

Chapter 2

ℓ_1 -Induced Controller Design for Positive Systems

In this chapter, the problem of ℓ_1 -induced state-feedback controller design is investigated for discrete-time positive systems based on linear Lyapunov functions. For a given positive linear system, the objective is to design a state-feedback controller such that the closed-loop system is positive, asymptotically stable and satisfies a prescribed ℓ_1 -induced performance. In detail, the exact value of ℓ_1 -induced norm is firstly computed with the use of an analytical method. A novel stability and ℓ_1 -induced performance characterization is then proposed. Based on such a characterization, necessary and sufficient conditions are derived for the existence of desired controllers. To solve the conditions, a corresponding iterative convex optimization algorithm is developed. Secondly, the controller synthesis problem for single-input multiple-output (SIMO) positive systems is investigated. Specifically speaking, an analytical method is established to obtain the optimal ℓ_1 -induced controller for the special case. Moreover, some links to the spectral radius of the closed-loop systems are provided. Finally, the effectiveness of the obtained theoretical results in this chapter are illustrated through a numerical example.

The organization of this chapter is as follows. Section 2.1 formulates the ℓ_1 -induced state-feedback controller synthesis problem for positive systems. In Sect. 2.2, a method to compute the exact value of ℓ_1 -induced norm is first proposed. An equivalent characterization is then developed under which the positive linear system is asymptotically stable and satisfies the ℓ_1 -induced performance. In Sect. 2.3, the controller design method for positive systems is put forward and an iterative algorithm is developed to solve the derived conditions. A novel condition for the existence of the state-feedback controller for SIMO positive systems is obtained in Sect. 2.4. Section 2.5 presents an illustrative example. A summary is given in Sect. 2.6.

2.1 Problem Formulation and Preliminaries

Consider a discrete-time positive linear system:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_w w(k), \\ y(k) = Cx(k) + Du(k) + D_w w(k), \end{cases} \quad (2.1)$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^l$, $w(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^r$ are the system state, control input, disturbance input and output, respectively; $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $B_w \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{r \times n}$, $D \in \mathbb{R}^{r \times l}$ and $D_w \in \mathbb{R}^{r \times m}$ are nonnegative.

In this chapter, the state-feedback controller will be designed in the form of

$$u(k) = Kx(k). \quad (2.2)$$

Then, the following closed-loop system is obtained:

$$\begin{cases} x(k+1) = (A + BK)x(k) + B_w w(k), \\ y(k) = (C + DK)x(k) + D_w w(k). \end{cases} \quad (2.3)$$

For the system in (2.3), we say that it has ℓ_1 -induced performance at the level γ if, under zero initial conditions, for all nonzero $w \in \ell_1$,

$$\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}, \quad (2.4)$$

or equivalently,

$$\sum_{k=0}^{\infty} \|y(k)\|_1 < \gamma \sum_{k=0}^{\infty} \|w(k)\|_1, \quad (2.5)$$

where $\gamma > 0$ is a given performance level.

In this investigation, apart from the ℓ_1 -induced performance in (2.4), positivity is required to be preserved. Then, the problem to be addressed in this chapter is described as follows.

Problem PPL1CD (Positivity-Preserving ℓ_1 -Induced Controller Design): Given the positive system (2.1), the control objective is to find a controller $u(k) = Kx(k)$ such that the closed-loop system (2.3) is positive, asymptotically stable, and satisfies the ℓ_1 -induced performance in (2.4) under zero initial conditions.

2.2 Performance Analysis

Consider a discrete-time positive linear system:

$$\begin{cases} x(k+1) = Ax(k) + B_w w(k), \quad x(0) = x_0, \\ y(k) = Cx(k) + D_w w(k), \end{cases} \quad (2.6)$$

where $x(k) \in \mathbb{R}^n$, $w(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^r$ are the system state, input, and output, respectively; A , B_w , C and D_w are nonnegative matrices. In this section, we first propose a method to compute the exact value of ℓ_1 -induced norm for system (2.6). Then a characterization is proposed under which the positive linear system is asymptotically stable and satisfies the performance in (2.4).

2.2.1 Exact Computation of ℓ_1 -Induced Norm

In the following theorem, we establish an analytical characterization through which the value of ℓ_1 -induced norm of system (2.6) is computed directly.

Theorem 2.1 *For an asymptotically stable positive linear system given in (2.6), denote the system operator $G : \ell_1 \rightarrow \ell_1$ for input w to output y . Then, the exact value of the ℓ_1 -induced norm is given by*

$$\|G\|_{(\ell_1, \ell_1)} = \|C(I - A)^{-1}B_w + D_w\|_1. \quad (2.7)$$

Proof From system (2.6), we know that

$$\begin{aligned} y(0) &= Cx(0) + D_w w(0), \\ y(1) &= CAx(0) + CB_w w(0) + D_w w(1), \\ y(2) &= CA^2x(0) + CB_w w(1) + CAB_w w(0) + D_w w(2), \\ &\vdots \\ y(k) &= CA^k x(0) + C \sum_{m=1}^k A^{k-m} B_w w(m-1) + D_w w(k). \end{aligned}$$

Under the assumption that $x(0) = 0$, we have

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(s) \end{bmatrix} = Q \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ \vdots \\ w(s) \end{bmatrix}, \quad (2.8)$$

where

$$\begin{aligned} Q &= \begin{bmatrix} D_w & 0 & \cdots & 0 \\ CB_w & D_w & & \\ CAB_w & CB_w D_w & \ddots & \\ \vdots & & & \\ CA^{s-1}B_w & \cdots & CAB_w CB_w D_w \end{bmatrix} \\ &= [q_{ij}] \in \mathbb{R}^{(s+1)r \times (s+1)m}. \end{aligned}$$

Taking the 1-norm of both sides of (2.8) yields $\sum_{k=0}^s \|y(k)\|_1 \leq \|Q\|_1 \sum_{k=0}^s \|w(k)\|_1$ with

$$\|Q\|_1 = \max_j \left(\sum_{i=1}^{(s+1)r} q_{ij} \right),$$

that is,

$$\|Q\|_1 = \|CB_w + CAB_w + \cdots + CA^{s-1}B_w + D_w\|_1.$$

Since A is asymptotically stable, as $s \rightarrow \infty$, $\|Q\|_1 \rightarrow \|C(I - A)^{-1}B_w + D_w\|_1$, which leads to $\|y\|_{\ell_1} \leq \|C(I - A)^{-1}B_w + D_w\|_1 \|w\|_{\ell_1}$.

In the following, we investigate the condition under which $\frac{\|y\|_{\ell_1}}{\|w\|_{\ell_1}}$ reaches its supreme value. Suppose all the components of vector $w(k)$ are equal to zero except at time $k = 0$. We have

$$\begin{aligned} y(0) &= D_w w(0), \\ y(1) &= CB_w w(0), \\ y(2) &= CAB_w w(0), \\ &\vdots \\ y(s) &= CA^{s-1}B_w w(0), \end{aligned} \tag{2.9}$$

which yields $\sum_{k=0}^s \|y(k)\|_1 = \mathbf{1}^T (CB_w + CAB_w + \cdots + CA^{s-1}B_w + D_w)w(0)$.

Since A is asymptotically stable, as $s \rightarrow \infty$,

$$\|y\|_{\ell_1} = \mathbf{1}^T (C(I - A)^{-1}B_w + D_w)w(0) \tag{2.10}$$

$$\leq \|C(I - A)^{-1}B_w + D_w\|_1 \|w\|_{\ell_1}. \tag{2.11}$$

Now denote $\bar{Q} = [\bar{q}_j] \triangleq \mathbf{1}^T (C(I - A)^{-1}B_w + D_w)$. Without loss of generality, we assume that the norm $\|\bar{Q}\|_1 = \max_j (\bar{q}_j)$ achieves its supreme value at the j_{th}^* column and all the components of vector $w(0)$ are equal to zero except at the j_{th}^* row, we have that $\frac{\|y\|_{\ell_1}}{\|w\|_{\ell_1}}$ achieves its supremal value $\|C(I - A)^{-1}B_w + D_w\|_1$. $\square$

2.2.2 Novel ℓ_1 -Induced Performance Characterization

In this subsection, our objective is to develop a novel characterization under which system (2.6) is asymptotically stable and satisfies the performance in (2.4).

When the input w is taken into account, we can derive the following result which provides a fundamental characterization on the stability of system (2.6) with the performance in (2.4).

Theorem 2.2 *The positive linear system in (2.6) is asymptotically stable and satisfies $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$ if and only if there exists a vector $p \succeq 0$ satisfying*

$$\mathbf{1}^T C + p^T A - p^T \ll 0, \quad (2.12)$$

$$p^T B + \mathbf{1}^T D - \gamma \mathbf{1}^T \ll 0, \quad (2.13)$$

Proof Sufficiency: First, we assume that there exists an integer k such that $x(k) \neq 0$. From (2.12), we can see that $p^T A - p^T \ll 0$ holds, and thus the asymptotic stability of system (2.6) is proved.

Consider the linear Lyapunov function candidate $V(x) = p^T x$, computing the Lyapunov difference yields

$$\Delta V(k) = p^T (Ax(k) + Bw(k)) - p^T x(k).$$

Let

$$\begin{aligned} J &= \|y(k)\|_1 - \gamma \|w(k)\|_1 \\ &= \sum_{i=1}^r y_i(k) - \gamma \sum_{i=1}^m w_i(k) \\ &= \left[\sum_{i=1}^r y_i(k) - \gamma \sum_{i=1}^m w_i(k) + \Delta V(k) \right] - \Delta V(k) \\ &= [\mathbf{1}^T Cx(k) + \mathbf{1}^T Dw(k) - \gamma \mathbf{1}^T w(k) \\ &\quad + p^T (Ax(k) + Bw(k)) - p^T x(k)] - \Delta V(k) \\ &= [\mathbf{1}^T C + p^T (A - I)]x(k) \\ &\quad + (\mathbf{1}^T D + p^T B - \gamma \mathbf{1}^T)w(k) - \Delta V(k) \\ &= [\mathbf{1}^T C + p^T (A - I) + \varepsilon \mathbf{1}^T]x(k) + (\mathbf{1}^T D \\ &\quad + p^T B - \gamma \mathbf{1}^T)w(k) - \varepsilon \mathbf{1}^T x(k) - \Delta V(k), \end{aligned} \quad (2.14)$$

where $\varepsilon > 0$ is sufficiently small.

From (2.12) and (2.13), we have

$$\sum_{k=0}^s \sum_{i=1}^r y_i(k) + \varepsilon \sum_{k=0}^s \sum_{i=1}^n x_i(k) < \gamma \sum_{k=0}^s \sum_{i=1}^m w_i(k) - V(s+1). \quad (2.15)$$

Since the system is asymptotically stable, we have

$$\sum_{k=0}^{\infty} \sum_{i=1}^r y_i(k) + \varepsilon \sum_{k=0}^{\infty} \sum_{i=1}^n x_i(k) \leq \gamma \sum_{k=0}^{\infty} \sum_{i=1}^m w_i(k), \quad (2.16)$$

which implies

$$\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}. \quad (2.17)$$

Next we consider the case with $x(k) = 0$. From (2.12), the asymptotic stability of system (2.6) is proved. It is easy to see that if $x(k) = 0$, we have $y(k) = Dw(k)$ and from (2.13), $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$ holds. This proves sufficiency.

Necessity: Assume that system (2.6) is asymptotically stable and satisfies the performance $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$. Now it follows that (2.16) holds, that is, under zero initial conditions,

$$\sum_{k=0}^{\infty} \left(\sum_{i=1}^r y_i(k) - \gamma \sum_{i=1}^m w_i(k) \right) < 0, \quad (2.18)$$

which is equal to

$$(\mathbf{1}^T D + \mathbf{1}^T C(I - A)^{-1} B - \gamma \mathbf{1}^T) \sum_{k=0}^{\infty} w(k) < 0. \quad (2.19)$$

As $0 \neq w \in \ell_1$ is arbitrary, inequality (2.19) implies

$$\mathbf{1}^T D + \mathbf{1}^T C(I - A)^{-1} B - \gamma \mathbf{1}^T \ll 0. \quad (2.20)$$

Define $\tilde{p} \triangleq (\mathbf{1}^T C(I - A)^{-1})^T \geq 0$ and $p \triangleq \tilde{p} + \epsilon \alpha \gg 0$, where $\alpha \gg 0$ satisfies $\alpha^T(I - A) \gg 0$, and $\epsilon > 0$ is sufficiently small. We have

$$\begin{aligned} \mathbf{1}^T C + p^T A - p^T &= \mathbf{1}^T C - (\tilde{p}^T + \epsilon \alpha^T)(I - A) \\ &= \mathbf{1}^T C - \mathbf{1}^T C - \epsilon \alpha^T(I - A) \\ &= -\epsilon \alpha^T(I - A) \\ &\ll 0. \end{aligned} \quad (2.21)$$

On the other hand,

$$\mathbf{1}^T D + p^T B - \gamma \mathbf{1}^T = \mathbf{1}^T D + \mathbf{1}^T C(I - A)^{-1} B + \epsilon \alpha^T B - \gamma \mathbf{1}^T. \quad (2.22)$$

From (2.20) and ϵ is sufficiently small, we have that (2.13) holds. $\square$

Remark 2.1 The condition obtained in Theorem 2.2 is necessary and sufficient in terms of linear programming which can be verified and solved efficiently. Hence, we can easily obtain γ and feasible solution p with the help of convex optimization techniques.

Remark 2.2 The result developed in Theorem 2.2 has also appeared in [1] and been proven by virtue of dissipativity theory. Theorem 2.2 gives an alternative proof and the result was obtained independently of the paper [1].

2.3 Controller Synthesis and Algorithm

In this section, the synthesis results of state-feedback controllers for positive system is derived with linear Lyapunov functions. Based on the analysis in Sect. 2.2, a necessary and sufficient condition for the existence of a solution to Problem PPL1CD is obtained. Then, an iterative convex optimization approach is developed to compute the controller matrices accordingly.

Corollary 2.1 *The closed-loop system (2.3) is positive, asymptotically stable and satisfies $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$ if and only if there exist a matrix K and a vector $p \gg 0$ satisfying*

$$A + BK \geq 0, \quad (2.23)$$

$$C + DK \geq 0, \quad (2.24)$$

$$\mathbf{1}^T(C + DK) + p^T(A + BK) - p^T \ll 0, \quad (2.25)$$

$$p^T B_w + \mathbf{1}^T D_w - \gamma \mathbf{1}^T \ll 0. \quad (2.26)$$

Remark 2.3 Note that conditions (2.25) and (2.26) are LMIs and have the polynomial time complexity [2]. However, in (2.25), the Lyapunov vector p is coupled with the controller matrix K . It should be noted that such a problem is generally a bilinear matrix inequality (BMI) problem and known to be NP-hard, which means that it is rather unlikely to find a polynomial time algorithm for solving general BMI problems [3].

In the following, our aim is to derive a numerically tractable mean to synthesize the required controllers with the help of convex optimization. It is noted that when matrix K is fixed, (2.25) turns out to be linear with respect to the other variables. Therefore, a natural way is to fix K , and solve (2.23)–(2.26) by linear programming. Thus, the following iterative algorithm can be proposed to solve the problem (see [4]).

Algorithm PPL1CD

- Step 1. Set $i = 1$. Select an initial matrix K_1 such that system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_w w(k), \\ y(k) = Cx(k) + Du(k) + D_w w(k), \end{cases}$$

with

$$u(k) = K_1 x(k) \quad (2.27)$$

is asymptotically stable and $A + BK_1 \geq 0$ and $C + DK_1 \geq 0$.

- Step 2. For fixed K_i , solve the following optimization problem for p_i and γ_i .
OP2: Minimize γ_i subject to the following constraints:

$$\begin{aligned} \mathbf{1}^T(C + DK) + p_i^T(A + BK_i) - p_i^T &\ll 0, \\ p_i^T B_w + \mathbf{1}^T D_w - \gamma_i \mathbf{1}^T &\ll 0, \\ p_i &\geq 0. \end{aligned}$$

Denote γ_i^* , p_i as the solution to the optimization problem. If $|(\gamma_i^* - \gamma_{i-1}^*) / \gamma_i^*| \leq \varepsilon_1$, where ε_1 is a prescribed bound, then $K = K_i$, $p = p_i$. STOP.

- Step 3. For fixed p_i , solve the following optimization problem for K_i .
OP2: Minimize γ_i subject to the following constraints:

$$\begin{aligned} A + BK_i &\geq 0, \\ C + DK_i &\geq 0, \\ \mathbf{1}^T(C + DK) + p_i^T(A + BK_i) - p_i^T &\ll 0. \end{aligned}$$

Denote γ_i^* as the solution to the optimization problem.

- Step 4. If $|(\gamma_i^* - \gamma_{i-1}^*) / \gamma_i^*| \leq \varepsilon_2$, where ε_2 is a prescribed tolerance, STOP; else set $i = i + 1$ and $K_i = K_{i-1}$, then go to Step 2.

Remark 2.4 The selection of K_1 in Step 1 can be made easily. In fact, from [5], we know that system (2.1) with (2.27) is positive and asymptotically stable if and only if there exist a diagonal matrix $P \triangleq \text{diag}(p_1, p_2, \dots, p_n)$ and $Q \triangleq [q_{ij}] \in \mathbb{R}^{l \times n}$ such that

$$\begin{bmatrix} -P & AP + BQ \\ * & -P \end{bmatrix} < 0. \quad (2.28)$$

$$a_{ij}p_j + \sum_{z=1}^l b_{iz}q_{zj} \geq 0. \quad (2.29)$$

Under this condition, an initial choice of K can be given by $K_1 = QP^{-1}$.

Remark 2.5 The parameter γ can be optimized iteratively. Notice that $\gamma_{i+1}^* \leq \gamma_i^*$ since the corresponding parameters obtained in Step 3 will be utilized as the initial values in Step 2 to derive a smaller γ . Therefore, the convergence of the iterative process is naturally guaranteed. Moreover, it follows from Step 1 that if one cannot find such a matrix K_1 , then it can be concluded immediately that there does not exist a solution to Problem PPL1CD.

Remark 2.6 Note that problem in Step 1 is an LMI problem, which has a polynomial time complexity [6]. For LMI problems, the number $\mathcal{N}(\varepsilon)$ of flops needed to compute an ε -accurate solution is bounded by $O(\mathcal{M}\mathcal{N}^3 \log(\mathcal{V}/\varepsilon))$, where $\mathcal{M}$ is the total row size of the LMI system, $\mathcal{N}$ is the total number of scalar decision variables, $\mathcal{V}$ is a data-dependent scaling factor, and ε is the relative accuracy set for the algorithm. Every problem in Steps 2–3 is an LP problem and it is less computational demanding than LMI problems.

2.4 Analytical Method for Special Case

This section is devoted to the synthesis of the state-feedback controller for SIMO systems. It is noted that in the general case, the result in Corollary 2.1 is based on the introduction of a nonnegative vector p . Different from the synthesis in Sect. 2.3, a novel design approach is given, and some links to the convergence rate of the closed-loop systems are provided. Moreover, we extend the results to the multiple-input multiple-output (MIMO) systems by deriving the minimal K under a structural constraint, which may provide an upper bound of the ℓ_1 -induced performance when the closed-loop system is stable.

First, we give the following result which is valid for general MIMO systems.

Theorem 2.3 *The closed-loop system (2.3) is positive, asymptotically stable and satisfies $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$ if and only if there exists a matrix K satisfying*

$$A + BK \geq 0, \quad (2.30)$$

$$C + DK \geq 0, \quad (2.31)$$

$$\rho(A + BK) < 1, \quad (2.32)$$

$$\|(C + DK)(I - (A + BK))^{-1}B_w + D_w\|_1 < \gamma. \quad (2.33)$$

Proof From (2.30)–(2.32), it is easy to see that closed-loop system (2.3) is positive and asymptotically stable.

According to Theorem 2.1, the ℓ_1 -induced norm of system (2.3) is

$$\|\tilde{G}\|_{\ell_1 \rightarrow \ell_1} = \|(C + DK)(I - (A + BK))^{-1}B_w + D_w\|_1. \quad (2.34)$$

From (2.33), we have $\|y\|_{\ell_1} < \gamma \|w\|_{\ell_1}$ holds.

The necessity can be obtained by reversing the above procedure and the details are omitted here. $\square$

In the following, we present an analytic solution to the controller design problem.

When (2.32) holds, we have that

$$(I - (A + BK))^{-1} = \sum_{i=0}^{\infty} (A + BK)^i. \quad (2.35)$$

If the elements of $A + BK \geq 0$ are sufficiently small, (2.32) is more likely to be satisfied and from (2.34), the ℓ_1 -induced norm of system (2.3) will also be small when all the elements of $A + BK \geq 0$ and $C + DK \geq 0$ are sufficiently small. For SIMO systems, the problem can be analytically solved.

For ease of reference when addressing SIMO systems, we have B and D as vectors and write

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad D = d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_r \end{bmatrix},$$

where $b \in \bar{\mathbb{R}}_+^n$ and $d \in \bar{\mathbb{R}}_+^r$. In this case, K is a row vector given by

$$K = [k_1 \ k_2 \ \dots \ k_n].$$

We will show in the following theorem that for SIMO systems, the optimal K can be computed analytically. Moreover, if such an optimal K gives an asymptotically stable closed-loop system, it achieves the minimal performance level γ^* and the closed-loop system has the fastest convergence rate.

Theorem 2.4 *For an SIMO system with $b \neq 0$, the optimal K such that the elements of $A + bK \geq 0$ and $C + dK \geq 0$ are jointly minimized is given by*

$$K^* = [k_1^* \ k_2^* \ \dots \ k_n^*] \quad (2.36)$$

where

$$k_j^* = - \min_{\substack{i=1,2,\dots,n \\ h=1,2,\dots,r}} \left\{ \frac{a_{ij}}{b_i}, \frac{c_{hj}}{d_h} \right\} \quad (2.37)$$

with the corresponding term inside $\{\cdot\}$ disappears if $b_i = 0$ or $d_h = 0$. Moreover, if $\rho(A + bK^*) < 1$, then $K^* = \arg \min \|(C + dK)(I - (A + bK))^{-1}B_w + D_w\|_1$ subject to $A + bK \geq 0$ and $C + dK \geq 0$ and the closed-loop system has the fastest convergence rate.

Proof Observe that the constraints $A + bK \geq 0$ and $C + dK \geq 0$ on k_j come from the j th column of the two matrices only. With $A = [a_{ij}] \geq 0$ and $C = [c_{hj}] \geq 0$, these constraints on k_j can be written explicitly as

$$\begin{aligned} a_{ij} + b_i k_j &\geq 0, & i = 1, 2, \dots, n, \\ c_{hj} + d_h k_j &\geq 0, & h = 1, 2, \dots, r. \end{aligned}$$

It is easy to see that if $b_i = 0$ or $d_h = 0$, the corresponding constraint is slack. In fact, the remaining constraints become

$$k_j \geq \max_{\substack{i=1,2,\dots,n \\ h=1,2,\dots,r}} \left\{ -\frac{a_{ij}}{b_i}, -\frac{c_{hj}}{d_h} \right\} = - \min_{\substack{i=1,2,\dots,n \\ h=1,2,\dots,r}} \left\{ \frac{a_{ij}}{b_i}, \frac{c_{hj}}{d_h} \right\}. \quad (2.38)$$

Thus, to minimize the elements of $A + bK$ and $C + dK$, the choice of k_j should be the minimum value subject to constraints in (2.38) which is given by k_j^* in (2.37). Hence, the first part of the result has been established.

When $\rho(A + bK^*) < 1$, the system is asymptotically stable and

$$(I - (A + bK^*))^{-1} = \sum_{i=0}^{\infty} (A + bK^*)^i.$$

For any $K \geq K^*$ satisfying $\rho(A + bK) < 1$, we have

$$(C + dK)(I - (A + bK))^{-1}B_w + D_w = \sum_{i=0}^{\infty} (C + dK)(A + bK)^i B_w + D_w.$$

Since $0 \leq A + bK^* \leq A + bK$ and $0 \leq C + dK^* \leq C + dK$, we have, from Lemma 1.6, that

$$0 \leq (C + dK^*)(A + dK^*)^i \leq (C + dK)(A + dK)^i$$

and consequently,

$$\begin{aligned} 0 &\leq (C + dK^*)(I - (A + bK^*))^{-1}B_w + D_w \\ &\leq (C + dK)(I - (A + bK))^{-1}B_w + D_w. \end{aligned}$$

The 1-norm of these matrices thus gives

$$\begin{aligned} &\|(C + dK^*)(I - (A + bK^*))^{-1}B_w + D_w\|_1 \\ &\leq \|(C + dK)(I - (A + bK))^{-1}B_w + D_w\|_1. \end{aligned}$$

As K^* gives the smallest K elementwise satisfying the constraints $A + bK \geq 0$ and $C + dK \geq 0$, the value of the ℓ_1 -induced performance corresponds to its minimal value.

Finally, from Lemma 1.3, we have $\rho(A + bK^*) \leq \rho(A + bK)$ for any $A + bK \geq 0$. The convergence rate is characterized by the spectral radius. Hence, the closed-loop system with K^* has the fastest convergence rate. The proof is completed. $\square$

Remark 2.7 For SIMO systems with $b \neq 0$, Theorem 2.4 gives a complete solution for the minimal ℓ_1 -induced performance problem when $\rho(A + bK^*) < 1$. Notice that the optimal K in (2.36) gives the fastest convergence rate when the closed-loop system is stable, but it does not minimize the spectral radius of $A + bK$ due to the presence of the constraint $C + dK \geq 0$.

It can be easily seen that if $d = 0$, then the constraint $C + dK \geq 0$ is naturally satisfied and

$$k_j^* = - \min_{i=1,2,\dots,n} \left\{ \frac{a_{ij}}{b_i} \right\} \quad (2.39)$$

with the corresponding term inside $\{\cdot\}$ disappears if $b_i = 0$.

In this case, we have the following corollary in view of Remark 2.7.

Corollary 2.2 *For an SIMO system with $b \neq 0$ and $d = 0$, the optimal K given by (2.36) with k_j^* in (2.39) minimizes $\rho(A + bK)$ subject to $A + bK \geq 0$. If $\rho(A + bK^*) < 1$, then the closed-loop system achieves the minimal ℓ_1 -induced performance.*

We note that actually Theorem 2.3 serves as a characterization of the existence of a state-feedback controller for general MIMO systems. Unfortunately, for multi-input systems, the problem of jointly minimizing all the elements of $A + BK$ and $C + DK$ is not generally well-posed. Under a structural constraint, we present a method to derive the minimal K for MIMO systems.

Suppose $K \in \mathbb{R}^{m \times n}$ takes the following structure:

$$K = L\Delta \quad (2.40)$$

where $L \triangleq [l_1 \ l_2 \ \dots \ l_n] \in \bar{\mathbb{R}}_+^{m \times n}$ is a fixed ‘direction matrix’ with $l_j \geq 0$, $j = 1, 2, \dots, n$, being unit vectors, and $\Delta \triangleq \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$ with δ_j to be determined. For convenience, we write

$$B = \begin{bmatrix} \bar{b}_1^T \\ \bar{b}_2^T \\ \vdots \\ \bar{b}_n^T \end{bmatrix}, \quad D = \begin{bmatrix} \bar{d}_1^T \\ \bar{d}_2^T \\ \vdots \\ \bar{d}_r^T \end{bmatrix}$$

and there hold $\bar{b}_i^T l_j \geq 0$ and $\bar{d}_h^T l_j \geq 0$.

The following proposition provides a characterization of the optimal structural K which jointly minimizes the constraints $A + BK \geq 0$ and $C + DK \geq 0$. The proof is similar to that of Theorem 2.4 and hence omitted.

Proposition 2.1 *For an MIMO system with $Bl_j \neq 0$, the optimal structural K in (2.40) such that the elements of $A + BK \geq 0$ and $C + DK \geq 0$ are jointly minimized has Δ given by*

$$\Delta^* = \text{diag}(\delta_1^*, \delta_2^*, \dots, \delta_n^*)$$

where

$$\delta_j^* = - \min_{\substack{i=1,2,\dots,n \\ h=1,2,\dots,r}} \left\{ \frac{a_{ij}}{\bar{b}_i^T l_j}, \frac{c_{hj}}{\bar{d}_h^T l_j} \right\}$$

with the corresponding term inside $\{\cdot\}$ disappears if $\bar{b}_i^T l_j = 0$ or $\bar{d}_h^T l_j = 0$.

Remark 2.8 It is noted that $A + BK \geq 0$ and $C + DK \geq 0$ are jointly minimized with a given direction matrix L . When $\rho(A + BL\Delta^*) < 1$, the ℓ_1 -induced performance will be minimized under the structurally constrained K . However, if

the closed-loop system is unstable, one has to choose another direction matrix L to search a possible K to stabilize the system. The selection of an appropriate direction matrix L to minimize the ℓ_1 -induced performance represents a future research direction.

2.5 Numerical Example

In this section, we present an illustrative example to demonstrate the applicability of the proposed results.

Among different age-structural population models, the Leslie model is the most classical and widely used in population dynamics and control. In a Leslie model, the survival rates and fertility rates depend exclusively on age [7, 8]. From [9], we know that under some conditions, for example, there is an ample food supply, plenty of space and a freedom from predators in the area, it might be assumed for theoretical purposes that some age specific rates of fertility and mortality would remain approximately constant over a period of time.

In this example, we investigate the structural population dynamics of a certain pest described by a Leslie model. An external disturbance is brought into consideration and our aim is to annihilate the pests in a certain area. We consider the following Leslie model:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + B_w w(k), \\ y(k) = Cx(k) + Du(k), \end{cases} \quad (2.41)$$

where

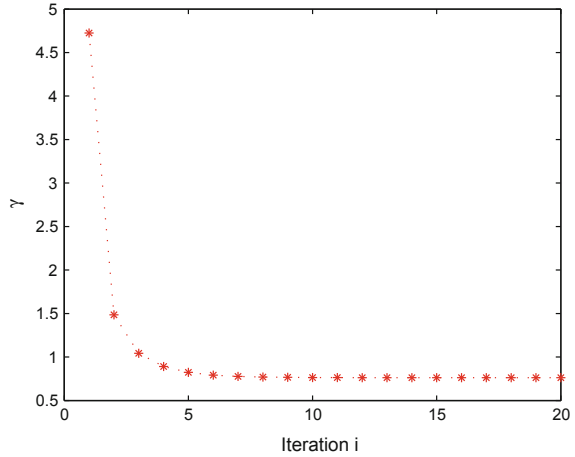
$$A = \begin{bmatrix} 0.2 & 0.3 & 2 \\ 0.8 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.1 \\ 0.05 \\ 0.1 \end{bmatrix}, \quad C = [1 \ 1 \ 1], \quad D = 0.5.$$

In this model, $x(k) = [x_1(k) \ x_2(k) \ x_3(k)]^T$ where $x_i(k)$ represents the number of individuals of age i in year k before the reproduction season. The external input, denoted as $w(k)$, is regarded as a measure of the population of the pests from other regions that migrates into the area of interest. The output $y(k)$ denotes the sum of the number of pests in the area.

Since the Leslie model in (2.41) is a single-input system, we first use the method proposed in Sect. 2.4 to compute the optimal K . By resorting to Theorem 2.4, we can easily obtain the controller gain matrix

Fig. 2.1 Variation of γ_i^* with iteration i . Reprinted from Ref. [10], Copyright 2013, with permission from Elsevier



$$K = [-0.4 \ -0.6 \ -2], \quad (2.42)$$

which yields

$$A + BK = \begin{bmatrix} 0 & 0 & 1 \\ 0.8 & 0 & 0 \\ 0 & 0.7 & 0 \end{bmatrix}, \quad \rho(A + BK) = 0.8243, \quad C + DK = [0.8 \ 0.7 \ 0].$$

With the introduction of controller (2.42), the minimal value γ^* is 0.7614 and the convergence rate of the closed-loop system is the fastest.

On the other hand, by solving conditions in Corollary 2.1 using Algorithm PPL1CD, we obtain an upper bound of γ^* as 0.7623 after 5 iterations and a feasible solution is achieved with

$$p = [3.0931 \ 2.8658 \ 3.0934]^T,$$

which yields the controller gain matrix as

$$K = [-0.3999 \ -0.5999 \ -1.9999].$$

Figure 2.1 shows the variation of γ_i^* with iteration i . From Fig. 2.1 one can clearly see that γ_i^* is monotonically decreasing with respect to iteration i .

The performance of the open-loop and the closed-loop system is evaluated via simulation. The initial condition used in the simulation is

$$x(0) = [50 \ 10 \ 30]^T.$$

Fig. 2.2 Open-loop unforced response. Reprinted from Ref. [10], Copyright 2013, with permission from Elsevier

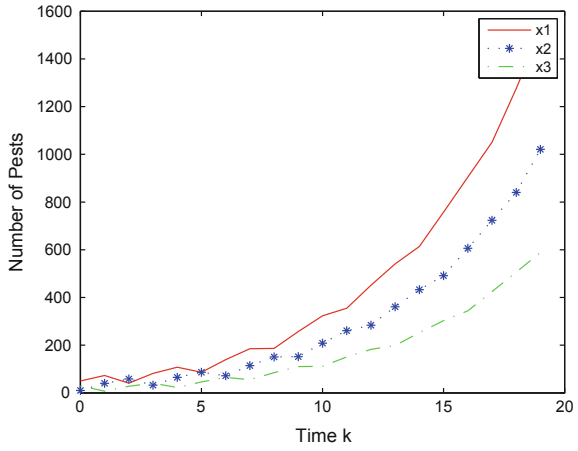


Fig. 2.3 Closed-loop unforced response. Reprinted from Ref. [10], Copyright 2013, with permission from Elsevier

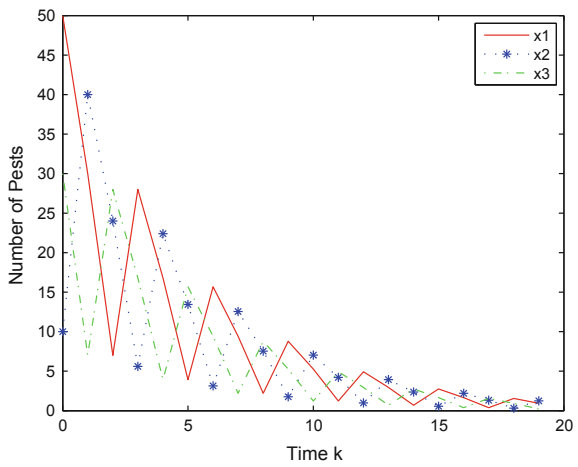


Figure 2.2 shows the state response of open-loop system and Fig. 2.3 shows the state response of the closed-loop system when the external disturbance $w(k) \equiv 0$, from which we can see that the state converges to zero. To illustrate the disturbance attenuation performance, the external disturbance $w(k)$ is assumed to be

$$w(k) = \begin{cases} 50, & 5 \leq k \leq 10, \\ 0, & \text{otherwise.} \end{cases} \quad (2.43)$$

Figure 2.4 shows the response of closed-loop system with $w(k)$ and Fig. 2.5 shows the signal $w(k)$.

It is noted that for SIMO systems, we can easily compute the optimal K and the exact value of the minimal γ by resorting to Theorem 2.4. Moreover, we see that the

Fig. 2.4 Closed-loop forced response. Reprinted from Ref. [10], Copyright 2013, with permission from Elsevier

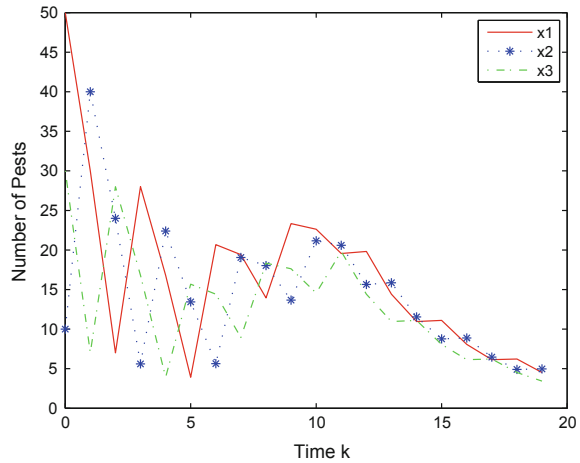
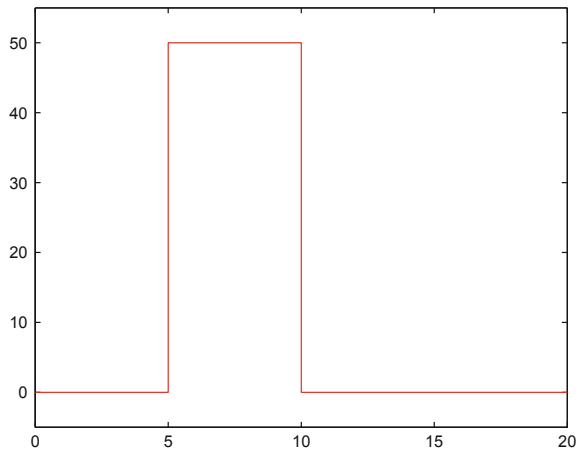


Fig. 2.5 Signal $w(k)$. Reprinted from Ref. [10], Copyright 2013, with permission from Elsevier



values of K and γ^* obtained by two approaches are very close, which implies that Corollary 2.1 also provides a good solution to the controller synthesis problem.

From this example, we know by applying the state-feedback control, even under the influx of pests from other regions, the pests in this region can be annihilated eventually. In practice, the spraying of pesticide is usually one of the pest control methods. Here, the number of insects of all ages can be reduced by spraying pesticide, which corresponds to the introduction of state-feedback controller $u(k) = Kx(k)$ where K represents the amount of pesticide.

2.6 Summary

In this chapter, the problem of state-feedback controller design for linear positive systems with the use of linear Lyapunov function has been studied. A method has been derived to compute the exact value of ℓ_1 -induced norm for positive systems. A characterization has been proposed to ensure the asymptotic stability of the controlled system with a prescribed ℓ_1 -induced performance level. A necessary and sufficient condition for the existence of a desired controller has been established accordingly. Then, an iterative convex optimization algorithm has been developed to solve the design conditions. In addition, an analytical design approach has been put forward to investigate the synthesis of the state-feedback controller for SIMO positive systems. Finally, one example has been presented to illustrate the effectiveness of the theoretical results.

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and L_1 Performance

Chen, X.

2017, XIX, 116 p. 37 illus. in color., Hardcover

ISBN: 978-981-10-2226-5