

Chapter 2

Application Background of Circle Integrals

On many occasions we repeatedly emphasize that application of engineering mathematics has the same importance as mathematics itself. Here we start the discussion from circle integral, e.g.,

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} \quad (2.1)$$

Circle integral is related to *capacitance C of conducting circle disk*. We first consider a conducting circle disk with an infinitely small thickness shown in Fig. 2.1.

Assume charge density $\sigma(r)$ on the conducting circle disk. According to the symmetry of the disk, $\sigma(r)$ is irrelevant to azimuthal angle φ . Charge density on the periphery of the disk is distributed in the form of inverse square root, i.e.,

$$\sigma(r) = \frac{\sigma_0}{\sqrt{1 - \left(\frac{r}{a}\right)^2}} \quad (2.2)$$

in which σ_0 denotes a maximum magnitude of the charge density. As shown in Fig. 2.2, on the periphery of the disk the charge density has the inverse square root distribution.

The capacitance of the circle disk C is denoted as

$$\boxed{C = \frac{Q}{V}} \quad (2.3)$$

Fig. 2.1 A conducting disk with an infinitely small thickness

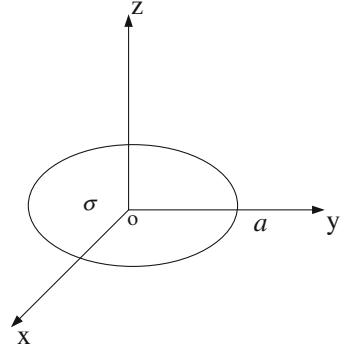
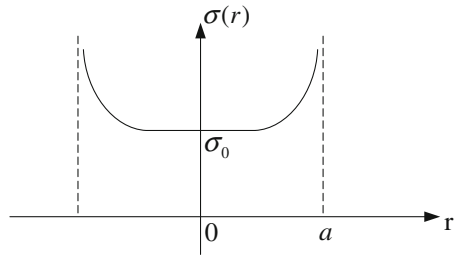


Fig. 2.2 The distribution of the charge density over a circle disk



in which Q is total charge of the disk, and V is the electric potential. Hence we have

$$Q = \iint_s \sigma(r) r dr d\phi \quad (2.4)$$

$$V = \iint_s \frac{\sigma(r)}{4\pi\epsilon r} r dr d\phi \quad (2.5)$$

Substituting Eq. (2.2) into Eq. (2.4), we have

$$\begin{aligned} Q &= \iint_s \frac{\sigma_0}{\sqrt{1 - \left(\frac{r}{a}\right)^2}} r dr d\phi \\ &= 2\pi\sigma_0 a^2 \int_0^a \frac{-\frac{1}{2} d\left[1 - \left(\frac{r}{a}\right)^2\right]}{\sqrt{1 - \left(\frac{r}{a}\right)^2}} \\ &= 2\pi\sigma_0 a^2 \frac{\frac{1}{2} \sqrt{1 - \left(\frac{r}{a}\right)^2}}{\frac{1}{2}} \bigg|_a^0 \\ &= 2\pi\sigma_0 a^2 \end{aligned} \quad (2.6)$$

Considering Eq. (2.6) and charge distribution shown in Fig. 2.2, we can completely think that the average charge density on the circle disk is

$$\bar{\sigma} = 2\sigma_0 \quad (2.7)$$

We rewrite Eq. (2.6) as

$$Q = \bar{\sigma}S = \bar{\sigma}\pi a^2 = 2\sigma_0\pi a^2 \quad (2.8)$$

On the other hand, the electric potential V can be expressed as

$$\begin{aligned} V &= \iint_s \frac{\sigma_0 r dr d\phi}{4\pi\epsilon r \sqrt{1 - \left(\frac{r}{a}\right)^2}} \\ &= 2\pi\sigma_0 a \int_0^a \frac{d\left(\frac{r}{a}\right)}{4\pi\epsilon \sqrt{1 - \left(\frac{r}{a}\right)^2}} \\ &= \frac{2\pi\sigma_0 a}{4\pi\epsilon} \sin^{-1}\left(\frac{r}{a}\right) \Big|_0^a \\ &= \frac{\pi\sigma_0 a}{4\epsilon} \end{aligned} \quad (2.9)$$

According to Eq. (2.3), we can obtain the capacitance as follows:

$$\boxed{C = \frac{Q}{V} = 8\epsilon a} \quad (2.10)$$

Further, if we express the electric potential V in an integral form, we have

$$Q = 2\pi\sigma_0 a^2 \quad (2.11)$$

$$V = \frac{2\pi\sigma_0 a}{4\pi\epsilon} \int_0^1 \frac{dx}{\sqrt{1-x^2}} \quad (2.12)$$

Therefore, we can get

$$\boxed{C = \frac{4\pi\epsilon a}{\int_0^1 \frac{dx}{\sqrt{1-x^2}}}} \quad (2.13)$$

It can be clearly seen that capacitance of the conducting circle disk is tightly related to the circle integral $\int_0^1 dx / \sqrt{1-x^2}$.

In addition, according to the above discussion, we can know that there are not circle integrals of the first and the second kinds. This is because factor $\sqrt{1-x^2}$ is

always in the denominator of the studied problem. The second important application of the circle integral is to calculate the perimeter of the circle, i.e.,

$$L = 4a \int_0^1 \frac{dx}{\sqrt{1-x^2}} = 4a \left(\frac{\pi}{2} \right) = 2\pi a. \quad (2.14)$$

Notes on the Ellipsoidal Function

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