

Chapter 2

Theoretical Preliminaries

Abstract Over the last two decades, different algorithms have been proposed to address the stabilization problem of the underactuated systems. Joint research endeavors from academia and industry have led to the propositions of several advanced control algorithms. Nonetheless, among those numerous control strategies only a few could stand out as strong contenders. In this chapter, a few relevant control algorithms are being discussed to provide the prerequisite knowledge to the readers. All being well, after going through this chapter reader will not face any kind of difficulties in comprehending the contents of subsequent chapters.

2.1 Feedback Linearization

Feedback linearization is a control system design methodology especially applicable for nonlinear systems. The key idea of this approach is to algebraically transform a nonlinear system dynamics into an equivalent linear system, such that linear control technologies can be applied on the transformed system. The feedback linearization methodology is entirely different from the Jacobian linearization technique. Indeed, it can be viewed as a methodology to transform a complex nonlinear dynamics into *an equivalent simpler linear model*. The technique has two subsections one is input-state feedback linearization and the other is Input–Output Feedback Linearization [8].

2.1.1 Input-State Feedback Linearization

A generic single input nonlinear system is described by the following equation:

$$\dot{x} = f(x) + g(x)u \quad (2.1)$$

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where $x \in \mathbb{R}^n$, $f(x): \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g(x) \in \mathbb{R}^{n \times 1}$, u is a single dimensional control input.

Basically the input-state linearization is a two-step control design methodology. At the onset it employs a state transformation to convert the system into a linear one. Thereafter, a linear control technique is being used to design a linear control law to obtain the desired performance from the system. For the sake of better comprehensibility, following second-order nonlinear system is considered to explain the design procedure of input-state feedback linearization

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_1^3 + x_2 \\ \dot{x}_2 &= x_1 - x_2 + bu\end{aligned}\tag{2.2}$$

In the above state model, b is nonzero system parameters. A careful observation of the above equations reveals that the equilibrium point of the system is located at $(0, 0)$. Linearization of the above state model using Jacobian linearization yields the following linear state model of the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix} u\tag{2.3}$$

Linear control law can be exploited to design a control law that would give a satisfactory performance around the equilibrium point. However, performance of the system gets worse as soon as the system states start deviating from the equilibrium point. This is not at all a surprising outcome, a careful observation on the first equation reveals that the dynamics of x_1 state deviates a lot from the linearized one when $|x_1| \geq 1$. Consequently, it is very easy to understand that linear control laws fail to generate a satisfactory performance when operating point is located away from the equilibrium.

In this context, feedback linearization can give us a plausible solution. Basically the input-state linearization is a two-step control design methodology. At the onset it finds a state transform $z = z(x)$ and an input transform $u = u(x, v)$, which transform the original nonlinear system into an equivalent linear system. Thereafter, in the second step it uses some linear control design techniques to design v .

However, before start discussion on feedback linearization, let us have a close look on the system of (2.2). Minute observation reveals that the nonlinear terms not only appear in the dynamics of x_2 , but it also appears in the dynamics of x_1 , which further complicates the design procedure. Now if all the nonlinear terms appear in the second equation, then it would become an easier task for the designer to cancel all the nonlinearities using state feedback in a single step. However, in this case, at first a state transformation is required to convert the system model in such a way that all nonlinear terms would only appear with input u , then only one can easily utilize a state feedback to cancel all the nonlinear terms. In order to accomplish the task, following state transformation can be defined as shown in Eq. (2.4) below

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= -x_1^3 + x_2\end{aligned}\tag{2.4}$$

Above state transformation yields

$$\begin{aligned}\dot{z}_1 &= -z_1 + z_2 \\ \dot{z}_2 &= 2z_1^3 - 3z_1^2 z_2 + z_1 - z_2 + bu\end{aligned}\quad (2.5)$$

Now the following state feedback control law can be employed to make the system a linear one:

$$u = \frac{1}{b} (-2z_1^3 + 3z_1^2 z_2 + v) \quad (2.6)$$

That yields

$$\begin{aligned}\dot{z}_1 &= -z_1 + z_2 \\ \dot{z}_2 &= z_1 - z_2 + v\end{aligned}\quad (2.7)$$

Now the system would behave like a linear system in z_1, z_2 coordinate, and numerous linear control laws are there which can give us a satisfactory response from this system. Poles of the system (2.7) are located at $-2, 0$. The above system behaves like a badly damped linear system; now further inspection on the linear model of Eq. (2.7) reveals that the system is fully state controllable at origin. Therefore, one can design a state feedback control law to place the pole of the linear systems at desired location. Hence, if we want that the closed system will behave like a linear system with damping ratio $\zeta = 0.7$, and natural frequency $\omega_n = 5$ rad/s, then the pole must be placed at $(-3.5 + 3.75j)$ and $(-3.5 - 3.75j)$. Now, this can be easily achieved by designing a state feedback law with gain matrix $k = [19.999 \ 5]$. Readers are encouraged to verify this design by their own. Hence, the feedback law will take the form

$$v = -[19.999 \ 5] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (2.8)$$

Which yields the following stable dynamics:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -18.999 & -6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (2.9)$$

Therefore, the control input u will take the form

$$u = \frac{1}{b} (-19.999z_1 - 6z_2 - 2z_1^3 + 3z_1^2 z_2) \quad (2.10)$$

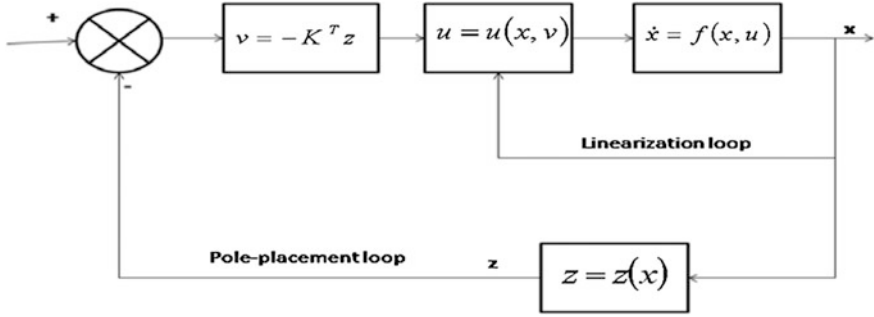


Fig. 2.1 Schematic diagram of the control law

In x_1 and x_2 coordinate, input u will take the following form:

$$u = \frac{1}{b} (-19.999x_1 - 6x_2 + 4x_1^3 - 3x_1^5 + 3x_1^2x_2) \quad (2.11)$$

A pictorial representation of the control law is given in Fig. 2.1.

Important Observations

- The input-state linearization process consisting of state transformation and input transformation, with state feedback used in both. So this is a linearization by feedback and that is why this methodology is termed as feedback linearization. It is fundamentally different from a Jacobian linearization which linearized a nonlinear system around a small region of the state plane.
- In order to implement the control law, the new state components (z_1, z_2) must be available. If they are not physically meaningful or cannot be measured directly, the original state x must be measured, and used to compute them from (2.4). In the next section, concept of output feedback linearization will be discussed in Sect. 2.1.2.

2.1.2 Output Feedback Linearization

In the preceding discussions, it has been assumed that all the states are readily available for measurement, and they can be utilized to design a state feedback according to the designer's disposal. However, most of the times practical systems fail to satisfy this requirement of state feedback linearization. Most of the cases, only output from the systems become available to the designers. In such a situation, design of input-state feedback linearization is not possible [8]. However, output from the system can be utilized to design a feedback law for the system. More precisely, output signal can be used to design a feedback law to cancel out the

nonlinear terms of systems equations. Consider the following nonlinear system as shown below in Eqs. (2.12.a, 2.12.b):

$$\dot{x} = f(x) + g(x)u \quad (2.12.a)$$

$$y = h(x) \quad (2.12.b)$$

Prime objective of the tracking control algorithm is to make the output $y(t)$, to track a desired trajectory $y_d(t)$, while keeping all the states bounded. It is assumed that the time derivatives of the reference signal (up to a sufficiently high order) are known and bounded. In order to elaborate the concept with a design problem, tracking control for the nonlinear system of Eq. (2.13) is considered.

$$\begin{aligned} \dot{x}_1 &= \cos x_2 + (x_2 + 1)x_3 \\ \dot{x}_2 &= x_1^3 + x_3 \\ \dot{x}_3 &= x_1^2 + u \\ y &= x_1 \end{aligned} \quad (2.13)$$

As stated above, prime objective of the controller design is to find out a control law that will ensure guaranteed tracking of the reference signal $y_d(t)$. However, the main challenge of this tracking control is that the input to the system has got no direct relationship with the output y . Nonetheless, successive differentiation of output with respect to time may yield an explicit relationship between output and input. Thereafter it is always possible to manipulate the input u that will ensure the guaranteed tracking of the output signal. Hence, the output signal is differentiated twice with respect to time to establish an explicit relation between input and output,

$$\ddot{y} = (x_2 + 1)u + f_1(\vec{x}) \quad (2.14)$$

where $f_1(\vec{x})$ is a function of state vector defined by

$$f_1(\vec{x}) = (x_1^3 + x_3)(x_3 - \sin x_2) + (x_2 + 1)x_1^2 \quad (2.15)$$

Clearly, Eq. (2.14) depicts an explicit relationship between output y and input u . Now a feedback control law can be used to cancel out all the nonlinearities present in the Eq. (2.14) as shown in Eq. (2.16) below

$$u = \frac{1}{x_2 + 1} (v - f_1(\vec{x})) \quad (2.16)$$

where v is an equivalent input to be designed according to the design requirement. Application of the above-mentioned control input u yields a simple double integrator relationship between equivalent input v with the output y as shown in Eq. (2.17)

$$\ddot{y} = v \quad (2.17)$$

At this stage, it is very easy to deploy standard linear control algorithm to get a desired tracking performance from the system. Tracking error e can be defined as $e = y - y_d$, now it is easy to relate the tracking error e and new control input v as shown in the following Eq. (2.18)

$$v = y_d - k_1 e - k_2 \dot{e} \quad (2.18)$$

with k_1 and k_2 are two positive design constants, which yield the following equation for closed loop error dynamics as shown in Eq. (2.19) below:

$$\ddot{e} + k_2 \dot{e} + k_1 e = 0 \quad (2.19)$$

which represents an exponential stable error dynamics. Therefore, perfect tracking can be achieved in almost all cases except the singular point that is located at $x_2 = -1$. Otherwise, this control algorithm ensures perfect tracking of a smooth reference signal. Since in this case output signal is being used to convert the system in linear form, this type of linearization is known as *input–output linearization*.

For an n th order nonlinear system at the time of *Input–Output Linearization*, if r differentiation is required to establish an explicit relationship between *input* u and *output* y , then the system can be termed as a system with **relative degree r** [1]. Therefore, it is easy to conclude that the system of Eq. (2.13) has a relative degree equal to two. At the time of *Input–Output Linearization* algorithm one only considers a part of closed-loop system. Frequently a part of the system dynamics is rendered *unobservable* at the time of *Input–Output Linearization*. In general, this part of the dynamics of the system is termed as the **Internal Dynamics** of the original system. Indeed, it cannot be seen from the external relationship, between equivalent input and output relationship of the system. For the tracking control of the system shown in Eq. (2.13), the internal dynamics is represented by the following equation,

$$\dot{x}_3 = x_1^2 + \frac{1}{x_2 + 1} (\ddot{y}_d - k_1 e - k_2 \dot{e} + f_1(\vec{x})) \quad (2.20)$$

In case of *Input–output linearization* it is possible to design tracking control algorithm, only when internal dynamics of the system is stable.

Zero Dynamics: The *Zero Dynamics* is defined to be the internal dynamics of the system when the system output is kept at zero by the input [1, 3]. In case of a linear system the poles of the zero dynamics are exactly the zeros of the original system. The zero dynamics is an intrinsic property of the nonlinear system, while stability of the same ensures the stability of the internal dynamics of the nonlinear system in local sense.

2.1.3 Partial Feedback Linearization

It is a special type of feedback linearization method that is being considered as one of the most useful control technique for underactuated mechanical systems [UMS] [5–7]. This method provides a natural global change of coordinates that transforms the system into a strict feedback form, and thereafter the conventional control method can be easily applied to the transformed system [9]. However, in case of the UMS neither input feedback linearization nor input–output linearization can provide a satisfactory result. Let us consider the Lagrangian model of UMS as shown in Eq. (2.21)

$$\begin{aligned} m_{11}(\mathbf{q})\ddot{\mathbf{q}}_1 + m_{12}(\mathbf{q})\ddot{\mathbf{q}}_2 + \mathbf{h}_1(\mathbf{q}, \mathbf{p}) &= 0 \\ m_{21}(\mathbf{q})\ddot{\mathbf{q}}_1 + m_{22}(\mathbf{q})\ddot{\mathbf{q}}_2 + \mathbf{h}_2(\mathbf{q}, \mathbf{p}) &= B(\mathbf{q})\tau \end{aligned} \quad (2.21)$$

where, $\mathbf{q}_1 \in \mathbb{R}^{n_1}$, $\mathbf{q}_2 \in \mathbb{R}^{n_2}$ and $\mathbf{q} = \text{col}(\mathbf{q}_1, \mathbf{q}_2) \in \mathbb{R}^n$ is the configuration vector of n th degree of freedom underactuated mechanical system. In the above representation, $\mathbf{q}_1$ represents passive joint configuration variables, and $\mathbf{q}_2$ represents active joint configuration variables. The dimension of the overall configuration manifold is $n_1 + n_2 = n$. The time derivative of the configuration vector $\mathbf{q}$ is expressed as $\mathbf{p} = \text{col}(\mathbf{p}_1, \mathbf{p}_2) \in \mathbb{R}^n$, and $\tau \in \mathbb{R}^{n_2}$ is the control input. In the above representation, $\mathbf{h}_1(\mathbf{q}, \mathbf{p}): \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n_1}$ and $\mathbf{h}_2(\mathbf{q}, \mathbf{p}): \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n_2}$ contain the coriolis, centrifugal, and gravity terms. Whereas, $m_{ij}(\mathbf{q})$ $q = 1, 2$, represents the components of the $n \times n$ inertia matrix, which is symmetric and positive definite for all $\mathbf{q}$. $B(\mathbf{q}) \in \mathbb{R}^{n_2 \times n_2}$ represents a full rank matrix. Due to Spong, there exists an invertible change of the control input τ as shown in Eq. (3.19)

$$\tau = B^{-1}(\mathbf{q})((\mathbf{h}_2(\mathbf{q}, \mathbf{p}) - m_{21}(\mathbf{q})m_{11}^{-1}(\mathbf{q})\mathbf{h}_1(\mathbf{q}, \mathbf{p})) + (m_{22}(\mathbf{q}) - m_{21}(\mathbf{q})m_{11}^{-1}(\mathbf{q})m_{12}(\mathbf{q}))\mathbf{u}) \quad (2.22)$$

that transforms the Lagrangian model of Eq. (2.23) into

$$\begin{aligned} \dot{\mathbf{q}}_1 &= \mathbf{p}_1 \\ \dot{\mathbf{q}}_2 &= \mathbf{p}_2 \\ \dot{\mathbf{p}}_1 &= \mathbf{f}(\mathbf{q}, \mathbf{p}) + \mathbf{g}(\mathbf{q})\mathbf{u} \\ \dot{\mathbf{p}}_2 &= \mathbf{u} \end{aligned} \quad (2.23)$$

where, $\mathbf{f}(\mathbf{q}, \mathbf{p}): \mathbb{R}^{2n} \rightarrow \mathbb{R}^{n_1}$, $\mathbf{g}(\mathbf{q}) \in \mathbb{R}^{n_1 \times n_2}$ are given by

$$\begin{aligned} \mathbf{f}(\mathbf{q}, \mathbf{p}) &= -m_{11}^{-1}(\mathbf{q})\mathbf{h}_1(\mathbf{q}, \mathbf{p}) \\ \mathbf{g}(\mathbf{q}) &= -m_{11}^{-1}(\mathbf{q})m_{12}(\mathbf{q}) \end{aligned} \quad (2.24)$$

The state vector can be expressed as $X = [q_1 \ p_1 \ q_2 \ p_2] \in \mathbb{R}^{2n}$. Now it is easy to understand that control law of Eq. (2.22) can only ensure partial linearization of the system. State model of Eq. (2.23) is partially linearized. The q_I and p_I subsystem represents a nonlinear subsystem as shown in Eq. (2.25), whereas Eq. (2.26) represents linear subsystem

$$\begin{aligned}\dot{q}_1 &= p_1 \\ \dot{p}_1 &= f(q, p) + g(q)u\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}\dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\tag{2.26}$$

Therefore, it is defined as partial feedback linearization. There are two PFL techniques that have been mentioned in the literature of control engineering; one is termed as collocated feedback linearization, and other one is referred as non-collocated feedback linearization.

(a) **Collocated feedback linearization**

Collocated linearization refers to a feedback law, which linearizes the non-linear equations of the UMS associated with the actuated degrees of freedom [5–7]. This method globally transforms all the UMSs model in the form of two parallel connected fully actuated system, where the new control input appears in the dynamics of both subsystems. Collocated linearization has been extensively applied to the control of the different underactuated system like Acrobot, the three-link pendulum, rotational pendulum, etc. [9].

(b) **Non-Collocated feedback linearization**

Non-collocated linearization refers to a special class of partial feedback linearization method that linearizes the system with respect to the passive degrees of freedom. Nonetheless, the non-collocated feedback linearization is not a global transformation technique like collocated feedback linearization. It is only applicable for “Strongly Inertially Coupled” underactuated mechanical system. “Strongly Inertially coupled” system is a special class of UMSs, where the number of unactuated configuration variables is less than or equal to the number of actuated configuration variables (Spong M.W.). The major advantage of the collocated and the non-collocated linearization is that they produce a structural simplification of the control problems of any UMSs. Consequently, they are always utilized as an initial simplifying step for reduction and control of the UMSs.

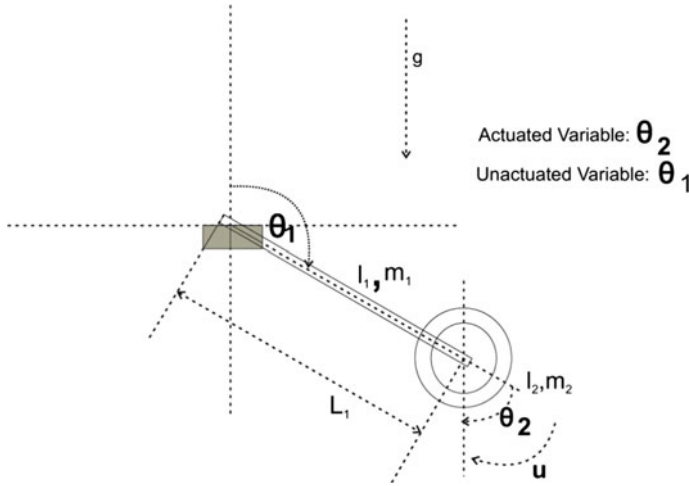


Fig. 2.2 Schematic diagram of inertia wheel pendulum

In order to explain the collocated PFL and non-collocated PFL, let us consider the Lagrangian model of inertia wheel pendulum, which is expressed in Eq. (2.27) below

$$\begin{aligned} m_{11}(q)\ddot{q}_1 + m_{12}(q)\ddot{q}_2 + h_1(q, p) &= 0 \\ m_{21}(q)\ddot{q}_1 + m_{22}(q)\ddot{q}_2 + h_2(q, p) &= \tau \end{aligned} \quad (2.27)$$

Schematic diagram of the system is shown in Fig. 2.2.

Indeed, collocated linearization is a kind of input-state feedback linearization for UMS. Now from the above Eq. (2.27), one can write

$$\ddot{q}_1 = -m_{11}^{-1}(q)(m_{12}(q)\ddot{q}_2 + h_1(q, p)) \quad (2.28)$$

Now, second actuated dynamics equation of (2.27) can be rewritten after replacing the value of $\ddot{q}_1$ by the above equation

$$(m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q))\ddot{q}_2 + (h_2(q, p) - m_{11}^{-1}(q)h_1(q, p)) = \tau \quad (2.29)$$

The above equation can be linearized by applying the following control input of Eq. (2.30)

$$\tau = (m_{22}(q) - m_{21}(q)m_{11}^{-1}(q)m_{12}(q))(u + (h_2(q, p) - m_{11}^{-1}(q)h_1(q, p))) \quad (2.30)$$

In the above Eq. (2.30) u is actually a new control input that has to be defined according to the requirement of control designer. Therefore, equation of (2.27) takes the following form:

$$\begin{aligned} m_{11}(q)\ddot{q}_1 + h_1(q, p) &= -m_{12}(q)u \\ \ddot{q}_2 &= u \end{aligned} \quad (2.31)$$

Eventually, the above Eq. (2.31) can be rewritten in a simplified form as shown below:

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= f(q, p) + g(q)u \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u \end{aligned} \quad (2.32)$$

where

$$\begin{aligned} f(q, p) &= -m_{11}^{-1}(q)h_1(q, p) \\ g(q) &= -m_{11}^{-1}(q)m_{12}(q) \end{aligned} \quad (2.33)$$

Indeed, it is one of the most common forms of feedback linearization. Please note that the collocated linearization actually linearizes the actuated degree of freedom, and decouples it from the unactuated degrees of freedom. However, non-collocated linearization is kind of input output feedback linearization. Now, if one wants to linearize the unactuated configuration variable that is q_1 , an obvious choice of control input is

$$v = m_{11}^{-1}(h_1 - m_{12}u) \quad (2.34)$$

That yields

$$\begin{aligned} \ddot{q}_1 &= v \\ \ddot{q}_2 &= -m_{11}^{-1}(q)(m_{12}(q)v - h_1(q, p)) \end{aligned} \quad (2.35)$$

Consequently, the state model for the IWP can be expressed as

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= v \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= \frac{v - f}{g} \end{aligned} \quad (2.36)$$

Therefore, it is clear from the above-stated model that non-collocated linearization yields decoupling of unactuated degrees of freedom. However, the non-collocated linearization is not widely used to treat the control problems of UMSs, while the collocated linearization is very popular to reshape the Lagrangian

model of UMSs in a partially feedback linearized form. However, feedback linearization is not enough to address the control problem of nonlinear systems. In the next section, authors will describe a few shortcomings of feedback linearization, and then they will describe a successor of feedback linearization, namely backstepping.

2.2 Control Lyapunov Function

Before describing the concept of backstepping, authors would like to discuss a few relevant topics that would give the readers a prerequisite knowledge, which will help them to comprehend the concepts of backstepping design. Best of the authors' knowledge, Lyapunov stability criterion is one the most famous and reliable stability analysis technique that has been used to analyze the stability of any autonomous dynamical systems. However, the naïve approach of Lyapunov stability analysis does not lead to any constructive approach that could be utilized to devise a control law for complicated nonlinear systems. Therefore, a new Lyapunov function approach has been conceived to assist the design procedure for nonlinear nonautonomous control systems. Let us consider a nonautonomous time invariant nonlinear system as shown in Eq. (2.37)

$$\dot{x} = f(x, u); f(0, 0) = 0 \quad (2.37)$$

It is easy to understand from that the equilibrium point of the nonlinear system mentioned in Eq. (2.37) is located at $\mathbf{X} = 0$. Nonetheless, if someone wants to devise a state feedback control law $u = \alpha(x)$ such that the origin of the system becomes a global asymptotic stable system, then a modified Lyapunov function approach may yield a conducive solution. It is not a difficult task to construct a positive definite, radially unbounded, continuous and smooth scalar function $V(\mathbf{X})$ as a Lyapunov function candidate. Moreover, an additional requirement can be placed on its time derivative along $f(x, \alpha(x))$ as shown in Eq. (2.38) below

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} f(x, \alpha(x)) < 0; \quad x \neq 0 \quad (2.38)$$

Now, a state feedback control law may be designed to achieve the aforesaid condition of Eq. (2.38). Careful observation reveals that in this context the Lyapunov function $V(\mathbf{X})$ is not used to analyze stability of the system, rather it is used to find out a conducive control law that yields the condition of Eq. (2.38). *Since here Lyapunov function is used to design a control law for the nonautonomous systems, it is known as Controlled Lyapunov Function (clf) [2].*

Let us consider the scalar system of the Eq. (2.39) below

$$\dot{x} = -\sin x - x^3 + u \quad (2.39)$$

Input-State feedback stabilization control law will yield a control law that cancel out all the nonlinear entries of the above equation

$$u = \sin x + x^3 - kx \quad (2.40)$$

Application of the control input of Eq. (2.40) makes the system's origin a global asymptotically stable equilibrium. However, such inveterate approach of feedback linearization always raises a few questions, which are very important and need to be answered before practical implementation. First, what is the essence of canceling out the term x^3 , while it can assist the stabilization process of the overall system? Second, what is the price a designer has to pay for introducing the x^3 term in feedback equation? Third, what are the alternative approaches that could be utilized to address the above-mentioned control problem?

Indeed, one can design $u = \alpha(x, u) = -kx + \sin x$ and may select a simple clf like $V(X) = \frac{1}{2}x^2$ that will yield

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) = x(-x^3 - kx) = -x^4 - kx^2 \quad (2.41)$$

which is negative definite. The same negative definiteness can also be achieved with the control law of Eq. (2.40), which will yield the following time derivative for the same clf

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) = x(-kx) = -kx^2 \quad (2.42)$$

However, let us first consider the control law $u = \alpha(x, u) = -kx + \sin x$ and corresponding time derivative of the clf as shown in Eq. (2.41). A careful observation will reveal that the control signal increases linearly with x . Furthermore, the time derivative of Lyapunov Function (as shown in 2.41) decreases at a faster rate due to the presence of $-x^4$. *Actually, the term $-x^3$ of the original system assists the stabilization process, and thereby it is termed as the beneficial nonlinearities or useful nonlinearities.* Conversely, application of the control law of Eq. (2.40) yields a different time derivative of the clf that is shown in Eq. (2.42). Needless to say that here the amplitude of control signal is quite high due to the presence of $+x^3$ term in Eq. (2.40). Moreover, in this case the rate of decay of the time derivative of the clf is very poor (as shown in Eq. 2.42). In addition, it should be noted that the presence of term $+x^3$ in the feedback path makes the control law more sensitive with respect to slight change of systems parameter.

2.3 Integrator Backstepping

In order to find out an alternative control approach to overcome the shortcomings of feedback linearization, designers have conceived the concept of backstepping control law [3]. Indeed, the simplicity for first-order design can easily be extended to find out the control law for higher order nonlinear systems. In a recursive manner, one can easily construct the clf for n dimensional systems. For the sake of understandability, the authors have augmented the system of (2.39) by a simple integrator. Second-order state model of the resultant system is shown below in Eqs. (2.43.a, 2.43.b)

$$\dot{x} = -\sin x - x^3 + z \quad (2.43.a)$$

$$\dot{z} = u \quad (2.43.b)$$

Now, if the intended control objective is to ensure the regulation of $x(t)$, that is $x(t) \rightarrow 0$, as $t \rightarrow \infty$, for all $x(0)$ and $z(0)$. The state variable $z(t)$ must remain bounded during the stabilization process. Indeed, it is clear from Eqs. (2.43.a, 2.43.b) that the equilibrium of the system is located at $(0, 0)$. Simply stated the design objective is to find out a feedback control input u for the system such that this equilibrium becomes a GAS one. The system is shown in the block diagram form in Fig. 2.3.

To construct a clf for system of Eqs. (2.43.a, 2.43.b), one can first construct a clf for its subsystem shown in the dashed box (dynamics of x). Instead of u , if z is being considered as the control input to the system of Eq. (2.43.a), then the system of Eqs. (2.43.a, 2.43.b) would become identical to that of the first-order system of Eq. (2.39). Therefore, one can easily construct a clf that is just a quadratic function of state variable x . That is $V_x = 0.5x^2$

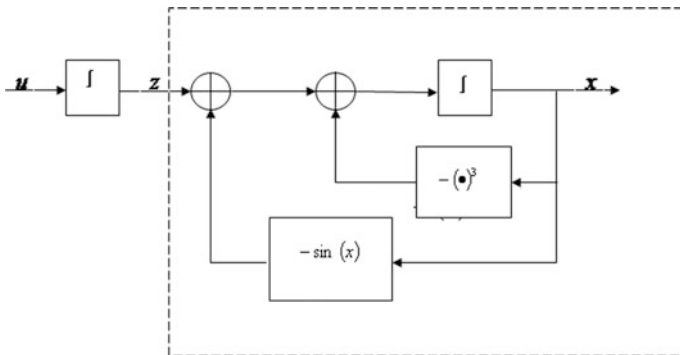


Fig. 2.3 Schematic diagram of Eqs. (2.43.a, 2.43.b)

According to the previous discussion of our last section, one can devise a feedback law $-c_1x + \sin x$ to ensure the proper regulation of the state variable x . If Eq. (2.43.a) is only considered then z appears as an input to the system as shown below

$$z = -c_1x + \sin x \quad (2.44)$$

However, one should not forget that z is not the actual control input; rather it is just an ordinary state variable. Simply stated it is not possible for the designers to shape the signal z according to their own requirements. One can only define the ideal value of the z as shown below in Eq. (2.45)

$$z_{\text{des}} = -c_1x + \sin x \cong \alpha_s(x) \quad (2.45)$$

Indeed, z is not the actual control input to the system; it only plays the role of a *virtual input* to the system of Eq. (2.43.a). The desired value of virtual control is defined as the stabilization function for the system of Eq. (2.43.a) [4]. Error can be defined as the difference between the virtual control and stabilization function as shown below

$$e = z - z_{\text{des}} = z - \alpha_s(x) = z + c_1x - \sin x \quad (2.46)$$

Now, one can rewrite the system Eqs. (2.43.a, 2.43.b) in (x, e) coordinate that will make the controller design task quite easy as illustrated in Fig. 2.4.

$$\dot{x} = -\sin x - x^3 + (z + c_1x - \sin x) - c_1x + \sin x = -c_1x - x^3 + e \quad (2.47.a)$$

$$\dot{e} = \dot{z} - \dot{\alpha}_s(x) = \dot{z} + (c_1 - \cos x)\dot{x} = u + (c_1 - \cos x)(-c_1x - x^3 + e) \quad (2.47.b)$$

The *stabilizing function* $\alpha_s(x)$ can be added up and accordingly subtracted from the $\dot{x}_1$ equation as shown in Fig. 2.4.

Then the signal $\alpha_s(x)$ is being used as a feedback control inside the dashed box and “backstep” $-\alpha_s(x)$ through the integrator as shown in Fig. 2.5.

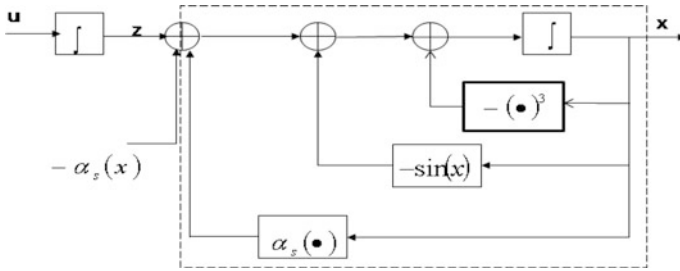


Fig. 2.4 Addition and subtraction of same stabilizing function in $\dot{x}$ equation

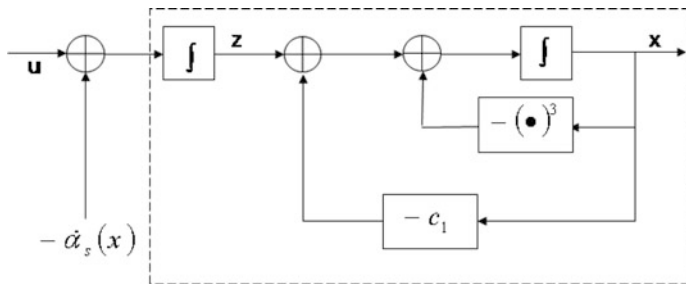


Fig. 2.5 Backstepping signal $\dot{\alpha}_s(x)$

One of the key features of backstepping is that it does not require any differentiator to realize the time derivative of $\dot{\alpha}_s(x)$ [3, 4]. Since it is a known function, it is always possible to compute its derivative analytically as shown in Eq. (2.48) below

$$\dot{\alpha}_s(x) = \frac{\partial \alpha_s}{\partial x} \dot{x} = -(c_1 - \cos x)(-c_1 x - x^3 + e) \quad (2.48)$$

A clf for the overall nonlinear system [as shown in Eqs. (2.43.a, 2.43.b)] must be constructed to design the control input for the system of Eqs. (2.43.a, 2.43.b). One obvious way of constructing the clf for the entire system is to augment a quadratic term of error variable e with it as shown in Eq. (2.49) below

$$V_a(x, z) = V(x) + \frac{1}{2}e^2 = \frac{1}{2}x^2 + \frac{1}{2}(z + c_1 x - \sin x)^2 \quad (2.49)$$

Differentiation of V_a with respect to time yields

$$\begin{aligned} \dot{V}_a(x, e, u) &= x(-c_1 x - x^3 + e) + e(u + (c_1 - \cos x)(-c_1 x - x^3 + e)) \\ &= -c_1 x^2 - x^4 + e(x + u + (c_1 - \cos x)(-c_1 x - x^3 + e)) \end{aligned} \quad (2.50)$$

Assuming that $\dot{V}_a$ be an analytic function, the design goal is to construct a control input in such a way that it could ensure the negative definiteness of $\dot{V}_a$. In order to achieve the control objective, a feedback control law can be designed to cancel out the cross-term xe , and the undesirable nonlinear terms. This is possible because u is multiplied by e due to chosen form of augmented Lyapunov function V_a . This is another salient feature of the backstepping. Hence, control input u can be chosen to make $\dot{V}_a$ negative definite in x and e . One obvious design choice is to make the bracketed term of the last Eq. (2.50) equal to $-c_2 e$, where $c_2 > 0$

$$\begin{aligned} u &= -c_2 e - x - (c_1 - \cos x)(-c_1 x - x^3 + e) \\ &= -c_2(z + c_1 x - \sin x) - x - (c_1 - \sin x)(z - \sin x - x^3) \end{aligned} \quad (2.51)$$

With this control input the derivative of the clf becomes

$$\dot{V}_a = -c_1x^2 - x^4 - c_2e^2 \quad (2.52)$$

Now the above definition of $\dot{V}_a$ proves that in (x, e) coordinate system the equilibrium point $(0, 0)$ is globally asymptotically stable. Now, $e = 0$ implies that $z = z_{\text{des}}$ that is virtual control becomes equal to stabilizing function, and it will ensure asymptotic stabilization of x . A careful observation of Eq. (2.45) reveals that $x = 0$ implies $z = 0$. Hence, the equilibrium point of the original system in (x, z) coordinate system $(0, 0)$ is also globally asymptotically stable. The resulting closed loop system in (x, e) coordinate is

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} -c_1 - x^2 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (2.53)$$

In the above equation, we are representing a nonlinear system in linear-like form. An important structural property of this system is that it is nonlinear “system matrix.” This is another noteworthy feature of backstepping, it only cancels out undesirable nonlinear entries, whereas it does not to pay any extra effort to eliminate helpful nonlinearities from the system equation. However, another flexibility of the design is explained in the following example and consider the following system (2.54)

$$\begin{aligned} \dot{x} &= x^2 + xz \\ \dot{z} &= u \end{aligned} \quad (2.54)$$

Careful observation reveals that the system is uncontrollable at origin. Needless to say that feedback linearization fails to answer the control problems of the uncontrollable systems. However, backstepping can yield a fruitful solution to the above-mentioned problem. Now, if the equation of $\dot{x}$ is considered, then it turns out to be $f(x) = x^2$ and $g(x) = x$. Hence, clf can be constructed as $V(x) = \frac{1}{2}x^2$. Therefore, stabilizing function α_s can be designed to ensure the stabilization

$$\alpha_s(x) = -c_1x^2 - x \quad (2.55)$$

Indeed, like the previous example here also z is playing the role of virtual control. Similar to the previous case error function can be defined as shown below

$$e = z - \alpha_s = z + c_1x^2 + x \quad (2.56)$$

Similar to the previous case the system state model is again to be constructed in (x, e) coordinate

$$\begin{aligned}\dot{x} &= -c_1 x^3 + ex \\ \dot{e} &= u + (1 + 2c_1 x)(x^2 + xz)\end{aligned}\quad (2.57)$$

Now, the derivative of the augmented Lyapunov function $V_a = \frac{1}{2}x^2 + \frac{1}{2}e^2$ is shown below

$$\dot{V}_a = -c_1 x^4 + e(u + (1 + 2c_1 x)(x^2 + xz) + x^2) \quad (2.58)$$

The control law which makes the derivative of V_a negative definite is given by

$$u = -c_2 e - (1 + 2c_1 x)(x^2 + xz) - x^2 \quad (2.59)$$

The resulting system in the (x, z) coordinate is

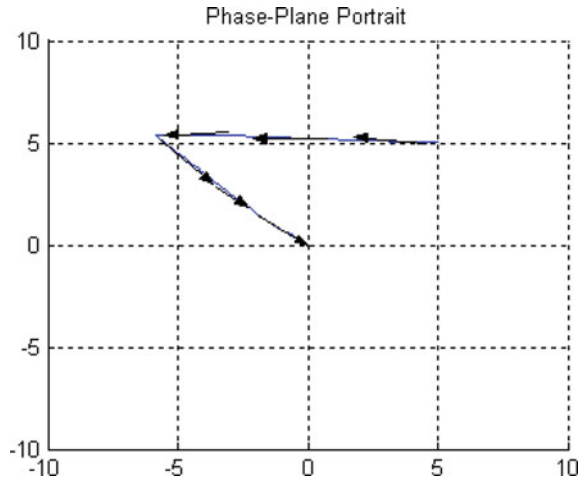
$$\dot{x} = x^2 + xz \quad (2.60)$$

$$\dot{z} = -c_2 z - c_2 x - (c_1 c_2 + 2)x^2 - zx - 2c_1 x^2 z - 2c_1 x^3 \quad (2.61)$$

And its equilibrium $(0, 0)$ is global asymptotic stable (Fig. 2.6).

It is clear from the Fig. 2.6 that an unstable and uncontrollable system can be stabilized using integrator backstepping. However, only one shortcoming of the backstepping control is that it requires the system to be in strict feedback system. However, if the system is not in strict feedback form, block backstepping can yield a plausible solution to such control problem [4]. Concept of block backstepping will be presented in the next section.

Fig. 2.6 Stabilization of an unstable system with integrator backstepping control design



2.4 Block Backstepping Design

Before proceeding to the actual concept of block backstepping, let us consider the state model of inertia wheel pendulum as shown in Fig. 2.2. As discussed above, collocated feedback linearization yields the following state model as shown in Eq. (2.62):

$$\begin{aligned}\dot{q}_1 &= p_1 \\ \dot{p}_1 &= C \sin q_1 - Du \\ \dot{q}_2 &= p_2 \\ \dot{p}_2 &= u\end{aligned}\tag{2.62}$$

where C and D are just two design constants (detailed description of C and D as well as the derivation of the state model are available in Appendix A.1). It is very easy to understand that the above-stated model is not in strict feedback form, and thereby restricts the application of conventional integrator backstepping. Therefore, one obvious choice left for the designers is to convert the state model into a form that is suitable for application of backstepping algorithm. Therefore, one can rely on algebraic state transformation as shown below

$$z_1 = q_2 - k(q_1 + p_1 + Dp_2)\tag{2.63}$$

Now the time derivative of z_1 yields

$$\dot{z}_1 = p_2 - k(p_1 + C \sin q_1)\tag{2.64}$$

However, the stabilizing function for Eq. (2.63) can be defined according to the following Eq. (2.65):

$$\alpha = -c_1 z_1 + k(p_1 + C \sin q_1)\tag{2.65}$$

Now, second error variable z_2 can be defined as shown in Eq. (2.66) below

$$z_2 = p_2 - \alpha\tag{2.66}$$

Above definition of the z_2 yields

$$\dot{z}_1 = z_2 - c_1 z_1\tag{2.67}$$

Differentiation of z_2 with respect to time

$$\dot{z}_2 = u - c_1(-c_1 z_1 + z_2) + k(2C \sin q_1 - Du)\tag{2.68}$$

Now, desired dynamics of z_2 is

$$\dot{z}_2 = -z_1 - c_2 z_2 \quad (2.69)$$

Therefore, one can select the following control input to ensure the desired dynamics for z_2 as shown below

$$u = \frac{1}{1 - kD} (-c_1^2 z_1 - c_1 z_2 - 2kC \sin q_1) \quad (2.70)$$

The control law u transforms the system into a block-strict feedback system

$$\begin{aligned} \dot{z}_1 &= -c_1 z_1 + z_2 \\ \dot{z}_2 &= -z_1 - c_2 z_2 \end{aligned} \quad (2.71)$$

However, it is not possible to reduce the order of the actual system. Therefore, it is easy to understand that the two states of the system comprise internal dynamics that has become unobservable from input output relationship of the system, where u is being considered as the input to the system and z is considered to be the output from the system. Therefore, two states of the original system can be chosen to represent the internal dynamics of the transformed system. Nevertheless, in order to derive the zeroing input for the system, let us start the derivation considering z_1 as the output from the system. Now the first-order time derivative of z_1 is shown below

$$\dot{z}_1 = p_2 - k(p_1 + C \sin q_1) \quad (2.72)$$

Subsequent differentiation of z_1 with respect to time yields

$$\ddot{z}_1 = u(1 + kD) - k(C \sin q_1 + C \cos q_1 p_1) \quad (2.73)$$

Hence, $\ddot{z}_1 = 0$ yields

$$u = \frac{k}{(1 + kD)} (C \sin q_1 + C \cos q_1 p_1) \quad (2.74)$$

That is zeroing input to the system, so when zeroing input to the system is applied to the system then the zero dynamics of the transformation can be expressed as shown below

$$\begin{aligned} \dot{q}_1 &= p_1 \\ \dot{p}_1 &= C \sin q_1 - \frac{kD}{(1 + kD)} (C \sin q_1 + C \cos q_1 p_1) \end{aligned} \quad (2.75)$$

Now, k should be selected in a judicial manner to ensure the stability of the entire nonlinear system. However, detailed discussions of block backstepping is beyond the scope of this book; for a comprehensive treatment on the said topic, readers may refer the book *Nonlinear and Adaptive Control Design* by Krstic et al.

2.5 Notes

Last few decades, the literature of nonlinear control engineering has been enriched by the potential contributions from applied mathematics and practicing engineers. However, it is impossible to discuss about all the potential design approaches of nonlinear control engineering in a single volume, as well as it is beyond the scope of this presentation. However, for detailed in-depth discussions on the said topic researchers may go through the references.

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