

Preface

Algebraic and topological structures compatibly placed on the same underlying set lead to the notions of topological semigroups, groups and vector spaces, among others. It is then natural to consider concepts such as continuous homomorphisms and continuous linear transformations between above-said objects. By an ‘operator’, we mean a continuous linear transformation of a normed linear space into itself.

Functional analysis was developed around the turn of the last century by the pioneering work of Banach, Hilbert, von Neumann, Riesz and others. Within a few years, after an amazing burst of activity, it was well developed as a major branch of mathematics. It is a unifying framework for many diverse areas such as Fourier series, differential and integral equations, analytic function theory and analytic number theory. The subject continues to grow and attracts the attention of some of the finest mathematicians of the era.

A generalisation of the methods of vector algebra and calculus manifests itself in the mathematical concept of a Hilbert space, named after the celebrated mathematician Hilbert. It extends these methods from two-dimensional and three-dimensional Euclidean spaces to spaces with any finite or infinite dimension. These are inner product spaces, which allow the measurement of angles and lengths; once completed, they possess enough limits in the space so that the techniques of analysis can be used. Their diverse applications attract the attention of physicists, chemists and engineers alike in good measure.

Chapter 1 establishes notations used in the text and collects results from vector spaces, metric spaces, Lebesgue integration and real analysis. No attempt has been made to prove the results included under the above topics. It is assumed that the reader is familiar with them. Appropriate references have, however, been provided.

Chapter 2 includes in some details the study of inner product spaces and their completions. The space $L^2(X, M, \mu)$, where X , M and μ denote, respectively, a nonempty set, a σ -algebra of subsets of it and an extended nonnegative real-valued measure, has been studied. The theorem of central importance in the analysis due to Riesz and Fischer, namely that $L^2(X, M, \mu)$ is a complete metric space, has been proved. So has been the result, namely, the space $A(\Omega)$ of holomorphic functions defined on a bounded domain Ω is complete. To make the book useful to

probabilists, statisticians, physicists, chemists and engineers, we have included many applied topics: Legendre, Hermite, Laguerre polynomials, Rademacher functions, Fourier series and Plancherel's theorem. Such applications of the abstract theory are also of significance for the pure mathematician who wants to know the origin of the subject. This chapter also contains the study of linear functionals on Hilbert spaces; more specifically, Riesz Representation Theorem, the dual of a Hilbert space is itself a Hilbert space and the fact that these spaces constitute important examples of reflexive normed linear spaces. Applications of Hilbert space theory to different branches of mathematics, such as approximation theory (Müntz' Theorem), measure theory (Radon–Nikodým Theorem), Bergman kernel and conformal mapping (analytic function theory), are included in Chap. 2.

A major portion of this book is devoted to the study of operators in Hilbert spaces. It is carried out in Chaps. 3 and 4. The set of operators in a Hilbert space H , equipped with the uniform norm, is denoted by $\mathcal{B}(H)$. Some well-known classes of operators have been defined. Under compact operators, Fredholm theory has been discussed. The Mean Ergodic Theorem has been proved as an application at the end of Chap. 3. Spectrum of an operator is the key to the understanding of the operator. Properties of the spectrum of different classes of operators, such as normal operators, self-adjoint operators, unitaries, isometries and compact operators, have been discussed under appropriate headings. Here, the properties of the spectrum specific to the class of operators under consideration are studied. A large number of examples of operators together with their spectrum and its splitting into point spectrum, continuous spectrum, residual spectrum, approximate point spectrum and compression spectrum have been painstakingly worked out. It is expected that the treatment will aid the understanding of the reader. The treatment of polar decomposition of an operator is different from the ones available in books. Numerical range and numerical radius of an operator have been defined. The spectral radius and the numerical radius of an operator have been compared. Professor Ajit Iqbal Singh deserves special thanks for the help she rendered while this part was being written. Spectral theorems, which reveal almost everything about the operators, have been accorded special treatment in the text. After proving the spectral theorem for compact normal operators, spectral theorems for self-adjoint operators and normal operators have been proved. Here, we have been guided by the fundamental principle of pedagogy that repetition helps in imbibing rather subtle techniques needed for proving the spectral theorems. A bird's eye view of invariant subspaces with special attention to the Volterra operator is included. We close the chapter with a brief introduction to unbounded operators.

Chapter 5 contains important theorems followed by applications from Banach spaces.

The final chapter contains hints and solutions to the 166 problems listed under various sections. These are over and above the numerous detailed examples scattered all over the text.

Elements of Hilbert Spaces and Operator Theory

Vasudeva, H.L.

2017, XIII, 522 p. 5 illus., Hardcover

ISBN: 978-981-10-3019-2