Chapter 2
What Is Generalized Continuum Mechanics (GCM)?

Introduction

We classify under the title "generalized continuum mechanics" all what is not covered in the restricted framework of the Cauchy model exposed in the prerequisite Chap. 1 under the title of "classical continuum mechanics". In a structured overview this generalization can be presented through the successive abandonment of the basic working hypotheses of standard continuum mechanics of Cauchy: that is, introduction of a density of bulk couple, of a rigidly rotating microstructure and *couple stresses* (Cosserat continua or *micropolar* bodies, nonsymmetric stresses), introduction of a truly deformable microstructure (*micromorphic* bodies), "weak" *nonlocalization* with *gradient theories* and the notion of *hyperstresses*, and the introduction of characteristic lengths, "strong" *nonlocalization* with space functional constitutive equations and the loss of the Cauchy notion of stress, and finally giving up the Euclidean and even Riemannian material background. We peruse these steps in this overview, referring the reader to specialized entries for technical details.

Asymmetric Stress

This asymmetry may be due to the existence of *body couples*; the only known physical example of these couples relates to the case of electromagnetic deformable continua where the volume magnetization is not aligned with the local magnetic field $\mathbf{M}$, or the dielectric polarization $\mathbf{P}$ is not aligned with the local electric field creating thus couples per unit volume in the form of vector products $\mathbf{M} \times \mathbf{H}$ or $\mathbf{P} \times \mathbf{E}$ in an obvious notation. Accounting for such terms in Eq. (1.7) will result in a deviation from the symmetry condition (1.2) with the existence of a nonzero skew part of the stress given by

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$$t_{[ji]} = M_{[i}H_{j]} \quad or \quad P_{[i}E_{j]}. \tag{2.1}$$

In many materials this is strictly zero in reason of the proportionality of the field **M** in **H** or of **P** in **E**. Also, the situation described by Eq. (2.1) may be only transient as **M** may rapidly align with **H** or **P** with **E**. Of course interaction of electromagnetic fields with deformable matter may be much more complicated than that described by Eq. (2.1) involving *both* couple and force of electromagnetic origin, and an import of a specific energy. For a full development of this aspect in Galilean or relativistic dynamics we recommend the treatise of Eringen and Maugin (1990; reprint 2012).

Surface Couples

This concept may be harder to imagine physically. But there is no opposition of principle to introduce in strict parallel with an applied surface traction (in the Cauchy model), an applied surface couple $\mathbf{C}^d$ per unit surface. This is an axial vector. A reasoning *à la Cauchy* will yield the introduction of the notion of couple stress **m** such that

$$n_j m_{ji} = C_i^d. \tag{2.2}$$

The object of induced component m_{ji} still is "axial" in its second index i. Accordingly, we can introduce a geometrical object with three indices, m_{jik}, such that

$$m_{kji} = m_{k[ji]} = m_{kp}\varepsilon_{pji}, \tag{2.3}$$

where ε_{pji} is Levi-Civita alternating symbol. Inclusion of a surface contribution involving the expression in Eq. (1.7) will transform the local statement of the balance of moment of momentum (1.2) in the following more general form:

$$t_{[ji]} + m_{kji,k} = 0. \tag{2.4}$$

If in addition there exits a distribution of body couples per unit mass (rewritten as a skewsymmetric tensor $C_{ji} = -C_{ij}$), then Eq. (2.4) will be generalized to the following local equilibrium of couples:

$$t_{[ji]} + \rho\, C_{ij} + m_{kji,k} = 0. \tag{2.5}$$

Furthermore, if this additional effect is related to the existence of a true internal degree of freedom (of rotation) giving rise to some spin, then an inertial term will be added in the right-hand side of Eq. (2.5) that then becomes a true dynamic equation:

$$t_{[ji]} + \rho\, C_{ij} + m_{kji,k} = \rho\, \dot{S}_{ji}, \tag{2.6}$$

where $S_{ij} = -S_{ji}$ is a spin (i.e., an angular momentum).

Equation (2.5) was the equation to which the Cosserats (1909) were naturally led—together with the static form of (1.1)—by applying an invariance requirement (Euclidean action) that represents a pioneer's application of group theory in continuum mechanics, and requires from translation and possible rotational degrees of freedom of a material point to be on the same a priori footing. Of course, the expression of constitutive equations in order to close the system of field equations demands an elaboration of the associated generalized kinematics (see entry on Cosserat continua). Furthermore, Eringen (1966) formulated a law of conservation of micro-inertia that complements the usual conservation of mass in the dynamical case. Equation (2.6) corresponds to a rewriting of the global conservation law in the following generalized form:

$$\frac{d}{dt}\int\limits_{B} \rho(\mathbf{v}\times\mathbf{x}+\mathbf{s})\,dv = \int\limits_{B} \rho(\mathbf{f}\times\mathbf{x}+\mathbf{c})\,dv + \int\limits_{\partial B} \left(\mathbf{T}^{d}\times\mathbf{x}+\mathbf{C}^{d}\right)da, \tag{2.7}$$

where $\mathbf{s}$ and $\mathbf{c}$ are the axial vectors dual of the skew tensors $\mathbf{S}$ and $\mathbf{C}$, respectively.

Eringen-Mindlin Micromorphic Model of Microstructured Continua

In the previous section no mention of any microscopic definition of the newly introduced quantities was given. But we can well imagine in agreement with the original vision of Voigt, Duhem and the Cosserats that a material point that normally experiences a translation is now assimilated to a small rigid body that can also rotate, and is thus able to respond to local couples. A more refined vision would be to see this point itself as a small deformable body, hence exhibiting six degrees of freedom in addition to translation. This corresponds to the model of **micromorphic media** devised by Eringen and Suhubi in 1964; the model of **microstructured media** devised by Mindlin (1964) in the same year is equivalent (representing a homogeneous deformation within the small body) (see entry on Eringen-Mindlin model).

The local balance of equilibrium in a micromorphic body can be written as

$$\mu_{kij,k} + t_{ji} - s_{ji} + l_{ij} = 0, \quad t_{ji} = t_{(ji)} + t_{[ji]}, \quad s_{[ji]} = 0, \quad l_{ji} = C_{ji} + l_{(ji)} \tag{2.8}$$

where μ_{kji} is the called the hyperstress tensor, s_{ji} is the so-called symmetric micro-stress, and l_{ij} is the body-moment tensor of which the skew part represents a body couple $C_{ji} = -C_{ij}$:

$$\mu_{kij,k} + t_{ji} - s_{ji} + l_{ij} = 0, \quad t_{ji} = t_{(ji)} + t_{[ji]}, \quad s_{[ji]} = 0, \quad l_{ji} = C_{ji} + l_{(ji)}; \tag{2.9}$$

then the Cosserat or micropolar model is obtained by taking the skew part of the first of Eq. (2.9) and setting $m_{kji:} = \mu_{k[ji]}$.

Bodies with microstretch (Eringen 1969). This is a further reduction of the model Eq. (2.9) obtained by noting m_k the intrinsic dilatational stress or microstretch vector; l the body microstretch force such that $l_{(ij)} = (l/3)\delta_{ij}$, and t and s are intrinsic and micro scalar forces, so that we have

$$\mu_{klm} = \frac{1}{3}\, m_k \delta_{lm} - \frac{1}{2}\, \varepsilon_{lmr} m_{kr}, \tag{2.10}$$

hence

$$m_{kl,k} + \varepsilon_{lmn} t_{mn} + C_l = 0, \quad m_{k,k} + t - s + l = 0. \tag{2.11}$$

Note that an additional natural boundary condition involving the new higher-order stresses μ_{kij} and m_{ji} must complement the standard Cauchy condition of the Prequisite Chap. 1, e.g.,

$$n_k\, \mu_{kij} = C^d_{ij} \quad or \quad n_j\, m_{ji} = C^d_i, \tag{2.12}$$

where C^d_i is akin to a surface couple.

Finally, we note the further case of **dilatational elasticity** (Cowin and Nunziato 1983) [only the second of Eq. (2.11) is relevant]:

$$m_{k,k} + t - s + l = 0. \tag{2.13}$$

Here the additional natural boundary condition will be of the form

$$n_k\, m_k = M^d, \tag{2.14}$$

where M^d is akin to a *tension*.

All these equations are given here in Cartesian components in order to avoid any misunderstanding that can be created by a direct intrinsic notation: μ_{kij} is a new internal force having the nature of a third-order tensor. It has to start with no specific symmetry in Eq. (2.8) and it may be referred to as a ***hyperstress***. In the case of Eq. (2.10) this quantity is skewsymmetric in its last two indices and a second order tensor—called a *couple stress*—of components m_{ji} can be introduced having *axial* nature with respect to its second index. The fields s_{ji} and l_{ij} are, respectively, a symmetric second-order tensor and a general second-order tensor. The former is an *intrinsic interaction stress*, while the latter refers to an external source of *both* stress and couple according to the last of Eq. (2.9). Only the skew part of the later remains in the special case of micropolar materials. The skewsymmetric C_{ji} can be of electromagnetic origin, and more rarely of pure mechanical origin. Equations (2.10) and (2.13) represent a kind of intermediate case between micromorphic and micropolar materials. The case of dilatational elasticity in Eq. (2.13) appears as a further reduction of that in Eq. (2.11). This will be useful in describing the mechanical behaviour of media exhibiting a distribution of holes or cavities in evolution.

Weakly Nonlocal Modelling

Cases examined in the preceding paragraphs do not question the notion of contiguity of Euler and Cauchy. They just add new fields of internal forces that still satisfy the same contiguity argument. Totally different is the viewpoint that envisages a more analytically precise definition of a classical quantity such as the elastic displacement. This is best emphasized by extending the obvious limited expansion of the power on internal forces considered in the first of Eq. (2.15) to higher order spatial gradients of the velocity field, for instance as

$$p_{int} = -(t_{ji}v_{i,j} + m_{kji}v_{i,jk} + \cdots), \tag{2.15}$$

where $m_{kji} = m_{(kj)i}$ may be called a stress of higher order or *hyperstress*. This a priori has at most eighteen independent components. In terms of the geometry of a bounding surface (so-called natural boundary condition) this new concept will require the consideration of the second-order geometrical description of the surface, hence the curvature. This destroys the standard Euler-Cauchy notion of contiguity. In pure elasticity, the effect of the contribution of the hyperstress will be of importance wherever the strain is not spatially uniform, and obviously where one observes a rapid variation of the elastic displacement, e.g. in boundary layers. This vision is quite different from the one considered in the preceding section, since now only one standard field, the displacement, or the velocity in the case of fluids, is involved. The Euler-Cauchy framework maybe referred to as a first—gradient theory—when referred to the expression of the power of internal forces. The theory described by Eq. (2.15) with an expansion limited to second-order is called a second-gradient theory. One can generalize this in principle to an nth-gradient theory (cf. Maugin 1980). The second-gradient is well exposed in Germain (1973). Such theories are often referred to as *weakly nonlocal theories*. The only complications are the statement of the relevant boundary conditions, and the obviously large number of material coefficients to be measured, save in carefully selected simple geometries. Analytically, the resulting problems will be *stiffer* than standard ones, but they may be approached by some approximations such as singular perturbations (as exemplified by boundary-value problems involving matched asymptotic expansions between inner and outer expansions).

Historically, the roots of the gradient theories may probably be found in the general presentation by G. Piola (in the 1840s–1860s), and more precisely in Barré de Saint-Venant (1869) and the original works of Le Roux (1911, 1913). In the 1960s we note the works of Mindlin and co-workers and Toupin who revived this approach in the modern framework (Mindlin and Eshel 1968; Mindlin and Tiersten 1962; Toupin 1962). Note that this modelling is sometimes mistaken for the Cosserat model even by the best authors. This may come from the fact that if one assumes in a Cosserat continuum that the rotational velocity of the microstructure is constrained to follow the usual rate of rotation of the Cauchy continuum, then we are led to a degenerate theory of the second-gradient type, which should be called the *constrained Cosserat continuum*. This appears to be badly conditioned for dynamical properties.

Strongly Nonlocal Modelling

Although the basic idea may be mentioned in Duhem (1893), a true development of this modelling took place in the 1960s with the works of Kröner and Datta (1966), Kunin (1966), and Rogula (1965). Later on Eringen and Edelen (1972) elaborated a more abstract formulation. Synthesis works on the subject are by Kunin (1982) and Eringen (2002). Technically, the Cauchy construct does *not* apply anymore since contiguity is lost altogether. In principle, only the case of infinite bodies should be considered as any cut would destroy the prevailing long-range ordering. Constitutive equations become integral expressions over space, perhaps with a more or less rapid attenuation with distance of the spatial kernel. This, of course, inherits from the action-at-a-distance dear to the Newtonians, while adapting the disguise of a continuous framework. This view is justified by the approximation of an infinite crystal lattice: the relevant kernels can be justified through this discrete approach. But this approach raises the matter of solving integro-differential equations—not always a pleasant task—instead of partial-differential equations. What about boundary conditions that are in essence foreign to this representation of matter-matter interaction? There remains a possibility of the existence of a “weak-nonlocal” limit by the approximation by gradient models. Typically one would consider in the linear elastic case a stress constitutive equation in the form

$$t_{ji}(\mathbf{x}) = \int_{all\ space} C_{jikl}(|\mathbf{x}-\mathbf{x}'|)\ e_{kl}(\mathbf{x}')\ d^3\mathbf{x}', \tag{2.16}$$

where the constitutive functions C_{jikl} decreases markedly with the distance between material points $\mathbf{x}'$ and $\mathbf{x}$, that are equivalent with an obvious reciprocity. Note that standard local linear elasticity follows from Eq. (2.16) by considering the special case

$$C_{jikl}(|\mathbf{x}-\mathbf{x}'|) = C^0_{jikl}\,\delta(|\mathbf{x}-\mathbf{x}'|), \tag{2.17}$$

where δ is Dirac's delta generalized function, and the tensorial coefficient C^0_{jikl} depends at most on the point $\mathbf{x}$ alone (for inhomogeneous materials).

In one-dimensional space, a constitutive equation such as Eq. (2.16) will provide a balance of linear momentum in the following integro-differential form:

$$\rho_0 \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[\int_{-\infty}^{+\infty} E\,\alpha(|x-x'|)\,\frac{\partial u(x',t)}{\partial x'}\,dx'\right] = 0, \tag{2.18}$$

or

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial}{\partial x}\left[\alpha * \frac{\partial u}{\partial x}\right] = 0, \tag{2.19}$$

where the symbol $*$ stands for the convolution product (in space) and we have set $c_0^2 = E/\rho_0$. One needs a sensible expression for the kernel (or influence function or weight function) α.

The historical moment in the recognition of the usefulness of strongly nonlocal theories was the EUROMECH colloquium on nonlocality organized by Dominik Rogula in Warsaw in 1977. Note in conclusion to this point that any field theory can be generalized to a nonlocal one while saving the notions of linearity and anisotropy, but losing the usual notion of flux.

The Loss of Euclidean Structure

In classical continuum mechanics the arena of regular deformations is the physical Euclidean space E^3 which is assimilated to R^3. That is fine for regular displacement fields. But in some materials such as metals there exists a huge quantity of dislocations, lines along which the displacement suffers a discontinuity measured by the so-called Burgers vector. There also exist other kinds of singularities such as disclinations (lines along which the rotation vector does not close up in a round circuit), and cavities (vacancies in the case of crystals) or micro inclusions (foreign atoms in the case of atoms). The existence of what may be called *defects* questions the generally accepted idea to represent a material manifold—the set of material points—as a simple Euclidean space. Something more sophisticated must be envisaged. This was achieved in the second half of the twentieth century with no unique answer. But the most frequent one seems to consider a more adapted geometric background that will be non-Euclidean or even non-Riemannian. This is exemplified by a manifold without curvature but with affine connection, or an Einstein-Cartan space with both torsion and curvature, etc. With this one enters a true "geometrization" of continuum mechanics of which conceptual difficulties compare favourably with those met in modern theories of gravitation. Pioneers in the field in the years 1950–70 were Kondo (1955) in Japan, Kröner (1958) in Germany, Bilby (1955) and his group in the UK, Stojanovic (1969) in what was then Yugoslavia, and Noll (1967) and Wang (1967) in the USA. Main properties of this type of approach are: (i) the relationship to the multiple decomposition of finite strains (Bilby, Kröner, Lee) and (ii) the generalization of theories such as the theory of volumetric growth or the theory of phase transitions within a unified approach to local structural rearrangements (local evolution of reference).

Another complication may be the intrinsic difficulty to define analytically some fields, in particular gradients, when the material itself is viewed as a fractal set. This constitutes the last avatar of continuum mechanics with a possible relationship to fractional derivatives (see the dictionary entry "Fractal continua").

General references on generalized continuum mechanics are: Altenbach and Eremeyev (2013), Altenbach et al. (2011, 2013), Maugin (2010, 2011), Maugin and Metrikine (2010), and the historical proceedings (Kröner 1968).

Cross references in the dictionary part: Cosserat continua, Couple stress, Directors theory, Electromagnetic continua, Eringen-Mindlin medium, Fractal continua, Gradient elasticity, Higher-order gradient theories, Hyperstresses, Le Roux elasticity, Micromorphic continua, Microstructure, Non-locality (strong), Non-locality (weak).

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