Chapter 2
Complex Function Theory

We next turn to functions of a complex variable, that are perhaps best known for solving integrals by contour deformation in the complex plane. It also provides a starting point to special functions and linear transform methods, some to be introduced in subsequent chapters. Complex functions also give a concise description of idealized two-dimensional fluids, in the approximation of incompressible and irrotational flows. These developments derive from *analyticity*. If a function $f(z)$ of a complex variable z is differentiable at some z, we say that $f(z)$ is analytic at z. Extended over an open region in the complex plane, this property has broad implications.

2.1 Analytic Functions

Complex function theory considers maps $w = f(z)$ given by functions

$$f(z) = u(x, y) + iv(x, y) \tag{2.1}$$

from the complex plane to itself, where we decompose $f(z)$ into its real and imaginary parts $u(x, y)$ and $v(x, y)$, respectively. We define the complex conjugate $\bar{z} = x - iy$, whereby

$$x = \mathrm{Re}\, z = \frac{1}{2}\left(z + \bar{z}\right), \quad y = \mathrm{Im}\, z = \frac{1}{2i}\left(z - \bar{z}\right). \tag{2.2}$$

Likewise, we have

$$\begin{aligned} u(x, y) &= \mathrm{Re} f(z) = \tfrac{1}{2}\left(f(z) + \overline{f(z)}\right), \\ v(x, y) &= \mathrm{Im} f(z) = \tfrac{1}{2i}\left(f(z) - \overline{f(z)}\right). \end{aligned} \tag{2.3}$$

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Our focus will be on functions that are differentiable. We say that $f(z)$ is differentiable at $z = z_0$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2.4}$$

exists. It means that the limit gives a unique finite answer, regardless of the way z_0 is approached. Equivalently, we formulate (2.4) in terms of sequences $\{z_n\}_{n=0}^{\infty}$ with $z_n \to z_0$ in the limit as n approaches infinity, insisting that

$$\lim_{n \to \infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} \tag{2.5}$$

exists for any such sequence approaching z_0. Of particular interest are the two alternatives of approaching z_0 along the x and y directions, i.e., $z = z_0 + h$ and $z = z_0 + ih$, respectively. The limit (2.4) exists iff

$$A(h) = \frac{f(z_0 + h) - f(z_0)}{h}, \quad B(h) = \frac{f(z_0 + ih) - f(z_0)}{ih} \tag{2.6}$$

approach finite limits which are the same as h approaches zero. In the notation of (2.1),

$$A_0 \equiv \lim_{h \to 0} A(h), \quad B_0 \equiv \lim_{h \to 0} B(h) \tag{2.7}$$

gives

$$A_0 = u_x(x_0, y_0) + i v_x(x_0, y_0), \quad B_0 = v_y(x_0, y_0) - i u_y(x_0, y_0). \tag{2.8}$$

where we used $1/i = -i$. Equating $A_0 = B_0$, we obtain the *Cauchy-Riemann relations*

$$u_x = v_y, \quad u_y = -v_x \tag{2.9}$$

at the point $z = z_0$ in the complex plane.

If (2.9) holds true throughout some region (an open domain) D in the complex plane, we say that $f(z)$ is *analytic* in D. (A more restricted focus would be analyticity along a curve.)

Example 2.1. Consider $f(z) = z$ with $u(x, y) = x$ and $v(x, y) = y$. The Cauchy-Riemann relations (2.9) are satisfied by inspection, so $f(z)$ is analytic for all z, i.e., $f(z)$ is entire. For $f(z) = z^2$, we have $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$, and hence

$$u_x = 2x, \quad v_y = 2x, \quad u_y = -2y, \quad v_x = 2y. \tag{2.10}$$

It follows that (2.9) holds for all z, i.e., also $f(z) = z^2$ is entire. Summarizing, we have

$$f(z) = z \to f'(z) = 1, \quad f(z) = z^2 \to f'(z) = 2z. \tag{2.11}$$

The latter can also be seen from

$$f'(z) = \lim_{\xi \to z} \frac{\xi^2 - z^2}{\xi - z} = \lim_{\xi \to z} \frac{(\xi - z)(\xi + z)}{\xi - z} = 2z. \tag{2.12}$$

Analyticity is a strong condition. Functions of a complex variable that are not analytic are easily found. For instance, $f(z) = z\bar{z} = x^2 + y^2$ is *not* analytic. Since $f(z)$ is constant along circles concentric at the origin, its tangential derivative is zero, while its normal derivative is $2|z|$. The requirement that the derivative of $f(z)$ be the same irrespective of the direction is hereby not satisfied and the Cauchy-Riemann relations (2.9) do not hold. Also, writing $f(z) = u(x, y) + iv(x, y)$ into its real and imaginary parts shows

$$\Delta u(x, y) = 4, \quad \Delta v(x, y) = 0. \tag{2.13}$$

Its real part is not harmonic, and hence $f(z)$ is not analytic.

When functions $f(z)$ and $g(z)$ are analytic in a common domain, then the following holds:

$$\begin{aligned} (f(z)g(z))' &= f'(z)g(z) + f(z)g'(z), \\ (f(z)/g(z))' &= \left[f'(z)g(z) - f(z)g'(z)\right]/g(z)^2 \end{aligned} \tag{2.14}$$

and if $g(z)$ is analytic on a domain D and $f(z)$ is analytic on the image $D' = g(D)$, then

$$[f(g(z))]' = f'(g(z))g'(z). \tag{2.15}$$

These results may be seen as *analytic continuations of existing identities on the real line.* By elementary considerations, we have

$$\begin{array}{c} z^n \to nz^{n-1} \;\; (n \in \mathbb{Z}), \\ e^{az} \to ae^{az} \;\; (a, z \in \mathbb{C}) \\ \sin z, \; \cos z, \; \sinh z, \; \cosh z \to \cos z, \; -\sin z, \; \cosh z, \; \sinh z \\ \arctan z \to \frac{1}{1+z^2} \;\; (z \neq \pm i). \end{array} \tag{2.16}$$

2.2 Cauchy's Integral Formula

The strength of the condition of analyticity comes about mostly importantly in *Cauchy's Theorem*: if $f(z)$ is analytic in a region D,[1] then

$$\int_\gamma f(z)dz = 0 \tag{2.17}$$

for any closed contour γ in D.[2] Here, the integral is defined according to

$$\int_\gamma (u+iv)(dx+idy) = \int_\gamma (udx - vdy) + i\int_\gamma (vdx + udy). \tag{2.18}$$

If, by analyticity, the Cauchy-Riemann relations (2.9) hold true throughout some open domain containing γ, then by Green's theorem

$$\begin{array}{ll} \int_\gamma (udx - vdy) = -\int_{I(\gamma)} (u_y + v_x)dxdy = 0, \\ \int_\gamma (vdx + udy) = \quad \int_{I(\gamma)} (u_x - v_y)dxdy = 0, \end{array} \tag{2.19}$$

giving (2.17).

A fundamental proof of Cauchy's Theorem is due to Goursat, assuming only the existence of $f'(z)$, i.e., (2.9) with no a priori condition on continuity. Goursat's proof applies to regions D which are simply connected (no punctures or holes), as illustrated in Figs. 2.1 and 2.2. In particular, the unit disk is simply connected, but the unit disk with the origin removed is not.

As a consequence of (2.17), the integral

$$F(z) = \int_{z_0}^{z} f(z)dz \tag{2.20}$$

is a well-defined for z and z_0 in D. Any two paths $\gamma_{1,2}$ connecting them give rise to a loop γ by following γ_1 and traversing γ_2 in reverse, so that by (2.17)

$$0 = \int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz, \tag{2.21}$$

whereby (2.20) is *path independent*. By the fundamental theorem of calculus, $F'(z) = f(z)$, whereby (2.9) applies to $F(z) = U(x, y) + iV(x, y)$.

[1] Here, D is an open subset of the complex plane which is simply connected.

[2] A closed contour γ divides the complex plane into a region $I(\gamma)$ within γ and an unbounded region outside, according to the Jordan curve theorem.

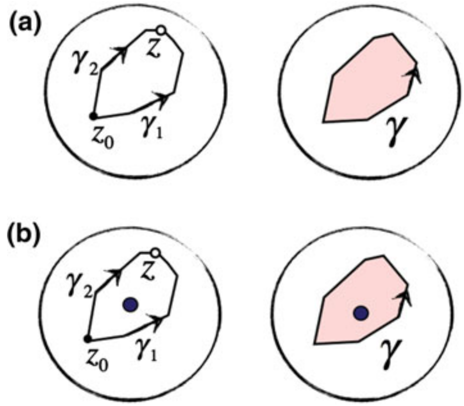

Fig. 2.1 **a** The unit disk $|z| < 1$ is a simply connected domain: any two points z_0 and z can be connected by a path γ_1 within, and choosing another such path γ_2 obtains a closed loop γ as shown, whose interior $I(\gamma)$ lies within $|z| < 1$ also). **b** The punctured disk obtained by removing the origin fails to have this property. The interior of the closed loop γ shown contains the origin, which is not part of the punctured disk. The punctured disk is not simply connected

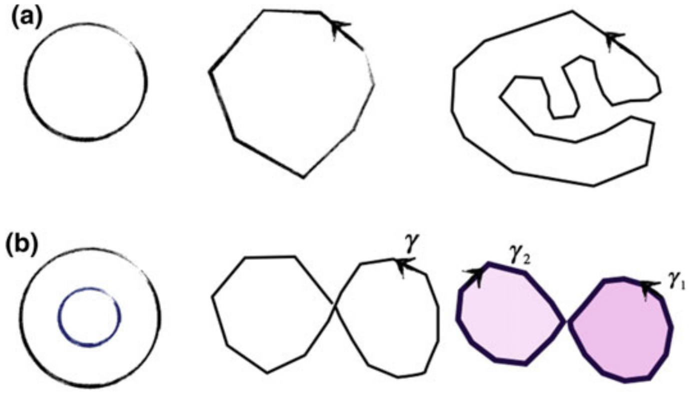

Fig. 2.2 **a** Simply connected domains such as the unit disk can be deformed by continuous deformations. Preserving no self-intersecting boundaries, their boundaries remain simple contours. **b** An annulus obtains by making a hole in the unit disk. Like the punctured disk, it is not simply connected. The figure eight exemplifies a self-intersection curve, whose interior forms two disjoint simply connected domains. There are no paths connecting points z_0 in one and z in the other domain. The figure eight is not a simple contour

Example 2.2. The function $f(z) = 1/(z - a)$ is analytic everywhere in the complex plane except at the isolated pole $z = a$. Let γ be a contour whose interior $I(\gamma)$ does not contain a. The orientation of γ is taken to go around a once in a counter clock-wise direction, as defined by tracking the change in

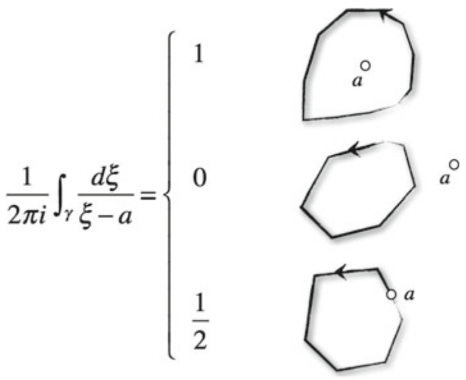

Fig. 2.3 Contour integration of $1/(z-a)$ as shown equals 1 or 0 if a is in or out of $I(\gamma)$. The integral equals $\frac{1}{2}$ if a is on γ, when taking the principle value of the integral, provided a is on a smooth section of γ. If a is located at a corner of γ, e.g., a polygon, the result is $\alpha/(2\pi)$, where α denotes the interior angle at that corner

argument of $\log(z-a)$ as z traverses γ from a point $z_0 \epsilon \gamma$ back to itself. Then $f(z)$ is analytic everywhere in $I(\gamma)$. By (2.9), we have

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = 0. \tag{2.22}$$

Next, let $a \epsilon I(\gamma)$, so that (2.9) does not apply. Specializing to a circle C: $|z-a| = \rho$ (with counter-clockwise orientation and winding number 1), we put $z = a + \rho e^{i\theta}$, so that $dz = i\rho e^{i\theta}d\theta$. As a result,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{1}{2\pi i}\int_0^{2\pi} i d\theta = 1. \tag{2.23}$$

The above generalizes to γ traversing a N times, by which

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = N \tag{2.24}$$

measures the winding number of γ around $z = a$. For simple contours (no self-crossing), (2.22) and (2.24) are summarized in Fig. 2.3.

For a more general γ, consider *contour deformation* following Fig. 2.4. Here, we give γ a detour branching from $z_0 \epsilon \gamma$ to the circle C : $|z-a| = \rho$, with ρ sufficiently such that C lies in $I(\gamma)$, and back to z_0. A single path P connecting C with z_0 may be used twice (acting as a bridge), in going from z_0 to C and back. The new contour Γ consisting of γ, P and C circumvents the singularity $z = a$, whereby $f(z)$ is analytic in $I(\Gamma)$. It follows that

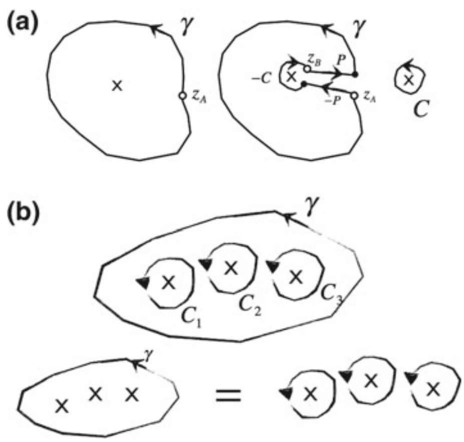

Fig. 2.4 **a** z_A on a simple contour γ can be connected to a contour $-C$ within via a bridge P between z_A and z_B on $-C$. Here, $-C$ refers to C with opposite orientation. The interior of this new contour does not contain X shown. Integration of $f(z)$ that is analytic in a domain containing γ, except possibly at X, then vanishes by Cauchy's theorem. Since integration over the bridge forth and back produces zero, the integral of $f(z)$ over γ equals the integral of $f(z)$ over C. This procedure is exemplifies *contour deformation*. **b** The same applies to multiple isolated singularities. Integration of $f(z)$ over γ is then equivalent to the sum of the integrations of $f(z)$ over contours around each singularity individually. The result gives the *residue theorem*: the net result of integration is defined by the coefficients A_i in $A_i/(z-z_i)$ in the expansion of $f(z)$ in partial fractions up to an arbitrary function which is analytic throughout $I(\gamma)$

$$0 = \int_\Gamma f(z)dz = \int_\gamma f(z)dz - \int_C f(z)dz \tag{2.25}$$

if we define the orientation of C also to be counter clock wise. Here, we used the fact that integration over P forth and back between z_0 and C produces zero. The result is a deformation of γ into C:

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_C f(z)dz. \tag{2.26}$$

The arguments from Example 2.2 give *Cauchy's integral formula*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}d\xi \tag{2.27}$$

with the same conditions on γ as before. In fact, for $f(z)$ analytic in D containing γ,

$$f(\xi) = f(z) + (f'(z) + A(\xi, z))(\xi - z) \tag{2.28}$$

where $A(\xi) \to 0$ as $\xi \to z$, by the existence of the derivative of $f(\xi)$ at $\xi = z$. The result (2.27) obtains directly following a contour deformation of γ to a circle C with radius ρ as above, when we let ρ approach zero.

By Cauchy's integral formula, the n-th derivative satisfies

$$\frac{1}{n!} f^{(n)}(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi, \tag{2.29}$$

showing that *if $f(z)$ is analytic, then $f(z)$ is infinitely differentiable.* An important corollary is that analyticity of a function in a domain D implies the existence of a power series with a finite radius of convergence. That is, $f(z)$ has a *Taylor series in z about $z_0 \in D$*,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{2.30}$$

where the a_n (that depend on the choice of z_0) are defined by (2.29).

2.3 Evaluation of a Real Integral

As an application of the above, consider the integral

$$K = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \lim_{R \to \infty} K_R, \quad K_R = \int_{-R}^{R} \frac{dx}{1 + x^2}. \tag{2.31}$$

First, we consider the analytic extension

$$f(x) = \frac{1}{1 + x^2} \to f(z) = \frac{1}{1 + z^2} \tag{2.32}$$

by taking x into the complex plane. Here, we note the poles $z = \pm i$ and the associated partial fraction expansion,

$$f(z) = \frac{A_1}{z - i} + \frac{A_2}{z + i} = \frac{1}{2i} \left(\frac{1}{z - i} - \frac{1}{z + i} \right). \tag{2.33}$$

Next, we consider a contour given by $[-R, R]$ and the semi-circle $C: \ z = Re^{i\theta}$, $0 \le \theta \le \pi$ (Fig. 2.5). It encloses the singularity $z = i$ but not $z = -i$. By (2.24) and (2.22) (see also Fig. 2.3), we have

$$\frac{1}{2\pi i} \left[\int_{-R}^{R} f(x) dx + \int_C f(z) dz \right] = A_1 = \frac{1}{2i}. \tag{2.34}$$

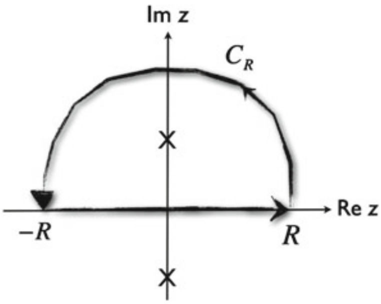

Fig. 2.5 Shown is a contour comprising $[-R, R]$ and the semi-circle C_R in the upper half plane with counter clockwise orientation, to evaluate the integral $A = \int dx/(1+x^2)$ over the real line in terms of $A_R = \int dx/(1+x^2)$ over $[-R, R]$, following an extension of x to complex values z and taking R to infinity. As a function of z, $f(z) = 1/(1+z^2)$ has two poles at $z = i, -i$. The contour shown can be deformed to a circle around $z = i$, to give $2\pi i$ times the residue $1/(2i)$ of $f(z)$ at $z = i$, while the integral over C_R approaches zero in the limit as R approaches infinity. As a result, $A = \pi$

Here,

$$\left|\int_C f(z)dz\right| \leq \int_C |f(z)|\,|dz| \leq \int_0^\pi \frac{1}{R^2-1} R d\theta = O\left(\frac{1}{R}\right). \tag{2.35}$$

Taking R to infinity in (2.34) hereby gives

$$K = 2\pi i \times \frac{1}{2i} = \pi. \tag{2.36}$$

2.4 Residue Theorem

Consider contour integration of the function

$$f(z) = \frac{1}{z^2(z-\frac{1}{2})} \tag{2.37}$$

in light of its isolated singularities at $z_1 = 0$ and $z_2 = \frac{1}{2}$. The singularity at $z = 0$ is of second order. and the singularity at $z_2 = \frac{1}{2}$ is a simple pole. Integration over a contour around either one of them will pick up contributions according to their *residues* according to either one of the following.

1. By partial fractions, we have the expansion

$$f(z) = \frac{1}{z} \times \frac{1}{z(z-\frac{1}{2})} = \frac{2}{z} \times \left(\frac{1}{z-\frac{1}{2}} - \frac{1}{z}\right) = -\frac{2}{z^2} - \frac{4}{z} + \frac{4}{z-\frac{1}{2}} \quad (2.38)$$

2. In the disk $D: \ |z| < \frac{1}{2}$, we have, alternatively, the expansion

$$f(z) = \frac{1}{z^2(z-\frac{1}{2})} = -\frac{2}{z^2} \times \frac{1}{1-2z} = -\frac{2}{z^2}\left[1 + 2z + 4z^2 g(z)\right], \quad (2.39)$$

where $g(z) = 1 + 2z + 4z^2 + \cdots$ is analytic in D. We thus have the *Laurent expansion*

$$f(z) = \frac{1}{z^2(z-\frac{1}{2})} = -\frac{2}{z^2} - \frac{4}{z} - 8g(z). \quad (2.40)$$

3. We compute

$$\lim_{z\to 0} \frac{d}{dz}\left[z^2 f(z)\right] = -4. \quad (2.41)$$

According to (1), integration of $f(z)$ over an anti-clockwise contour γ gives

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = -4, \ 4, \ 0 \quad (2.42)$$

according to whether γ encloses (a) $z_1 = 0$, (b) $z_2 = \frac{1}{2}$ or (c) both $z = z_1$ and $z = z_2$. Here, we use the fact that (cf. Fig. 2.3)

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z^n} = \delta_{n0}, \quad (2.43)$$

where $\delta_{ij} = 1$ $(i = j)$ and $\delta_{ij} = 0$ $(i \neq j)$ denotes the Kronecker delta symbol and γ is an anti-clock wise oriented contour around the origin (see Exercise 2.11).

According to (2–3), the result for (a) obtains by noting that the remainder $8u(z)$ is analytic in D, which is immaterial by Cauchy's theorem. Only the term A_{-1}/z matters in view of the identity (2.43). For a contour in D, it thereby follows that

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = A_{-1} = -4. \quad (2.44)$$

The same arguments can be readily extended to functions with a number of isolated singularities (Fig. 2.4):

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{N} \text{Res}_{z=z_k} f(z) \tag{2.45}$$

in terms of

$$\text{Res}_{z_k} f(z) = A_k, \quad f(z) = \frac{A_k}{z-z_k} + g_{(k)}(z) \tag{2.46}$$

where $g_{(k)}(z)$ denotes a remainder which is analytic within γ, except perhaps at $z = z_m$, $m \neq k$.

To further illustrate Cauchy's integral formula (2.27), we calculate the following contour integrals. The first two are over C, given by a circle of radius $\rho > 2$ around the origin, described by $z = \rho e^{i\theta}$, $0 \leq \theta \leq 2\pi$.

Example 2.3. Consider the contour integral

$$I = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^2+2z+2}dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)(z-z_1)}dz, \tag{2.47}$$

where $z_0 = -1+i$ and $z_1 = -1-i$ denote the zeros of the polynomial $p(z) = z^2+2z+2$. Proceeding by partial fractions, we have

$$I = \frac{1}{2\pi i}\frac{1}{z_0-z_1}\int_C e^z\left[\frac{1}{z-z_0} - \frac{1}{z-z_1}\right]dz. \tag{2.48}$$

By Cauchy's integral formula (2.27)

$$I = \frac{1}{2\pi i}\frac{1}{2i}\left[\int_C \frac{e^z}{z-z_0}dz - \int_C \frac{e^z}{z-z_1}dz\right] = e^{-1}\frac{1}{2i}(e^i - e^{-i}) = e^{-1}\sin(1). \tag{2.49}$$

Example 2.4. By (2.29), the integral

$$I = \frac{1}{2\pi i}\int_C \frac{e^{2z}}{(z+1)^4}dz \tag{2.50}$$

is $1/3!$ times the third derivative of

$$g(w) = \frac{1}{2\pi i}\int_C \frac{e^{2z}}{z-w}dz = e^{2w} \tag{2.51}$$

evaluated at $w = -1$. Therefore,

$$I = \frac{1}{6}g^{(3)}(-1) = \frac{4}{3}e^{-2}. \tag{2.52}$$

Example 2.5. Let C_1 denote the unit circle $|z| = 1$. By (2.27)

$$I = \int_{C_1} \frac{e^{kz}}{z}dz = 2\pi i \tag{2.53}$$

is readily evaluated. With $dz = ie^{i\theta}d\theta$, the left hand side can be expanded in real and imaginary parts

$$I = \int_0^{2\pi} e^{k(\cos\theta + i\sin\theta)} i d\theta = i\int_0^{2\pi}\left[e^{k\cos\theta}(\cos(k\sin\theta) + i\sin(k\sin\theta))\right]d\theta \tag{2.54}$$

so that

$$\int_0^{2\pi} e^{k\cos\theta}\cos(k\sin\theta)d\theta = 2\pi, \quad \int_0^{2\pi} e^{k\cos\theta}\sin(k\sin\theta)d\theta = 0. \tag{2.55}$$

These relations hold for all k.

Example 2.6. The Poisson integral

$$I = \int_{\mathbb{R}} e^{-x^2}\cos(2bx)dx = e^{-b^2}\sqrt{\pi} \tag{2.56}$$

is an exact result, illustrating that a smooth function, here e^{-x^2}, integrated against the oscillatory function $\cos(2bx)$ goes to zero as the frequency k

approaches infinity.[3] To see this, we first consider the finite integral over the finite interval $\Lambda: \ -L \leq x \leq L$,

$$I_L = \int_\Lambda e^{-x^2} \cos(2bx) dx = \frac{1}{2} e^{-b^2} \int_\Lambda \left[e^{-(x-ib)^2} + e^{-(x+ib)^2} \right]. \quad (2.57)$$

To treat the integration of $e^{-(x+ib)^2}$ on the right hand side of (2.57), let Γ_L denote the segment $z = x + ib$ with $x \epsilon \Lambda$ and complete it to a clock-wise oriented closed contour γ containing Λ as follows,

$$\int_{\Gamma_L} e^{-z^2} dx - i \int_L^{L+ib} e^{-z^2} dy - \int_\Lambda e^{-x^2} dx + i \int_{-L}^{-L+ib} e^{-z^2} dy = 0, \quad (2.58)$$

where the sum vanishes since e^{-z^2} is entire. On the sides at $x = \pm L$ of γ, we have

$$\left| e^{-(L \pm iy)^2} \right| = \left| e^{-(L^2 - y^2)} e^{\pm 2iyL} \right| = e^{-L^2 - y^2} \quad (2.59)$$

and so

$$\left| \int_L^{L+ib} e^{-z^2} dz \right| \leq \int_L^{L+ib} e^{-(L^2 - b^2)} dy \leq b e^{L^2 - b^2} \to 0 \quad (2.60)$$

in the limit as L approaches infinity. Therefore,

$$\lim_{L \to \infty} \int_{\Gamma_L} e^{-z^2} dx = \int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}. \quad (2.61)$$

Since (2.61) is independent of b, the result of integration of $e^{-(x-ib)^2}$ on the right hand side of (2.57) will be the same, thus showing (2.56).

Example 2.7. Compute $f'(1)$ of

$$f(z) = \int_{|\xi|=3} \frac{\xi^2 + 2\xi + 2}{\xi - z} d\xi. \quad (2.62)$$

By partial fractions, there exist constants A, B and C such that

[3] Known as the Riemann-Lebesgue theorem in Fourier transforms, see e.g., van Putten, M.H.P.M., 1998, SIAM Rev., 40(2), 333.

$$f'(z) = \int_{|\xi|=3} \left[\frac{A}{(\xi - z)^2} + \frac{B}{\xi - z} + C \right] d\xi = 2\pi i B(z). \tag{2.63}$$

We can extract B from the numerator $p = \xi^2 + 2\xi + 2 = C(\xi - z)^2 + B(\xi - z) + A$ in (2.62) as the residue

$$B = \mathrm{Res}_{\xi=z} \left[\frac{\xi^2 + 2\xi + 2}{(\xi - z)^2} \right] = \lim_{\xi \to z} p'(\xi) = 2z + 2. \tag{2.64}$$

Evaluated at $z = 1$, we conclude $f'(1) = 8\pi i$.

2.5 Morera's Theorem

Morera's theorem[4] states that given a continuous function $f(z)$ in some domain D, if

$$\int_C f(z)dz = 0 \tag{2.65}$$

for every closed curve C in D, then $f(z)$ is analytic (holomorphic) in D. Morera's theorem is hereby a converse to Cauchy's theorem of Sect. 2.2. It follows that

$$F(z) = \int_{z_0}^{z} f(\xi)d\xi \tag{2.66}$$

is a well-defined primitive of $f(z)$, independent of the path of integration from z_0 to z. Thus, $F'(z) = f(z)$ and hence $F(z)$ is analytic in D. For complex functions, this implies that $F(z)$ is infinitely differentiable and in particular, $F''(z)$ exists, showing that $f(z)$ is analytic in D.

Write $f(z) = u + iv$ in its real and imaginary parts as before. If we assume that u and v are differentiable, then

$$\int_C f(z)dz = \int_C (u + iv)(dx + idy) = \int_C (udx - vdy) + i \int_C (udx + vdy). \tag{2.67}$$

implies by Green's theorem as before in (2.19)

[4]Giacinto Morera 1856–1909.

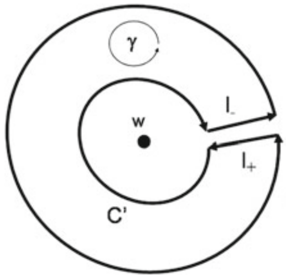

Fig. 2.6 Shown is an oriented path γ formed out of an outer contour C, a counter-oriented inner contour C' and a bridge I that is traversed twice, from C to C' over I_+ and back over I_-. The inner contour C' encircles a point w

$$\int_C f(z)dz = -\int_{I(C)} (u_x + v_y)dA + i\int_{I(C)} (u_x - v_y)dA. \tag{2.68}$$

Implied by (2.65) is that the Cauchy-Riemann relations hold

$$u_x = v_y, \quad u_y = -v_x, \tag{2.69}$$

previously derived as conditions for the derivative of $f(z)$ to be the same regardless of direction of differentiation,

$$\frac{df(z)}{dx} = u_x + iv_x, \quad \frac{df(z)}{idy} = -iu_y + v_x. \tag{2.70}$$

Let $f(z)$ satisfy Morera's condition (2.65) and consider

$$g(z) = \frac{f(z)}{z-w} \tag{2.71}$$

for some w in D. Then $g(z)$ satisfies Morera's condition (2.65) in the punctured domain D' with w removed. Consider integration over the contour γ shown in Fig. 2.6

$$0 = \int g(z)dz = \int_C + \int_{I_+} + \int_{C'} + \int_{I_-} = \int_C + \int_{C'}, \tag{2.72}$$

where $-C'$ is the clockwise oriented curve obtained by reversing orientation of the counter-clockwise oriented C', so that

$$\int_C g(z)dz = \int_{-C'} g(z)dz. \tag{2.73}$$

Note that no specific shape of C or C' has been used; C can be deformed to any other C' (with the same orientation), provided that in a continuous deformation the region traced out enclosed between C and C' is within D'. Allow C' to become a small circle around w, i.e., $z - w = \epsilon e^{i\theta}$. If $f(z)$ is continuous at w, then $f(z) = f(w) + \eta(z, w)$ with $\eta(z, w)$ approaching zero in the limit as $z \to w$. By (2.73) and $dz = ie^{i\theta}d\theta$, we have

$$\int_{-C'} g(z) = \int_0^{2\pi} \frac{f(w+\epsilon e^{i\theta})}{\epsilon e^{i\theta}} i e^{i\theta} d\theta = i \int_0^{2\pi} f(w+\epsilon e^{i\theta}) d\theta \to 2\pi f(w) \quad (2.74)$$

in the limit as $\epsilon \to 0$. Thus, (2.73) gives (2.27) once more. An immediate consequence is that $f(z)$ is infinitely times differentiable, and the n-th derivative of $f(z)$ satisfies (2.29). *A continuous complex function satisfying Morera's theorem is infinitely times differentiable.* Finally, consider the circle C: $z - w = \rho e^{i\theta}$. A similar calculation to (2.74) shows the *Mean Value Theorem*

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + \rho e^{i\theta}) d\theta. \quad (2.75)$$

Viewed as a mean value over a boundary, it immediately follows that the real and imaginary parts u and v of $f(w) = u + iv$ are bounded by the extrema of u and v on C. This so-called *minmax theorem* holds true for boundaries of arbitrary shape.

2.6 Liouville's Theorem

If $f(z)$ is entire *and* $f(z)$ is bounded, i.e., $f(z) \leq M$ for all $z \epsilon \mathbb{C}$, then $f(z)$ reduces to a constant. This is *Liouville's theorem*, and it follows readily from Cauchy's integral formula (2.27),

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^2} dz \quad (2.76)$$

where C is a simple closed contour whose interior contains z. We may choose C to be a circle $\xi - z = Re^{i\theta}$ of radius R. Taking the modulus gives

$$\left|f'(z)\right| \leq \frac{1}{2\pi} \int_C \frac{|f(\xi)|}{R^2} R\, d\theta = \frac{M}{R}. \quad (2.77)$$

Since R was arbitrary, we may take it to infinity, and hence $|f'(z)| = 0$. It follows that $f(z)$ is a constant.

Liouville's theorem can be used to show that every polynomial

$$p_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n \quad (a_n \neq 0) \quad (2.78)$$

of degree n has exactly n roots, $p_n(z_i) = 0, i = 1, 2, \cdots, n$, including multiplicities. To see this, let

$$h(z) = \frac{1}{p_n(z)}. \quad (2.79)$$

If $p_n(z)$ has no zeros anywhere in $\mathbb{C}$, then $h(z)$ is entire and reduces to a constant by Liouville's theorem. Then $p_n(z) = 1/h(z)$ is a constant, which contradicts our assumption that $a_n \neq 0$. It follows that $p_n(z)$ has at least one zero, say, z_1. Next, consider $h_1(z) = (z - z_1)/p_n(z)$. Repeating, we encounter additional zeros. This procedure terminates when the constant $h_n(z) = a_n$ is reached, thus retrieving n zeros.

As the examples show, we often encounter polynomials (2.78) with real coefficients. In this event, *roots come in pairs of complex conjugates*. If $p_n(z) = 0$, then

$$0 = \overline{p_n(z)} = p_n(\bar{z}) \tag{2.80}$$

so $\bar{z}$ is also a root. If the root z is real, then this produces no new root, but if it is complex, then conjugation produces a genuine second root.

2.7 Poisson Kernel

On a circle C of radius a, Cauchy's integral formula for a function $f(z)$ reduces to an analytic extension of by a real kernel into the region in C. To see this, consider $z = re^{i\theta}$ inside of C and a point

$$z^* = \frac{a^2}{\bar{z}} \tag{2.81}$$

symmetric with respect to C. Since z^* is outside, we have

$$f(z) = \frac{1}{2\pi i}\int_C f(\xi)\left[\frac{1}{\xi - z} - \frac{1}{\xi - z^*}\right]d\xi. \tag{2.82}$$

Since $a^2 = \xi\bar{\xi}$, (2.82) reduces to

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f\left(ae^{i\alpha}\right)\frac{a^2 - r^2}{a^2 + r^2 - 2ra\cos(\alpha - \theta)}d\alpha. \tag{2.83}$$

Since the Poisson kernel $(a^2 - r^2)/(a^2 + r^2 - 2ar\cos(\alpha - \theta))$ is real, (2.83) defines analytic continuation of the real and imaginary parts of $f(z)$ separately, i.e., giving analytic continations of harmonic functions on the disk. This result is closely connected to the Fourier transform.[5] To see this, specialize to $a = 1$ and let $u(r, \theta)$ denote the real part of $f(z)$ with Fourier coefficients

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} u_0(\alpha)e^{-in\alpha}d\alpha \quad (n\epsilon\mathbb{Z}) \tag{2.84}$$

[5]More on this in Chap. 6.

of data $u_0(\alpha) = u(a, \alpha)$ on S^1. Since $u(r, \theta)$ is real, $C_{-n} = \bar{C}_n$, and hence

$$u(r, \theta) = C_0 + \sum_{n \geq 1} r^n \left(C_n e^{in\theta} + \bar{C}_n e^{-in\theta} \right) \quad (r \leq 1) \tag{2.85}$$

defines the harmonic extension of $u_0(\alpha)$ into $r \leq 1$. Using (2.84), we have

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} u_0(\alpha) \left[1 + \sum_{n \geq 1} \left(r^n e^{-in(\alpha - \theta)} + r^n e^{in(\alpha - \theta)} \right) \right] d\alpha. \tag{2.86}$$

With $\sum_{n \geq 1} z^n = z/(1-z)$, $z = re^{-i(\alpha - \theta)}$, $r < 1$, the kernel in (2.86) evaluates to

$$1 + \frac{z}{1-z} + \frac{\bar{z}}{1-\bar{z}} = \frac{1 - z\bar{z}}{1 + z\bar{z} - z - \bar{z}} = \frac{1 - r^2}{1 + r^2 - 2r\cos(\alpha - \theta)}, \tag{2.87}$$

thus recovering the Poisson kernel in (2.83).

2.8 Flux and Circulation

The equations of motion of fluids derive from conservation laws of mass, energy and momentum. Solutions critically depend on the Reynolds number, a dimensionless ratio of convective to diffusive momentum transport,[6] due to randow walks of molecules or atoms that make up the fluid. For large Reynolds numbers, flows are generally complex and inherently time-dependent, that rarely permit analytical solutions. Even so, some key aspects of conservation of mass and momentum are amenable to analytical solutions, particularly for *solenoidal* (incompressible) and *irrotational* (vanishing vorticity) flows. In two dimensions, these properties are described by a *complex velocity potential*

$$w(z) = \phi(x, y) - i\psi(x, y) \tag{2.88}$$

representing a flow velocity

$$\mathbf{u} = u\mathbf{i} + v\mathbf{j} = \phi_x\mathbf{i} + \phi_y\mathbf{j}, \tag{2.89}$$

satisfying

$$u_x + v_y = 0, \quad u_y - v_x = 0. \tag{2.90}$$

[6] An excellent introduction to classical fluid dynamics is [1].

by virtue of the Cauchy-Riemann relations (2.9) *with a change in sign in v*. The power of complex function theory can hereby be introduced to describe this class of two-dimensional flows.

Consider a smooth curve Γ connecting two points P and Q. We denote the unit tangent vector by τ and the unit normal by $\mathbf{n}$. In two dimensions, we can fix $\mathbf{n}$ uniquely by defining it to be a rotation over $-\pi/2$ of τ. With this convention, $\mathbf{n}$ is the outer normal for a simply positively connected curve, i.e.,

$$\boldsymbol{\tau} = \tau_x\mathbf{i} + \tau_y\mathbf{j}, \quad \mathbf{n} = \tau_y\mathbf{i} - \tau_x\mathbf{j}. \tag{2.91}$$

Thus, the tangent and normal are related by a clockwise rotation over $\pi/2$,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{\tau}. \tag{2.92}$$

For a smooth flow, consider the path integrals

$$I_A = \int_\Gamma \mathbf{u} \cdot \mathbf{n}\, ds, \quad I_B = \int_\Gamma \mathbf{u} \cdot \boldsymbol{\tau}\, ds \tag{2.93}$$

over a curve Γ from P to Q. In general, these two integrals are path dependent. Consider two alternative paths Γ_1 and Γ_2 as shown in Fig. 2.7. Then Γ_1 as shown followed by Γ_2 in the opposite direction produces a closed loop γ with positive orientation. By Green's theorem, we have

$$\begin{aligned} \mathcal{A} &= \int_\gamma \mathbf{u} \cdot \mathbf{n}\, ds = \int_{I(\gamma)} (u_x + v_y) dA, \\ \mathcal{B} &= \int_\gamma \mathbf{u} \cdot \boldsymbol{\tau}\, ds = \int_{I(\gamma)} (v_x - u_y) dA, \end{aligned} \tag{2.94}$$

where $I(\gamma)$ denotes the interior of γ. The first integral in (2.94) vanishes when the flow is solenoidal, and the second vanishes when the flow is irrotational. If these two conditions are satisfied, then the integrals (2.93) are *path independent*. They are hereby well-defined as functions of P and Q without specifying the path connecting them. Exceptions may still arise, when $I(\gamma)$ is not simply connected, e.g., when $I(\gamma)$

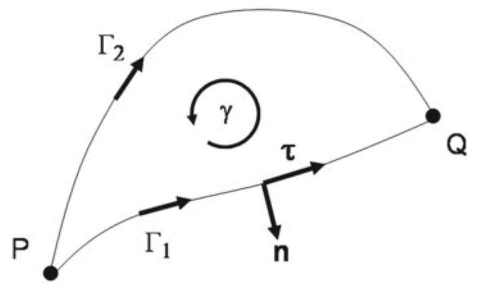

Fig. 2.7 Two paths Γ_1 and Γ_2 from P to Q form a closed loop γ. The loop γ is positively oriented, when formed buy traversing along Γ_1 and traversing Γ_2 in the opposite direction

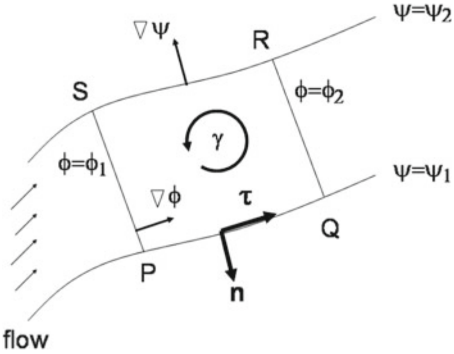

Fig. 2.8 In solenoidal and irrotational flows, streamlines are level curves of the stream function ψ, which are orthogonal to level curves of the flow potential ϕ. The separation between two streamlines forms a duct of *flux*, that is conserved as measured across level curves of the flow potential at $P - S$ and $Q - R$. Circulation obtains from integration over γ, which vanishes for irrotational flow as generated by a flow potential ϕ

is an annulus. In this event, the winding number of γ becomes relevant. We will not consider these possibilities here.

For a flow in a simply connected region, we now consider the *stream function* and the *flow potential* given by integration over a path from A to B,

$$\psi(B) = \psi(A) - \int_A^B \mathbf{u} \cdot \mathbf{n}\, ds, \quad \varphi(B) = \varphi(A) + \int_A^B \mathbf{u} \cdot \boldsymbol{\tau}\, ds. \tag{2.95}$$

As illustrated in Fig. 2.8, ψ is constant along the streamlines of $\mathbf{u}$. Consider a closed loop γ as indicated. Integration of $\mathbf{u} \cdot \mathbf{n}$ over γ receives contributions from segments $Q - R$ and $S - P$ only, and so

$$\Delta\psi = [\psi]_Q^R = [\psi]_P^S. \tag{2.96}$$

It expresses a conserved flux passing through a level curve of ϕ between P to S as well as between Q to R, i.e., the flux between two streamlines $\psi_1 = \psi(Q)$ and $\psi_2 = \psi(R)$. Integration of $\mathbf{u} \cdot \tau$ over γ expresses the *circulation* of the flow over γ. It vanishes based on the assumption of irrotational flow. When the segments $Q - R$ and $S - P$ are orthogonal ($\mathbf{u} \cdot \tau = 0$) to the flow, integration receives contributions from segments $P - Q$ and $R - P$ only, and so

$$\Delta\phi = [\phi]_P^Q = [\phi]_S^R \tag{2.97}$$

measures the strength of the flow along the streamlines.

With ψ and ϕ defined by (2.95) with (2.96), we have $u = \phi_x = \psi_y$, $v = \phi_y = -\psi_x$, that is, the Cauchy-Riemann relations for the real and imaginary part of an analytic function

$$w(z) = \phi + i\psi, \quad w'(z) = u - iv, \quad \phi(z) = \mathrm{Re}\, w(z), \quad \psi(z) = \mathrm{Im}\, w(z) \tag{2.98}$$

as a function of $z = x + iy$.

2.9 Examples of Potential Flows

A uniform flow with unit velocity has a complex velocity potential

$$w(z) = z, \tag{2.99}$$

whose streamlines of constant $\psi(x, y) = \mathrm{Im}\, w(z)$ are parallel to the x-axis. A potential $w(z) = e^{-i\alpha} z$ describes a uniform flow at an inclination angle α to the x-axis.

In a potential flow past a solid body, the surface of the body is a streamline, along which $\psi(x, y) = \mathrm{Im}\, w(z)$ is constant. If the same flow is uniform at infinity, then $w(z) \sim z$ at large z, perhaps up to a complex constant that defines the direction of the flow. The potential

$$w(z) = z + \frac{1}{z} \tag{2.100}$$

hereby describes a flow past a cylinder of unit radius, since $w(z) = \cos\theta$ and hence $\mathrm{Im}\, w(z) = 0$ on $z = e^{i\theta}$. The potential (2.100) has two stagnation points at $z = \pm 1$, where the velocity

$$w'(z) = \frac{z^2 - 1}{z^2} \tag{2.101}$$

vanishes. One can envision sliding the location of these stagnation points over the cylinder, by modifying the zeros of $w'(z)$ to, e.g., $z_1 = e^{-i\alpha}$ and $z_2 = e^{i(\pi+\alpha)}$, changing the numerator in (2.101) to

$$(z - z_1)(z - z_2) = z^2 - 2i \sin\alpha z - 1. \tag{2.102}$$

The associated velocity potential

$$w(z) = z + \frac{1}{z} + 2i \sin\alpha \ln z, \tag{2.103}$$

where the logarithm adds ciruclation to the flow past a cylinder.

The flow potentials

$$w_1(z) = \frac{Q}{2\pi} \ln z, \quad w_2(z) = \frac{\Gamma}{2\pi i} \ln z \tag{2.104}$$

describe point sources of flow with streamlines emanating radially from and, respectively, concentric about the origin. The total flux and circulation integrals (2.93) over $S^1 : z = e^{i\theta}$ give

$$\int_C w_1'(z)(-idz) = Q, \quad \int_C w_2'(z)dz = \Gamma, \tag{2.105}$$

where we used the correspondences $\mathbf{n}ds = -idz$ and $\boldsymbol{\tau}\, ds = dz$. With $\Gamma = -4\pi \sin\alpha$, the potential (2.103) obtains in standard form

$$w(z) = z + \frac{1}{z} + \frac{\Gamma}{2\pi i} \ln z. \tag{2.106}$$

As a map, $\zeta = z + 1/z$ in (2.101) is known as the Joukowski transformation. This map is conformal (having non-zero derivative) away from $z = \pm 1$. It generates Joukowski airfoil profiles as images of circles C with radius $a > 1$ that pass through $z = 1$. These images of C are smooth except for a cusp at $\zeta(1) = 2$, where the map fails to be conformal. The cusp sets location of the corresponding stagnation point on C, to fix circulation and to avoid flow separation and the shedding of vorticies. Let $w(z)$ denote a complex flow potential about a C, such as (2.103). Then $W(\zeta) \equiv w(z(\zeta)))$ is a flow potential past the Joukowski airfoil with velocity

$$W'(\zeta) = w'(z)\left(\frac{d\zeta}{dz}\right)^{-1} = \frac{z^2 w'(z)}{z^2 - 1}. \tag{2.107}$$

Table 2.1 summarizes this discussion.

2.10 Exercises

2.1. Show by explicit evaluation that the real and imaginary part of e^z with $z = i\varphi$ in the defining Taylor series of the exponential function recovers the Taylor series expressions for $\cos\varphi$ and $\sin\varphi$.

2.2. Show that an analytic function $f(z)$ is a conformal map whenever $f'(z) \neq 0$, that is, if α is the angle between the tangents of two curves intersecting at $z = z_0$, then α is also the angle between the tangents to the images of these two curves under $f(z)$.

Table 2.1 Complex function theory

1. Cauchy's integral formula gives a representation of functions $f(z)$ analytic in some domain D according to $f(z) = (1/2\pi i)\int_\gamma f(\xi)/(\xi - z)d\xi$, where γ is a simple counterclockwise oriented contour in D that encloses $z \epsilon D$.

2. The winding number of a contour γ about z satisfies $N = (1/2\pi i)\int_\gamma d\xi/(\xi - z) \epsilon \mathbb{Z}$. $N = 0$, $1/2$ or 1 depending on whether, respectively, z is outside, on or inside γ.

3. Contour integrals of $f(z)$ are determined by residues of $f(z)$ at isolated singularities $z = z_0$, given by the coefficient A_{-1} in expansion $f(z) = \cdots + A_{-1}/(z - z_0) + \cdots$.

4. Analytic functions describe the complex velocity potential $w(z) = \phi + i\psi$ of irrotational solenoidal flows in two dimensions, whereby $w'(z) = u - iv$. The real part ϕ is the velocity potential $\mathbf{u} = \nabla\phi$ and the imaginary part ψ is the stream function.

5. Streamlines are level curves of ψ along which the gradient in ϕ expresses the flow velocity.

2.3. Verify by explicit calculation that the Cauchy-Riemann relations for $f(z) = z^n$ for the three cases $n = 0, 1, 2$.

2.4. Prove (2.16).

2.5. We say that $f(z)$ is analytic at infinity if $g(w) = f(1/w)$ is analytic at $w = 0$. Consider

$$f(z) = \frac{1}{z^2 + 1}. \tag{2.108}$$

Show that $f(z)$ is analytic at infinity. What is the radius of convergence of the Taylor series of $g(w)$?

2.6. Obtain the Laurant series expansion of

$$f(z) = \frac{1}{(z^2 + 1)(4 - z^2)} \tag{2.109}$$

valid in the annulus $1 < |z| < 2$.

2.7. Derive (2.43) by contour deformation to $z = e^{i\theta}$.

2.8. Calculate the integrals

$$(a) \quad \int_\gamma \frac{dz}{z^n}, \qquad (b) \quad \int_\gamma \log(1+z)dz \tag{2.110}$$

where n is an integer and γ is a small loop around the origin.

2.9. Obtain the partial fractions of

$$\text{(a) } \frac{1}{z^2-1}, \text{ (b) } \frac{2z}{z^2+1}, \text{ (c) } \frac{1}{z^2+1}, \text{ (d) } \frac{z-2}{z^2+z}. \tag{2.111}$$

2.10. Obtain the partial fractions of

$$\text{(a) } \frac{1}{z^2(z-\frac{1}{2})(z^2+4)} \quad \text{(b) } \frac{1}{z^n(z+1)} \ (n = 1, 2, 3) \tag{2.112}$$

2.11. Obtain the integrals

$$\text{(a) } \int_{1+i}^{3-2i} \sin z dz, \ \text{ (b) } \int_\gamma z^n dz \ (n \in \mathbb{N}), \ \text{ (c) } \int_\gamma \log(1+z)dz \tag{2.113}$$

where γ is a contour in the unit disk.

2.12. Let γ be the unit circle $|z| = 1$ with counter clockwise orientation. Compute the following complex integrals

1. $\int_\gamma \frac{dz}{(z-a)^n}$ for $a = 0, 2$ and $n = 1, 2$.
2. $\int_\gamma \frac{dz}{\cosh^2 z}$
3. $\int_\gamma \frac{\sin(\pi z^2)}{(z-1/2)(z-2)} dz$.

2.13. Obtain the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{\sin z}{2z+i} \tag{2.114}$$

where γ is a contour which encloses $z = -\frac{1}{2i}$.

2.14. Find all the *branch points*, where the following function does *not* satisfy the

Cauchy-Riemann relations:

$$f(z) = \sqrt{e^z + 1}. \tag{2.115}$$

2.15. Prove the minmax theorem based on (2.75).

2.16. Consider the functions

$$f(z) = \frac{2}{z} + 3 + 4z, \quad g(z) = \frac{1}{z^2} + f(z) \tag{2.116}$$

and the residues defined by the contour integrals

$$\mathrm{Res}_{z=0}\, f(z) = \frac{1}{2\pi i}\int_\gamma f(z)dz, \quad \mathrm{Res}_{z=0}\, g(z) = \frac{1}{2\pi}\int_\gamma g(z)dz \tag{2.117}$$

over a contour γ that encloses the origin $z = 0$. Show that

$$\mathrm{Res}_{z=0}\, f(z) = \lim_{z\to 0} zf(z), \quad \mathrm{Res}_{z=0}\, g(z) = \lim_{z\to 0}\frac{d}{dz}\left[z^2 g(z)\right]. \tag{2.118}$$

2.17. The Möbius transformation $w(z) = (z - i)/(z + i)$ maps the real line onto the unit circle. Cauchy's integral formula on the unit circle hereby transforms to one on the real line. Follow steps similar to those in deriving the Poisson kernel (2.83), now with the point $z^* = \bar{z}$ symmetric with respect to the real line, to obtain

$$u(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u_0(x')}{(x' - x)^2 + y^2}dx'. \tag{2.119}$$

2.18. Sketch the streamlines including the direction of flow of the following complex velocity potentials

$$\text{(a)}\ \ w(z) = z^k \ \left(k = \frac{1}{2}, 2, 3\right), \quad w(z) = \frac{1}{z}. \tag{2.120}$$

$$\text{(b)}\ \ w(z) = \frac{1}{2\pi}\ln(z^2 - 1), \quad w(z) = z + \frac{1}{2\pi}\ln z. \tag{2.121}$$

2.19. Sketch the field lines of the complex velocity potential of the dipole

$$w(z) = \frac{1}{2\pi\epsilon}\left[\ln(z + \epsilon) - \ln(z - \epsilon)\right] \tag{2.122}$$

and determine the limit as ϵ approaches zero.

2.20. Show that

$$f(z) = (1+z)^{\frac{1}{z}} \tag{2.123}$$

has a removable singularity at the origin by deriving a Taylor series expansion $f(z) = e\Sigma_{m=0}^{\infty} c_m z^m$ about $z = 0$. Note that the radius of convergence is 1 in view of the singularity at $z = -1$. Match the behavior of $f(z)$ as z approaches -1 from the right to the Taylor series expansion to show that the (rational) coefficients c_m approach $e^{-1}(-1)^m$. [*Hint.* Use $e^{z^{-1}\log(1+z)} = e^{1-\frac{1}{2}z+\frac{1}{3}z^2-\frac{1}{4}z^3+\cdots}$.]

2.21. Consider an asymptotically flat black hole spacetime. Viewed as an analytic function of radius $z \in \mathbb{C}$, the metric satisfies $\eta_{ab} + O(z^{-1})$, where η_{ab} denotes the Minkowski metric. Use Liouville's theorem to argue that the metric must have at least one singularity in $\mathbb{C}$ [2].

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