{i}\theta} x\right),$$

which implies that $e^{-2\mathrm{i}\theta}\lambda$ is a coneigenvalue of A for any $\theta \in \mathbb{R}$ if λ is a coneigenvalue of A. This shows that a matrix may have infinitely many coneigenvalues. On the other hand, if $A\overline{x} = \lambda x$, then

$$A\overline{A}x = A(\overline{A\overline{x}}) = A\left(\overline{\lambda x}\right) = \overline{\lambda} A\overline{x} = \overline{\lambda}\lambda x = |\lambda|^2 x,$$

so a scalar λ is a coneigenvalue of A only if $|\lambda|^2$ is an eigenvalue of $A\overline{A}$. The example

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

for which $A\overline{A} = -I$ has no nonnegative eigenvalues, shows that there are matrices that have no coneigenvalues at all. It is known, however, that if $A \in \mathbb{C}^{n\times n}$ and n is odd, then A must have at least one coneigenvalue, a result analogous to the fact that every real matrix of odd order has at least one real eigenvalue.

Thus, in contrast to the theory of ordinary eigenvalues, a matrix may have infinitely many distinct coneigenvalues or it may have no coneigenvalues at all. If a matrix has a coneigenvalue, it is sometimes convenient to select from among the coneigenvalues of equal modulus the unique nonnegative one as a representative.

The necessary condition just observed for the existence of a coneigenvalue is also sufficient as stated below.

Proposition 2.8 *Let $A \in \mathbb{C}^{n\times n}$, and let $\lambda \geq 0$ be given. Then λ is an eigenvalue of $A\overline{A}$ if and only if $\sqrt{\lambda}$ is a coneigenvalue of A.*

Proof If $\lambda \geq 0$, $\sqrt{\lambda} \geq 0$, and $A\overline{x} = \sqrt{\lambda}x$ for some $x \neq 0$, then

$$A\overline{A}x = A(\overline{A\overline{x}}) = A\left(\overline{\sqrt{\lambda}x}\right) = \sqrt{\lambda}A\overline{x} = \sqrt{\lambda}\sqrt{\lambda}x = \lambda x.$$

Conversely, if $A\overline{A}x = \lambda x$ for some $x \neq 0$, there are two possibilities:

(a) $A\overline{x}$ and x are dependent; or
(b) $A\overline{x}$ and x are independent.

In the former case, there is some $\mu \in \mathbb{C}$ such that $A\overline{x} = \mu x$, which says that μ is a coneigenvalue of A. But then

$$\lambda x = A\overline{A}x = A(\overline{A\overline{x}}) = A\left(\overline{\mu x}\right) = \overline{\mu}A\overline{x} = \overline{\mu}\mu x = |\mu|^2 x,$$

so $|\mu| = \sqrt{\lambda}$. Since $e^{-2\mathrm{i}\theta}\mu$ is a coneigenvalue associated with the coneigenvector $e^{\mathrm{i}\theta}x$ for any $\theta \in \mathbb{R}$, one can conclude that $+\sqrt{\lambda}$ is a coneigenvalue of A. Notice that

$$A\overline{A}(A\overline{x}) = A\left(\overline{A\overline{A}x}\right) = A\left(\overline{\lambda x}\right) = \lambda\left(A\overline{x}\right)$$

and $A\overline{A}x = \lambda x$, so if λ is a simple eigenvalue of $A\overline{A}$, (a) must always be the case.

In the latter case (b) (which could occur if λ is a multiple eigenvalue of $A\overline{A}$), the vector

$$y = A\overline{x} + \sqrt{\lambda}x$$

is nonzero and is a coneigenvector corresponding to the coneigenvalue $\sqrt{\lambda}$ since

$$A\overline{y} = A\overline{A}x + \sqrt{\lambda}A\overline{x} = \lambda x + \sqrt{\lambda}A\overline{x} = \sqrt{\lambda}(A\overline{x} + \sqrt{\lambda}x) = \sqrt{\lambda}y. \qquad \blacksquare$$

It has been seen that for each distinct nonnegative eigenvalue of $A\overline{A}$ there corresponds a coneigenvector of A, a result analogous to the ordinary theory of eigenvectors. The following result extends this analogy a bit further.

Proposition 2.9 *Let $A \in \mathbb{C}^{n\times n}$ be given, and let $x_1, x_2, \ldots, x_k$ be coneigenvectors of A with corresponding coneigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. If $|\lambda_i| \neq \left|\lambda_j\right|$ whenever $1 \leq i, j \leq k$ and $i \neq j$, then $\{x_1, \ldots, x_k\}$ is an independent set.*

Proof Each x_i is an eigenvector of $A\overline{A}$ with associated eigenvalue $|\lambda_i|^2$. The vectors x_i, $i \in \mathbb{I}[1, k]$, are independent because they are eigenvectors of the matrix $A\overline{A}$ and their associated eigenvalues $|\lambda_i|^2$, $i \in \mathbb{I}[1, k]$, are distinct by assumption. $\blacksquare$

2.8 Real Linear Spaces and Real Linear Mappings

Let us first recall the concept of mapping. Let U and V be two nonempty sets of elements. A rule T that assigns some unique element $y \in V$ for each $x \in U$ is called a mapping of U into V, and it is denoted by $T : U \to V$. If T assigns $y \in V$ for $x \in U$, then y is said to be the image of x under T, and is denoted by $y = T \circledcirc x$.

Most readers are very familiar with the concept of linear spaces and linear mappings. It is well-known that for a given $A \in \mathbb{C}^{m\times n}$, the mapping $T : x \mapsto Ax$ is a linear mapping. However, the simple mapping $M : x \mapsto A\overline{x}$ is not a linear mapping since

$$M \circledcirc (cx) = \overline{c}A\overline{x} = \overline{c}\left(M \circledcirc x\right) \neq c\left(M \circledcirc x\right)$$

for an arbitrarily given complex scalar $c \in \mathbb{C}$. Thus, some conclusions on linear spaces are not easy to be applied to the simple mapping $M : x \mapsto A\overline{x}$. Nevertheless, the mapping $M : x \mapsto A\overline{x}$ has some nice properties. For example, $M \circledcirc (cx) = c\,(M \circledcirc x)$ for any real scalar $c \in \mathbb{R}$. This is the motivation to investigate real linear spaces.

2.8.1 Real Linear Spaces

The concept of real linear spaces is first given as follows.

Definition 2.8 Let V be a set on which a closed binary operation $(+)$ is defined. In addition, let a binary operation (scalar multiplication) be defined from $V \times \mathbb{R}$ to V. The set V is said to be a real linear space if for any $x, y, z \in V$, and any $\alpha, \beta \in \mathbb{R}$, the following axioms are satisfied:

A1. $x + y$ is a unique vector of the set V;
A2. $x + y = y + x$;
A3. $(x + y) + z = x + (y + z)$;
A4. There is a vector 0 such that $x + 0 = x$;
A5. For every $x \in V$ there exists a vector $-x$ such that $x + (-x) = 0$.
S1. αx is a unique vector of the set V;
S2. $\alpha\,(\beta x) = (\alpha\beta)\,x$;
S3. $\alpha\,(x + y) = \alpha x + \alpha y$;
S4. $(\alpha + \beta)\,x = \alpha x + \beta x$;
S5. $1x = x$.

An element in a real linear space V is called a vector.

It is easily checked that the set $\mathbb{C}^n$ equipped with the ordinary vector addition is both a linear space and a real linear space. Now, consider the following set

$$V_0 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle|\; a \in \mathbb{C}, b \in \mathbb{R} \right\}. \tag{2.77}$$

For this set, the addition is defined as follows

$$\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}.$$

In addition, for $\alpha \in \mathbb{C}$ the scalar multiplication is defined as follows

$$\alpha \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha a \\ \alpha b \end{bmatrix}.$$

It is obvious that the set V in (2.77) is not a linear space over $\mathbb{C}$, since for $x \in V$, $\mathrm{i}\,x$ may not belong to V. However, it is easily checked that the set V in (2.77) is a real linear space by the axioms in Definition 2.8.

Let V be a real linear space, and consider a subset V_0 of elements from V. The operations of addition and scalar multiplication are defined for all elements of V and, in particular, for those belonging to V_0. If V_0 is a real linear space with the same addition and scalar multiplication, then V_0 is said to be a real linear subspace of V. However, it is not necessary to check whether a subset is a real linear subspace by the axioms in Definition 2.8.

Theorem 2.9 *A nonempty subset V_0 of a real linear space V is a real linear subspace of V if for every $x, y \in V_0$, and any $\alpha \in \mathbb{R}$, there hold*

(1) $x + y \in V_0$;
(2) $\alpha x \in V_0$.

Example 2.1 The set V_0 in (2.77) is a subset of $\mathbb{C}^2$. Previously, it has been checked that V_0 is a real linear space according to the definition of real linear spaces. However, since V_0 is a subset of $\mathbb{C}^2$, and $\mathbb{C}^2$ is a real linear space, one can use the preceding theorem to determine whether V_0 is a real linear space. In fact, for the subset V_0 in (2.77) it is easily checked that $x + y \in V_0$ and $\alpha x \in V_0$ for every $x, y \in V_0$ and any $\alpha \in \mathbb{R}$. Therefore, V_0 is a real linear subspace of $\mathbb{C}^2$, and thus is a real linear space.

For real linear subspaces, the following conclusion can be proven easily.

Lemma 2.26 *If V_i, $i = 1, 2$, are real linear subspaces of a real linear space V, then the set $V_1 \cap V_2$ is also a real linear subspace of V.*

In a real linear space V, given a set of vectors $\{x_1, x_2, \ldots, x_k\}$, if a vector x can be written as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k$$

for some real scalars α_i, $i \in \mathbb{I}[1, k]$, then x is called a real linear combination of the vectors $x_1, x_2, \ldots, x_k$. A set of vectors $\{x_1, x_2, \ldots, x_k\}$ is said to be real linearly dependent if there exit real coefficients α_i, $i \in \mathbb{I}[1, k]$, not all 0, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0.$$

A set of vectors that is not real linearly dependent is said to be real linearly independent.

Example 2.2 In $\mathbb{C}^2$, the set of $\left\{ \begin{bmatrix} 1+\mathrm{i} & 1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} -2 & -1+\mathrm{i} \end{bmatrix}^{\mathrm{T}} \right\}$ is linearly dependent over the field $\mathbb{C}$ of complex numbers since

$$-(1-\mathrm{i}) \begin{bmatrix} 1+\mathrm{i} \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ -1+\mathrm{i} \end{bmatrix} = 0.$$

However, the set of $\left\{ \begin{bmatrix} 1+\mathrm{i} & 1 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} -2 & -1+\mathrm{i} \end{bmatrix}^{\mathrm{T}} \right\}$ is real linearly independent.

Example 2.3 In $\mathbb{C}^n$, e_i represents the vector whose i-th element is 1, and all the other elements are zero. Obviously, e_i and $\mathrm{i}e_i$ are linearly dependent since $e_i + \mathrm{i}\,(\mathrm{i}e_i) = 0$. However, if $\mathbb{C}^n$ is viewed as a real linear space with ordinary vector addition and scalar multiplication, then the vectors e_i and $\mathrm{i}e_i$ are real linearly independent.

Example 2.3 implies that the concept of linear dependence is closely relevant to the scalar field. For the same vector sets, even if the definitions of addition and scalar multiplication are identical, a set of vectors to be linearly dependent over some field may be linearly independent over another field.

It can be easily verified that the set of all real linear combinations of the vectors $a_1, a_2, \ldots, a_n$ belonging to a real linear space V generates a real linear subspace. In order to distinguish from the notation of conventionally generated subspace, this real linear subspace is denoted by $\mathrm{RLspan}\,\{a_1, a_2, \ldots, a_n\}$, that is,

$$\mathrm{RLspan}\,\{a_1, a_2, \ldots, a_n\} = \left\{ \sum_{i=1}^{n} \alpha_i a_i \;\middle|\; \alpha_i \in \mathbb{R}, i \in \mathbb{I}[1, n] \right\}.$$

Example 2.4 For a set S of complex vectors a_i, $i \in \mathbb{I}[1, n]$, the linear subspace generated by S (over complex field $\mathbb{C}$) is denoted by

$$\mathrm{span}\,\{a_1, a_2, \ldots, a_n\} = \left\{ \sum_{i=1}^{n} \alpha_i a_i \;\middle|\; \alpha_i \in \mathbb{C}, i \in \mathbb{I}[1, n] \right\}.$$

Let e_i represent the n-dimensional vector whose i-th element is 1, and all the other elements are 0, then

$$\begin{aligned}
\mathrm{RLspan}\,\{e_1, e_2, \ldots, e_n\} &= \mathbb{R}^n, \\
\mathrm{span}\,\{e_1, e_2, \ldots, e_n\} &= \mathbb{C}^n, \\
\mathrm{RLspan}\,\{\mathrm{i}e_1, \mathrm{i}e_2, \ldots, \mathrm{i}e_n\} &= \mathrm{i}\mathbb{R}^n = \{\mathrm{i}\eta \mid \eta \in \mathbb{R}^n\}, \\
\mathrm{span}\,\{\mathrm{i}e_1, \mathrm{i}e_2, \ldots, \mathrm{i}e_n\} &= \mathbb{C}^n.
\end{aligned}$$

Let V be a real linear space, a finite set of vectors

$$\{a_1, a_2, \ldots, a_n\} \tag{2.78}$$

is said to be a real basis of V if they are real linearly independent, and every vector $x \in V$ is a real linear combination of the vectors in (2.78):

$$x = \sum_{i=1}^{n} \alpha_i a_i,\ \alpha_i \in \mathbb{R},\ i \in \mathbb{I}[1, n].$$

In other words, the vectors in (2.78) form a real basis of V if

$$V = \text{RLspan}\{a_1, a_2, \ldots, a_n\}$$

and no a_i ($i \in \mathbb{I}[1, n]$) can be discarded. In this case, the vectors in (2.78) are referred to as real basis vectors of V.

Example 2.5 For $\mathbb{C}^n$, it is well-known that the set $\{e_1, e_2, \ldots, e_n\}$ with e_i defined as in Example 2.4 is a basis of V if $\mathbb{C}^n$ is regarded as a linear space over field $\mathbb{C}$. However, $\{e_1, e_2, \ldots, e_n\}$ is not a real basis of $\mathbb{C}^n$. One can choose the following set of vectors as a real basis of $\mathbb{C}^n$:

$$\{e_1, e_2, \ldots, e_n, \mathrm{i}e_1, \mathrm{i}e_2, \ldots, \mathrm{i}e_n\}.$$

Example 2.6 Consider the set P_n of complex polynomials of s with degree less than or equal to n. If the set P_n is viewed as a linear space over field $\mathbb{C}$, it is well-known that the set $\left\{1, s, s^2, \ldots, s^n\right\}$ is a basis of P_n. However, a real basis of P_n is

$$\left\{1, s, s^2, \ldots, s^n, \mathrm{i}, \mathrm{i}s, \mathrm{i}s^2, \ldots, \mathrm{i}s^n\right\}.$$

Example 2.7 Consider the set

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \,\middle|\, x \in \mathbb{C}^n, y \in \mathbb{R}^n \right\}.$$

It is easily verified that the set V with ordinary vector addition and scalar multiplication is not a linear space over field $\mathbb{C}$. However, the set V with ordinary vector addition and scalar multiplication is a real linear space. The following set forms a real basis of V:

$$\left\{ \begin{bmatrix} e_1 \\ 0_n \end{bmatrix}, \begin{bmatrix} e_2 \\ 0_n \end{bmatrix}, \ldots, \begin{bmatrix} e_n \\ 0_n \end{bmatrix}, \begin{bmatrix} 0_n \\ e_1 \end{bmatrix}, \begin{bmatrix} 0_n \\ e_2 \end{bmatrix}, \ldots, \begin{bmatrix} 0_n \\ e_n \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} \mathrm{i}e_1 \\ 0_n \end{bmatrix}, \begin{bmatrix} \mathrm{i}e_2 \\ 0_n \end{bmatrix}, \ldots, \begin{bmatrix} \mathrm{i}e_n \\ 0_n \end{bmatrix} \right\}.$$

Similarly to normal linear spaces, a real basis necessarily consists of only finitely many vectors. Accordingly, the real linear spaces they generate are said to be finite dimensional. The following property holds for finite dimensional real linear spaces.

Proposition 2.10 *Any vector of a finite dimensional real linear space can be expressed uniquely as a real linear combination of the vectors of a fixed real basis.*

Similarly to the case of normal linear spaces, there are concepts of coordinates in real linear spaces. Let x belong to a real linear space V with a real basis $\{a_1, a_2, \ldots, a_n\}$. The column vector $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$ such that the decomposition

$$x = \sum_{i=1}^{n} \alpha_i a_i$$

holds is said to be the representation of x with respect to the real basis $\{a_1, a_2, \ldots, a_n\}$. The real scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, are referred to as the coordinates of x with respect to the real basis $\{a_1, a_2, \ldots, a_n\}$.

Example 2.8 Consider the real linear space

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| \, x \in \mathbb{R}, y \in \mathbb{C} \right\},$$

with the ordinary vector addition and scalar multiplication. It is easily known that

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathrm{i} \end{bmatrix} \right\} \tag{2.79}$$

forms a real linear basis. For any $\begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}} \in V$, one has

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (\operatorname{Re} y) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (\operatorname{Im} y) \begin{bmatrix} 0 \\ \mathrm{i} \end{bmatrix}.$$

This relation implies that $\alpha = \begin{bmatrix} x & \operatorname{Re} y & \operatorname{Im} y \end{bmatrix}^{\mathrm{T}}$ is the representation of $\begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}$ with respect to the real basis in (2.79).

Theorem 2.10 *All real bases for a finite dimensional real linear space have the same number of vectors.*

The proof of this theorem is very analogous to the case of normal linear spaces, and thus is omitted.

For a finite dimensional real linear space V, the number of vectors in a real basis is a characteristic of the real linear space that is invariant under different choice of real bases. In this book, this number is formally defined as the real dimension of the real linear space V, and is denoted by $\operatorname{rdim} V$. With this notation, one has

$$\operatorname{rdim} \mathbb{C}^n = 2n.$$

Remark 2.4 In this subsection, some concepts have been given for real linear spaces. In fact, the so-called real linear space is only a special case of normal linear spaces. Remember that the involved field is the field of real numbers. Nevertheless, it should be pointed out that, in most textbooks on matrices or linear algebra, the set $\mathbb{C}^n$ is only viewed as a linear space over the field of $\mathbb{C}$.

In the next subsection, it will be seen that the theory on real linear spaces can be conveniently applied to some complex mappings.

2.8.2 Real Linear Mappings

Let us begin this subsection with the concept of image of a mapping. Let U and V be two nonempty sets of elements, and T be a mapping of U into V. The image of the mapping T is defined as

$$\operatorname{Image} T = \{y \in V \mid y = T \circledcirc x, x \in U\} .$$

In addition, if V has a zero element, the kernel of T is defined as follows:

$$\operatorname{Ker} T = \{x \in U \mid T \circledcirc x = 0\} .$$

Now, it is required that U and V are two real linear spaces. A mapping $T : U \to V$ is real linear if for any two vectors $x_1, x_2 \in U$, and any real scalar $\alpha \in \mathbb{R}$, there hold

$$\begin{aligned} T \circledcirc (x_1 + x_2) &= T \circledcirc x_1 + T \circledcirc x_2, \\ T \circledcirc (\alpha x_1) &= \alpha (T \circledcirc x_1) . \end{aligned}$$

Example 2.9 Let $U = \mathbb{C}^m$ and $V = \mathbb{C}^n$. Obviously, both U and V are linear spaces over the field of $\mathbb{C}$ with the ordinary vector addition and scalar multiplication. Moreover, both U and V are also real linear spaces. Let $A \in \mathbb{C}^{n\times m}$. It is well-known that the mapping $T : U \to V$ defined by $T \circledcirc x = Ax$ is linear. Moreover, this mapping is also real linear.

Example 2.10 Let $U = \mathbb{C}^m$ and $V = \mathbb{C}^n$. For a given matrix $A \in \mathbb{C}^{n\times m}$, the mapping $T : U \to V$ defined by $T \circledcirc x = A\overline{x}$, which is an antilinear mapping, is often encountered in the context of consimilarity. For this mapping, there holds $T \circledcirc (\alpha x) = \overline{\alpha} (T \circledcirc x)$. Generally, for an arbitrary complex number α, $A\overline{\alpha x} \neq \alpha A\overline{x}$. Therefore, this mapping is not linear over the field of $\mathbb{C}$. However, this mapping is real linear.

Example 2.11 Let $U = \mathbb{C}^m$ and $V = \mathbb{C}^n$. For two given matrices $M, M_\# \in \mathbb{C}^{n\times m}$, consider the mapping $T : U \to V$ defined by $T \circledcirc x = Mx + M_\#\overline{x}$. This mapping is neither linear, nor antilinear over the field of $\mathbb{C}$. However, this mapping is real linear.

Example 2.12 Given two matrices $A \in \mathbb{C}^{n\times n}$ and $F \in \mathbb{C}^{p\times p}$, consider the mapping $T : \mathbb{C}^{n\times p} \to \mathbb{C}^{n\times p}$ defined by

$$T \circledcirc X = AX - \overline{X}F.$$

In this book, this mapping will be referred to as a con-Sylvester mapping. It is easily checked that this mapping is real linear.

Next, the image and kernel of real linear mappings are investigated.

Theorem 2.11 *Let U and V be two real linear spaces, and T be a real linear mapping of U into V. Then,* Image T *is a real linear subspace of V.*

Proof Let $y_1, y_2 \in \operatorname{Image} T$. Then, there are $x_1, x_2 \in U$ such that $y_i = T \circledcirc x_i$, $i = 1, 2$. Since T is a real linear mapping, then for any $\alpha \in \mathbb{R}$

$$\begin{aligned} y_1 + y_2 &= T \circledcirc x_1 + T \circledcirc x_2 = T \circledcirc (x_1 + x_2); \\ \alpha y_1 &= \alpha (T \circledcirc x_1) = T \circledcirc (\alpha x_1). \end{aligned}$$

Since U is a real linear space, then $x_1 + x_2 \in U$ and $\alpha x_1 \in U$. Therefore, it follows from the preceding two relations that $y_1 + y_2 \in \operatorname{Image} T$, and $\alpha y_1 \in \operatorname{Image} T$. By Theorem 2.9, $\operatorname{Image} T$ is a real linear subspace of V. ■

Theorem 2.12 *Let U and V be two real linear spaces, and T be a real linear mapping of U into V. Then,* $\operatorname{Ker} T$ *is a real linear subspace of U.*

The proof of this theorem is very simple, and thus is omitted.

Theorem 2.13 *Let U and V be two real linear spaces, and T be a real linear mapping of U into V. Then,*

$$\operatorname{rdim}(\operatorname{Image} T) + \operatorname{rdim}(\operatorname{Ker} T) = \operatorname{rdim} U.$$

Proof Let $r_1 = \operatorname{rdim}(\operatorname{Ker} T)$, and $r = \operatorname{rdim} U$. Then, there exist a set of real linearly independent vector x_i, $i \in \mathbb{I}[1, r_1]$, such that

$$\operatorname{Ker} T = \operatorname{RLspan}\left\{x_1, x_2, \ldots, x_{r_1}\right\}.$$

In this case, there exist additional vectors $x_{r_1+1}, x_{r_1+2}, \ldots, x_r$ such that $\{x_1, x_2, \ldots, x_r\}$ is a real basis of U. Thus, one has

$$\begin{aligned} \operatorname{Image} T &= \left\{ T \circledcirc \left(\sum_{i=1}^{r} \alpha_i x_i \right) \,\middle|\, \alpha_i \in \mathbb{R}, i \in \mathbb{I}[1, r] \right\} \\ &= \left\{ T \circledcirc \left(\sum_{i=1}^{r_1} \alpha_i x_i \right) + T \circledcirc \left(\sum_{i=r_1+1}^{r} \alpha_i x_i \right) \,\middle|\, \alpha_i \in \mathbb{R}, i \in \mathbb{I}[1, r] \right\} \\ &= \left\{ \left(\sum_{i=1}^{r_1} \alpha_i (T \circledcirc x_i) \right) + \left(\sum_{i=r_1+1}^{r} \alpha_i (T \circledcirc x_i) \right) \,\middle|\, \alpha_i \in \mathbb{R}, i \in \mathbb{I}[1, r] \right\} \\ &= \left\{ \left(\sum_{i=r_1+1}^{r} \alpha_i (T \circledcirc x_i) \right) \,\middle|\, \alpha_i \in \mathbb{R}, i \in \mathbb{I}[r_1 + 1, r] \right\}. \end{aligned}$$

Now, let us show that $T \circledcirc x_i$, $i \in \mathbb{I}[r_1 + 1, r]$, are real linearly independent. Let β_i, $i \in \mathbb{I}[r_1 + 1, r]$, be real scalars such that

$$\sum_{i=r_1+1}^{r} \beta_i (T \circledcirc x_i) = 0.$$

Since T is real linear, one has

$$T \circledcirc \left(\sum_{i=r_1+1}^{r} \beta_i x_i \right) = 0,$$

which implies that

$$\sum_{i=r_1+1}^{r} \beta_i x_i \in \operatorname{Ker} T.$$

Therefore, $\beta_i = 0$, $i \in \mathbb{I}[r_1 + 1, r]$. With the preceding relations, one has

$$\operatorname{rdim}\,(\operatorname{Image} T) = r - r_1.$$

The proof is thus completed. ∎

2.9 Real Inner Product Spaces

An inner product space is a vector space with an additional structure called an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors. For a general complex vector space, the complex inner product is defined as follows.

Definition 2.9 A complex inner product space is a vector space V over the complex field $\mathbb{C}$ together with an inner product, i.e., with a map

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

satisfying the following three axioms for all vectors $x, y, z \in V$, and all scalars $a \in \mathbb{C}$.

(1) Conjugate symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
(2) Linearity in the first argument:

$$\langle ax, y \rangle = a \langle x, y \rangle ;$$
$$\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle ;$$

(3) Positive-definiteness: $\langle x, x \rangle > 0$ for all $x \neq 0$.

Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v \rangle = 0$.

In Definition 2.9, conjugate symmetry gives

$$\langle x, ay \rangle = \overline{\langle ay, x \rangle} = \overline{a \langle y, x \rangle} = \overline{a}\overline{\langle y, x \rangle} = \overline{a} \langle x, y \rangle ;$$
$$\langle x, y + z \rangle = \overline{\langle y + z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle .$$

An inner product space V with the inner product $\langle\cdot,\cdot\rangle$ has a naturally defined norm:

$$\|x\| = \sqrt{\langle x, x\rangle}, \text{ for any } x \in V.$$

For an n-dimensional inner product space V with the inner product $\langle\cdot,\cdot\rangle$, there exists a special basis in which all the vectors are orthogonal and have unit norm. Such a basis is called orthonormal. In details, a basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ is orthonormal if $\left\langle\varepsilon_i, \varepsilon_j\right\rangle = 0$, for $i \neq j$ and $\langle\varepsilon_i, \varepsilon_i\rangle = \|\varepsilon_i\| = 1$ for each i. In an n-dimensional complex vector space $\mathbb{C}^n$, the often encountered inner product is given by

$$\langle x, y\rangle = y^{\mathrm{H}} x$$

for any $x, y \in \mathbb{C}^n$. For the matrix space $\mathbb{C}^{m\times n}$, one can equip it with the following inner product:

$$\langle X, Y\rangle = \operatorname{tr}\left(Y^{\mathrm{H}} X\right)$$

for any $X, Y \in \mathbb{C}^{m\times n}$.

For a real linear space, one can equip it with a real inner product.

Definition 2.10 [297] A real inner product space is a real linear space V equipped with a real inner product, i.e., with a map

$$\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$$

satisfying the following three axioms for all vectors $x, y, z \in V$, and all scalars $a \in \mathbb{R}$.

(1) Symmetry: $\langle x, y\rangle = \langle y, x\rangle$;
(2) Linearity in the first argument:

$$\begin{aligned} \langle ax, y\rangle &= a\langle x, y\rangle ; \\ \langle x + y, z\rangle &= \langle x, z\rangle + \langle y, z\rangle ; \end{aligned}$$

(3) Positive-definiteness: $\langle x, x\rangle > 0$ for all $x \neq 0$.

Two vectors $u, v \in V$ are said to be orthogonal if $\langle u, v\rangle = 0$.

The following theorem defines a real inner product on space $\mathbb{C}^n$.

Theorem 2.14 *In the real linear space $\mathbb{C}^n$, a real inner product can be defined as*

$$\langle x, y\rangle = \operatorname{Re}\left(y^{\mathrm{H}} x\right) \tag{2.80}$$

for $x, y \in \mathbb{C}^n$. This real inner product space is denoted as $(\mathbb{C}^n, \mathbb{R}, \langle\cdot,\cdot\rangle)$.

Proof (1) For $x, y \in \mathbb{C}^n$, one has

$$\begin{aligned}\langle x, y\rangle &= \mathrm{Re}\left(y^{\mathrm{H}}x\right) = \mathrm{Re}\left(\left(y^{\mathrm{H}}x\right)^{\mathrm{T}}\right) = \mathrm{Re}\left(x^{\mathrm{T}}\overline{y}\right) \\ &= \mathrm{Re}\left(\overline{x^{\mathrm{T}}\overline{y}}\right) = \mathrm{Re}\left(x^{\mathrm{H}}y\right) \\ &= \langle y, x\rangle .\end{aligned}$$

(2) For a real number a, and $x, y, z \in \mathbb{C}^n$, it is easily obtained that

$$\begin{aligned}\langle ax, y\rangle &= \mathrm{Re}\left(y^{\mathrm{H}}(ax)\right) = \mathrm{Re}\left(ay^{\mathrm{H}}x\right) \\ &= a\,\mathrm{Re}\left(y^{\mathrm{H}}x\right) = a\langle x, y\rangle ,\end{aligned}$$

$$\begin{aligned}\langle x + y, z\rangle &= \mathrm{Re}\left[z^{\mathrm{H}}(x + y)\right] \\ &= \mathrm{Re}\left(z^{\mathrm{H}}x\right) + \mathrm{Re}\left(z^{\mathrm{H}}y\right) \\ &= \langle x, z\rangle + \langle y, z\rangle .\end{aligned}$$

(3) It is well-known that $\mathrm{tr}\left(x^{\mathrm{H}}x\right) = \|x\|^2 > 0$ for all $x \neq 0$. Thus, $\langle x, x\rangle = \mathrm{Re}\left[\mathrm{tr}\left(x^{\mathrm{H}}x\right)\right] > 0$ for all $x \neq 0$.

According to Definition 2.10, all the above arguments reveal that the space $\mathbb{C}^n$ with the real inner product defined in (2.80) is a real inner product space. ■

When the matrix space $\mathbb{C}^{m\times n}$ is viewed as a real linear space, the following theorem provides a real inner product for this space.

Theorem 2.15 *In the real linear space $\mathbb{C}^{m\times n}$, a real inner product can be defined as*

$$\langle A, B\rangle = \mathrm{Re}\left[\mathrm{tr}\left(A^{\mathrm{H}}B\right)\right] \tag{2.81}$$

for $A, B \in \mathbb{C}^{m\times n}$. This real inner product space is denoted as $\left(\mathbb{C}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle\right)$.

Proof (1) For $A, B \in \mathbb{C}^{m\times n}$, according to the properties of the trace of a matrix one has

$$\begin{aligned}\langle A, B\rangle &= \mathrm{Re}\left[\mathrm{tr}\left(A^{\mathrm{H}}B\right)\right] = \mathrm{Re}\left[\mathrm{tr}\left(B^{\mathrm{T}}\overline{A}\right)\right] = \mathrm{Re}\left[\overline{\mathrm{tr}\left(B^{\mathrm{T}}\overline{A}\right)}\right] \\ &= \mathrm{Re}\left[\mathrm{tr}\left(\overline{B^{\mathrm{T}}\overline{A}}\right)\right] = \mathrm{Re}\left[\mathrm{tr}\left(B^{\mathrm{H}}A\right)\right] \\ &= \langle B, A\rangle .\end{aligned}$$

(2) For a real number a, and $A, B, C \in \mathbb{C}^{m\times n}$, it can be derived that

$$\begin{aligned}\langle aA, B\rangle &= \mathrm{Re}\left[\mathrm{tr}\left((aA)^{\mathrm{H}} B\right)\right] = \mathrm{Re}\left[\mathrm{tr}\left(aA^{\mathrm{H}}B\right)\right] \\ &= \mathrm{Re}\left[a\,\mathrm{tr}\left(A^{\mathrm{H}}B\right)\right] = a\,\mathrm{Re}\left[\mathrm{tr}\left(A^{\mathrm{H}}B\right)\right] \\ &= a\langle A, B\rangle,\end{aligned}$$

$$\begin{aligned}\langle A+B, C\rangle &= \mathrm{Re}\left[\mathrm{tr}\left((A+B)^{\mathrm{H}} C\right)\right] \\ &= \mathrm{Re}\left[\mathrm{tr}\left(\left(A^{\mathrm{H}}+B^{\mathrm{H}}\right) C\right)\right] \\ &= \mathrm{Re}\left[\mathrm{tr}\left(A^{\mathrm{H}}C\right)\right] + \mathrm{Re}\left[\mathrm{tr}\left(B^{\mathrm{H}}C\right)\right] \\ &= \langle A, C\rangle + \langle B, C\rangle.\end{aligned}$$

(3) It is well-known that $\mathrm{tr}\left(A^{\mathrm{H}}A\right) > 0$ for all $A \neq 0$. Thus, $\langle A, A\rangle = \mathrm{Re}\left[\mathrm{tr}\left(A^{\mathrm{H}}A\right)\right] > 0$ for all $A \neq 0$.

According to Definition 2.10, all the above arguments reveal that the space $\mathbb{C}^{m\times n}$ with the real inner product defined in (2.81) is a real inner product space. ■

The real inner product space defined in Theorem 2.14 is different from the complex Hilbert space, and is a very interesting inner product space. In $V = \mathbb{C}^2$, let

$$v_1 = \begin{bmatrix} 1+\mathrm{i} \\ \mathrm{i} \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -\mathrm{i} \end{bmatrix}.$$

By simple computations, one has $v_1^{\mathrm{H}}v_2 = -\mathrm{i}$, so $\mathrm{Re}\left[\mathrm{tr}\left(v_1^{\mathrm{H}}v_2\right)\right] = 0$. Thus, in the real inner product space $\left(\mathbb{C}^2, \mathbb{R}, \langle\cdot,\cdot\rangle\right)$, the vectors v_1 and v_2 are orthogonal. However, v_1 and v_2 are not orthogonal in Hilbert space. In space $\mathbb{C}^n$, the symbol e_i is used to denote the vector whose i-th element is 1, and the other elements are zero. In the real inner product space $(\mathbb{C}^n, \mathbb{R}, \langle\cdot,\cdot\rangle)$ with $\langle\cdot,\cdot\rangle$ defined in (2.80), it is easily verified that

$$e_i, \mathrm{i}e_i, i \in \mathbb{I}[1, n],$$

are orthogonal. In addition, it is easily known that for any vector $v \in \mathbb{C}^n$ there must exist $2n$ real numbers a_i, $i \in \mathbb{I}[1, 2n]$, such that

$$v = \sum_{i=1}^{n} a_i e_i + \sum_{i=1}^{n} a_{n+i}\mathrm{i}e_i.$$

These facts reveal that e_i, $\mathrm{i}e_i$, $i \in \mathbb{I}[1, n]$, form a group of orthogonal real basis of the real inner product space $(\mathbb{C}^n, \mathbb{R}, \langle\cdot,\cdot\rangle)$. So the real dimension of the real inner product space $(\mathbb{C}^n, \mathbb{R}, \langle\cdot,\cdot\rangle)$ is $2n$.

Similarly, it is easily verified that the real inner product space $\left(\mathbb{C}^{m\times n}, \mathbb{R}, \langle\cdot,\cdot\rangle\right)$ with $\langle\cdot,\cdot\rangle$ defined in (2.81) is $2mn$-dimensional. In general, the real linear space $\mathbb{C}^{m_1\times n_1} \times \mathbb{C}^{m_2\times n_2} \times \cdots \times \mathbb{C}^{m_N\times n_N}$ with the real inner product defined as

$$\langle (R_1, R_2, \ldots, R_N), (S_1, S_2, \ldots, S_N) \rangle = \operatorname{Re}\left[\sum_{i=1}^{N} \operatorname{tr}\left(R_i^{\mathrm{H}} S_i\right)\right]$$

for $(R_1, R_2, \ldots, R_N), (S_1, S_2, \ldots, S_N) \in \mathbb{C}^{m_1 \times n_1} \times \mathbb{C}^{m_2 \times n_2} \times \cdots \times \mathbb{C}^{m_N \times n_N}$, is of real dimension $2\sum_{i=1}^{N} m_i n_i$.

2.10 Optimization in Complex Domain

In this section, some results are introduced on the optimization of real-valued functions with respect to complex variables. First, the partial derivative of a function with respect to a complex variable is introduced.

Let z be a complex variable, and denote it as

$$z = x + \mathrm{i}y, x, y \in \mathbb{R}.$$

Then, the formal derivatives for z and its conjugate $\overline{z}$ are defined as, respectively

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \tag{2.82}$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right). \tag{2.83}$$

With the definitions in (2.82) and (2.83), it is easily checked that

$$\frac{\partial z}{\partial z} = \frac{\partial \overline{z}}{\partial \overline{z}} = 1, \tag{2.84}$$

$$\frac{\partial z}{\partial \overline{z}} = \frac{\partial \overline{z}}{\partial z} = 0. \tag{2.85}$$

The expressions (2.84) and (2.85) describe a basic result in complex analysis: the complex variable z and its conjugate $\overline{z}$ are independent when partial derivatives need to be computed.

Let $f(z, \overline{z})$ be a scalar function with complex variable z. In addition, the partial derivatives of $f(z, \overline{z})$ with respect to the real part x and the imaginary part y of z is continuous. Thus, there hold

$$\frac{\partial f(z, \overline{z})}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \mathrm{i}\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial f(z, \overline{z})}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \mathrm{i}\frac{\partial f}{\partial y}\right).$$

With these basic definitions, the gradient and conjugate gradient of a scalar function with respect to a vector are given.

Definition 2.11 Let $\omega = \left[\omega_1\ \omega_2\ \cdots\ \omega_n\right]^{\mathrm{T}} \in \mathbb{C}^n$, and $f(\omega)$ be a complex-valued scalar function of ω. Then the gradient of $f(\omega)$ with respect to ω is defined as

$$\frac{\partial f(\omega)}{\partial \omega} = \nabla_{\omega} f(\omega) = \left[\frac{\partial f(\omega)}{\partial \omega_1}\ \frac{\partial f(\omega)}{\partial \omega_2}\ \cdots\ \frac{\partial f(\omega)}{\partial \omega_n}\right]^{\mathrm{T}}.$$

In addition, the gradient of $f(\omega)$ with respect to the conjugate $\overline{\omega}$ of the vector variable ω, which is also called the conjugate gradient of $f(\omega)$ with respect to ω, is defined as

$$\frac{\partial f(\omega)}{\partial \overline{\omega}} = \nabla_{\overline{\omega}} f(\omega) = \left[\frac{\partial f(\omega)}{\partial \overline{\omega_1}}\ \frac{\partial f(\omega)}{\partial \overline{\omega_2}}\ \cdots\ \frac{\partial f(\omega)}{\partial \overline{\omega_n}}\right]^{\mathrm{T}}.$$

In the following, the definitions are given on the gradient and conjugate gradient of a vector function with respect to a scalar.

Definition 2.12 If $\omega \in \mathbb{C}$, and $f(\omega) = \left[f_1(\omega)\ f_2(\omega)\ \cdots f_m(\omega)\right]^{\mathrm{T}} \in \mathbb{C}^m$ is a complex-valued vector function. Then, the gradient of $f(\omega)$ with respect to ω is defined as

$$\frac{\partial f(\omega)}{\partial \omega} = \nabla_{\omega} f(\omega) = \left[\frac{\partial f_1(\omega)}{\partial \omega}\ \frac{\partial f_2(\omega)}{\partial \omega}\ \cdots\ \frac{\partial f_m(\omega)}{\partial \omega}\right],$$

and the conjugate gradient of $f(\omega)$ with respect to ω is defined as

$$\frac{\partial f(\omega)}{\partial \overline{\omega}} = \nabla_{\overline{\omega}} f(\omega) = \left[\frac{\partial f_1(\omega)}{\partial \overline{\omega}}\ \frac{\partial f_2(\omega)}{\partial \overline{\omega}}\ \cdots\ \frac{\partial f_m(\omega)}{\partial \overline{\omega}}\right].$$

Similarly to the preceding definitions, the gradient of a vector function with respect to a vector can be defined. Let $f(\omega) = \left[f_1(\omega)\, f_2(\omega) \cdots f_m(\omega)\right]^{\mathrm{T}}$ is an m-dimensional vector function of $\omega = \left[\omega_1\ \omega_2\ \cdots\ \omega_n\right]^{\mathrm{T}} \in \mathbb{C}^n$, then the gradient of $f(\omega)$ with respect to ω is defined as

$$\frac{\partial f(\omega)}{\partial \omega} = \nabla_{\omega} f(\omega) = \left[\frac{\partial f_1(\omega)}{\partial \omega}\ \frac{\partial f_2(\omega)}{\partial \omega}\ \cdots\ \frac{\partial f_m(\omega)}{\partial \omega}\right],$$

which can be written as

$$\frac{\partial f(\omega)}{\partial \omega} = \begin{bmatrix} \frac{\partial f_1(\omega)}{\partial \omega_1} & \frac{\partial f_2(\omega)}{\partial \omega_1} & \cdots & \frac{\partial f_m(\omega)}{\partial \omega_1} \\ \frac{\partial f_1(\omega)}{\partial \omega_2} & \frac{\partial f_2(\omega)}{\partial \omega_2} & \cdots & \frac{\partial f_m(\omega)}{\partial \omega_2} \\ \cdots & \cdots & & \cdots \\ \frac{\partial f_1(\omega)}{\partial \omega_n} & \frac{\partial f_2(\omega)}{\partial \omega_n} & \cdots & \frac{\partial f_m(\omega)}{\partial \omega_n} \end{bmatrix}.$$

Similarly, the conjugate gradient of $f(\omega)$ with respect to ω is defined as

$$\frac{\partial f(\omega)}{\partial \overline{\omega}} = \nabla_{\overline{\omega}} f(\omega) = \left[\frac{\partial f_1(\omega)}{\partial \overline{\omega}} \; \frac{\partial f_2(\omega)}{\partial \overline{\omega}} \; \cdots \; \frac{\partial f_m(\omega)}{\partial \overline{\omega}} \right],$$

which can be expressed as

$$\frac{\partial f(\omega)}{\partial \overline{\omega}} = \begin{bmatrix} \frac{\partial f_1(\omega)}{\partial \overline{\omega_1}} & \frac{\partial f_2(\omega)}{\partial \overline{\omega_1}} & \cdots & \frac{\partial f_m(\omega)}{\partial \overline{\omega_1}} \\ \frac{\partial f_1(\omega)}{\partial \overline{\omega_2}} & \frac{\partial f_2(\omega)}{\partial \overline{\omega_2}} & \cdots & \frac{\partial f_m(\omega)}{\partial \overline{\omega_2}} \\ \cdots & \cdots & & \cdots \\ \frac{\partial f_1(\omega)}{\partial \overline{\omega_n}} & \frac{\partial f_2(\omega)}{\partial \overline{\omega_n}} & \cdots & \frac{\partial f_m(\omega)}{\partial \overline{\omega_n}} \end{bmatrix}.$$

By the preceding definitions, some often-encountered formulae are given on gradient and conjugate gradient.

(1) Given $A \in \mathbb{C}^{n\times n}$, for the quadratic function $f(x) = x^{\mathrm{H}} A x$ with $x \in \mathbb{C}^n$ there holds

$$\frac{\partial f(x)}{\partial x} = A^{\mathrm{T}}\overline{x}, \; \frac{\partial f(x)}{\partial \overline{x}} = Ax.$$

(2) (The chain rule) If $y(x) \in \mathbb{C}^n$ is a complex vector-valued function of $x \in \mathbb{C}^n$, then

$$\frac{\partial f(y(x))}{\partial \overline{x}} = \frac{\partial y(x)}{\partial \overline{x}} \frac{\partial f(y)}{\partial y}.$$

(3) Given $A \in \mathbb{C}^{n\times m}$, for the function $f(x) = Ax$ with respect to $x \in \mathbb{C}^n$ there holds

$$\frac{\partial f(x)}{\partial x} = A^{\mathrm{T}}, \; \frac{\partial f(x)}{\partial \overline{x}} = 0.$$

Let $f(\omega)$ be a complex-valued function with respect to $\omega = \left[\omega_1 \; \omega_2 \; \cdots \; \omega_n \right]^{\mathrm{T}} \in \mathbb{C}^n$. The Hessian matrix $H(f(\omega))$ of the function $f(\omega)$ is a square matrix of second-order partial derivatives, and has the following form

$$H(f(\omega)) = \begin{bmatrix} \frac{\partial^2 f(\omega)}{\partial \overline{\omega_1} \partial \omega_1} & \frac{\partial^2 f(\omega)}{\partial \overline{\omega_1} \partial \omega_2} & \cdots & \frac{\partial^2 f(\omega)}{\partial \overline{\omega_1} \partial \omega_n} \\ \frac{\partial^2 f(\omega)}{\partial \overline{\omega_2} \partial \omega_1} & \frac{\partial^2 f(\omega)}{\partial \overline{\omega_2} \partial \omega_2} & \cdots & \frac{\partial^2 f(\omega)}{\partial \overline{\omega_2} \partial \omega_n} \\ \cdots & \cdots & & \cdots \\ \frac{\partial^2 f(\omega)}{\partial \overline{\omega_n} \partial \omega_1} & \frac{\partial^2 f(\omega)}{\partial \overline{\omega_n} \partial \omega_2} & \cdots & \frac{\partial^2 f(\omega)}{\partial \overline{\omega_n} \partial \omega_n} \end{bmatrix}.$$

It is easily seen that the transpose of the Hessian matrix of $f(\omega)$ is the gradient of the conjugate gradient of the function $f(\omega)$. That is,

$$H^{\mathrm{T}}(f(\omega)) = \frac{\partial f(\omega)}{\partial \overline{\omega} \partial \omega} = \frac{\partial}{\partial \omega} \left[\frac{\partial f(\omega)}{\partial \overline{\omega}} \right].$$

Due to this relation, the Hessian matrix will be denoted by $\nabla^2_\omega f(\omega)$. Thus, one has

$$\nabla^2_\omega f(\omega) = \left[\frac{\partial^2 f(\omega)}{\partial \overline{\omega} \partial \omega}\right]^{\mathrm{T}} = \left[\frac{\partial}{\partial \omega}\left[\frac{\partial f(\omega)}{\partial \overline{\omega}}\right]\right]^{\mathrm{T}} = \left[\frac{\partial f(\omega)}{\partial \overline{\omega_i} \partial \omega_j}\right]_{n \times n}.$$

The Hessian matrix plays a vital role in the solution to minimization problems.

At the end of this section, a sufficient condition and a necessary condition are provided for the local minimum point of a real-valued function $f(\omega)$.

Lemma 2.27 *Let $f(\omega)$ be a real-valued function with respect to complex vector variable $\omega \in \mathbb{C}^n$. Then, $\omega = \omega_0$ is a strictly local minimum point of $f(\omega)$ if*

$$\left.\frac{\partial f(\omega)}{\partial \overline{\omega}}\right|_{\omega=\omega_0} = 0, \ \left.\frac{\partial^2 f(\omega)}{\partial \overline{\omega} \partial \omega}\right|_{\omega=\omega_0} > 0.$$

Lemma 2.28 *Let $f(\omega)$ be a real-valued function with respect to complex vector variable $\omega \in \mathbb{C}^n$. If $\omega = \omega_0$ is a local minimum point of $f(\omega)$, then*

$$\left.\frac{\partial f(\omega)}{\partial \overline{\omega}}\right|_{\omega=\omega_0} = 0, \ \left.\frac{\partial^2 f(\omega)}{\partial \overline{\omega} \partial \omega}\right|_{\omega=\omega_0} \geq 0.$$

For a convex function $f(\omega)$ with respect to the complex vector ω, its arbitrary local minimum point $\omega = \omega_0$ is the global minimum point. If the convex function $f(\omega)$ is differentiable, then the stationary point $\omega = \omega_0$ satisfying $\left.\frac{\partial f(\omega)}{\partial \bar{\omega}}\right|_{\omega=\omega_0} = 0$, is the global minimum point.

2.11 Notes and References

In this chapter, some mathematical tools which will be used in the sequel chapters of this book are introduced. The main materials in this chapter are well-known, and can be found in some textbooks. Of course, there are some of the authors' own research results. These include the result on the characteristic polynomial of the real representation of a complex matrix in Sect. 2.6, and the real inner product given in Theorem 2.14.

In Sect. 2.1, the operation of Kronecker products for matrices is introduced. The materials in Sect. 2.1 are mainly taken from [143, 172]. Some further properties on Kronecker products can be found in literature and some textbooks. For example, an identity on the determinant of the Kronecker product of two matrices was given in [172]; eigenvalues of Kronecker products were also investigated in [172]. The Kronecker product plays an important role in a variety of areas, such as, signal processing, image processing and semidefinite programming. A new method for estimating high dimensional covariance matrices was presented in [230] based on a Kronecker product series expansion of the true covariance matrix. In [207], the application of Kronecker products in discrete unitary transformations was introduced, and some characteristics of Hadamard transformations were derived in the framework of Kronecker products. A brief survey on Kronecker products can be referred to [232].

In Sect. 2.2, the celebrated Leverrier algorithm is introduced. The proof of this algorithm is taken from [10]. In this proof, an appropriate companion matrix is used. Another proof was given in [144] by establishing a simple formula related to the trace of the resolvent and the characteristic polynomial of a matrix. The Leverrier algorithm is also called Leverrier-Faddeev algorithm, and has been widely applied. In [137], it was used to solve Lyapunov and Sylvester matrix equations.

In Sect. 2.3, a generalization of the Leverrier algorithm is introduced. The generalized Leverrier algorithm for $(sM - A)$ was given in [186] in order to calculate the transfer function for a descriptor linear system. The further extension for this algorithm was given in [174]. In fact, the result in Theorem 2.4 is a direct corollary of the main result in [174]. In [305], this result was also obtained by directly using the Leverrier algorithm. The content of this section is mainly taken from [186, 305]. It should be pointed out that the idea in [186] has been extended to obtain algorithms for solving determinant of some general polynomial matrices [234].

In Sect. 2.4, the concepts of singular values and singular vectors are first introduced, and then the well-known singular value decomposition (SVD) of a given complex matrix is given. Up till now, it has been recognized that the process of obtaining a singular value decomposition for a given matrix is numerically reliable. In addition, the SVD is one of the most important tools in numerical linear algebra, because it contains a lot of information about a matrix, including rank, distance to singularity, column spaces, row spaces, and null spaces. Therefore, the SVD has become a popular technique, and can be found in many textbooks [81, 143, 172]. Another feature of the SVD is that it involves only two unitary or orthogonal matrices. This makes it a very favorable tool in many applications. For example, in [261] the SVD was used to derive the I-controllablizability condition, and give a general expression of I-controllablizing controller for square descriptor linear systems. Such a method was generalized in [259, 260] to investigate the impulsive-mode controllablizability of nonsquare descriptor linear systems. In [112], the SVD was also used to solve the problem of dynamical order assignment of descriptor linear systems. In [63], the robust pole assignment problem via output feedback was solved for linear systems with the SVD as a tool. Besides, SVD techniques also play important roles in image processing [7, 151].

In Sect. 2.5, the vector norms and operator norms for matrices are introduced. Vector norms are generalizations of ordinary length, and are used to measure the size of a vector in vector spaces. In the first subsection of this section, an axiomatic definition is first given for vector norms, and then the celebrated p-norms are introduced. These materials can be found in most textbooks on matrix theory or numerical analysis [142, 172]. The proof on p-norms is also presented in this subsection. During the procedure of the proof, the celebrated Young's Inequality and Hölder's Inequality are used. In fact, the triangle inequality on p-norms is the celebrated Minkowski's Inequality. For more information on Hölder's Inequality and Minkowski's Inequality, one can refer to [184, 187]. In the second subsection of Sect. 2.5, operator norms of matrices are introduced. The definition of operator norms is first given, and then some properties of operator norms are established. Moreover, some often encountered operator norms are introduced. The operator norm in this section can be found in [132]. It is worth pointing out that the operator norm given in this section is different from that in most textbooks on matrix theory, where the operator norm for a square matrix is defined as [142, 172, 187]

$$\|A\| = \max_{x \neq 0} \left\{ \frac{\|Ax\|}{\|x\|} \right\}.$$

In this definition, the two vector norms are required to be identical. However, the definition of operator norms in Sect. 2.5 does not have this requirement. In addition, in Sect. 2.5 it is not required that the matrix A is square. On general operator norms introduced in Sect. 2.5, one can refer to [62, 132, 209]. Table 2.1 lists some $p \to q$ induced norms for a matrix

$$A = \left[a_{ij}\right]_{m \times n} = \begin{bmatrix} a_{1\cdot}^{\mathrm{T}} \\ a_{2\cdot}^{\mathrm{T}} \\ \cdots \\ a_{m\cdot}^{\mathrm{T}} \end{bmatrix} = \left[a_1 \; a_2 \; \cdots \; a_n \right].$$

These results can be found in [62, 209]. In Table 2.1, $\{-1, 1\}^n$ represents the Cartesian product of the set $\{-1, 1\}$. That is,

$$\begin{aligned} \{-1, 1\}^n &= \{-1, 1\} \times \{-1, 1\} \times \cdots \times \{-1, 1\} \\ &= \left\{ \left[a_1 \; a_2 \; \cdots \; a_n \right]^{\mathrm{T}} \middle| \, a_i \in \{-1, 1\}, i \in \mathbb{I}[1, n] \right\}. \end{aligned}$$

In Sect. 2.6, the concept of a real representation of complex matrices is introduced, and some properties of this real representation are given. The real representation in Sect. 2.6 was firstly proposed by Jiang and Wei in [155] to solve the con-Kalman-Yakubovich matrix equation $X - A\overline{X}B = C$. In [258], it was used to give an explicit solution of normal con-Sylvester matrix equations. This real representation was also applied to investigate some other complex conjugate matrix equations [265, 281]. In addition, it was also used in [154] to investigate the consimilarity of complex matrices. In Sect. 2.6, the results in Lemmas 2.13 and 2.14 are taken from [155]. The result of Lemma 2.15 is taken from [265]. The results of Theorems 2.7 and 2.8 were given in [258] and [280]. However, in Subsection 2.6.2 an alternative proof for Theorem 2.7 was provided.

In Sect. 2.7, the consimilarity of complex matrices is simply introduced. The main content of this section is taken from [142]. Besides similarity, consimilarity is another equivalence relation between complex matrices. Consimilarity firstly appeared as a change of basis for the matrix representation of semilinear transformations [20, 150]. In Sect. 2.7, the concepts of consimilarity, coneigenvalues and coneigenvectors are given, and a relation between coneigenvalues and eigenvalues is established.

Table 2.1 Some $p \to q$ induced norms

Norm	Expression	Norm	Expression
$\|A\|_{1\to 1}$	$\max\limits_{j\in\mathbb{I}[1,n]}\left\{\sum\limits_{i=1}^{m}\left\|a_{ij}\right\|\right\} = \max\limits_{j\in\mathbb{I}[1,n]}\left\{\left\|a_j\right\|_1\right\}$	$\|A\|_{\infty\to 1}$	$\max\limits_{s\in\{-1,1\}^n}\{\|As\|_{1\to 1}\}$
$\|A\|_{1\to 2}$	$\max\limits_{j\in\mathbb{I}[1,n]}\left\{\sum\limits_{i=1}^{m}\left\|a_{ij}\right\|^2\right\} = \max\limits_{j\in\mathbb{I}[1,n]}\left\{\left\|a_j\right\|_2\right\}$	$\|A\|_{\infty\to 2}$	$\max\limits_{i\in\mathbb{I}[1,m]}\left\{\sum\limits_{j=1}^{n}\left\|a_{ij}\right\|^2\right\} = \max\limits_{i\in\mathbb{I}[1,m]}\{\|a_{i\cdot}\|_2\}$
$\|A\|_{1\to\infty}$	$\max\limits_{i\in\mathbb{I}[1,m],\ j\in\mathbb{I}[1,n]}\left\{\left\|a_{ij}\right\|\right\}$	$\|A\|_{\infty\to\infty}$	$\max\limits_{i\in\mathbb{I}[1,m]}\left\{\sum\limits_{j=1}^{n}\left\|a_{ij}\right\|\right\} = \max\limits_{i\in\mathbb{I}[1,m]}\{\|a_{i\cdot}\|_1\}$
$\|A\|_{2\to 1}$	$\max\limits_{s\in\{-1,1\}^m}\left\{\sigma_{\max}\left(s^{\mathrm{T}}A\right)\right\}$	$\|A\|_{2\to 2}$	$\sigma_{\max}(A)$

It can be seen that the theory of coneigenvalues is much more complicated than that of eigenvalues. A matrix may have infinitely many distinct coneigenvalues or it may have no coneigenvalues at all. There have been many results on consimilarity available in literature. In [141], a criterion of consimilarity between A and B was established by the similarity between $A\overline{A}$ and $B\overline{B}$ combined with the alternating-product rank condition. Moreover, some canonical forms were given in [141] under consimilarity.

Section 2.8 first introduces the concept of real linear spaces, and then gives some properties of real linear mappings. In fact, the concept of real linear spaces is only a special case of normal linear spaces. However, in most books the entries of the involved vector in a linear space over the field $\mathbb{R}$ are all real. In this section, the real linear space is emphasized in order to deal with some special spaces. Some concepts in normal linear spaces are specialized to the context of real linear spaces. These concepts include real bases, real dimensions, and real linear dependence. In the second subsection of this section, a result on the real dimensionality of image and kernel of a real linear mapping is given. This result will be used in Sect. 6.1 to investigate the solvability of the so-called normal con-Sylvester matrix equation.

In Sect. 2.9, the real inner product space is introduced. A real inner product for the matrix space $\mathbb{C}^{m\times n}$ is given, which will be used in Chap. 5 to investigate finite iterative algorithms for some complex conjugate matrix equations.

In Sect. 2.10, some basic results are introduced for complex optimization. The definition of the derivative with respect to a complex variable is first given, and then the concepts of gradient and conjugate gradient are presented. Based on these preliminaries, some conditions for minimum point are given for a real-valued function with respect to complex vector variables. The result in this section will be used in Chap. 12. The main content of this section is taken from [297].

Springer

http://www.springer.com/978-981-10-0635-7

Complex Conjugate Matrix Equations for Systems and Control
Wu, A.-G.; Zhang, Y.
2017, XVIII, 487 p. 13 illus., Hardcover
ISBN: 978-981-10-0635-7