Chapter 2
Consensus of Homogeneous Multi-agent Systems with Time Delays

2.1 Motivation

With the help of various analysis methods, including frequency-domain analysis, Lyapunov method, properties of stochastic matrix theory, etc., delay effect on consensus seeking multi-agent systems, especially for the homogeneous multi-agent systems, has been extensively analyzed in the past decade. With nonnegligible communication delay, however, consensus results are mainly for the first-order, second-order, and high-order multi-agent systems driven by stationary consensus algorithms with diverse communication delays, dynamical consensus algorithm in synchronously coupled form, etc. Besides, consensus problem with identical input delay has also been extensively studied, but consensus analysis of multi-agent systems with diverse input delays only attracted a little attention.

Compared with synchronously coupled consensus algorithm, collective behaviors of dynamical consensus algorithm in asynchronously coupled seem more interesting and complicated. Furthermore, consensus convergence subject to both input delay and communication delay is still worth investigation, and the relationship between communication delay and input delay needs intensive study.

2.2 Multiple First-Order Agents

Firstly, two first-order agents (1.15) with input delay are investigated

$$\dot{x}_i(t) = u_i(t - T), i = 1, 2, \tag{2.1}$$

where $T > 0$ is the identical input delay of each agent. For the intercoupling agents (2.1), asynchronously coupled consensus algorithm with communication delay is given by

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$$u_1(t) = \kappa(x_2(t-\tau) - x_1(t)), \tag{2.2}$$

$$u_2(t) = \kappa(x_1(t-\tau) - x_2(t)), \tag{2.3}$$

where $\kappa > 0$, $\tau > 0$ is the identical communication delay. With (2.2) and (2.3), the closed-loop form of agents (2.1) is formulated as

$$\dot{x}_1(t) = \kappa(x_2(t-T-\tau) - x_1(t-T)), \tag{2.4}$$

$$\dot{x}_2(t) = \kappa(x_1(t-T-\tau) - x_2(t-T)). \tag{2.5}$$

Some results for two coupled agents (2.4) and (2.5) are listed firstly.

Theorem 2.1 ([1]) *The coupled agents (2.4) and (2.5) without communication delay, i.e., $\tau = 0$, reach an asymptotic stationary consensus, i.e., $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = c$, where c is a constant, if and only if*

$$\kappa < \frac{\pi}{4T}. \tag{2.6}$$

Theorem 2.2 ([2]) *The coupled agents (2.4) and (2.5) converge to a stationary consensus asymptotically, if*

$$\kappa < \frac{1}{2T}. \tag{2.7}$$

Remark 2.1 Evidently, communication delay-independent condition (2.7) in Theorem 2.2 is relatively conservative, because the simulation results in [2] demonstrated that consensus convergence of multi-agent systems with both communication delays and input delays is dependent on the communication delays.

Now, we make further investigation on delay-dependent consensus criterion of the coupled agents (2.4) and (2.5).

Theorem 2.3 *If*

$$\kappa < \frac{\pi}{2(2T+\tau)}, \tag{2.8}$$

the coupled agents (2.4) and (2.5) achieve an asymptotic stationary consensus.

Proof Let $\bar{x}(t) = x_1(t) - x_2(t)$, and we get

$$\dot{\bar{x}}(t) = -\kappa(\bar{x}(t-T-\tau) + \bar{x}(t-T)). \tag{2.9}$$

The characteristic equation of system (2.9) is given by $1 + g(s) = 0$, where $g(s) = \kappa\frac{e^{-\tau s}+1}{s}e^{-Ts}$. By computing, we obtain

$$g(j\omega) = \kappa\frac{2\cos(\frac{\omega\tau}{2})}{\omega}e^{-j(\omega T + \frac{\omega\tau}{2} + \frac{\pi}{2})}.$$

Thus, $|g(\mathrm{j}\omega)| = |\kappa \frac{2\cos(\frac{\omega\tau}{2})}{\omega}|$. When $\omega \in (0, +\infty)$, $g(\mathrm{j}\omega)$ crosses the negative real axis for the first time at $\omega_{c1} = \frac{\pi}{2T+\tau}$. $g(\mathrm{j}\omega)$ crosses the negative real axis for the kth time at $\omega_{ck}, k \geq 2$, and we get $\omega_{ck} > \omega_{c1}$. Thus, if $\frac{2\kappa}{\omega_{c1}} < 1$, i.e., $\kappa < \frac{\pi}{2(2T+\tau)}$, $|g(\mathrm{j}\omega_{ck})| < 1$ holds for $k = 1, \ldots, +\infty$. Hence, $g(\mathrm{j}\omega)$ does not enclose the point $(-1, \mathrm{j}0)$. Therefore, the system (2.9) is asymptotically stable, i.e., the coupled agents (2.4) and (2.5) converge to a stationary consensus asymptotically. □

Remark 2.2 Obviously, the upper bound of κ in (2.8) is larger than that in (2.7) when $\tau < \tau_c$ with $\tau_c = T(\pi - 2)$. For the coupled agents (2.4) and (2.5), the control parameters can be designed by utilizing the conditions (2.7) and (2.8) comprehensively. When $0 < \tau < \tau_c$, the control parameter κ is designed based on Theorem 2.3, while the control parameter κ is designed based on Theorem 2.2 when $\tau \geq \tau_c$.

Example 2.1 Consider two coupled agents (2.4) and (2.5) with the input delay $T = 0.2\,(\mathrm{s})$. From Theorem 2.1, two agents without communication delay reach an asymptotic consensus if and only if $\kappa < \frac{\pi}{0.8}$. When $\kappa = \frac{\pi}{0.8}$, arbitrary positive communication delay makes the agents' states diverge (see Fig. 2.1). For the agents (2.4) and (2.5) with both input and communication delays, we get $\kappa < 2.5$ from Theorem 2.2, and the asymptotic consensus convergence is independent of the communication delay, i.e., consensus convergence is robust to arbitrary communication delay. Based on the condition (2.8) in Theorem 2.3, upper bound of κ with different communication delay is obtained (see Fig. 2.2). When the communication delay increases, the maximum value of κ decreases and converges to zero finally (see Fig. 2.2). Within the bound of κ, two coupled agents converge to the stationary consensus asymptotically (see Fig. 2.3). As mentioned in Remark 2.2, the design of the control parameter κ in real engineering application is comprehensively completed according to Theorems 2.2 and 2.3. In Fig. 2.2, κ is designed from (2.8) when $\tau < \tau_c = 0.2283\,(\mathrm{s})$, while it is designed based on (2.7) when $\tau \geq \tau_c$.

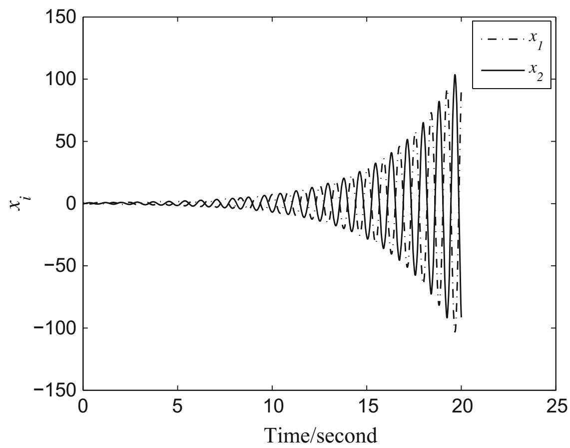


Fig. 2.1 Divergence of coupled first-order agents' states

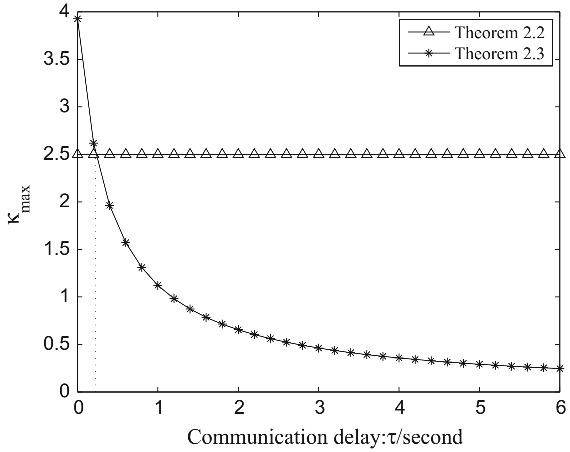


Fig. 2.2 Upper bound of control parameter κ

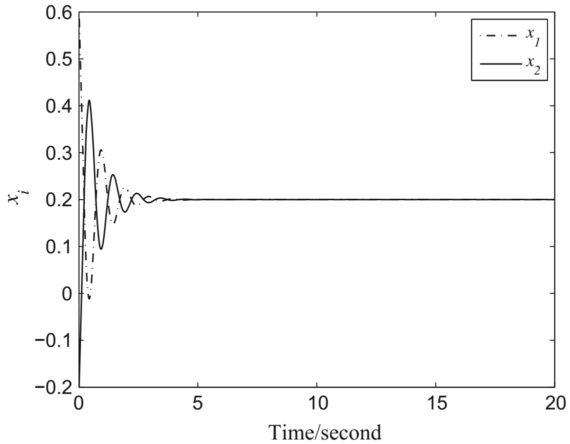


Fig. 2.3 Consensus convergence of coupled first-order agents

2.3 Multiple Second-Order Agents

Consensus algorithm (1.6) of second-order multi-agent systems is simply classified into two types including stationary consensus algorithm with velocity stabilization and dynamical consensus algorithm composed of the position and velocity consensus coordination control parts.

2.3.1 Stationary Consensus Algorithms with Time Delays

Second-order dynamic agents (1.5) with velocity damping part are modeled by [3]

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -\alpha_i v_i(t) + u_i(t - T_i),\ i \in \mathscr{I}, \end{aligned} \tag{2.10}$$

where $T_i > 0$ is the input delay of agent i, respectively, and $-\alpha_i v_i$ with $\alpha_i > 0$ denotes the velocity damping term caused by the resistance, e.g., the friction.

For system (2.10), a simple consensus protocol is given by

$$u_i(t) = \kappa_i \sum_{j \in N_i} a_{ij}(x_j(t - \tau_{ij}) - x_i(t)), \tag{2.11}$$

where $\kappa_i > 0$, $a_{ij} > 0$, $j \in N_i$, and $\tau_{ij} > 0$ is the communication delay.

With (2.11), the closed-loop form of system (2.10) is

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -\alpha_i v_i(t) + \kappa_i \sum_{j \in N_i} a_{ij}(x_j(t - T_i - \tau_{ij}) - x_i(t - T_i)),\ i \in \mathscr{I}. \end{aligned} \tag{2.12}$$

2.3.1.1 Consensus Convergence with Diverse Input Delays

Consensus seeking is analyzed for the multi-agent systems (2.12) only with diverse input delays as follows

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -\alpha_i v_i(t) + \kappa_i \sum_{j \in N_i} a_{ij}(x_j(t - T_i) - x_i(t - T_i)),\ i \in \mathscr{I}. \end{aligned} \tag{2.13}$$

Before presenting consensus criteria for (2.13), some useful lemmas are listed as follows.

Lemma 2.1 (Generalized Nyquist stability criterion [4]) $G_r(s) \in R(s)^{N\times N}$ *is proper, where* N *is a positive integer, and the zeros of* $\Delta(s) = \det(I + G_r(s))$ *lie on the open left half complex plane if and only if the number of counterclockwise encirclements of* $(-1, \mathrm{j}0)$ *by the eigenloci of* $G_r(\mathrm{j}\omega)$ *as* ω *goes from* $-\infty$ *to* $+\infty$ *is equivalent to the total number of right half-plane poles of* $G_r(s)$.

Lemma 2.2 ([5]) *Let* $Q \in C^{n\times n}$, $Q = Q^\star \geq 0$ *and* $T = \operatorname{diag}\{t_i, t_i \in C\}$. *Then*

$$\lambda(QT) \in \rho(Q)Co(0 \cup \{t_i\}),$$

where $\lambda(\cdot)$ *denotes matrix eigenvalue,* $\rho(\cdot)$ *denotes the matrix spectral radius, and* $\mathrm{Co}(\cdot)$ *denotes the convex hull.*

Besides, we get the following lemma from Remark 4 and Claim 1 in [6].

Lemma 2.3 *Assume that*

$$(T_i\alpha_i - T_j\alpha_j)(T_i - T_j) \leq 0,\ \forall i, j \in \mathscr{I}, i \neq j \tag{2.14}$$

hold for the frequency response of a family of systems given by

$$G_i(\mathrm{j}\omega) = \frac{M_i}{s^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega},\quad i \in \mathscr{I},$$

where M_i *is the gain margin of transfer function* $w_i(s) = \frac{1}{s^2+\alpha_i s}\mathrm{e}^{-sT_i}$*. Then,* $\delta Co(0 \cup \{G_i(\mathrm{j}\omega), i \in \mathscr{I}\})$ *does not contain the point* $(-1, \mathrm{j}0)$ *for any real number* $\delta \in [0, 1)$ *and any* $\omega \in (-\infty, \infty)$.

Theorem 2.4 *Consider the agents (2.13) with a static interconnection topology* $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ *that is undirected (or bidirectional), symmetric (*$a_{ij} = a_{ji}$*), and connected. With (2.14), then, all the agents in system (2.13) asymptotically converge to a stationary consensus, if*

$$d_i < \frac{M_i}{2\kappa_i}, \forall i \in \mathscr{I}, \tag{2.15}$$

where $d_i = \sum_{j\in N_i} a_{ij}$ *and* M_i *is the gain margin of transfer function* $w_i(s) = \frac{1}{s^2+\alpha_i s}\mathrm{e}^{-sT_i}$.

Proof The characteristic equation of system (2.13) about $x = [x_1, \ldots, x_n]^T$ is obtained as

$$\det(\mathrm{diag}\{s^2 + \alpha_i s, i \in \mathscr{I}\} + \mathrm{diag}\{\kappa_i \mathrm{e}^{-T_i s}, i \in \mathscr{I}\}L) = 0.$$

Taking $\mathscr{D}(s) = \det(\mathrm{diag}\{s^2 + \alpha_i s, i \in \mathscr{I}\} + \mathrm{diag}\{\kappa_i \mathrm{e}^{-T_i s}, i \in \mathscr{I}\}L)$, $\mathscr{D}(0) = \det(\mathrm{diag}\{0^2 + \alpha_i 0, i \in \mathscr{I}\} + \mathrm{diag}\{\kappa_i \mathrm{e}^{-T_i 0}, i \in \mathscr{I}\}L) = \det(\mathrm{diag}\{\kappa_i, i \in \mathscr{I}\}) \det(L)$. Since the interconnection topology $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ is undirected and connected, it follows from Lemma 1.1 that 0 is a simple eigenvalue of L, i.e., $\mathrm{rank}(L) = n - 1$. Hence, $\mathscr{D}(s)$ has only one zero at $s = 0$.

Let $\mathscr{F}(s) = \det(I + \mathrm{diag}\{\frac{\kappa_i}{s^2+\alpha_i s}\mathrm{e}^{-T_i s}, i \in \mathscr{I}\}L)$ with $s \neq 0$. From Lemma 2.1, the zeros of $\mathscr{F}(s)$ are on the open left half complex plane, if $\lambda(\mathrm{diag}\{\frac{\kappa_i}{-\omega^2+\mathrm{j}\omega\alpha_i}\mathrm{e}^{-\mathrm{j}\omega T_i}, i \in \mathscr{I}\}L)$ does not enclose the point $(-1, \mathrm{j}0)$ for $\omega \in R$.

Based on the definition of Laplacian matrix, $L = L^T \geq 0$ holds for symmetric weights ($a_{ij} = a_{ji}$). It follows from Lemma 2.2 that

$$
\begin{aligned}
&\lambda\left(\mathrm{diag}\left\{\frac{\kappa_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}\right\}L\right)\\
&=\lambda\left(\mathrm{diag}\left\{\frac{M_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}\right\}\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}L\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}\right)\\
&\in\rho\left(\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}L\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}\right)Co\left(0\cup\frac{M_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}\right).
\end{aligned}
$$

In addition, the condition (2.15) yields

$$
\begin{aligned}
&\rho\left(\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}L\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}\right)\\
&=\rho\left(\mathrm{diag}\left\{\kappa_i M_i^{-1}\right\}L\right)\\
&\le\max_{i\in\mathscr{I}}\frac{2\kappa_i d_i}{M_i}\\
&<1.
\end{aligned}
$$

With the help of Lemma 2.3, thus, it is obvious that

$$
(-1,0)\notin\rho\left(\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}L\mathrm{diag}\left\{\sqrt{\kappa_i M_i^{-1}}\right\}\right)Co\left(0\cup\frac{M_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}\right),
$$

i.e., $\lambda(\mathrm{diag}\{\frac{\kappa_i}{-\omega^2+\mathrm{j}\omega\alpha_i}\mathrm{e}^{-\mathrm{j}\omega T_i}, i\in\mathscr{I}\}L)$ does not enclose the point $(-1,\mathrm{j}0)$ for $\omega\in R$. Hence, the zeros of $\mathscr{F}(s)$ all lie on the open left half complex plane.

Now, we have proved that $\mathscr{D}(s)$ has its zeros on the open left half complex plane except for one zero at $s=0$. Thus, we obtain from (2.13) that $\lim_{t\to\infty}x_i(t)=x_i^*$, $i\in\mathscr{I}$ and $\lim_{t\to\infty}v_i(t)=0$, $\forall i\in\mathscr{I}$. In turn, we get $L[x_1^*,\dots,x_n^*]^T=0$ from (2.13). Since $\mathrm{rank}(L)=n-1$ and $L[1,\dots,1]^T=0$ hold for the Laplacian matrix L of a connected graph (or digraph), the roots of $Lx^*=0$ is expressed as $x^*=c[1,\dots,1]^T$, where c is a constant. Theorem 2.4 is proved. □

Remark 2.3 From Lemma 2.3 and the proof of Theorem 2.4, consensus condition (2.15) is strictly dependent on the precondition (2.14) and the symmetry of interconnection topology.

Example 2.2 Consider a multi-agent network of six second-order dynamic agents given by (2.13), and the topology in Fig. 2.4 is undirected and connected. The symmetric weights are chosen as $a_{12}=a_{21}=0.2$, $a_{14}=a_{41}=0.3$, $a_{23}=a_{32}=0.5$, $a_{24}=a_{42}=0.1$, $a_{35}=a_{53}=0.2$, $a_{36}=a_{63}=0.6$, $a_{45}=a_{54}=0.4$, and $a_{56}=a_{65}=0.7$. The agents' input delays are as follows: $T_1=0.5\,(\mathrm{s})$, $T_2=0.6\,(\mathrm{s})$, $T_3=0.4\,(\mathrm{s})$, $T_4=0.3\,(\mathrm{s})$, $T_5=0.2\,(\mathrm{s})$, and $T_6=0.25\,(\mathrm{s})$. The agents' damping coefficients are given by $\alpha_1=1$, $\alpha_2=\frac{2}{3}$, $\alpha_3=1.75$, $\alpha_4=2.5$, $\alpha_5=4.5$, and $\alpha_6=3.2$. Thus, the condition (2.14) holds for all the agents.

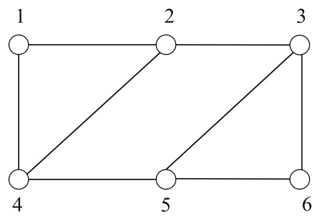


Fig. 2.4 Undirected graph with symmetric weights

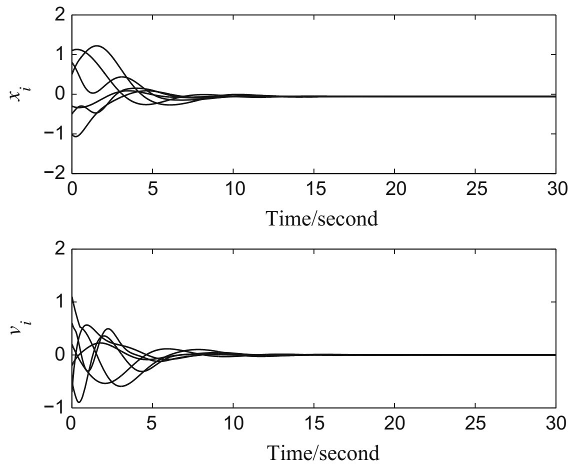


Fig. 2.5 Consensus convergence with input delays

By numerical computing with the help of MATLAB simulator, the gain margins of $\omega_i(s) = \frac{e^{-sT_i}}{s^2+\alpha_i s}$, $i = 1, \ldots, 6$ are given by $M_1 \simeq 2.15$, $M_2 \simeq 1.18$, $M_3 \simeq 4.81$, $M_4 \simeq 9.22$, $M_5 \simeq 25.29$, and $M_6 \simeq 14.24$. According to the condition (2.15), the control parameters κ_i satisfy: $\kappa_1 \in (0, 2.15)$, $\kappa_2 \in (0, 0.74)$, $\kappa_3 \in (0, 1.85)$, $\kappa_4 \in (0, 5.76)$, $\kappa_5 \in (0, 9.73)$, and $\kappa_6 \in (0, 5.48)$, and we choose $\kappa_1 = 2$, $\kappa_2 = 0.7$, $\kappa_3 = 0.3$, $\kappa_4 = 2$, and $\kappa_5 = 3$. Then, the agents in system (2.13) converge to a stationary consensus asymptotically (see Fig. 2.5).

2.3.1.2 Consensus Convergence with Diverse Input And Communication Delays

From the previous analysis, it is obvious that asymmetric connections and diverse communication delays destroy the symmetry of frequency-domain Laplacian matrix, so Lemmas 2.2 and 2.3 cannot be applied in these cases. Fortunately, Gershgorin's disk theorem is an effective method for analyzing the consensus seeking of second-order dynamic agents (2.12) with both heterogeneous input delays and communication delays under general directed topology.

Theorem 2.5 *Suppose that the interconnection topology $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ of the agents (2.12) has a spanning tree. If*

$$d_i < \frac{\alpha_i^2}{2\kappa_i(1 + \alpha_i T_i)}, \ \forall i \in \mathscr{I}, \tag{2.16}$$

where $d_i = \sum_{j \in N_i} a_{ij}$, all the agents (2.12) reach a stationary consensus asymptotically.

Proof The characteristic equation of system (2.12) on $x = [x_1, \ldots, x_n]^T$ is described as

$$\det(\mathrm{diag}\{s^2 + \alpha_i s, i \in \mathscr{I}\} + \mathrm{diag}\{\kappa_i \mathrm{e}^{-T_i s}, i \in \mathscr{I}\} L(s)) = 0,$$

where $L(s) = \{l_{ij}(s)\} \in C^{n \times n}$ is defined by

$$l_{ij}(s) = \begin{cases} -a_{ij}\mathrm{e}^{-\tau_{ij}s}, & j \in N_i; \\ \sum_{j \in N_i} a_{ij}, & j = i; \\ 0, & \text{otherwise}, \end{cases}$$

and $L(0) = L$.

Let $\tilde{\mathscr{D}}(s) = \det(\mathrm{diag}\{s^2 + \alpha_i s, i \in \mathscr{I}\} + \mathrm{diag}\{\kappa_i \mathrm{e}^{-T_i s}, i \in \mathscr{I}\} L(s))$. Analogous to the proof of Theorem 2.4, $\tilde{\mathscr{D}}(s)$ has only one zero at $s = 0$, since the topology $\mathscr{G} = (\mathscr{V}, \mathscr{E}, A)$ has a spanning tree.

With $s \neq 0$, define $\tilde{\mathscr{F}}(s) = \det(I + \mathrm{diag}\{\frac{\kappa_i}{s^2 + \alpha_i s}\mathrm{e}^{-T_i s}, i \in \mathscr{I}\} L(s))$. It follows the Gershgorin's disk theorem that

$$\begin{aligned} &\lambda\left(\mathrm{diag}\left\{\frac{\kappa_i}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}, i \in \mathscr{I}\right\} L(s)\right) \\ &\in \bigcup_{i \in \mathscr{I}} \left\{\zeta : \zeta \in C, |\zeta - \frac{\kappa_i(\sum_{j \in N_i} a_{ij})}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}| \leq \sum_{j \in N_i} |\frac{\kappa_i a_{ij}}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}(T_i + \tau_{ij})\omega}|\right\} \\ &= \bigcup_{i \in \mathscr{I}} \left\{\zeta : \zeta \in C, |\zeta - \frac{\kappa_i d_i}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}| \leq |\frac{\kappa_i d_i}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}|\right\} \end{aligned}$$

holds for $\omega \in R$. $\lambda(\mathrm{diag}\{\frac{\kappa_i}{(\mathrm{j}\omega)^2 + \mathrm{j}\omega\alpha_i}\mathrm{e}^{-\mathrm{j}\omega T_i}, i \in \mathscr{I}\} L(\mathrm{j}\omega))$ does not enclose the point $(-1, \mathrm{j}0)$ for $\omega \in R$, as long as the point $(-a, \mathrm{j}0)$ with $a \geq 1$ does not in the disk $\{\zeta : \zeta \in C, |\zeta - \frac{\kappa_i d_i}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}| \leq |\frac{\kappa_i d_i}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}|\}$ for all $\omega \in R$, i.e., $| - a + \mathrm{j}0 - \frac{\kappa_i d_i}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}| > |\frac{\kappa_i d_i}{-\omega^2 + \mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}|$ holds for all $\omega \in R$ and $a \geq 1$. By calculating, we obtain

$$
\begin{aligned}
&|-a+\mathrm{j}0-\frac{\kappa_i d_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}|^2-|\frac{\kappa_i d_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}|^2\\
&=a\left(a-2\kappa_i d_i\frac{\cos(\omega T_i)+\alpha_i\frac{\sin(\omega T_i)}{\omega}}{\omega^2+\alpha_i^2}\right).
\end{aligned}
$$

Since $\cos(\omega T_i)\leq 1$ and $\frac{\sin(\omega T_i)}{\omega}\leq T_i$ hold for $\omega\in R$, we get from (2.16) that

$$
2\kappa_i d_i\frac{\cos(\omega T_i)+\alpha_i\frac{\sin(\omega T_i)}{\omega}}{\omega^2+\alpha_i^2}\leq\frac{2\kappa_i d_i(1+\alpha_i T_i)}{\alpha_i^2}<1.
$$

Thus, $|-a+\mathrm{j}0-\frac{\kappa_i d_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}|>|\frac{\kappa_i d_i}{-\omega^2+\mathrm{j}\alpha_i\omega}\mathrm{e}^{-\mathrm{j}T_i\omega}|$ holds for all $\omega\in R$ and $a\geq 1$. According to Lemma 2.1, the zeros of $\tilde{\mathscr{F}}(s)$ all have negative real parts. Then, $\tilde{\mathscr{D}}(s)$ has its zeros on the open left half complex plane except for a zero at $s=0$. Similar to the proof of Theorem 2.4, therefore, the system (2.12) converges to a stationary consensus asymptotically. Theorem 2.5 is proved. □

Remark 2.4 By using Gershgorin's disk theorem, a general decentralized frequency-domain consensus condition has been obtained for the multi-agent systems with agents' dynamics modeled by strictly stable linear systems under diverse communication delays [7]. In fact, the system (2.12) can be expressed as a special case of the system studied in [7]. However, the consensus condition (2.16) in Theorem 2.5 provides a concrete algebraic criterion for designing the consensus algorithm conveniently.

Remark 2.5 Based on the inequality (24) in [8], the condition (2.15) in Theorem 2.4 satisfies $\frac{1}{2\kappa_i M_i^{-1}}>\frac{\alpha_i}{2\kappa_i T_i}>\frac{\alpha_i^2}{2\kappa_i(1+\alpha_i T_i)}$, so the condition (2.16) in Theorem 2.5 is more conservative than the condition (2.15) in Theorem 2.4.

Example 2.3 Investigate a directed interconnection topology composed of six agents modeled by (2.12) (see Fig. 2.6). Obviously, the topology in Fig. 2.6 has a spanning tree. The weights of the directed edges are as follows: $a_{15}=0.3$, $a_{21}=0.25$, $a_{32}=0.15$, $a_{36}=0.6$, $a_{41}=0.45$, $a_{54}=0.1$, $a_{51}=0.4$, $a_{56}=0.5$, and $a_{63}=0.15$, and the corresponding communication delays are as follows: $\tau_{15}=0.1\,(\mathrm{s})$, $\tau_{21}=0.15\,(\mathrm{s})$, $\tau_{32}=0.15\,(\mathrm{s})$, $\tau_{36}=0.1\,(\mathrm{s})$, $\tau_{41}=0.15\,(\mathrm{s})$, $\tau_{54}=0.15\,(\mathrm{s})$, $\tau_{51}=0.1\,(\mathrm{s})$, $\tau_{56}=0.05\,(\mathrm{s})$, and $\tau_{63}=0.15$. The velocity damping coefficients of the agents

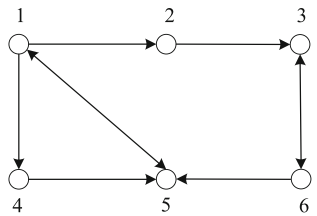


Fig. 2.6 Digraph composed of six agents

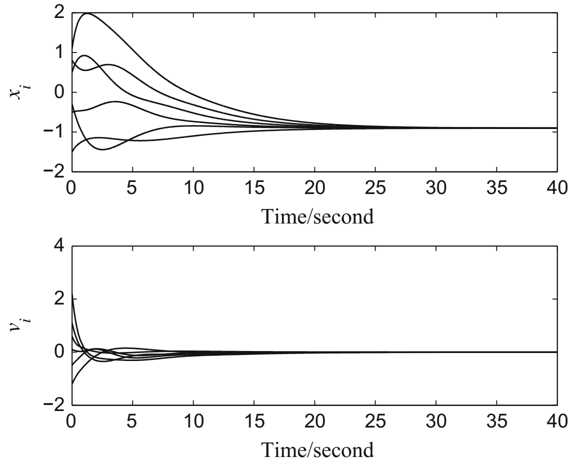


Fig. 2.7 Consensus convergence with input and communication delays

are as follows: $\alpha_1 = 1.5$, $\alpha_2 = 2$, $\alpha_3 = 1$, $\alpha_4 = 2$, $\alpha_5 = 3$, and $\alpha_6 = 3$. Choosing the control parameters: $\kappa_1 = 1.5$, $\kappa_2 = 2$, $\kappa_3 = 0.3$, $\kappa_4 = 2$, $\kappa_5 = 1.5$, and $\kappa_6 = 2.5$, we obtain from the condition (2.16) that the constraints on the input delays are as follows: $T_1 \in (0, 1)$ (s), $T_2 \in (0, 1.5)$ (s), $T_3 \in (0, 1.22)$ (s), $T_4 \in (0, 0.61)$ (s), $T_5 \in (0, 0.67)$ (s), and $T_6 \in (0, 3.67)$. With $T_1 = 0.7$ (s), $T_2 = 0.8$ (s), $T_3 = 0.6$ (s), $T_4 = 0.2$ (s), $T_5 = 0.6$ (s), and $T_6 = 0.5$ (s), the agents in the system (2.12) converge to a stationary consensus asymptotically (see Fig. 2.7).

2.3.2 *Dynamical Consensus Algorithms with Time Delays*

Different from stationary consensus algorithms, collective behaviors caused by dynamical consensus algorithm are quite distinct between the synchronously coupled form and asynchronously coupled.

Two continuous-time second-order agents (1.5) are modeled by

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= u_i(t), i = 1, 2, \end{aligned} \tag{2.17}$$

and the dynamical consensus algorithm is presented as

$$u_1(t) = \kappa((x_2(t) - x_1(t)) + \gamma(v_2(t) - v_1(t))),$$

$$u_2(t) = \kappa((x_1(t) - x_2(t)) + \gamma(v_1(t) - v_2(t))),$$

where $\kappa > 0$ and $\gamma > 0$.

2.3.2.1 Consensus Convergence with Communication Delay

Asynchronously coupled form of the dynamical consensus algorithms with communication delay is expressed as

$$u_1(t) = \kappa((x_2(t-\tau) - x_1(t)) + \gamma(v_2(t-\tau) - v_1(t))), \tag{2.18}$$

$$u_2(t) = \kappa((x_1(t-\tau) - x_2(t)) + \gamma(v_1(t-\tau) - v_2(t))), \tag{2.19}$$

where $\tau > 0$ is the communication delay.

Driven by (2.18) and (2.19), the second-order agents' dynamics become

$$\begin{aligned} \dot{x}_1(t) &= v_1(t), \\ \dot{v}_1(t) &= \kappa((x_2(t-\tau) - x_1(t)) + \gamma(v_2(t-\tau) - v_1(t))), \end{aligned} \tag{2.20}$$

$$\begin{aligned} \dot{x}_2(t) &= v_2(t), \\ \dot{v}_2(t) &= \kappa((x_1(t-\tau) - x_2(t)) + \gamma(v_1(t-\tau) - v_2(t))). \end{aligned} \tag{2.21}$$

Proposition 2.1 *Two coupled agents (2.20) and (2.21) achieve a stationary consensus asymptotically, if and only if*

$$\frac{2\kappa}{\omega_{c1}^2} < 1, \tag{2.22}$$

where ω_{c1} satisfies

$$\frac{\omega_{c1}\tau}{2} = \arctan(\gamma\omega_{c1}), \ \omega_{c1} > 0. \tag{2.23}$$

Before proving Proposition 2.1, we give two useful lemmas firstly.

Lemma 2.4 *The following equation*

$$s^2 + \kappa(\gamma s + 1)(1 + \mathrm{e}^{-s\tau}) = 0, \tag{2.24}$$

where $\kappa > 0$, $\gamma > 0$ and $\tau > 0$, has its roots lying on the open left half complex plane, if and only if (2.22) holds.

Proof The Eq. (2.24) equals

$$1 + g(s) = 0,$$

where $g(s) = \frac{\kappa(\gamma s+1)}{s^2}(1 + \mathrm{e}^{-s\tau})$. Based on the Nyquist stability criterion, the roots of (2.24) lie on the open left half complex plane, if and only if $g(\mathrm{j}\omega)$ does not enclose the point $(-1, \mathrm{j}0)$ for $\omega \in R$. By computing, we get

$$g(\mathrm{j}\omega) = \frac{2\kappa\cos(\frac{\omega\tau}{2})\sqrt{\gamma^2\omega^2+1}}{\omega^2}\mathrm{e}^{-\mathrm{j}(\pi + \frac{\omega\tau}{2} - \arctan(\gamma\omega))}$$

and

$$\arg(g(\mathrm{j}\omega)) = \begin{cases} -\pi - \frac{\omega\tau}{2} + \arctan(\gamma\omega), & 2n\pi < \frac{\omega\tau}{2} < 2n\pi + \frac{\pi}{2}; \\ -\frac{\omega\tau}{2} + \arctan(\gamma\omega), & 2n\pi + \frac{\pi}{2} < \frac{\omega\tau}{2} < 2n\pi + \frac{3\pi}{2}; \\ -\pi - \frac{\omega\tau}{2} + \arctan(\gamma\omega), & 2n\pi + \frac{3\pi}{2} < \frac{\omega\tau}{2} < 2(n+1)\pi, \end{cases}$$

where $n \geq 0$ is nonnegative integer.

For $\omega \in (0, +\infty)$, $g(\mathrm{j}\omega)$ crosses the negative real axis for the first time at ω_{c1} satisfying

$$\frac{\omega_{c1}\tau}{2} = \arctan(\gamma\omega_{c1})$$

and

$$|g(\mathrm{j}\omega_{c1})| = 2\frac{\kappa}{\omega_{c1}^2}.$$

$g(\mathrm{j}\omega)$ crosses the negative real axis for the second time at ω_{c2} satisfying

$$\frac{\omega_{c2}\tau}{2} = \pi + \arctan(\gamma\omega_{c2})$$

and

$$|g(\mathrm{j}\omega_{c2})| = 2\frac{\kappa}{\omega_{c2}^2}.$$

Apparently, $|g(\mathrm{j}\omega_{c1})| > |g(\mathrm{j}\omega_{c2})|$ for $\omega_{c1} < \omega_{c2}$. In the same way, we obtain that $g(\mathrm{j}\omega)$ crosses the negative real axis for the kth time at ω_{ck} and $|g(\mathrm{j}\omega_{ck})| = 2\frac{\kappa}{\omega_{ck}^2}$, $i = 3, \ldots, \infty$. Thus, $|g(\mathrm{j}\omega_{c1})| > |g(\mathrm{j}\omega_{ck})|$, $k = 3, \ldots, \infty$.

Therefore, the roots of (2.24) are all on the open left half complex plane, if and only if $|g(\mathrm{j}\omega_{c1})| < 1$ that is equivalent to (2.22). Lemma 2.4 is proved. □

Lemma 2.5 *The following equation*

$$s^2 + \kappa(\gamma s + 1)(1 - \mathrm{e}^{-s\tau}) = 0, \tag{2.25}$$

where $\kappa > 0$, $\gamma > 0$ and $\tau > 0$, has its roots lying on the open left half complex plane except for one root at $s = 0$, if and only if

$$\frac{2\kappa}{\hat{\omega}_{c1}^2} < 1, \tag{2.26}$$

where $\hat{\omega}_{c1}$ satisfies

$$\frac{\hat{\omega}_{c1}\tau}{2} = \frac{\pi}{2} + \arctan(\gamma\hat{\omega}_{c1}), \hat{\omega}_{c1} > 0.$$

Proof Referring to Taylor series expression of $\mathrm{e}^{-s\tau}$ at $s = 0$, we rewrite (2.25) as

$$s(s+\kappa(\gamma s+1)\left(\tau-\sum_{k=2}^{\infty}\left(\frac{(-\tau)^k s^{k-1}}{k!})\right)\right)=0.$$

Obviously, the above equation has only one root at $s=0$ with $\tau>0$.

Let $f(s)=1+\hat{g}(s)$ for $s\neq 0$, and $\hat{g}(s)=\frac{\kappa(\gamma s+1)}{s^2}(1-\mathrm{e}^{-s\tau})$. Then, we obtain

$$\hat{g}(\mathrm{j}\omega)=\frac{2\kappa\sin(\frac{\omega\tau}{2})\sqrt{\gamma^2\omega^2+1}}{-\omega^2}\mathrm{e}^{-\mathrm{j}(\frac{\pi}{2}+\frac{\omega\tau}{2}-\arctan(\gamma\omega))}.$$

As the proof of Lemma 2.4, we can prove that the zeros of $f(s)$ lie on the open left half complex plane, if and only if (2.26) holds. Thus, the roots of (2.25) all lie on the open left half complex plane except for one root at $s=0$, if and only if (2.26) holds. Lemma 2.5 is proved. □

To prove Proposition 2.1, we proceed in two steps.

Step 1. Let $\bar{x}(t)=x_1(t)-x_2(t)$ and $\bar{v}(t)=v_1(t)-v_2(t)$, and we get

$$\begin{aligned}\dot{\bar{x}}(t)&=\bar{v}(t),\\ \dot{\bar{v}}(t)&=\kappa((-\bar{x}(t-\tau)-\bar{x}(t))+\gamma(-\bar{v}(t-\tau)-\bar{v}(t))).\end{aligned}\tag{2.27}$$

The characteristic equation of (2.27) about $\bar{x}(t)$ is given by

$$s^2+\kappa(\gamma s+1)(1+\mathrm{e}^{-s\tau})=0.$$

From Lemma 2.4, the system (2.27) is asymptotically stable, i.e., $\lim_{t\to\infty}(x_1(t)-x_2(t))=0$ and $\lim_{t\to\infty}(v_1(t)-v_2(t))=0$, if and only if (2.22) holds, which further implies that the agents (2.20) and (2.21) achieve the position consensus and velocity consensus asymptotically.

Step 2. Rewrite the coupled agents (2.20) and (2.21) in a multivariable form as follows

$$\begin{aligned}\dot{x}(t)&=v(t),\\ \dot{v}(t)&=-\kappa\gamma v(t)-\kappa x(t)+\kappa\gamma\begin{bmatrix}0&1\\1&0\end{bmatrix}v(t-\tau)+\kappa\begin{bmatrix}0&1\\1&0\end{bmatrix}x(t-\tau),\end{aligned}$$

where $x=[x_1,x_2]^T$ and $v=[v_1,v_2]^T$, and the characteristic equation about $x(t)$ is

$$\det\left(\begin{bmatrix}s^2+\kappa(\gamma s+1) & -\kappa(\gamma s+1)\mathrm{e}^{-s\tau}\\ -\kappa(\gamma s+1)\mathrm{e}^{-s\tau} & s^2+\kappa(\gamma s+1)\end{bmatrix}\right)=0,\tag{2.28}$$

which equals

$$s^2+\kappa(\gamma s+1)(1+\mathrm{e}^{-s\tau})=0;\tag{2.29a}$$

or

$$s^2 + \kappa(\gamma s + 1)(1 - \mathrm{e}^{-s\tau}) = 0. \tag{2.29b}$$

It follows from Lemma 2.4 that the roots of (2.29a) all lie on the open left complex plane if and only if (2.22) holds. When (2.22) holds, in addition, the roots of (2.29b) are on the open left half complex plane except for one root at $s = 0$ from Lemma 2.5.

Therefore, the roots of (2.28) are on the open left complex plane except for one root at $s = 0$. From the proof in Step 1, in turn, we obtain that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = c$, where c is a constant and $\lim_{t\to\infty} v_1(t) = \lim_{t\to\infty} v_2(t) = 0$.

Hence, the agents (2.20) and (2.21) achieve a stationary consensus asymptotically. Proposition 2.1 is proved. □

Remark 2.6 Necessary and sufficient conditions (2.22) and (2.23) in Proposition 2.1 are not convenient to gain the delay bound. From (2.23), we have $\tau = \frac{2\arctan(\gamma\omega_{c1})}{\omega_{c1}}$, and the derivative of τ with respect to ω_{c1} is

$$\frac{d\tau}{d\omega_{c1}} = 2\frac{\frac{\gamma\omega_{c1}}{1+\gamma^2\omega_{c1}^2} - \arctan(\gamma\omega_{c1})}{\omega_{c1}^2}.$$

Let $\psi(\omega_{c1}) = \frac{\gamma\omega_{c1}}{1+\gamma^2\omega_{c1}^2} - \arctan(\gamma\omega_{c1})$ and

$$\frac{d\psi(\omega_{c1})}{d\omega_{c1}} = -\frac{2\gamma^3\omega_{c1}^2}{(1+\gamma^2\omega_{c1}^2)^2} < 0$$

holds for $\omega_{c1} > 0$, i.e., $\psi(\omega_{c1})$ is monotonously decreasing for $\omega_{c1} > 0$. Since $\psi(0) = 0$, $\psi(\omega_{c1}) < 0$ holds for $\omega_{c1} > 0$, i.e., $\frac{d\tau}{d\omega_{c1}} < 0$ holds for $\omega_{c1} > 0$. According to (2.23), then, τ decreases when ω_{c1} increases. The inequality (2.22) is equivalent to

$$\omega_{c1} > \sqrt{2\kappa}.$$

Thus, the largest value of τ is determined by the conditions (2.22) and (2.23). Moreover, a simple algebraic condition, which only depends on the control parameters and the communication delay, provides alternative necessary and sufficient conditions in Proposition 2.1 and is shown as follows.

Theorem 2.6 *Two coupled agents (2.20) and (2.21) achieve a stationary consensus asymptotically, if and only if*

$$\tau < 2\frac{\arctan(\gamma\sqrt{2\kappa})}{\sqrt{2\kappa}}. \tag{2.30}$$

Remark 2.7 From (2.30) and the monotonicity analysis in Remark 2.6, it is obvious that large γ or small κ should be chosen to tolerate larger communication delay.

Remark 2.8 Evidently, the dynamical consensus algorithm in asynchronously coupled form can drive the second-order agents (2.20) and (2.21) to achieve a stationary consensus if the control parameters and communication delay satisfy some certain consensus conditions.

Remark 2.9 Even though consensus seeking problem has been analyzed for the second-order multi-agent systems with asynchronously coupled dynamical consensus algorithm, the conditions are much more conservative and the bound of communication delays cannot be easily calculated [9–11]. A simple case that two second-order agents are coupled with each other is investigated, but we provide a sufficient and necessary condition (2.30) in Theorem 2.6 to describe the exact relationship between the consensus convergence and the communication delay. This facilitates us to derive algebraic criterion for choosing the control parameters conveniently.

Example 2.4 Choose the control parameters $\kappa = 0.5$ and $\gamma = 1$ for two coupled agents (2.20) and (2.21). By calculating from the condition (2.30), we get $\tau \in (0, \tau_{\max})$(s) with $\tau_{\max} = \frac{\pi}{2}$. When $\tau < \tau_{\max}$, e.g., $\tau = 0.4$(s), the two agents reach a stationary consensus asymptotically (see Fig. 2.8). Two agents' states oscillate with $\tau = \tau_{\max}$ (see Fig. 2.9a) or diverge with $\tau > \tau_{\max}$ (see Fig. 2.9b).

The curved surface in Fig. 2.10 shows the largest tolerable communication delay $\tau_{\max}$ changing with respect to κ and γ, and the consensus-achieved area is below the curved surface. Obviously, $\tau_{\max}$ increases when γ increases with some certain κ, while $\tau_{\max}$ decreases when κ increases by fixing γ.

2.3.2.2 Consensus Convergence with Input and Communication Delay

Next, we investigate two coupled agents (2.17) with both communication delay and input delay as follows:

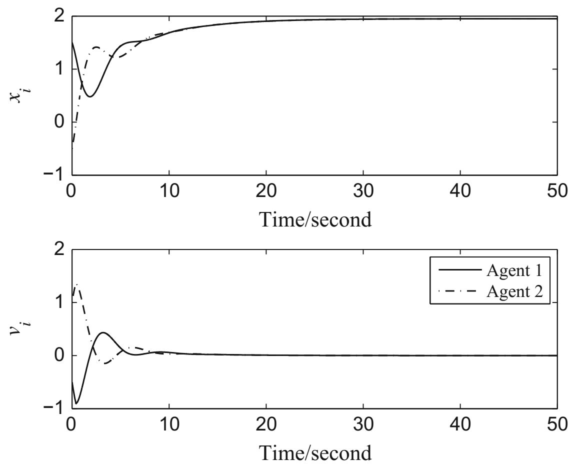
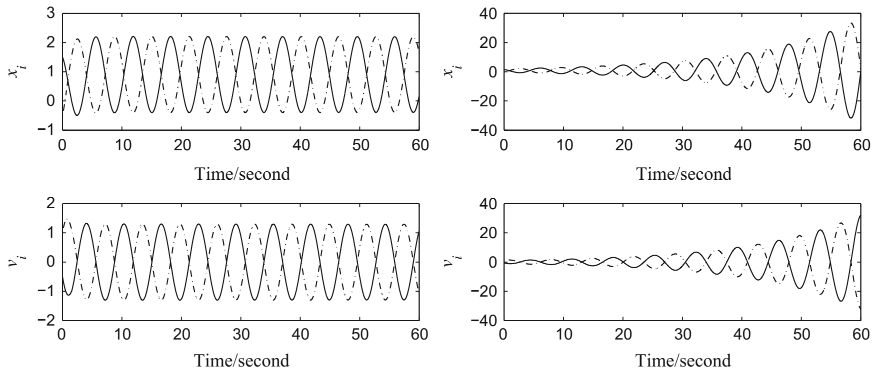


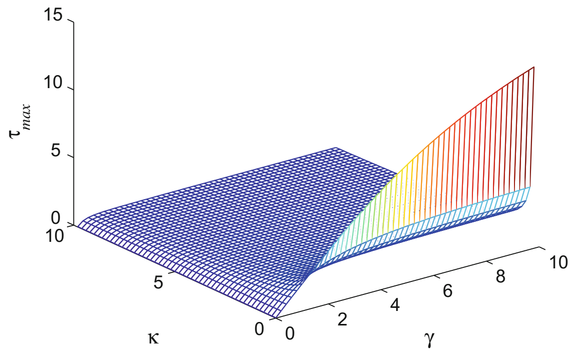
Fig. 2.8 Consensus convergence of coupled second-order agents



(a) Oscillations of agents' states
(b) Divergence of agents' states

Fig. 2.9 Instability of coupled second-order agents

Fig. 2.10 Bound of communication delay τ



$$\dot{x}_1(t) = \kappa((x_2(t - \tau - T) - x_1(t - T)) + \gamma(v_2(t - \tau - T) - v_1(t - T))), \tag{2.31}$$

$$\dot{x}_2(t) = \kappa((x_1(t - \tau - T) - x_2(t - T)) + \gamma(v_1(t - \tau - T) - v_2(t - T))), \tag{2.32}$$

where $\tau > 0$ and $T > 0$ are the communication delay and input delay, respectively.

Based on Theorem 2 in [12], we gain the following consensus condition of the coupled agents with only input delay.

Theorem 2.7 ([12]) *The coupled agents (2.31) and (2.32) without communication delay, i.e.,* $\tau = 0$, *reach a dynamical consensus asymptotically, if and only if*

$$T < \frac{\arctan(\gamma\sqrt{2\kappa^2\gamma^2 + 2\kappa\sqrt{\kappa^2\gamma^4 + 1}})}{\sqrt{2\kappa^2\gamma^2 + 2\kappa\sqrt{\kappa^2\gamma^4 + 1}}}. \tag{2.33}$$

Different from first-order agents, consensus convergence of second-order agents (2.31) and (2.32) is dependent on the communication delay regardless of the value of the control parameters and the input delay.

Then, we adopt the same analysis method as in Proposition 2.1 to obtain consensus criteria for two coupled agents (2.31) and (2.32).

Proposition 2.2 *Two coupled agents (2.31) and (2.32) converge to a stationary consensus asymptotically, if*

$$\frac{2\kappa\sqrt{1+(\gamma\omega_{c1})^2}}{\omega_{c1}^2} < 1 \tag{2.34}$$

holds, where ω_{c1} satisfies

$$\omega_{c1}T + \frac{\omega_{c1}\tau}{2} = \arctan(\gamma\omega_{c1}), \omega_{c1} > 0. \tag{2.35}$$

Similarly, two useful lemmas are listed firstly.

Lemma 2.6 *The roots of the following equation*

$$s^2 + \kappa(\gamma s + 1)(1 + \mathrm{e}^{-s\tau})\mathrm{e}^{-sT} = 0 \tag{2.36}$$

where $\kappa > 0$, $\gamma > 0$, $\tau > 0$ and $T > 0$, lie on the open left half complex plane, if and only if (2.34) and (2.35) hold.

Proof Set $\bar{g}(s) = \frac{\kappa(\gamma s+1)\mathrm{e}^{-sT}}{s^2}(1+\mathrm{e}^{-s\tau})$, and (2.36) is equivalent to $1 + \bar{g}(s) = 0$. For $\omega \in (0, +\infty)$, $\bar{g}(j\omega)$ crosses the negative real axis for the first time at ω_{c1} defined in (2.35), and for the kth time at $\omega_{ck} > \omega_{c1}$, $k \geq 2$. By computing, $|\bar{g}(\mathrm{j}\omega_{ck})| < \frac{2\kappa\sqrt{1+(\gamma\omega_{c1})^2}}{\omega_{c1}^2}$, $k = 1, \ldots, \infty$. Thus, $\hat{g}(j\omega)$ with $\omega \in (-\infty, +\infty)$ does not enclose the point $(-1, \mathrm{j}0)$ if (2.34) holds. Lemma 2.6 is proved. □

Lemma 2.7 *The roots of the following equation*

$$s^2 + \kappa(\gamma s + 1)(1 - \mathrm{e}^{-s\tau})\mathrm{e}^{-sT} = 0, \tag{2.37}$$

where $\kappa > 0$, $\gamma > 0$, $\tau > 0$ and $T > 0$, are on the open left half complex plane except for one root at $s = 0$, if and only if

$$\frac{2\kappa\sqrt{1+(\gamma\tilde{\omega}_{c1})^2}}{\tilde{\omega}_{c1}^2} < 1, \tag{2.38}$$

where $\tilde{\omega}_{c1}$ satisfies

$$\tilde{\omega}_{c1}T + \frac{\tilde{\omega}_{c1}\tau}{2} = \frac{\pi}{2} + \arctan(\gamma\tilde{\omega}_{c1}), \tilde{\omega}_{c1} > 0.$$

Proof Analogous to the proof of Lemma 2.5, firstly, we can prove the Eq. (2.37) has only one root at $s = 0$ with $\tau > 0$.

Define $\tilde{g}(s) = \frac{\kappa(\gamma s+1)e^{-sT}}{s^2}(1 - e^{-s\tau})$ for $s \neq 0$, and the Eq. (2.37) is rewritten as $1 + \tilde{g}(s) = 0$. Similar to the proof of Lemma 2.6, $\tilde{g}(j\omega)$ with $\omega \in (-\infty, +\infty)$ does not enclose the point $(-1, j0)$, if (2.38) holds. Lemma 2.7 is proved. □

Now, we prove Proposition 2.2.

Reformulate the agents (2.31) and (2.32) in a multivariable form as

$$\begin{aligned}\dot{x}(t) &= v(t),\\ \dot{v}(t) &= -\kappa\gamma v(t-T) + \kappa\gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} v(t-T-\tau)\\ &\quad -\kappa x(t-T) + \kappa \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t-T-\tau),\end{aligned} \tag{2.39}$$

where $x = [x_1, x_2]^T$ and $v = [v_1, v_2]^T$, and the characteristic equation about $x(t)$ is described as

$$\det\left(\begin{bmatrix} s^2 + \kappa(\gamma s+1)e^{-sT} & -\kappa(\gamma s+1)e^{-s(T+\tau)} \\ -\kappa(\gamma s+1)e^{-s(T+\tau)} & s^2 + \kappa(\gamma s+1)e^{-sT} \end{bmatrix}\right) = 0, \tag{2.40}$$

which is equivalent to

$$s^2 + \kappa(\gamma s+1)(1 + e^{-s\tau})e^{-sT} = 0; \tag{2.41a}$$

or

$$s^2 + \kappa(\gamma s+1)(1 - e^{-s\tau})e^{-sT} = 0. \tag{2.41b}$$

Based on Lemmas 2.6 and 2.7, the roots of (2.40) are all on the open left half complex plane except for one root at $s = 0$, i.e., $\lim_{t\to\infty} x(t) = [c_1, c_2]^T$ and $\lim_{t\to\infty} v(t) = 0$, where c_1 and c_2 are two constants. Thus,

$$-\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0,$$

i.e., $c_1 = c_2$. Therefore, two coupled agents (2.31) and (2.32) achieve a stationary consensus asymptotically if (2.34) holds. Proposition 2.2 is proved. □

Remark 2.10 Let $E = T + \frac{\tau}{2}$, (2.35) is equivalent to $E = \frac{\arctan(\gamma\omega_{ca})}{\omega_{c1}}$. Similar to the discussion in Remark 2.6, we can get from the condition (2.35) that E increases when ω_{c1} decreases. The condition (2.34) is rewritten as

$$\omega_{c1} > \sqrt{2\kappa^2\gamma^2 + 2\kappa\sqrt{\kappa^2\gamma^4 + 1}}.$$

Hence, the largest value of E is determined by the conditions (2.34) and (2.35). Then, the conditions in Proposition 2.2 can be re-expressed.

Theorem 2.8 *The coupled agents (2.31) and (2.32) reach a stationary consensus asymptotically, if*

$$T + \frac{\tau}{2} < \frac{\arctan(\gamma\sqrt{2\kappa^2\gamma^2 + 2\kappa\sqrt{\kappa^2\gamma^4 + 1}})}{\sqrt{2\kappa^2\gamma^2 + 2\kappa\sqrt{\kappa^2\gamma^4 + 1}}}. \tag{2.42}$$

Remark 2.11 Comparing the consensus condition (2.42) in Theorem 2.8 with (2.33) in Theorem 2.7 yields the conclusion that the existence of communication delay decreases the largest input delay that the system can tolerate.

2.4 Multiple Single-Input and Single-Output Linear Agents

2.4.1 Agents' Dynamics Description

In this section, we investigate a general linear multi-agent system, which consists of n identical single-input and single-output linear agents agentsmodeled by a strictly proper transfer function $G(s)$,

$$\begin{aligned} Y_i(s) &= G(s)U_i(s) = \frac{n(s)}{d(s)}\mathrm{e}^{-sT}U_i(s), i = 1, \ldots, n, \\ n(s) &= n_0 + n_1 s + \cdots + n_m s^m, \\ d(s) &= d_0 + d_1 s + \cdots + d_l s^l, \end{aligned} \tag{2.43}$$

where $Y_i(s)$ and $U_i(s)$ are the Laplace transforms of the output $y_i(t)$ and the input $u_i(t)$, respectively, $T > 0$ is the identical input delay, $n(s)$ and $d(s)$ are polynomials with real coefficients.

To avoid that the asymptotic consensus is reached only at the origin, the polynomial $d(s)$ usually has at least one root at the origin. Meanwhile, $l > m$ guarantees that $G(s)$ is strictly proper. Thus, the same assumption on $G(s)$ as that in [13] is made as follows.

Assumption 2.1 The polynomial of $d(s)$ has at least one root at the origin, $l > m$, and $n(s)$ and $d(s)$ are coprime.

Definition 2.1 If the agents' states under arbitrary initial values satisfy: $\lim_{t\to\infty} (y_i(t) - y_j(t)) = 0, \forall i, j \in \mathscr{I}$, we say that the agents (2.43) asymptotically converge to the *Output Consensus*.

For the agents (2.43), we adopt a simple output consensus algorithm as follows

$$u_i(t) = \frac{\kappa}{d_i} \sum_{j \in N_i} a_{ij}(y_j(t-\tau) - y_i(t)), \tag{2.44}$$

where $\kappa > 0$, $a_{ij} > 0$, $j \in N_i$, $d_i = \sum_{j \in N_i} a_{ij}$, and $\tau > 0$ is the identical communication delay. It is worth noting that the algorithm (2.44) satisfies Assumption 1.1 in the following consensus analysis.

With (2.44), the closed-loop form of system (2.43) is given by

$$Y_i(s) = \frac{\kappa n(s)\mathrm{e}^{-sT}}{d_i d(s)} \sum_{j \in N_i} a_{ij}(Y_j(s)\mathrm{e}^{-s\tau} - Y_i(s)), \tag{2.45}$$

which is rewritten as

$$Y_i(s) = \frac{\kappa n(s)\mathrm{e}^{-sT}}{d_i d(s)} \left(\sum_{j \in N_i} a_{ij}(Y_j(s) - Y_i(s))\mathrm{e}^{-s\tau} - d_i(1 - \mathrm{e}^{-s\tau})Y_i(s) \right), \tag{2.46}$$

whose characteristic equation about $y = [y_1, \ldots, y_n]^T$ is given by

$$\det(d(s)I + \kappa n(s)\mathrm{e}^{-sT}(D^{-1}L\mathrm{e}^{-s\tau} + (1 - \mathrm{e}^{-s\tau})I)) = 0. \tag{2.47}$$

Take the nonsingular transforms for $D^{-1}L$ with the transform matrix given by

$$Q = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 1 \end{bmatrix},$$

and we get

$$QD^{-1}LQ^{-1} = \begin{bmatrix} 0 & \mathbf{p}^T \\ \mathbf{0} & \bar{L} \end{bmatrix}, \tag{2.48}$$

where $\mathbf{0} = [0, 0, \ldots, 0]^T \in R^{n-1}$, $\mathbf{p} = [\frac{l_{12}}{d_1}, \frac{l_{13}}{d_1}, \ldots, \frac{l_{1n}}{d_1}]^T \in R^{n-1}$, and $\bar{L}$ is defined as

$$\bar{L} = \begin{bmatrix} \frac{l_{22}}{d_2} - \frac{l_{12}}{d_1} & \frac{l_{23}}{d_2} - \frac{l_{13}}{d_1} & \cdots & \frac{l_{2n}}{d_2} - \frac{l_{1n}}{d_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{l_{n2}}{d_n} - \frac{l_{12}}{d_1} & \frac{l_{n3}}{d_n} - \frac{l_{13}}{d_1} & \cdots & \frac{l_{nn}}{d_n} - \frac{l_{1n}}{d_1} \end{bmatrix}, \tag{2.49}$$

where l_{ij} is the element of the Laplacian matrix L.

From (2.48), therefore, the characteristic equation (2.47) equals

$$d(s) + \kappa n(s)\mathrm{e}^{-sT}(1 - \mathrm{e}^{-s\tau}) = 0.$$

or

$$\det(d(s)I + \kappa n(s)\mathrm{e}^{-sT}(\bar{L}\mathrm{e}^{-s\tau} + (1 - \mathrm{e}^{-s\tau})I)) = 0. \tag{2.50}$$

Besides, let $\bar{y}_i = y_i - y_1, i = 2, \ldots, n$, and the dynamics of $\bar{y}_i, i = 2, \ldots, n$ are

$$\begin{aligned}\bar{Y}_i(s) = \kappa \frac{n(s)\mathrm{e}^{-sT}}{d(s)} &\left(\frac{1}{d_i}\sum_{j\in N_i} a_{ij}(\bar{Y}_j(s) - \bar{Y}_i(s)\right)\mathrm{e}^{-s\tau} \\ &-\frac{1}{d_1}\sum_{j\in N_1} a_{1j}\bar{Y}_j(s)\mathrm{e}^{-s\tau} - (1 - \mathrm{e}^{-s\tau})\bar{Y}_i(s)).\end{aligned} \tag{2.51}$$

The consensus problem of system (2.45) turns to be the asymptotic stability problem of system (2.51). Moreover, (2.50) is just the characteristic equation of system (2.51) about $\bar{y} = [\bar{y}_2, \ldots, \bar{y}_n]^T$.

Again, the characteristic equation (2.47) is equivalent to

$$d(s) + \kappa n(s)\mathrm{e}^{-sT}(1 - \mathrm{e}^{-s\tau}) = 0. \tag{2.52}$$

or

$$d(s) + \kappa n(s)\mathrm{e}^{-sT}(\bar{\lambda}_i\mathrm{e}^{-s\tau} + (1 - \mathrm{e}^{-s\tau})) = 0, i = 2, \ldots, n. \tag{2.53}$$

where $\bar{\lambda}_i, i = 2, \ldots, n$ are the nonzero eigenvalues of $D^{-1}L$.

Remark 2.12 Based on the above discussion, the roots of the (2.53) determine whether the agents (2.45) could achieve an asymptotic consensus, while the root distribution of (2.52) determines the final consensus behavior.

2.4.2 Delay-Dependent Consensus Criterion

Final consensus behavior of multi-agent systems (2.45) is analyzed firstly.

Proposition 2.3 *The Eq. (2.52) just has one root at $s = 0$.*

Proof Based on the Taylor series expression of $\mathrm{e}^{-s\tau}$ at $s = 0$, the Eq. (2.52) is expressed as

$$d(s) + \kappa n(s)\mathrm{e}^{-sT}\left(\tau s - \sum_{k=2}^{\infty}\frac{(-\tau)^k s^k}{k!}\right) = 0, \tag{2.54}$$

which is rewritten as

$$s\left(\frac{d(s)}{s}+\kappa n(s)\mathrm{e}^{-sT}\left(\tau-\sum_{k=2}^{\infty}\frac{(-\tau)^k s^{k-1}}{k!}\right)\right)=0. \tag{2.55}$$

Evidently, the Eq. (2.55) just has one root at $s=0$. Proposition 2.3 is proved. □

Remark 2.13 Similar to the consensus analysis for second-order multiple agents in Sect. 2.3.2, asynchronously coupled form changes the original consensus behavior of the consensus algorithms without communication delays.

Consensus convergence criterion for the system (2.45) is presented as follows.

Proposition 2.4 *Assume that the system (2.45) without communication delay converges to a consensus asymptotically. Let*

$$\bar{m}_i(s)=\frac{\kappa\hat{G}(s)(1-\bar{\lambda}_i)}{1+\kappa\hat{G}(s)\bar{\lambda}_i}, i=2,\ldots,n, \tag{2.56}$$

where $\hat{G}(s)=\frac{n(s)}{d(s)}\mathrm{e}^{-sT}$ *and* $\bar{\lambda}_i, i=2,\ldots,n$ *are the nonzero eigenvalues of* $D^{-1}L$*. Then, all the agents (2.45) with both input delay and communication delay asymptotically converge to an asymptotic consensus, if*

$$2|\bar{m}_i(\mathrm{j}\omega)\sin\left(\frac{\omega\tau}{2}\right)|<1 \tag{2.57}$$

holds for $i=2,\ldots,n$*.*

Proof Under the assumption in Proposition 2.4, the system (2.45) without communication delay achieve an asymptotic consensus, i.e., the roots of the following equation

$$\det(d(s)I+\kappa n(s)\mathrm{e}^{-sT}\bar{L})=0.$$

all lie on the open left half complex plane, so we obtain $\det(\bar{L})\neq 0$, i.e., $D^{-1}L$ has one zero eigenvalue and other eigenvalues all have positive real parts. Hence, the roots of the following equation

$$1+\kappa\hat{G}(s)\lambda_i=0, i=2,\ldots,n,$$

all lie on the open left half complex plane.

Then, the Eq. (2.53) is reformulated as

$$1+\bar{m}_i(s)(1-\mathrm{e}^{-s\tau})=0, i=2,\ldots,n. \tag{2.58}$$

Obviously, $1-\mathrm{e}^{-s\tau}$ and $\bar{m}_i(s)=\frac{\kappa\hat{G}(s)(1-\lambda_i)}{1+\kappa\hat{G}(s)\lambda_i}$ both have no poles in the open right half complex plane from the assumption in Proposition 2.4.

From (2.57), we obtain that

$$\begin{aligned} & |\bar{m}_i(\mathrm{j}\omega)(1-\mathrm{e}^{-\mathrm{j}\omega\tau})| \\ = & 2|\bar{m}_i(\mathrm{j}\omega)\sin\left(\frac{\omega\tau}{2}\right)| \\ < & 1 \end{aligned}$$

holds for all $\omega \in R$.

Therefore, $1 + \bar{m}_i(s)(1 - \mathrm{e}^{-s\tau}) = 0, i = 2, \ldots, n$ is nonsingular for $\mathrm{Re}(s) \geq 0$, i.e., the roots of the Eq. (2.58) all lie on the open left half complex plane. Hence, the agents in the system (2.45) converge to a consensus asymptotically. □

Remark 2.14 Proposition 2.4 provides a consensus condition, but without pointing out the final consensus behavior, which is determined by the roots of the Eq. (2.52).

Theorem 2.9 *For the multi-agent systems (2.45) with both input delay and communication delay, it is assumed that the agents (2.45) without communication delay converge to a consensus asymptotically. Then, all the agents (2.45) asymptotically converge to a stationary consensus asymptotically, i.e.* $\lim_{t\to\infty} y_i(t) = c, \forall i \in \mathscr{I}$, *with c being a constant, if*

$$2|\bar{m}_i(\mathrm{j}\omega)\sin\left(\frac{\omega\tau}{2}\right)| < 1, i \in \mathscr{I} \tag{2.59}$$

hold, where $\bar{m}_i(s) = \frac{\kappa\hat{G}(s)(1-\bar{\lambda}_i)}{1+\kappa\hat{G}(s)\bar{\lambda}_i}$, $\hat{G}(s) = \frac{n(s)}{d(s)}\mathrm{e}^{-sT}$ *and* $\bar{\lambda}_i, i \in \mathscr{I}$ *are defined as that in Proposition 2.4.*

Proof From the proof of Proposition 2.4, the assumption and condition (2.59) with $\bar{\lambda}_i, i = 2, \ldots, n$ guarantee that the agents in (2.45) achieve an asymptotic consensus. Next, we just need to determine the root distribution of the Eq. (2.51).

Reformulate the Eq. (2.52) as follows

$$1 + \kappa\hat{G}(s)(1 - \mathrm{e}^{-s\tau}) = 0. \tag{2.60}$$

It is clear from the condition (2.59) with $\bar{\lambda}_1 = 0$ that the Eq. (2.60) has one root at $s = 0$ and other roots all have negative real parts. Therefore, the agents in the system (2.45) achieve a stationary consensus asymptotically. □

With $\max_{\omega\to\infty} |\frac{\mathrm{e}^{\mathrm{j}\omega\tau}-1}{\mathrm{j}\omega}| < \tau$, moreover, we provide the following consensus conditions which are more conservative but easier to calculate the communication delay bound.

Corollary 2.1 *Suppose that the agents (2.45) without communication delay converge to a consensus asymptotically. Then, all the agents (2.45) with both input delay and communication delay asymptotically converge to a stationary consensus asymptotically, if*

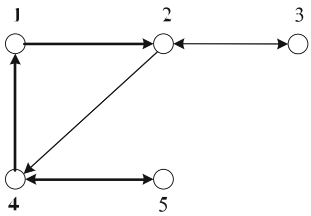


Fig. 2.11 A multi-agent network composed of five agents

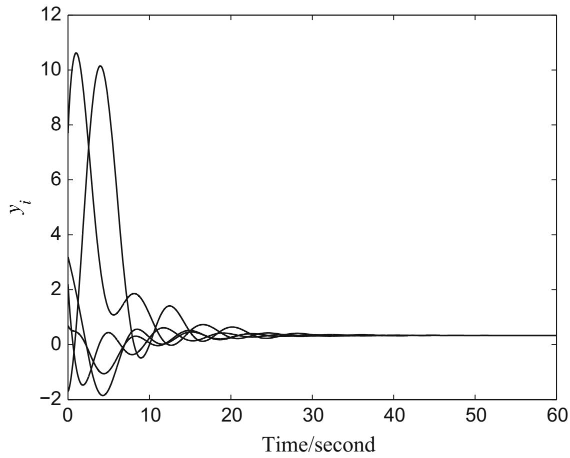


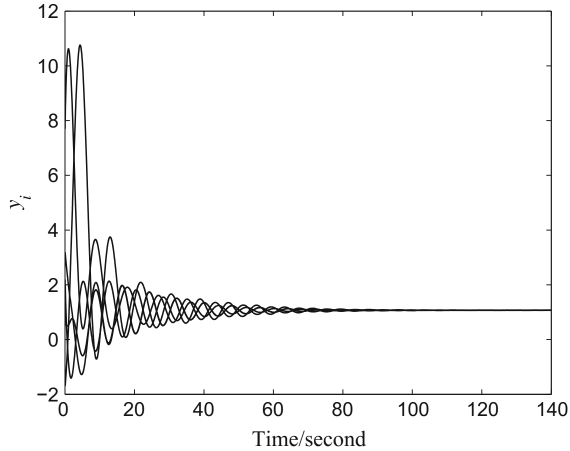
Fig. 2.12 Consensus convergence with $T = 0.5\,(\mathrm{s})$ and $\tau = 0\,(\mathrm{s})$

$$\tau < \frac{1}{|\omega \bar{m}_i(\mathrm{j}\omega)|}, i \in \mathscr{I} \tag{2.61}$$

hold, where $\bar{m}_i(s), i \in \mathscr{I}$ are the same as that in Theorem 2.9.

Example 2.5 A multi-agent system composed of five dynamic agents is investigated, and the interconnection topology containing a spanning tree is shown in Fig. 2.11. Agent's dynamics are modeled by $\frac{Y_i(s)}{U_i(s)} = \frac{s^2+2s+1}{s^4+6s^3+5s^2+2s}$, which satisfies Assumption 2.1. Choose the control gain: $\kappa = 1$ and the coupling weights: $a_{14} = 0.5, a_{21} = 0.8, a_{23} = 0.6, a_{32} = 1.2, a_{42} = 0.2, a_{45} = 1.0, and a_{54} = 0.5$. By numerical computing, the multi-agent systems (2.45) without communication delay converge to an asymptotic consensus when the input delay satisfies: $T < 1.07$(s) (see Fig. 2.12 with $T = 0.5$(s)). Choosing the input delay $T = 0.5$(s) for the system (2.45) under both communication and input delays, we get $\tau < 1.1223$(s) based on the condition (2.61), i.e., the agents (2.45) reach a stationary consensus asymptotically if $\tau < 1.1223$(s) under $T = 0.5$(s) (see Fig. 2.13).

Fig. 2.13 Consensus convergence with $T = 0.5\,(\mathrm{s})$ and $\tau = 0.8\,(\mathrm{s})$



2.4.3 Application to Second-Order Multi-agent Systems

The main results in above subsection will be applied into second-order multi-agent systems with input delay and communication delay.

For the second-order multi-agent system (1.14) with identical input delay $T_i = T > 0, i = 1, \ldots, n$, we adopt the following dynamical consensus algorithm in asynchronously coupled form as follows

$$u_i(t) = \frac{\kappa}{d_i} \sum_{j \in N_i} a_{ij}((x_j(t-\tau) - x_i(t)) + \gamma(v_j(t-\tau) - v_i(t))), i \in \mathscr{I}, \tag{2.62}$$

where $\kappa > 0$ and $\gamma > 0$ are the control parameters, $a_{ij} > 0$, $j \in N_i$, $d_i = \sum_{j \in N_i} a_{ij}$, and $\tau > 0$ is the communication delay.

Driven by the algorithm (2.62), the closed-loop form of system (1.14) with $T_i = T > 0, i \in \mathscr{I}$ is described as

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= \frac{\kappa}{d_i} \sum_{j \in N_i} a_{ij}((x_j(t-T-\tau) - x_i(t-T)) \\ &\quad + \gamma(v_j(t-T-\tau) - v_i(t-T))). \end{aligned} \tag{2.63}$$

Before presenting consensus criteria for system (2.63), sufficient and necessary consensus condition for the system (2.63) without communication delay under general directed graph is listed as follows.

Lemma 2.8 ([14]) *Suppose that the multi-agent network (2.63) contains a spanning tree and the following inequality*

$$\kappa\gamma^2 > \max_{2\le i\le n} \frac{\mathrm{Im}^2(\bar{\lambda}_i)}{\mathrm{Re}(\bar{\lambda}_i)(\mathrm{Re}^2(\bar{\lambda}_i) + \mathrm{Im}^2(\bar{\lambda}_i))} \tag{2.64}$$

holds, where $\bar{\lambda}_i$ are the nonzero eigenvalues of $D^{-1}L$, $i = 2, \ldots, n$. Then, the second-order agents (2.63) without communication delay reach a dynamical consensus asymptotically if and only if

$$T < \min_{2\le i\le n} \left\{ \frac{\theta_{i1}}{\omega_{i1}} \right\}, \tag{2.65}$$

where $\theta_{i1} \in [0, 2\pi)$ *satisfies* $\cos\theta_{i1} = \kappa \frac{\mathrm{Re}(\lambda_i) - \mathrm{Im}(\lambda_i)\omega_{i1}\gamma}{\omega_{i1}^2}$ *and* $\sin\theta_{i1} = \kappa \frac{\mathrm{Re}(\lambda_i)\omega_{i1}\gamma + \mathrm{Im}(\lambda_i)}{\omega_{i1}^2}$, $\omega_{i1} = \sqrt{\frac{||\lambda_i||^2\kappa^2\gamma^2 + \kappa\sqrt{||\lambda_i||^4\kappa^2\gamma^4 + 4||\lambda_i||^2}}{2}}$.

For the second-order multi-agent systems (2.63), the conditions (2.64) and (2.65) of Lemma 2.8 replace the assumptions that the communication free delay system can reach a consensus in Theorem 2.9, so we get the following corollary.

Corollary 2.2 *Assume that the interconnection topology of the second-order multi-agent network (2.63) has a spanning tree, and the conditions (2.64) and (2.65) hold. Let*

$$\tilde{m}(s) = \frac{\kappa\tilde{G}(s)(1 - \bar{\lambda}_i)}{1 + \kappa\tilde{G}(s)\bar{\lambda}_i}, \tag{2.66}$$

where $\tilde{G}(s) = \frac{(\gamma s+1)\mathrm{e}^{-sT}}{s^2}$, and $\bar{\lambda}_i$, $i \in \mathscr{I}$ are the nonzero eigenvalues of $D^{-1}L$. Then, all the agents in system (2.63) asymptotically converge to a stationary consensus, if

$$2|\tilde{m}_i(\mathrm{j}\omega)\sin\left(\frac{\omega\tau}{2}\right)| < 1, i \in \mathscr{I} \tag{2.67}$$

holds.

Proof Denote the output of second-order dynamic agent i as $y_i = x_i + \gamma v_i$, and the transfer function from the input to the output is

$$\frac{Y_i(s)}{U_i(s)} = \frac{\gamma s + 1}{s^2}\mathrm{e}^{-sT},$$

where $Y_i(s)$ and $U_i(s)$ are the Laplace transforms of $y_i(t)$ and $u_i(t)$, respectively. Then, the consensus algorithm (2.62) can be expressed as

$$u_i(t) = \frac{\kappa}{d_i}\sum_{j\in N_i} a_{ij}(y_j(t-\tau) - y_i(t)).$$

Hence, the second-order multi-agent system (2.63) is just a special case of the system (2.46), so Corollary 2.2 can be proved in the same way as that of Theorem 2.9. □

From Corollary 2.1, we can get the similar results for second-order multi-agent systems, but we omit here to avoid the repetition.

Remark 2.15 According to Corollary 2.2, the multi-agent systems composed of double integrators and high-order integrators can always be formulated as the special cases of the multi-agent systems (2.43). Thus, results in Proposition 2.3, Proposition 2.4, Theorem 2.9 and Corollary 2.1 can also be applied into the consensus analysis of high-order multi-agent systems.

Example 2.6 A symmetric and connected topology (see Fig. 2.14) is investigated for a second-order multi-agent network with six agents described by (2.63), and we choose the control parameters as $\kappa = 1$, $\gamma = 0.5$. For simplicity, all the coupling weights of the edges are assumed to be 1. When communication delays are negligible for the neighboring agents, we obtain from the conditions (2.64) and (2.65) of Lemma 2.8 that the agents in (2.63) without communication delay can achieve a dynamical consensus asymptotically (see Fig. 2.15 with $T = 0.2$(s)) when $T < 0.4402$(s).

Moreover, we choose the input delay $T = 0.2$(s) for the second-order agents (2.63) with both communication delay and input delay. From the condition (2.67) in Corollary 2.2, the agents (2.63) converge to a stationary consensus asymptotically

Fig. 2.14 A multi-agent network of six agents

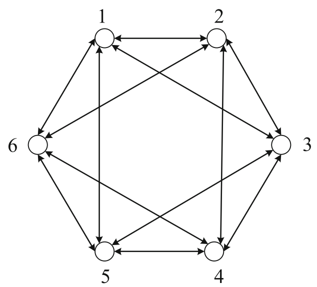
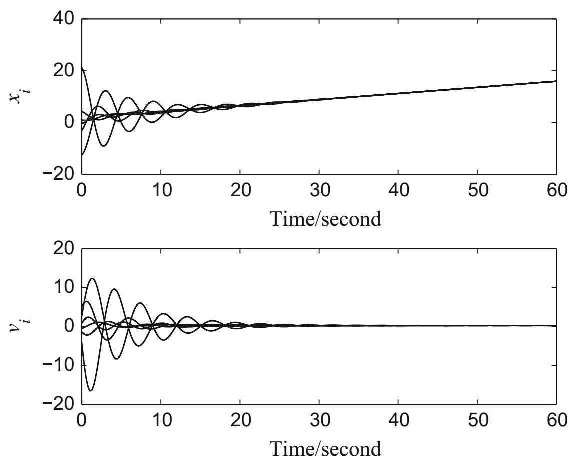


Fig. 2.15 Consensus convergence with $T = 0.2$ (s) and $\tau = 0$ (s)



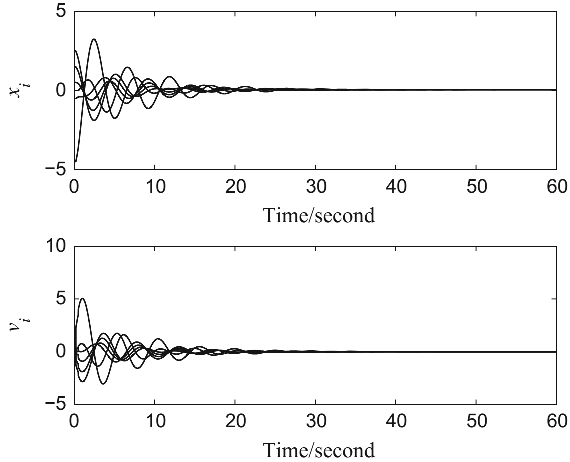


Fig. 2.16 Consensus convergence with $T = 0.2\,(\mathrm{s})$ and $\tau = 0.3\,(\mathrm{s})$

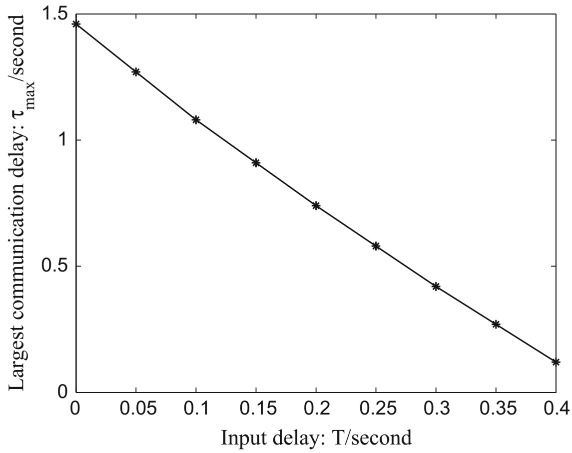


Fig. 2.17 Trade-off between input delay and communication delay

when $\tau < 0.74$(s) (see Fig. 2.16). With given control parameters κ and γ, the largest communication delay τ that the system (2.63) can tolerate decreases as the input delay T increases according to the condition (2.67) (see Fig. 2.17), i.e., there is a trade-off between the communication delay and the input delay.

2.5 Notes

Based on stability criteria in frequency-domain analysis, this chapter thoroughly analyzes the delay effect on consensus convergence for the first-order, second-order, and high-order homogeneous multi-agent systems. Different from the works aimed at general agents' dynamics and usual interconnection topology [5, 7, 13], we focus on simple agents' dynamics and intercoupling structure in order to get relatively conservative consensus conditions. The content of this chapter is mainly taken from our paper [15–18], but some results are further improved.

Input delay does not change the original consensus behavior of multi-agent systems without time delay, but it definitely affects the consensus converging time and whether the agents can reach an asymptotically consensus. Besides, communication delay in synchronously coupled consensus algorithms has the same impact on the consensus seeking as the input delay.

Specially, communication delay in asynchronously coupled consensus algorithms has distinct impacts on stationary and dynamical consensus problems, respectively. With proper control parameters and connected topology, stationary consensus algorithms can drive the agents to achieve an asymptotic consensus without any relationship with communication delay. But meanwhile, asymptotic consensus convergence of dynamical consensus algorithms is always delay-dependent strictly, and the original consensus behavior of the agents without communication delay must be changed under asynchronously coupled form.

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