Chapter 2
Hopf Bifurcation in Impulsive Systems

2.1 Hopf Bifurcation of a Discontinuous Limit Cycle

This chapter is organized in the following manner. In the first section, we give the description of the systems under consideration and prove the theorem of existence of foci and centers of the nonperturbed system. The main subject of Sect. 2.1.2 is the foci of the perturbed equation. The noncritical case is considered. In Sect. 2.1.3, the problem of distinguishing between the center and the focus is solved. Bifurcation of a periodic solution is investigated in Sect. 2.1.4. The last section consists of examples illustrating the bifurcation theorem.

2.1.1 The Nonperturbed System

Denote by $< x, y >$ the dot product of vectors $x, y \in \mathbb{R}^2$, and $||x|| =< x, x >^{\frac{1}{2}}$, the norm of a vector $x \in \mathbb{R}^2$. Moreover, let $\mathscr{R}$ be the set of all real-valued constant 2×2 matrices, and $\mathscr{I} \in \mathscr{R}$ be the identity matrix.

D_0-**system**. Consider the following differential equation with impulses

$$\begin{aligned} \frac{dx}{dt} &= Ax, \\ \Delta x|_{x\in \Gamma_0} &= B_0 x, \end{aligned} \tag{2.1.1}$$

where Γ_0 is a subset of $\mathbb{R}^2$, and it will be described below, $A, B_0 \in \mathscr{R}$.

The following assumptions will be needed throughout this chapter:

(C1) $\Gamma_0 = \cup_{i=1}^{p} s_i$, where p is a fixed natural number and half-lines s_i, $i = 1, 2, \ldots, p$, are defined by equations $< a^i, x >= 0$, where $a^i = (a_1^i, a_2^i)$ are constant vectors. The origin does not belong to the lines (see Fig. 2.1).

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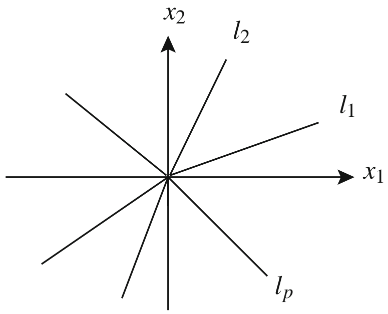


Fig. 2.1 The domain of the nonperturbed system (2.1.1) with a vertex which unites the straight lines s_i, $i = 1, 2, \ldots, p$

(C2)

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$;

(C3) there exists a regular matrix $Q \in \mathscr{R}$ and nonnegative real numbers k and θ such that

$$B_0 = kQ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} Q^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

We consider every angle for a point with respect to the positive half-line of the first coordinate axis. Denote $s_i^{'} = (\mathscr{I} + B_0)s_i$, $i = 1, 2, \ldots, p$. Let γ_i and ζ_i be angles of s_i and $s_i^{'}$, $i = 1, 2, \ldots, p$, respectively,

$$B_0 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

(C4) $0 < \gamma_1 < \zeta_1 < \gamma_2 < \cdots < \gamma_p < \zeta_p < 2\pi$, $(b_{11} + 1)\cos\gamma_i + b_{12}\sin\gamma_i \neq 0$, $i = 1, 2, \ldots, p$.

If conditions $(C1)$–$(C4)$ hold, then (2.1.1) is said to be a D_0-*system*.

Exercise 2.1.1 *Verify that the origin is a unique singular point of a* D_0*-system and (2.1.1) is not a linear system.*

Exercise 2.1.2 *Using the results of the last chapter, prove that* D_0*-system (2.1.1) provides a* B*-smooth discontinuous flow.*

If we use transformation $x_1 = r\cos(\phi)$, $x_2 = r\sin(\phi)$ in (2.1.1) and exclude the time variable t, we can find that the solution $r(\phi, r_0)$ which starts at the point $(0, r_0)$, satisfies the following system:

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda r, \\ \Delta r\,|_{\phi=\gamma_i(\mathrm{mod}\,2\pi)} &= k_i r, \end{aligned} \tag{2.1.2}$$

where $\lambda = \frac{\alpha}{\beta}$, the angle-variable ϕ is ranged over the set

$$R_\phi = \cup_{i=-\infty}^{\infty}[\cup_{j=1}^{p-1}(2\pi i + \zeta_j, 2\pi i + \gamma_{j+1}] \cup (2\pi i + \zeta_p, 2\pi(i+1) + \gamma_1]]$$

and $k_i = [((b_{11}+1)\cos(\gamma_i) + b_{12}\sin(\gamma_i))^2 + (b_{21}\cos(\gamma_i) + (b_{22}+1)\sin(\gamma_i))^2]^{\frac{1}{2}} - 1$. Equation (2.1.2) is 2π-periodic, so we shall consider just the section $\phi \in [0, 2\pi]$ in what follows. That is, the system

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda r, \\ \Delta r\,|_{\phi=\gamma_i} &= k_i r, \end{aligned} \tag{2.1.3}$$

is considered with $\phi \in [0, 2\pi]_\phi \equiv [0, 2\pi]\setminus \cup_{i=1}^{p}(\gamma_i, \zeta_i]$. System (2.1.3) is a sample of the timescale differential equation with transition condition [38]. We shall reduce (2.1.3) to an impulsive differential equation [6, 38] for the investigation's needs. Indeed, let us introduce a new variable $\psi = \phi - \sum_{0<\gamma_j<\phi}\theta_j, \theta_j = \zeta_j - \gamma_j$, with the range $[0, 2\pi - \sum_{i=1}^{p}\theta_i]$. We shall call this new variable ψ-substitution. It is easy to check that upon ψ-substitution, the solution $r(\phi, r_0)$ satisfies the following impulsive equation

$$\begin{aligned} \frac{dr}{d\psi} &= \lambda r, \\ \Delta r\,|_{\psi=\delta_j} &= k_j r, \end{aligned} \tag{2.1.4}$$

where $\delta_j = \gamma_j - \sum_{0<\gamma_i<\gamma_j}\theta_i$. Solving the last impulsive system and using the inverse of ψ-substitution, one can obtain that the solution $r(\phi, r_0)$ of (2.1.2) has the form

$$r(\phi, r_0) = \exp\left(\lambda\left(\phi - \sum_{0<\gamma_i<\phi}\theta_i\right)\right)\prod_{0<\gamma_i<\phi}(1+k_i)r_0, \tag{2.1.5}$$

if $\phi \in [0, 2\pi]_\phi$.

Denote

$$q = \exp\left(\lambda\left(2\pi - \sum_{i=1}^{p}\theta_i\right)\right)\prod_{i=1}^{p}(1+k_i). \tag{2.1.6}$$

Applying the Poincaré return map $r(2\pi, r_0)$ to (2.1.5), one can obtain that the following theorem follows.

Theorem 2.1.1 *If*

(1) $q = 1$, *then the origin is a center and all solutions of (2.1.1) are periodic with period* $T = (2\pi - \sum_{i=1}^{p}\theta_i)\beta^{-1}$*;*

(2) $q < 1$, *then the origin is a stable focus;*
(3) $q > 1$, *then the origin is an unstable focus of* D_0*-system.*

2.1.2 The Perturbed System

Theorem 2.1.1 of the last section implies that if conditions $(C1)$–$(C4)$ are valid, then each trajectory of (2.1.1) either spirals to the origin or is a discontinuous cycle. Moreover, if the trajectory spirals to the origin, then it spirals to infinity, too. That is, the asymptotic behavior of the trajectory is very similar to the behavior of trajectories of the planar linear system of ordinary differential equations with constant coefficients [91, 126]. In what follows, we will consider how a perturbation may change the phase portrait of the system.

D-**system**. Let us consider the following equation:

$$\begin{aligned} \frac{dx}{dt} &= Ax + f(x), \\ \Delta x|_{x\in\Gamma} &= B(x)x, \end{aligned} \tag{2.1.7}$$

in a neighborhood G of the origin.

The following is the list of conditions assumed for this system:

(C5) $\Gamma = \cup_{i=1}^{p} l_i$ is a set of curves which start at the origin and are determined by the equations $< a^i, x > +\tau_i(x) = 0, \quad i = 1, 2, \ldots, p$. The origin does not belong to the curves (see Fig. 2.2).

(C6)

$$B(x) = (k + \kappa(x))Q \begin{pmatrix} \cos(\theta + \upsilon(x)) & -\sin(\theta + \upsilon(x)) \\ \sin(\theta + \upsilon(x)) & \cos(\theta + \upsilon(x)) \end{pmatrix} Q^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$(\mathscr{I} + B(x))x \in G$ for all $x \in G$;

(C7) $\{f, \kappa, \nu\} \subset C^{(1)}(G), \{\tau_i, i = 1, 2, \ldots, p\} \subset C^{(2)}(G)$;

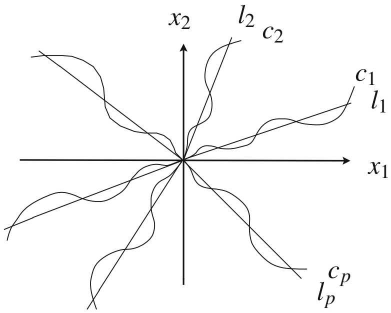


Fig. 2.2 The domain of the perturbed system (2.1.7) near a vertex which unites the curves l_i associated with the straight lines s_i, $i = 1, 2, \ldots, p$

(C8) $f(x) = o(||x||),\ \kappa(x) = o(||x||),\ \upsilon(x) = o(||x||),\ \tau_i(x) = o(||x||^2),\ i = 1, 2, \ldots, p$;
Moreover, we assume that the matrices A, Q, the vectors a^i, $i = 1, 2, \ldots, p$, and constants k, θ are the same as in (2.1.1); i.e.,

(C9) The system (2.1.1) is D_0-system associated with (2.1.7).

If conditions $(C1)$–$(C9)$ hold, then the system (2.1.7) is said to be a D-system. If G is sufficiently small, then conditions $(C4)$ and $(C8)$ imply that none of curves l_i intersect itself, they do not intersect each other, and the origin is a unique singular point of the D-system.

Exercise 2.1.3 *Using the results of the last chapter, and Example 2.1.2, prove that D-system defines a B-smooth discontinuous flow.*

Assume, without loss of generality, that $\gamma_i \neq \frac{\pi}{2} j,\ j = 1, 3,$ and transform the equations in $(C5)$ to the polar coordinates so that $l_i : a_i^1 r\cos(\phi) + a_i^2 r\sin(\phi) + \tau_i(r\cos(\phi),\ r\sin(\phi)) = 0$ or

$$\phi = \tan^{-1}\left(\tan\gamma_i - \frac{\tau_i}{a_i^2 r\cos(\phi)}\right).$$

Now, use Taylor's expansion to get that

$$l_i : \phi = \gamma_i + r\psi_i(r, \phi), \tag{2.1.8}$$

$i = 1, 2, \ldots, p$, where ψ_i are 2π-periodic in ϕ, continuously differentiable functions, and $\psi_i = O(r)$. If the point $x(t)$ meets the discontinuity curve l_i with an angle θ, then the point $x(\theta+)$ belongs to the curve $l_i' = \{z \in \mathbb{R}^2 | z = (\mathscr{I} + B(x))x,\ x \in l_i\}$. The following assertion is very important for the rest of the chapter.

Lemma 2.1.1 *Suppose $(C7)$ and $(C8)$ are satisfied. Then the curve $l_i^{'}$, $1 \le i \le p$, is placed between l_i and l_{i+1}, if G is sufficiently small.*

Proof Fix $i = 1, 2, \ldots, p$, and assume that $s_i,\ s_{i+1},\ l_i,\ l_{i+1}$ are transformed by the map $y = Q^{-1}x$ into lines $s_i^{''},\ s_{i+1}^{''},\ l_i^{''},\ l_{i+1}^{''}$, respectively. Set $L_i = \{z \in \mathbb{R}^2 |\ z = Q^{-1}(I + B(Qy))Qy,\ y \in l_i^{''}\}$, $\xi_i = Q^{-1}(I + B_0)Qs_i^{''}$, and let $\gamma_i^{'},\ \gamma_{i+1}^{'},\ \zeta_i^{'}$ be the angles of straight lines $s_i^{''},\ s_{i+1}^{''},\ \xi_i$. We may assume, without loss of generality, that $\gamma_i^{'} < \zeta_i^{'} < \gamma_{i+1}^{'}$. To prove the lemma, it is sufficient to check whether L_i lies between curves $l_i^{''},\ l_{i+1}^{''}$. Suppose that $0 < \gamma_i^{'} < \zeta_i^{'} < \gamma_{i+1}^{'} < \frac{\pi}{2}$. Otherwise, one can use a linear transformation, which does not change the relation of the curves. Let $c_1 y_1 + c_2 y_2 + l^*(y_1, y_2) = 0$ be the equation of the line $l_i^{''}$. Use the polar coordinates $y_1 = \rho\cos(\phi),\ y_2 = \rho\sin(\phi)$ and obtain $\phi = \gamma_i^{'} + \rho\psi^*(\rho, \phi)$, where $\psi^*(\rho, \phi) = O(\rho)$ and ψ^* is a 2π-periodic function. If $y = (y_1, y_2) \in l_i^{''}$, then the point

$$y^+ = Q^{-1}(B(Qy) + I)Qy, \tag{2.1.9}$$

where $y^+ = (y_1^+, y_2^+)$ belongs to L_i. Assume without loss of generality that $y_1^+ \neq 0$. Otherwise, use the condition $y_2^+ \neq 0$. If we set $\rho = (y_1^2 + y_2^2)^{\frac{1}{2}}$, $\phi = \tan^{-1}(\frac{y_2}{y_1})$, $\rho^+ = ((y_1^+)^2 + (y_2^+)^2)^{\frac{1}{2}}$, $\phi^+ = \tan^{-1}(\frac{y_2^+}{y_1^+})$, then (2.1.9) implies that

$$\rho^+ = k_i\rho + \rho\beta^*(\rho, \phi), \tag{2.1.10}$$

$$\phi^+ = \phi + \theta + \gamma^*(\rho, \phi), \tag{2.1.11}$$

where β^* and γ^* are 2π-periodic in ϕ functions and $\beta^* = O(\rho)$, $\gamma^* = O(\rho)$. Let $\sigma(y_1, y_2) = c_1y_1 + c_2y_2 + l^*(y_1, y_2)$. Then,

$$\sigma(y_1^+, y_2^+) = \rho^+(c_1\cos(\phi^+) + c_2\sin(\phi^+) + l^*(\rho^+\cos(\phi^+), \rho^+\sin(\phi^+)) =$$

$$\rho^+\sqrt{c_1^2 + c_2^2}\sin(\theta + \upsilon(\rho, \phi) - \rho\psi^*(\rho, \psi)) + l^*(\rho^+\cos(\phi^+), \rho^+\sin(\phi^+)),$$

where $\upsilon(\rho, \phi) = \upsilon(Qy)$. It is readily seen that the sign of $\sigma(\rho^+, \phi^+)$ is the same as of $\sin(\theta)$, if ρ is sufficiently small. Consequently, $\sigma(\rho^+, \phi^+) > 0$. Thus, the curve L_i is placed above the curve $l_i^{''}$ in the first quarter of the plane Ox_1x_2. Similarly, one can show that it is placed below $l_{i+1}^{''}$. The lemma is proved.

The last lemma guarantees that if G is sufficiently small, then every nontrivial trajectory of the system (2.1.7) meets each of the lines l_i, $i = 1, 2, \dots, p$, precisely once within any time interval of length T.

2.1.3 Foci of the D-System

Utilize the polar coordinates $x_1 = r\cos(\phi)$, $x_2 = r\sin(\phi)$ to reduce the differential part of (2.1.7) to the following form:

$$\frac{dr}{d\phi} = \lambda r + P(r, \phi).$$

It is known [60, 91, 173, 180] that $P(r, \phi)$ is 2π-periodic, continuously differentiable function, and $P = o(r)$. Set $x^+ = (x_1^+, x_2^+) = (\mathscr{I} + B(x))x$, $x^+ = r^+(\cos\phi^+, \sin\phi^+)$, $\tilde{x}^+ = (\tilde{x}_1^+, \tilde{x}_2^+) = (\mathscr{I} + B(0))x$, where $x = (x_1, x_2) \in l_i$, $i = 1, 2, \dots, p$. One can find that the inequality $||x^+ - \tilde{x}^+|| \le ||B(x) - B(0)||||x||$ implies $r^+ = r + k_ir + \omega(r, \phi)$. Use the relation between $\frac{x_2^+}{x_1^+}$ and $\frac{\tilde{x}_2^+}{\tilde{x}_1^+}$ and condition $(C5)$ to obtain that $\phi^+ = \phi + \theta_i + \gamma(r, \phi)$. Functions ω, γ are 2π-periodic in ϕ and $\omega = o(r)$, $\gamma(r, \phi) = o(r)$. Finally, (2.1.7) has the form

$$\begin{aligned}
\frac{dr}{d\phi} &= \lambda r + P(r, \phi),\\
\Delta r \mid_{(\rho,\phi)\in l_i} &= k_i r + \omega(r, \phi),\\
\Delta \phi \mid_{(\rho,\phi)\in l_i} &= \theta_i + \gamma(r, \phi).
\end{aligned} \tag{2.1.12}$$

It is convenient to introduce the following version of B-equivalence.

Introduce the following system:

$$\begin{aligned}
\frac{d\rho}{d\phi} &= \lambda \rho + P(\rho, \phi),\\
\Delta \rho \mid_{\phi=\gamma_i} &= k_i \rho + w_i(\rho),\\
\Delta \phi \mid_{\phi=\gamma_i} &= \theta_i,
\end{aligned} \tag{2.1.13}$$

where all elements, except w_i, $i = 1, 2, \ldots, p$, are the same as in (2.1.12) and the domain of (2.1.13) is $[0, 2\pi]_\phi$. Functions w_i will be defined below.

Let $r(\phi, r_0)$, $r(0, r_0) = r_0$, be a solution of (2.1.12) and ϕ_i be the angle where the solution intersects l_i. Denote by $\chi_i = \phi_i + \theta_i + \gamma(r(\phi_i, r_0), \phi_i)$ the angle of $r(\phi, r_0)$ after the jump.

We shall say that systems (2.1.12) and (2.1.13) are B-equivalent in G if there exists a neighborhood $G_1 \subset G$ of the origin such that for every solution $r(\phi, r_0)$ of (2.1.12) whose trajectory is in G_1, there exists a solution $\rho(\phi, r_0)$, $\rho(0, r_0) = r_0$, of (2.1.13) which satisfies the relation

$$r(\phi, r_0) = \rho(\phi, r_0), \phi \in [0, 2\pi]_\phi \setminus \cup_{i=1}^{p} \{[\phi_i, \hat{\gamma}_i,] \cup [\zeta_i, \hat{\chi}_i]\}, \tag{2.1.14}$$

and, conversely, for every solution $\rho(\phi, r_0)$ of (2.1.13) whose trajectory is in G_1, there exists a solution $r(\phi, r_0)$ of (2.1.12) which satisfies (2.1.14).

We will define functions w_i such that systems (2.1.12) and (2.1.13) are B-equivalent in G, if the domain is sufficiently small.

Fix i. Let $r_1(\phi, \gamma_i, \rho)$, $r_1(\gamma_i, \gamma_i, \rho) = \rho$, be a solution of the equation

$$\frac{dr}{d\phi} = \lambda r + P(r, \phi) \tag{2.1.15}$$

and $\phi = \eta_i$ be the meeting angle of $r_1(\phi, \gamma_i, \rho)$ with l_i. Then,

$$r_1(\eta_i, \gamma_i, \rho) = \exp(\lambda(\eta_i - \gamma_i))\rho + \int_{\gamma_i}^{\eta_i} \exp(\lambda(\eta_i - s))P(r_1(s, \gamma_i, \rho), s)ds.$$

Let $\eta_i^1 = \eta_i + \theta_i + \gamma(r_1(\eta_i, \gamma_i, \rho), \eta_i)$, $\rho^1 = (1 + k_i)r_1(\eta_i, \gamma_i, \rho) + \omega(r(\eta_i, \gamma_i, \rho), \eta_i)$, and $r_2(\phi, \eta_i^1, \rho^1)$ be the solution of system (2.1.15),

$$r_2(\zeta_i, \eta_i^1, \rho^1) = \exp(\lambda(\zeta_i - \eta_i^1))\rho^1 + \int_{\eta_i^1}^{\zeta_i} \exp(\lambda(\zeta_i - s))P(r_2(s, \eta_i^1, \rho^1), s)ds.$$

Introduce

$$w_i(\rho) = r_2(\zeta_i, \eta_i^1, \rho^1) - (1+k_i)\rho = \exp(\lambda(\zeta_i - \eta_i^1))[(1+k_i)(\exp(\lambda(\eta_i - \gamma_i))\rho +$$
$$\int_{\gamma_i}^{\eta_i} \exp(\lambda(\eta_i - s))P(r_1(s,\gamma_i,\rho),s)ds) + \omega(r_1(\eta_i,\gamma_i,\rho),\eta_i)] +$$
$$\int_{\eta_i^1}^{\zeta_i} \exp(\lambda(\zeta_i - s))P(r_2(s,\eta_i^1,\rho^1),s)ds - (1+k)\rho$$

or, if simplified,

$$\begin{aligned}
&w_i(\rho) = (1+k)[\exp(-\lambda\gamma(r_1(\eta_i,\gamma_i,\rho),\eta_i)) - 1]\rho + \\
&(1+k)\int_{\gamma_i}^{\eta_i} \exp(\lambda(\zeta_i - \theta_i - s - \rho\gamma(r_1(\eta_i,\gamma_i,\rho),\eta_i)))P(r_1(s,\gamma_i,\rho),s)ds + \\
&\int_{\eta_i^1}^{\zeta_i} \exp(\lambda(\zeta_i - s))P(r_2(s,\eta_i^1,\rho^1),s)ds + \\
&\exp(\lambda(\zeta_i - \eta_i^1))\omega(r_1(\eta_i,\gamma_i,\rho),\eta_i).
\end{aligned} \tag{2.1.16}$$

Differentiating (2.1.8) and (2.1.16), one can find that

$$\begin{aligned}
&\frac{d\eta_i}{d\rho} = \frac{\frac{\partial r_1}{\partial\rho}[\psi_i + r_1\frac{\partial\psi_i}{\partial r}]}{1 - (\lambda r_1 + P)[\psi_i + r_1\frac{\partial\psi_i}{\partial r}] - r_1\frac{\partial\psi_i}{\partial\phi}}, \quad \frac{d\eta_i^1}{d\rho} = \frac{d\eta_i}{d\rho}\left(1 + \frac{\partial\gamma}{\partial\phi}\right) + \frac{\partial\gamma}{\partial r}\frac{\partial r_1}{\partial\rho}, \\
&\frac{dw_i}{d\rho} = (1+k_i)[\mathrm{e}^{-\lambda\gamma} - 1] - \lambda(1+k_i)\mathrm{e}^{-\lambda\gamma}\left(\frac{\partial\gamma}{\partial r}\frac{\partial r_1}{\partial\rho} + \frac{\partial\gamma}{\partial\phi}\frac{d\eta_i}{d\rho}\right)\rho + \\
&(1+k_i)\mathrm{e}^{\lambda(\zeta_i - \theta_i - \eta_i - \gamma)}P\frac{d\eta_i}{d\rho} + \\
&(1+k_i)\int_{\gamma_i}^{\eta_i} \mathrm{e}^{\lambda(\zeta_i - \theta - s - \gamma)}\left\{-\lambda\left(\frac{\partial\gamma}{\partial r}\frac{\partial r_1}{\partial\rho} + \frac{\partial\gamma}{\partial\phi}\frac{d\eta_i}{d\rho}\right)P - \frac{\partial P}{\partial r}\frac{\partial r_1}{\partial\rho} - \frac{\partial P}{\partial\phi}\frac{d\eta_i}{d\rho}\right\}ds + \\
&\int_{\eta_i^1}^{\zeta_i} \mathrm{e}^{\lambda(\zeta_i - s)}\frac{\partial P(r_2(s,\eta_i^1,\rho^1),s)}{\partial r}\frac{\partial r_2}{\partial\rho}ds - \mathrm{e}^{\lambda(\zeta_i - \eta_i^1)}P(\rho^1,\eta_i^1)\frac{\partial\eta_i^1}{\partial\rho} + \\
&\mathrm{e}^{\lambda(\zeta_i - \eta_i^1)}\left[-\frac{\partial\eta_i^1}{\partial\rho}\omega + \frac{\partial\omega}{\partial r}\frac{\partial r_1}{\partial\rho} + \frac{\partial\omega}{\partial\phi}\frac{d\eta_i}{d\rho}\right].
\end{aligned} \tag{2.1.17}$$

Analyzing (2.1.16) and (2.1.17), one can prove that the following two lemmas are valid.

Lemma 2.1.2 *If conditions* $(C1)$–$(C5)$ *are valid then* w_i *is a continuously differentiable function, and* $w_i(\rho) = o(\rho)$, $i = 1, 2, \ldots, p$.

Lemma 2.1.3 *The systems (2.1.12) and (2.1.13) are* B*-equivalent if* G *is sufficiently small.*

Theorem 2.1.2 *Suppose that* $(C1)$–$(C6)$ *are satisfied and* $q < 1$ $(q > 1)$. *Then the origin is a stable (unstable) focus of system (2.1.7).*

Proof Let $r(\phi, r_0)$, $r(0, r_0) = r_0$, be the solution of (2.1.12), and $\rho(\phi, r_0)$, $\rho(0, r_0) = r_0$, be the solution of (2.1.13). Using ψ-substitution, one can obtain that

$$
\begin{aligned}
\rho(\phi, r_0) = \exp(\lambda\phi) \Bigg\{ & \Pi_{i=1}^{m}(1+k_i)\exp\left(-\lambda\sum_{s=1}^{m}\theta_s\right) r_0 + \\
& \Pi_{i=1}^{m}(1+k_i)\exp\left(-\lambda\sum_{s=1}^{m}\theta_s\right)\int_0^{\gamma_1}\exp(-\lambda u)P\,du + \\
& \Pi_{i=2}^{m}(1+k_i)\exp\left(-\lambda\sum_{s=2}^{m}\theta_s\right)\int_{\zeta_1}^{\gamma_2}\exp(-\lambda u)P\,du + \cdots \\
& \int_{\zeta_m}^{\phi}\exp(-\lambda u)P\,du + \Pi_{i=2}^{m}(1+k_i)\exp\left(-\lambda\sum_{s=2}^{m}\theta_s\right) w_1 + \\
& \Pi_{i=3}^{m}(1+k_i)\exp\left(-\lambda\sum_{s=3}^{m}\theta_s\right) w_2 \ldots + \exp(-\lambda\zeta_m)w_m \Bigg\},
\end{aligned} \tag{2.1.18}
$$

where $\phi \in [0, 2\pi]_\phi$, $P = P(\rho(\phi, r_0), \phi)$, $w_i = w_i(\rho(\gamma_i, r_0)$. Now, applying Theorem 6.1.1 in [1], conditions $(C4)$, $(C5)$ and Lemma 2.1.2, one can find that the solution $\rho(\psi, r_0)$ is differentiable in r_0 and the derivative $\frac{\partial\rho(\phi, r_0)}{\partial r_0}$ at the point $(2\pi, 0)$ is equal to q. Since (2.1.12) and (2.1.13) are B-equivalent, it follows that

$$
\frac{\partial r(2\pi, 0)}{\partial r_0} = q
$$

and the proof is completed.

2.1.4 The Center and Focus Problem

Throughout this section, we assume that $q = 1$. That is, the critical case is considered. Functions f, κ, v, τ_i, $i = 1, 2, \ldots, p$, are assumed to be analytic in G. By condition $(C8)$, Taylor's expansions of functions f, κ, and v start with members of order not less than 2, and the expansions of τ_i, $i = 1, 2, \ldots, p$, start with members of order not less than 3. First, we investigate the problem for (2.1.13) all of whose elements are analytic functions, if ρ is sufficiently small. Theorem 6.4.2 in [1] implies that w_i, $i = 1, 2, \ldots, p$, are analytic functions in ρ and the solution $\rho(\phi, r_0)$ of equation (2.1.13) has the following expansion:

$$\rho(\phi, r_0) = \sum_{i=0}^{\infty} \rho_i(\phi) r_0^i, \tag{2.1.19}$$

where $\phi \notin (\gamma_i, \zeta_i], i = 1, 2, \ldots, p, \rho_0(\phi) = 0, q = \rho_1(\phi) = 1$. One can define the Poincaré return map

$$\rho(2\pi, r_0) = \sum_{i=1}^{\infty} a_i r_0^i, \tag{2.1.20}$$

where $a_i = \rho_i(2\pi), i \geq 1, a_1 = q = 1$. The expansions exist, see Sect. 6.4 of the book [1], such that

$$\begin{aligned} P(\rho, \phi) &= \sum_{i=2}^{\infty} P_i(\phi) \rho^i, \\ w_j(\rho) &= \sum_{i=2}^{\infty} w_{ji} \rho^i, \end{aligned} \tag{2.1.21}$$

where $P_i(\phi), w_{ji}(\phi), j \geq 2$, are 2π-periodic functions which can be defined by using (2.1.12). The coefficient $\rho_j(\phi), j \geq 2$, is the solution of the system

$$\begin{aligned} &\frac{d\rho}{d\phi} = P_j(\phi), \\ &\Delta\rho \mid_{\phi \neq \gamma_i} = w_{ji}, \\ &\Delta\phi \mid_{\phi \neq \gamma_i} = \theta_i, \end{aligned} \tag{2.1.22}$$

with the initial condition $\rho_j(0) = 0$. Hence, coefficients of (2.1.20) are equal to

$$a_j = \int_0^{\gamma_1} P_j(\phi) d\phi + \sum_{i=1}^{p-1} \int_{\zeta_i}^{\gamma_{i+1}} P_j(\phi) d\phi + \int_{\zeta_p}^{2\pi} P_j(\phi) d\phi + \sum_{i=1}^{p} w_{ji}. \tag{2.1.23}$$

From (2.1.20) and (2.1.23), it follows that the following lemma is true.

Lemma 2.1.4 *Let $q = 1$ and the first nonzero element of the sequence $a_j, j \geq 2$, be negative (positive), then the origin is a stable (unstable) focus of (2.1.13). If $a_j = 0, j \geq 2$, then the origin is a center of (2.1.13).*

B-equivalence of systems (2.1.12) and (2.1.13) implies immediately that the following theorem is valid.

Theorem 2.1.3 *Let $q = 1$ and the first nonzero element of the sequence $a_j, j \geq 2$, be negative (positive), then the origin is a stable (unstable) focus of the equation (2.1.7). If $a_j = 0$ for all $j \geq 2$, then the origin is a center of (2.1.7).*

2.1.5 Bifurcation of a Discontinuous Limit Cycle

We consider the following system:

$$\begin{aligned}\frac{dx}{dt} &= Ax + f(x) + \mu F(x, \mu),\\ \Delta x|_{x\in\Gamma(\mu)} &= B(x, \mu)x.\end{aligned} \tag{2.1.24}$$

To establish the Hopf bifurcation theorem, we need the following assumptions:

(A1) The set $\Gamma(\mu) = \cup_{i=1}^{p} l_i(\mu)$ is a union of curves in G, which start at the origin and do not include it, $l_i : (a^i, x) + \tau_i(x) + \mu\nu(x, \mu) = 0,\ 1 \le i \le p$;
(A2) There exist a matrix $Q(\mu) \in \mathscr{R}$, $Q(0) = Q$, analytic in $(-\mu_0, \mu_0)$, and real numbers $\gamma,\ \chi$ such that $Q^{-1}(\mu)B(x, \mu)Q(\mu) =$

$$(k + \mu\gamma + \kappa(x))\begin{pmatrix}\cos(\theta + \mu\chi + \upsilon(x)) & -\sin(\theta + \mu\chi + \upsilon(x))\\ \sin(\theta + \mu\chi + \upsilon(x)) & \cos(\theta + \mu\chi + \upsilon(x))\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};$$

(A3) Associated with (2.1.24) systems

$$\begin{aligned}\frac{dx}{dt} &= Ax,\\ \Delta x|_{x\in\Gamma(0)} &= B_0 x,\end{aligned} \tag{2.1.25}$$

and

$$\begin{aligned}\frac{dx}{dt} &= Ax + f(x),\\ \Delta x|_{x\in\Gamma(0)} &= B(x, 0)x,\end{aligned} \tag{2.1.26}$$

are D_0-system and D-system, respectively;
(A4) Functions $\kappa, \upsilon : G \to \mathbb{R}^2$ and $F, \nu : G \times (-\mu_0, \mu_0) \to \mathbb{R}^2$ are analytic in $G \times (-\mu_0, \mu_0)$;
(A5) $F(0, \mu) = 0,\ \nu(0, \mu) = 0$, for all $\mu \in (-\mu_0, \mu_0)$.

Additionally, we shall need the following system:

$$\begin{aligned}\frac{dx}{dt} &= A(\mu)x,\\ \Delta x|_{x\in\Gamma_0(\mu)} &= B(0, \mu)x,\end{aligned} \tag{2.1.27}$$

where $A(\mu) = A + \mu\frac{\partial F(0,\mu)}{\partial x}$, and $\Gamma_0(\mu) = \cup_{i=1}^{p} m_i$ with

$$m_i : \quad \left(a^i + \mu\frac{\partial\nu(0, \mu)}{\partial x}, x\right) = 0, \quad i = 1, 2, \ldots, p.$$

The polar transformation takes (2.1.24) to the following form:

$$\begin{aligned}
&\frac{dr}{d\phi} = \lambda r + P(r, \phi, \mu),\\
&\Delta r \mid_{(r,\phi)\in l_i(\mu)} = k_i r + \omega(r, \phi, \mu),\\
&\Delta \phi \mid_{(r,\phi)\in l_i(\mu)} = \theta_i + r\gamma(r, \phi, \mu).
\end{aligned} \tag{2.1.28}$$

The functions $w_i(\rho, \mu)$ can be defined in the same manner as in (2.1.16) such that the system

$$\begin{aligned}
&\frac{d\rho}{d\phi} = \lambda \rho + P(\rho, \phi, \mu),\ \phi \neq \gamma_i(\mu),\\
&\Delta \rho \mid_{\phi=\gamma_i(\mu)} = k_i \rho + w_i(\rho, \mu),\\
&\Delta \phi \mid_{\phi=\gamma_i(\mu)} = \theta_i(\mu),
\end{aligned} \tag{2.1.29}$$

where $\gamma_i(\mu), i = 1, 2, \ldots, p$, are angles of m_i, is B-equivalent to (2.1.28).

Similar to (2.1.6), one can define the function

$$q(\mu) = \exp(\lambda(\mu)(2\pi - \sum_{j=1}^{p}(\zeta_j(\mu) - \gamma_j(\mu))\Pi_{j=p}^{1}(1 + k_j(\mu)) \tag{2.1.30}$$

for system (2.1.27). Theorem 6.4.2 of Chap. 6 in [1] implies that $q(\mu)$ is an analytic function.

Theorem 2.1.4 *Assume that $q(0) = 1$, $q^{'}(0) \neq 0$ and the origin is a focus of (2.1.26). Then, for sufficiently small r_0, there exists a continuous function $\mu = \delta(r_0), \delta(0) = 0$, such that the solution $r(\phi, r_0, \delta(r_0))$ of (2.1.28) is periodic function with period 2π. The period of the corresponding solution of (2.1.24) is $T = (2\pi - \sum_{i=1}^{p}\theta_i)\beta^{-1} + o(|\mu|)$. Moreover, if the origin is a stable focus of (2.1.26) then the closed trajectory is a limit cycle.*

Proof If $\rho(\phi, r_0, \mu)$ is a solution of (2.1.29), then by Theorem 6.4.2 in [1] we have that

$$\rho(2\pi, r_0, \mu) = \sum_{i=1}^{\infty} a_i(\mu) r_0^i,$$

where $a_i(\mu) = \sum_{j=0}^{\infty} a_{ij}\mu^j$, $a_{10} = q(0) = 1$, $a_{11} = q'(0) \neq 0$. Define the displacement function

$$\mathscr{V}(r_0, \mu) = \rho(2\pi, r_0, \mu) - r_0 = q'(0)\mu r_0 + \sum_{i=2}^{\infty} a_{i0} r_0^i + r_0\mu^2 G_1(r_0, \mu) + r_0^2\mu G_2(r_0, \mu),$$

where G_1, G_2 are functions analytic in a neighborhood of $(0, 0)$. The bifurcation equation is $\mathscr{V}(r_0, \mu) = 0$. Canceling by r_0, one can rewrite the equation as

$$\mathscr{H}(r_0, \mu) = 0, \tag{2.1.31}$$

where

$$\mathscr{H}(r_0, \mu) = q'(0)\mu + \sum_{i=2}^{\infty} a_{i0} r_0^{i-1} + \mu^2 G_1(r_0, \mu) + r_0 \mu G_2(r_0, \mu)$$

Since

$$\mathscr{H}(0, 0) = 0, \quad \frac{\partial \mathscr{H}(0, 0)}{\partial \mu} = q'(0) \neq 0,$$

for sufficiently small r_0, there exists a function $\mu = \delta(r_0)$ such that $r(\phi, r_0, \delta(r_0))$ is a periodic solution. If conditions $a_{i0} = 0$, $i = 2, \ldots, l-1$, and $a_{l0} \neq 0$ are valid, then one can obtain from (2.1.31) that

$$\delta(r_0) = -\frac{a_{l0}}{q'(0)} r_0^{l-1} + \sum_{i=l}^{\infty} \delta_i r_0^i. \tag{2.1.32}$$

By analysis of the latter expression, one can conclude that the bifurcation of periodic solutions emerges if the focus is stable with $\mu = 0$ and unstable with $\mu \neq 0$ and conversely. If $\rho(\phi) = \rho(\phi, \bar{r_0}, \bar{\mu})$ is a periodic solution of (2.1.29), then it is known that the trajectory is a limit cycle if

$$\frac{\partial \mathscr{V}(\bar{r_0}, \bar{\mu})}{\partial r_0} < 0. \tag{2.1.33}$$

We have that

$$\frac{\partial \mathscr{V}(r_0, \mu)}{\partial r_0} = q'(0)\mu + \sum_{i=2}^{\infty} i a_{i0} r_0^{i-1} + \mu^2 G_1(r_0, \mu) + 2 r_0 \mu G_2(r_0, \mu).$$

Let a_{l0} be the first nonzero element among a_{i0} and $a_{l0} < 0$. Using (2.1.32), one can obtain that

$$\frac{\partial \mathscr{V}(\bar{r_0}, \bar{\mu})}{\partial r_0} = (l-1) a_{l0} \bar{r}_0^{l-1} + Q(\bar{r_0}),$$

where Q starts with a member whose order is not less than l. Hence, (2.1.33) is valid. Now, B-equivalence of (2.1.28) and (2.1.29) proves the theorem.

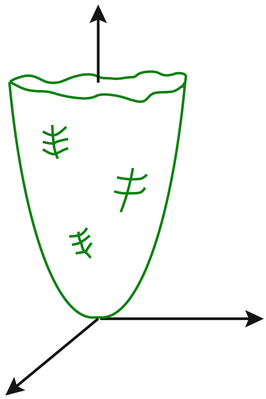


Fig. 2.3 A Hopf bifurcation diagram of an ordinary differential equation

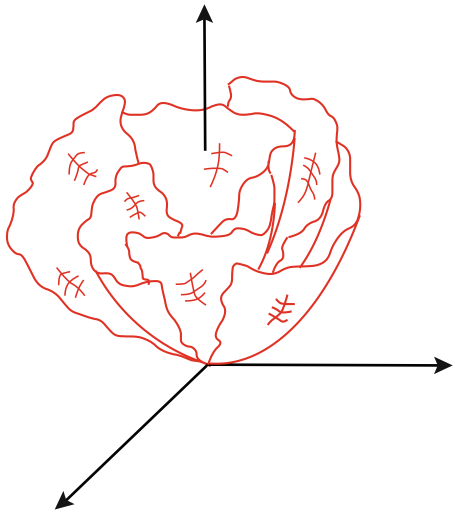


Fig. 2.4 A Hopf bifurcation diagram of a discontinuous dynamical system

Remark 2.1.1 (a). It is important to notice that the bifurcation theorem can be obtained by applying the results in [132] and theorems of Chap. 6 of [1]. We follow the approach which is focused on the expansions of solutions [173].

(b). To illustrate that discontinuous dynamical systems may provide more interesting opportunities than continuous dynamics, let us compare the bifurcation diagrams of an ordinary differential equation, Fig. 2.3, and a discontinuous dynamical system of type (2.1.24), Fig. 2.4. One can see that the first diagram resembles a bud, and the second one a rose. They demonstrate that a theory of differential equations flourishes if a discontinuity is involved in analysis.

2.1.6 Examples

Example 2.1.1 Consider the following system

$$
\begin{aligned}
x_1' &= (2+\mu)x_1 - x_2 + x_1^2 x_2,\\
x_2' &= x_1 + (2+\mu)x_2 + 3x_1^3 x_2,\\
\Delta x_1|_{x\in l} &= \left((\kappa+\mu^2)\cos\left(\frac{\pi}{6}\right) - 1\right)x_1 - (\kappa+\mu^2)\sin\left(\frac{\pi}{6}\right)x_2,\\
\Delta x_2|_{x\in l} &= (\kappa+\mu^2)\sin\left(\frac{\pi}{6}\right)x_1 + \left((\kappa+\mu^2)\cos\left(\frac{\pi}{6}\right) - 1\right)x_2,
\end{aligned} \tag{2.1.34}
$$

where $\kappa = e^{-\frac{11\pi}{6}}$, and the curve l is given by the equation $x_2 = x_1^3$, where $x_1 > 0$. One can define, using (2.1.30), that $q(\mu) = (\kappa+\mu^2)\exp((2+\mu)\frac{11\pi}{6})$, $q(0) = \kappa\exp(\frac{11\pi}{3}) = 1$, $q'(0) = -\frac{11\pi}{6} \neq 0$. Thus, by Theorem 2.1.4, system (2.1.34) has a periodic solution with period $\approx\frac{11\pi}{12}$ if $|\mu|$ is sufficiently small.

Example 2.1.2 Let the following system be given

$$
\begin{aligned}
x_1' &= (\mu-1)x_1 - x_2,\ x_2' = x_1 + (\mu-1)x_2,\\
\Delta x_1|_{x\in l} &= \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_1 - (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_2,\\
\Delta x_2|_{x\in l} &= (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_1 + \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_2,
\end{aligned} \tag{2.1.35}
$$

where l is a curve given by the equation $x_2 = x_1 + \mu x_1^2, x_1 > 0, \kappa = \exp(\frac{7\pi}{4})$. Using (2.1.30) one can find that $q(\mu) = \kappa\exp((\mu-1)\frac{7\pi}{4})$, $q(0) = \kappa\exp(-\frac{7\pi}{4}) = 1$, $q'(0) = \frac{7\pi}{4} \neq 0$. Moreover, one can see that for the associated D-system

$$
\begin{aligned}
x_1' &= -x_1 - x_2,\ x_2' = x_1 - x_2,\\
\Delta x_1|_{x\in s} &= \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_1 - (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_2,\\
\Delta x_2|_{x\in s} &= (\kappa - x_1^2 - x_2^2)\sin\left(\frac{\pi}{4}\right)x_1 + \left((\kappa - x_1^2 - x_2^2)\cos\left(\frac{\pi}{4}\right) - 1\right)x_2,
\end{aligned} \tag{2.1.36}
$$

where s is given by the equation $x_2 = x_1, x_1 > 0$, the origin is a stable focus. Indeed, using polar coordinates, denote by $r(\phi, r_0)$ the solution of (2.1.36) starting at the angle $\phi = \frac{\pi}{4}$. We can define that $r(\frac{\pi}{4} + 2\pi n, r_0) = (\kappa - r^2(\frac{\pi}{4} + 2\pi(n-1), r_0))\exp(-\frac{7\pi}{4})$. From the last expression, it is easily seen that the sequence $r_n = r(\frac{\pi}{4} + 2\pi n, r_0)$ is monotonically decreasing and there exists a limit of r_n. Assume that $r_n \to \sigma \neq 0$. Then, it implies that there exists a periodic solution of (2.1.36) and $\sigma = (\kappa - \sigma^2)\exp(-\frac{7\pi}{4})\sigma$ which is a contradiction. Thus, $\sigma = 0$. Consequently, the origin is a stable focus of (2.1.36), and by Theorem 2.1.4 the system (2.1.35) has a limit cycle with period $\approx\frac{7\pi}{4}$ if $\mu > 0$ is sufficiently small.

2.2 3D Discontinuous Cycles

We consider three-dimensional discontinuous dynamical systems with nonfixed moments of impacts. Existence of the center manifold is proved for the system. The result is applied for the extension of the planar Hopf bifurcation theorem in Sect. 2.1. Illustrative examples are constructed for the theory.

2.2.1 Introduction

Dynamical systems are used to describe the real-world motions using differential (continuous time) or difference (discrete time) equations. In the last several decades, the need for discontinuous dynamical systems has been increased because they, often, describe the model better when the discontinuous and continuous motions are mingled. This need has made scientists to improve and develop the theory of these systems. Many new results have raised. One must mention that namely systems with not prescribed time of discontinuities were, apparently, introduced for investigation of the real world firstly [73, 185], and this fact emphasizes very much the practical sense of the theory. The problem is one of the most difficult and interesting subjects of investigations [107, 117, 157, 158, 165, 177, 208]. It was emphasized in early stage of theory's development [176].

In the previous section, the Hopf bifurcation for the planar discontinuous dynamical system has been studied. Here, we extend this result to three-dimensional space based on the center manifold. The advantage is that we use the method of B-equivalence [1, 5] as well as the results of timescales which are developed in [38].

This section is organized as follows. In the next section, we start to analyze the nonperturbed system. Section 2.2.3 describes the perturbed system. The center manifold is given in Sect. 2.2.4. In Sect. 2.2.5, the bifurcation of periodic solutions is studied. Section 2.2.6 is devoted to examples in order to illustrate the theory.

2.2.2 The Nonperturbed System

Let $\mathbb{N}, \mathbb{R}$ be the sets of all natural and real numbers, respectively, and $\mathbb{R}^2$ be a real euclidean space. Denote by $\langle x, y\rangle$ the dot product of vectors $x, y \in \mathbb{R}^2$. Let $\|x\| = \langle x, x\rangle^{1/2}$ be the norm of a vector $x \in \mathbb{R}^2$, $\mathbb{R}^{2\times 2}$ be the set of real-valued constant 2×2 matrices, and $I \in \mathbb{R}^{2\times 2}$ be the identity matrix. We shall consider in $\mathbb{R}^3$ the following dynamical system:

$$\begin{aligned}
&\frac{dx}{dt} = Ax,\\
&\frac{dz}{dt} = \hat{b}z, \quad (x,z) \notin \Gamma_0,\\
&\Delta x \mid_{(x,z)\in\Gamma_0} = B_0 x,\\
&\Delta z \mid_{(x,z)\in\Gamma_0} = c_0 z,
\end{aligned} \tag{2.2.37}$$

where $A, B_0 \in \mathbb{R}^{2\times 2}$, $\hat{b}$, $c_0 \in \mathbb{R}$, Γ_0 is a subset of $\mathbb{R}^3$ and will be described below. The phase point of (2.2.37) moves between two consecutive intersections with the set Γ_0 along one of the trajectories of the system $x' = Ax$, $z' = \hat{b}z$. When the solution meets the set Γ_0 at the moment τ, the point $x(t)$ has a jump $\Delta x \mid_\tau := x(\tau+) - x(\tau)$ and the point $z(t)$ has a jump $\Delta z \mid_\tau := z(\tau+) - z(\tau)$. Thus, we suppose that the solutions are left continuous functions.

From now on, G denotes a neighborhood of the origin.

The following assumptions will be needed:

(C1) $\Gamma_0 = \bigcup_{i=1}^{p} \mathscr{P}_i$, $p \in \mathbb{N}$, where $\mathscr{P}_i = \ell_i \times \mathbb{R}$, ℓ_i are half-lines starting at the origin defined by $\langle a^i, x\rangle = 0$ for $i = 1, \ldots, p$ and $a^i = (a_1^i, a_2^i) \in \mathbb{R}^2$ are constant vectors;

(C2) $A = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, where $\beta \neq 0$;

(C3) There exists a regular matrix $Q \in \mathbb{R}^{2\times 2}$ and nonnegative real numbers k and θ such that

$$B_0 = kQ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} Q^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For the sake of brevity, in what follows, every angle for a point or a line is considered with respect to the half-line of the first coordinate axis in x-plane. Denote $\ell_i' = (I + B_0)\ell_i$, $i = 1, \ldots, p$. Let γ_i and ζ_i be the angles of ℓ_i and ℓ_i' for $i = 1, \ldots, p$, respectively, and $B_0 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$;

(C4) $0 < \gamma_1 < \zeta_1 < \gamma_2 < \cdots < \gamma_p < \zeta_p < 2\pi$, and $(b_{11} + 1)\cos\gamma_i + b_{12}\sin\gamma_i \neq 0$ for $i = 1, \ldots, p$.

In Fig. 2.5, the discontinuity set and a trajectory of the system (2.2.37) are shown. The planes $\mathscr{P}_i$ form the set Γ_0, and each $\mathscr{P}_i'$ is the image of $\mathscr{P}_i$ under the transformation $(I + B)x$.

The system (2.2.37) is said to be a D_0-*system* if conditions (C1)–(C4) hold. It is easy to see that the origin is a unique singular point of D_0-*system* and (2.2.37) is not linear.

Let us subject (2.2.37) to the transformation $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $z = z$ and exclude the time variable t. The solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ which starts at the point $(0, r_0, z_0)$ satisfies the following system in cylindrical coordinates:

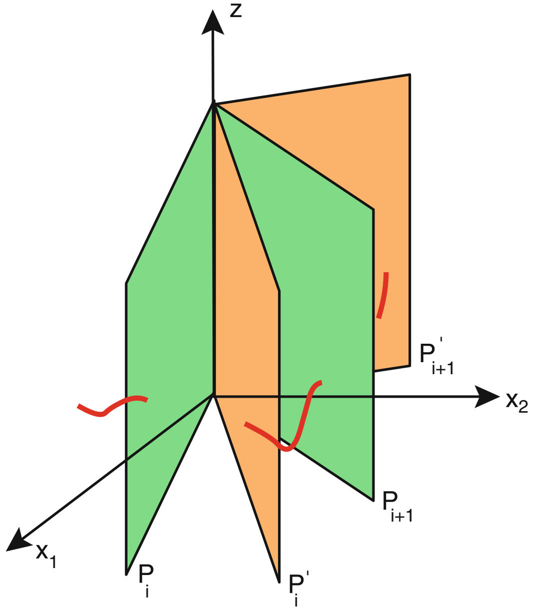


Fig. 2.5 The discontinuity set and a trajectory of (2.2.37)

$$\begin{aligned}
&\frac{dr}{d\phi} = \lambda r,\\
&\frac{dz}{d\phi} = bz, \quad \phi \neq \gamma_i \ (\mathrm{mod}\ 2\pi),\\
&\Delta r\ |_{\phi=\gamma_i\ (\mathrm{mod}\ 2\pi)} = k_i r,\\
&\Delta z\ |_{\phi=\gamma_i\ (\mathrm{mod}\ 2\pi)} = c_0 z,
\end{aligned} \tag{2.2.38}$$

where $\lambda = \alpha/\beta$, $b = \hat{b}/\beta$, and the variable ϕ is ranged over the timescale

$$\mathbb{R}_\phi = \mathbb{R} \setminus \bigcup_{i=-\infty}^{\infty} \bigcup_{j=1}^{p} (2\pi i + \gamma_j, 2\pi i + \zeta_j]$$

and

$$k_i = \left[((b_{11}+1)\cos\gamma_i + b_{12}\sin\gamma_i)^2 + (b_{21}\cos\gamma_i + (b_{22}+1)\sin\gamma_i)^2\right]^{1/2} - 1.$$

Equation (2.2.38) is 2π-periodic, so, in what follows, we shall consider just the section $[0, 2\pi]$. That is, the system

$$\begin{aligned}
&\frac{dr}{d\phi} = \lambda r,\\
&\frac{dz}{d\phi} = bz, \quad \phi \neq \gamma_i,\\
&\Delta r\mid_{\phi=\gamma_i} = k_i r,\\
&\Delta z\mid_{\phi=\gamma_i} = c_0 z,
\end{aligned} \tag{2.2.39}$$

is provided for discussion, where $\phi \in [0, 2\pi]_\phi = [0, 2\pi] \setminus \cup_{i=1}^{p}(\gamma_i, \zeta_i]$. System (2.2.39) is a sample of timescale differential equation. Let us use the ψ-*substitution*, $\varphi = \psi(\phi) = \phi - \sum_{0<\gamma_j<\phi} \theta_j, \theta_j = \zeta_j - \gamma_j$, which was introduced and developed in [6, 38]. The range of this new variable is $[0, 2\pi - \sum_{i=1}^{p} \theta_i]$.

It is easy to check that upon ψ-substitution (2.2.39) reduces to the following impulsive equations:

$$\begin{aligned}
&\frac{dr}{d\varphi} = \lambda r,\\
&\frac{dz}{d\varphi} = bz, \quad \varphi \neq \varphi_i,\\
&\Delta r\mid_{\varphi=\varphi_i} = k_i r,\\
&\Delta z\mid_{\varphi=\varphi_i} = c_0 z,
\end{aligned} \tag{2.2.40}$$

where $\varphi_i = \psi(\gamma_i)$. Solving (2.2.40) as an impulsive system [156, 215] and using ψ-substitution, one can obtain that a solution of (2.2.39) is of the form

$$r(\phi) = \exp\left(\lambda\left(\phi - \sum_{0<\gamma_i<\phi} \theta_i\right)\right)\left[\prod_{0<\gamma_i<\phi}(1+k_i)\right] r_0, \tag{2.2.41}$$

$$z(\phi) = \exp\left(b\left(\phi - \sum_{0<\gamma_i<\phi} \theta_i\right)\right)\left[\prod_{0<\gamma_i<\phi}(1+c_0)\right] z_0, \tag{2.2.42}$$

for $\phi \in [0, 2\pi]_\phi$. Denote

$$q_1 = \exp\left(\lambda\left(2\pi - \sum_{i=1}^{p} \theta_i\right)\right)\prod_{i=1}^{p}(1+k_i), \tag{2.2.43}$$

$$q_2 = \exp\left(b\left(2\pi - \sum_{i=1}^{p} \theta_i\right)\right)\prod_{i=1}^{p}(1+c_0). \tag{2.2.44}$$

Depending on q_1 and q_2, we may see that the following lemmas are valid.

Lemma 2.2.1 *Assume that* $q_1 = 1$. *Then, if*

(i) $q_2 = 1$ *then all solutions are periodic with period* $T = \left(2\pi - \sum_{i=1}^{p} \theta_i\right)\beta^{-1}$;
(ii) $q_2 = -1$ *then a solution that starts to its motion on* x_1x_2*-plane is* T*-periodic and all other solutions are* $2T$*-periodic;*

(iii) $\mid q_2 \mid > 1$ *then a solution that starts to its motion on* x_1x_2*-plane is* T*-periodic and all other solutions lie on the surface of a cylinder and they move away the origin (i.e., zero solution is unstable);*
(iv) $\mid q_2 \mid < 1$ *then a solution that starts to its motion on* x_1x_2*-plane is* T*-periodic and all other solutions lie on the surface of a cylinder and they move toward the* x_1x_2*-plane (i.e., zero solution is stable).*

Lemma 2.2.2 *Assume that* $q_1 < 1$*. Then, if*

(i) $\mid q_2 \mid < 1$ *all solutions will spiral toward the origin, i.e., origin is an asymptotically stable fixed point;*
(ii) $\mid q_2 \mid > 1$ *a solution that starts to its motion on* x*-plane spirals toward the origin and a solution that starts to its motion on* z*-axis will move away from the origin. In this case the origin is half stable (or conditionally stable);*
(iii) $q_2 = 1(q_2 = -1)$ *then a solution that starts to its motion on* z*-axis is periodic with period* $T(2T)$ *and all other solutions will approach to* z*-axis.*

Lemma 2.2.3 *Assume that* $q_1 > 1$*. Then, if*

(i) $\mid q_2 \mid < 1$ *then origin is a stable focus;*
(ii) $\mid q_2 \mid > 1$ *then origin is an unstable focus;*
(iii) $q_2 = 1(q_2 = -1)$ *then a solution that starts to its motion on* z*-axis is periodic with period* $T(2T)$ *and all other solutions will approach to* z*-axis.*

We note that when $q_2 = -1$ (this means z may be negative, too), the solutions starting their motion out of x_1x_2-plane will move above and below the x_1x_2-plane. More explicitly, if a solution starts to its motion above the x-plane, then after the time corresponding to an angle of T, it will be below the x-plane; in the next duration corresponding to an angle T, it will try to move above x-plane; and at the end of that duration, it will be above the x-plane, and so on.

From now on, we assume that $q_1 = 1$ and $\mid q_2 \mid < 1$.

2.2.3 The Perturbed System

Consider the system

$$\begin{aligned}
&\frac{dx}{dt} = Ax + f(x, z),\\
&\frac{dz}{dt} = \hat{b}z + g(x, z), \quad (x, z) \notin \Gamma,\\
&\Delta x \mid_{(x,z)\in\Gamma} = B(x)x,\\
&\Delta z \mid_{(x,z)\in\Gamma} = c(z)z,
\end{aligned} \tag{2.2.45}$$

where the followings are assumed to be true.

(C5) $\Gamma = \bigcup_{i=1}^{p} \mathscr{S}_i$, where $\mathscr{S}_i = s_i \times \mathbb{R}$ and the equation of s_i is given by s_i : $\langle a^i, x\rangle + \tau_i(x) = 0$, for $i = 1, \ldots, p$;

(C6)

$$B(x) = (k + \kappa(x))Q \begin{bmatrix} \cos(\theta + \Theta(x)) & -\sin(\theta + \Theta(x)) \\ \sin(\theta + \Theta(x)) & \cos(\theta + \Theta(x)) \end{bmatrix} Q^{-1} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $c(z) = c_0 + \tilde{c}(z)$;

(C7) Functions $f, g, \kappa, \tilde{c}$, and Θ are in C^1 and τ_i is in C^2;

(C8) $f(x,z) = \mathscr{O}(\|(x,z)\|)$, $g(x,z) = \mathscr{O}(\|(x,z)\|)$, $\kappa(x) = \mathscr{O}(\|x\|)$, $\Theta(x) = \mathscr{O}(\|x\|)$, $\tilde{c}(z) = \mathscr{O}(z)$, $\tau_i(x) = \mathscr{O}(\|x\|^2)$, $i = 1, \ldots, p$.

Moreover, it is supposed that the matrices A, Q, the vectors a^i, $i = 1, \ldots, p$, constants k, θ are the same as for (2.2.37), i.e.,

(C9) The associated with (2.2.45) is D_0 system.

Remark 2.2.1 Conditions (C5) and (C6) imply that surfaces $\mathscr{S}_i$ do not intersect each other except on z-axis and neither of them intersects itself.

The system (2.2.45) is said to be a D-*system* if the conditions $(C1)$–$(C8)$ hold.

In what follows, we assume without loss of generality that $\gamma_i \neq \frac{\pi}{2}j$, $j = 1, 2, 3$. Then, one can transform the equation in (C5) to the polar coordinates so that s_i : $a_i^1 r\cos\phi + a_i^2 r\sin\phi + \tau_i(r\cos\phi, r\sin\phi) = 0$ and, hence

$$\phi = \tan^{-1}\left(\tan\gamma_i - \frac{\tau_i(r\cos\phi, r\sin\phi)}{a_i^2 r\cos\phi}\right).$$

Using Taylor expansion gives that the previous equation can be written, for sufficiently small r, as

$$s_i : \phi = \gamma_i + r\Psi_i(r,\phi), i = 1, \ldots, p$$

where functions Ψ_i are 2π-periodic in ϕ, continuously differentiable and $\Psi_i = \mathscr{O}(r)$.

If the phase point $(x_1(t), x_2(t), z(t))$ meets the discontinuity surface $\mathscr{S}_i$ at the angle θ, then after the jump the point $(x_1(\theta+), x_2(\theta+), z(\theta+))$ will belong to the surface $\mathscr{S}_i' = \{(u, v) \in \mathbb{R}^3 : u = (I + B(x))x, v = (1 + c_0)z + c(z), (x, z) \in \mathscr{S}_i\}$. For the remaining part of this section, the following assertion is very important and the proof can be found in [6].

Lemma 2.2.4 *If the conditions (C7) and (C8) are valid then the surface $\mathscr{S}_i'$ is placed between the surfaces $\mathscr{S}_i$ and $\mathscr{S}_{i+1}$ for every i if G is sufficiently small.*

Using the cylindrical coordinates $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, $z = z$, one can find that the differential part of (2.2.45) has the following form:

$$\begin{aligned} \frac{dr}{d\phi} &= \lambda r + P(r,\phi,z), \\ \frac{dz}{d\phi} &= bz + Q(r,\phi,z), \end{aligned} \tag{2.2.46}$$

where, as is known [218], the functions $P(r, \phi, z)$ and $Q(r, \phi, z)$ are 2π-periodic in ϕ, continuously differentiable, and $P = \mathcal{O}(r, z)$, $Q = \mathcal{O}(r, z)$. Denote $x^+ = (x_1^+, x_2^+) = (I + B(x))x$, $x^+ = r^+(\cos\phi^+, \sin\phi^+)$, $\tilde{x}^+ = (\tilde{x}_1^+, \tilde{x}_2^+) = (I + B(0))x$, where $x = (x_1, x_2) \in s_i$, $i = 1, \ldots, p$. The inequality $\|x^+ - \tilde{x}^+\| \le \|B(x) - B(0)\| \cdot \|x\|$ implies that $r^+ = (1 + k_i)r + \omega(r, \phi)$. Moreover, using the relation $\frac{x_2^+}{x_1^+}$ and $\frac{\tilde{x}_2^+}{\tilde{x}_1^+}$ and condition (C5), one can conclude that $\phi^+ = \phi + \theta_i + \gamma(r, \phi)$. Functions ω and γ are 2π-periodic in ϕ and $\omega = \mathcal{O}(r)$, $\gamma = \mathcal{O}(r)$. Finally, transformed system (2.2.45) is of the following form:

$$
\begin{aligned}
&\frac{dr}{d\phi} = \lambda r + P(r, \phi, z),\\
&\frac{dz}{d\phi} = bz + Q(r, \phi, z), \quad (r, \phi, z) \notin \Gamma,\\
&\Delta r\,|_{(r,\phi)\in s_i} = k_i r + \omega(r, \phi),\\
&\Delta \phi\,|_{(r,\phi)\in s_i} = \theta_i + \gamma(r, \phi),\\
&\Delta z\,|_{(r,\phi)\in s_i} = c_0 z + \tilde{c}(z).
\end{aligned}
\tag{2.2.47}
$$

Let us introduce the following system besides (2.2.47):

$$
\begin{aligned}
&\frac{d\rho}{d\phi} = \lambda \rho + P(\rho, \phi, z),\\
&\frac{dz}{d\phi} = bz + Q(\rho, \phi, z), \quad \phi \ne \gamma_i,\\
&\Delta \rho\,|_{\phi=\gamma_i} = k_i \rho + W_i^1(\rho, z),\\
&\Delta \phi\,|_{\phi=\gamma_i} = \theta_i,\\
&\Delta z\,|_{\phi=\gamma_i} = c_0 z + W_i^2(\rho, z),
\end{aligned}
\tag{2.2.48}
$$

where all elements, except for $W_i = (W_i^1, W_i^2)$, $i = 1, \ldots, p$, are the same as in (2.2.47) and the domain of (2.2.48) is $[0, 2\pi]_\phi$. We shall define the functions W_i below.

Let $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ be a solution of (2.2.47) ϕ_i be the angle where the phase point intersects $\mathcal{S}_i$. Denote also by $\chi_i = \phi_i + \theta_i + \gamma(r(\phi_i, r_0, z_0), \phi_i)$ the angle where the phase point has to be after the jump.

Further $(\alpha, \hat{\,} \beta]$, $\{\alpha, \beta\} \subset \mathbb{R}$ denotes the oriented interval, that is

$$
(\alpha, \hat{\,}\beta] = \begin{cases} (\alpha, \beta] \text{ if } \alpha \le \beta, \\ (\beta, \alpha] \text{ otherwise.} \end{cases}
$$

Definition 2.2.1 We shall say that systems (2.2.47) and (2.2.48) are B-equivalent in G if for every solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (2.2.47) whose trajectory is in G for all $\phi \in [0, 2\pi]_\phi$ there exists a solution $(\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (2.2.48) which satisfies the relation

$$
r(\phi, r_0, z_0) = \rho(\phi, r_0, z_0), \quad \phi \in [0, 2\pi]_\phi \setminus \bigcup_{i=1}^{p} \{(\phi_i, \gamma_i] \hat{\cup} (\zeta_i, \hat{\,}\chi_i]\}, \tag{2.2.49}
$$

and, conversely, for every solution $(\rho(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (2.2.48) whose trajectory is in G, there exists a solution $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$ of (2.2.47) which satisfies (2.2.49).

Fix $i = 1, \ldots, p$. Let $(r_1(\phi), z_1(\phi))$, $(r_1(\gamma_i), z_1(\gamma_i)) = (\rho, z)$, be a solution of

$$\begin{aligned}\frac{dr}{d\phi} &= \lambda r + P(r, \phi, z),\\ \frac{dz}{d\phi} &= bz + Q(r, \phi, z),\end{aligned} \tag{2.2.50}$$

and let $\phi = \eta_i$ be the meeting angle of the solution with $\mathscr{P}_i$. Then

$$\begin{aligned}r_1(\eta_i) &= e^{\lambda(\eta_i-\gamma_i)}\rho + \int_{\gamma_i}^{\eta_i} e^{\lambda(\eta_i-s)}P(r_1(s), s, z_1(s))ds,\\ z_1(\eta_i) &= e^{b(\eta_i-\gamma_i)}z + \int_{\gamma_i}^{\eta_i} e^{b(\eta_i-s)}Q(r_1(s), s, z_1(s))ds.\end{aligned}$$

Let $\eta_i' = \eta_i + \theta_i + \gamma(r_1(\eta_i), \eta_i)$ and $(\rho', z') = ((1 + k_i)r_1(\eta_i) + \omega(r_1(\eta_i), \eta_i), (1 + c_0)z_1(\eta_i) + c(z_1(\eta_i))))$. Let $(r_2(\phi), z_2(\phi))$, $(r_2(\eta_i'), z_2(\eta_i')) = (\rho', z')$, be a solution of (2.2.50). Then,

$$\begin{aligned}r_2(\zeta_i) &= e^{\lambda(\zeta_i-\eta_i')}\rho' + \int_{\eta_i'}^{\zeta_i} e^{\lambda(\zeta_i-s)}P(r_2(s), s, z_2(s))ds,\\ z_2(\zeta_i) &= e^{b(\zeta_i-\eta_i')}z' + \int_{\eta_i'}^{\zeta_i} e^{b(\zeta_i-s)}Q(r_2(s), s, z_2(s))ds.\end{aligned}$$

We define that

$$\begin{aligned}W_i^1(\rho, z) &= r_2(\zeta_i) - (1 + k_i)\rho\\ &= e^{\lambda(\zeta_i-\eta_i')}\bigg[(1 + k_i)\bigg(e^{\lambda(\eta_i-\gamma_i)}\rho + \int_{\gamma_i}^{\eta_i} e^{\lambda(\eta_i-s)}P(r_1(s), s, z_1(s))ds\bigg)\\ &\quad +\omega(r_1(\eta_i), \eta_i)\bigg] + \int_{\eta_i'}^{\zeta_i} e^{\lambda(\zeta_i-s)}P(r_1(s), s, z_1(s))ds - (1 + k_i)\rho,\end{aligned}$$

or, if simplified

$$\begin{aligned}&W_i^1(\rho, z) = (1 + k_i)(e^{-\lambda\gamma(r_1(\eta_i),\eta_i)} - 1)\rho\\ &+ (1 + k_i)\int_{\gamma_i}^{\eta_i} e^{\lambda(\zeta_i-\theta_i-s-\gamma(r_1(\eta_i),\eta_i))}P(r_1(s), s, z_1(s))ds\\ &+ \int_{\eta_i'}^{\zeta_i} e^{\lambda(\zeta_i-s)}P(r_2(s), s, z_2(s))ds + e^{\lambda(\zeta_i-\eta_i')}\omega(r_1(\eta_i), \eta_i).\end{aligned} \tag{2.2.51}$$

We, similarly, define

$$
\begin{aligned}
W_i^2(\rho, z) &= z_2(\zeta_i) - (1 + c_0)z \\
&= e^{b(\zeta_i - \eta_i')}\bigg[(1 + c_0)\bigg(e^{b(\eta_i - \gamma_i)}z + \int_{\gamma_i}^{\eta_i} e^{b(\eta_i - s)}Q(r_1(s), s, z_1(s))ds\bigg) \\
&\quad + \tilde{c}(z_1(\eta_i))\bigg] + \int_{\eta_i'}^{\zeta_i} e^{b(\zeta_i - s)}Q(r_1(s), s, z_1(s))ds - (1 + c_0)z,
\end{aligned}
$$

or,

$$
\begin{aligned}
W_i^2(\rho, z) &= (1 + k_i)(e^{-b\gamma(r_1(\eta_i), \eta_i)} - 1)z \\
&\quad + (1 + c_0)\int_{\gamma_i}^{\eta_i} e^{(\zeta_i - \theta_i - s - \gamma(r_1(\eta_i), \eta_i))}Q(r_1(s), s, z_1(s))ds \\
&\quad + \int_{\eta_i'}^{\zeta_i} e^{b(\zeta_i - s)}Q(r_2(s), s, z_2(s))ds + e^{b(\zeta_i - \eta_i')}\tilde{c}(z_1(\eta_i)). \qquad (2.2.52)
\end{aligned}
$$

We note that there exists a Lipschitz constant ℓ and a bounded function $m(\ell)$ such that

$$
\|W_i^j(\rho_1, z_1) - W_i^j(\rho_2, z_2)\| \leq m(\ell)\ell(\|\rho_1 - \rho_2\| + \|z_1 - z_2\|), \qquad (2.2.53)
$$

for all $\rho_1, \rho_2, z_1, z_2 \in \mathbb{R}$, $j = 1, 2$. For detailed proof and explanation about (2.2.53), we refer to [1, 6, 38].

2.2.4 Center Manifold

Now, using ψ-substitution (2.2.48) reduces to the following system:

$$
\begin{aligned}
\frac{d\rho}{d\varphi} &= \lambda\rho + F(\rho, \varphi, z), \\
\frac{dz}{d\varphi} &= bz + G(\rho, \varphi, z), \quad \varphi \neq \varphi_i, \\
\Delta\rho\,|_{\varphi=\varphi_i} &= k_i\rho + W_i^1(\rho, z), \\
\Delta z\,|_{\varphi=\varphi_i} &= c_0 z + W_i^2(\rho, z),
\end{aligned} \qquad (2.2.54)
$$

where $\varphi = \psi(\phi)$, $\varphi_i = \psi(\gamma_i)$, $F(\rho, \varphi, z) = P(\rho, \psi^{-1}(\varphi), z)$, and $G(\rho, \varphi, z) = Q(\rho, \psi^{-1}(\varphi), z)$. Functions F and G are T-periodic in φ, with $T = \psi(2\pi)$, and satisfy

$$
\|F(\rho, \varphi, z) - F(\rho', \varphi, z')\| \leq k(\epsilon)(\|\rho - \rho'\| + \|z - z'\|), \qquad (2.2.55)
$$

$$
\|G(\rho, \varphi, z) - G(\rho', \varphi, z')\| \leq k(\epsilon)(\|\rho - \rho'\| + \|z - z'\|). \qquad (2.2.56)
$$

Following the methods given in [8], one can see that system (2.2.54) has two integral manifolds whose equations are given by:

$$\Phi_+(\varphi,\rho) = \int_{-\infty}^{\varphi} \pi_+(\varphi,s)G(\rho(s,\varphi,\rho),s,z(s,\varphi,\rho))ds \\ + \sum_{\varphi_i<\varphi} \pi_+(\varphi,\varphi_i^+)W_i^2(\rho(\varphi_i^+,\varphi,\rho),z(\varphi_i^+,\varphi,\rho)), \tag{2.2.57}$$

and

$$\Phi_-(\varphi,z) = -\int_{\varphi}^{\infty} \pi_-(\varphi,s)F(\rho(s,\varphi,z),s,z(s,\varphi,z))ds \\ + \sum_{\varphi_i<\varphi} \pi_-(\varphi,\varphi_i^+)W_i^1(\rho(\varphi_i^+,\varphi,z),z(\varphi_i^+,\varphi,z)), \tag{2.2.58}$$

where

$$\pi_+(\varphi,s) = e^{b(\varphi-s)} \prod_{s\le\varphi_j<\varphi}(1+c_0)$$

and

$$\pi_-(\varphi,s) = e^{\lambda(\varphi-s)} \prod_{s\le\varphi_j<\varphi}(1+k_j).$$

In (2.2.57), the pair $(\rho(s,\varphi,\rho), z(s,\varphi,\rho))$ denotes a solution of (2.2.54) satisfying $\rho(\varphi,\varphi,\rho)=\rho$. Similarly, $(\rho(s,\varphi,z), z(s,\varphi,z))$, in (2.2.58), is the solution of (2.2.54) with $z(\varphi,\varphi,z)=z$.

In [8], it was shown that there exist constants K_+, M_+, σ_+ such that Φ_+ satisfies:

$$\Phi_+(\varphi,0)=0, \tag{2.2.59}$$

$$\|\Phi_+(\varphi,\rho_1)-\Phi_+(\varphi,\rho_2)\| \le K_+\ell\|\rho_1-\rho_2\|, \tag{2.2.60}$$

for all ρ_1, ρ_2 such that a solution $w(\varphi)=(\rho(\varphi), z(\varphi))$ of (2.2.54) with $w(\varphi_0) = (\rho_0, \Phi_+(\varphi_0,\rho_0))$, $\rho_0\ge 0$, is defined on $\mathbb{R}$ and satisfies

$$\|w(\varphi)\| \le M_+\rho_0 e^{-\sigma_+(\varphi-\varphi_0)}, \quad \varphi\ge\varphi_0. \tag{2.2.61}$$

Similarly, it was shown that there exist constants K_-, M_-, σ_- such that Φ_- satisfies:

$$\Phi_-(\varphi,0)=0, \tag{2.2.62}$$

$$\|\Phi_-(\varphi,z_1)-\Phi_-(\varphi,z_2)\| \le K_-\ell\|z_1-z_2\|, \tag{2.2.63}$$

for all z_1, z_2 such that a solution $w(\varphi)=(\rho(\varphi), z(\varphi))$ of (2.2.54) with $w(\varphi_0) = (\Phi_-(\varphi_0,z_0), z_0)$, $z_0\in\mathbb{R}$, is defined on $\mathbb{R}$ and satisfies

$$\|w(\varphi)\| \leq M_-\|z_0\|e^{-\sigma_-(\varphi-\varphi_0)}, \quad \varphi \leq \varphi_0. \tag{2.2.64}$$

Set $S_+ = \{(\rho, \varphi, z) : z = \Phi_+(\varphi, \rho)\}$ and $S_- = \{(\rho, \varphi, z) : \rho = \Phi_-(\varphi, z)\}$. Here, S_+ is called the *center manifold* and S_- is called the *stable manifold*. A sketch of an arbitrary center manifold is shown in Fig. 2.6.

The analogues of the following two Lemma's together with their proofs can be found in [8].

Lemma 2.2.5 *If the Lipschitz constant ℓ is sufficiently small, then for every solution $w(\varphi) = (\rho(\varphi), z(\varphi))$ of (2.2.54) there exists a solution $\mu(\varphi) = (u(\varphi), v(\varphi))$ on the center manifold, S_+, such that*

$$\begin{aligned} &\|\rho(\varphi) - u(\varphi)\| \leq 2M_+\|\rho(\varphi_0) - u(\varphi_0)\|e^{-\sigma_+(\varphi-\varphi_0)}, \\ &\|z(\varphi) - v(\varphi)\| \leq M_+\|z(\varphi_0) - v(\varphi_0)\|e^{-\sigma_+(\varphi-\varphi_0)}, \quad \varphi \geq \varphi_0, \end{aligned} \tag{2.2.65}$$

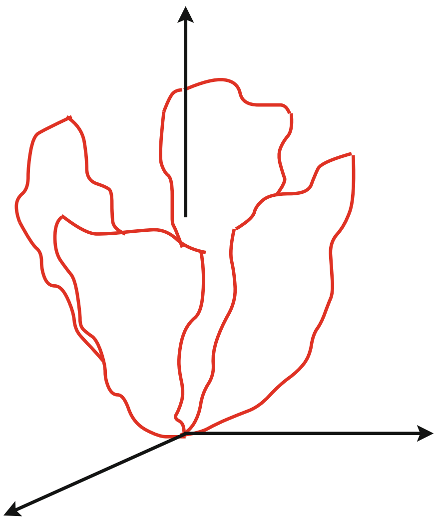
where M_+ and σ_+ are the constants used in (2.2.61).

Lemma 2.2.6 *For sufficiently small Lipschitz constant ℓ the surface S_+ is stable in large.*

On the local center manifold S_+, the first coordinate of the solutions of (2.2.54) satisfies the following system:

$$\begin{aligned} &\frac{d\rho}{d\varphi} = \lambda\rho + F(\rho, \varphi, \Phi_+(\varphi, \rho)), \quad \varphi \neq \varphi_i, \\ &\Delta\rho \mid_{\varphi=\varphi_i} = k_i\rho + W_i^1(\rho, \Phi_+(\varphi, \rho)). \end{aligned} \tag{2.2.66}$$

Fig. 2.6 The center manifold



Now, it is time to consider the reduction principle for which we need, together with the ones imposed before, the condition:

(C10) Functions $f(x,z)$ and $g(x,z)$ are continuously differentiable in x, z for all x, z, and

$$\frac{\partial f(0,0)}{\partial x_j} = 0, \quad \frac{\partial f(0,0)}{\partial z} = 0, \quad \frac{\partial g(0,0)}{\partial x_j} = 0, \quad \frac{\partial g(0,0)}{\partial z} = 0,$$

for $j = 1, 2$ where $x = (x_1, x_2)$.

Theorem 2.2.1 *Assume that conditions (C1)–(C10) are fulfilled. Then the trivial solution of (2.2.54) is stable, asymptotically stable or unstable if the trivial solution of (2.2.66) is stable, asymptotically stable or unstable, respectively.*

Using inverse of ψ-substitution and B-equivalence, one can see that the following theorem holds:

Theorem 2.2.2 *Assume that conditions (C1)–(C10) are fulfilled. Then the trivial solution of (2.2.45) is stable, asymptotically stable or unstable if the trivial solution of (2.2.66) is stable, asymptotically stable or unstable, respectively.*

2.2.5 Bifurcation of Periodic Solutions

This section is devoted to the bifurcation theorem of a periodic solution for the discontinuous dynamical system. Let us consider the system,

$$\begin{aligned}
\frac{dx}{dt} &= Ax + f(x,z) + \mu\tilde{f}(x,z,\mu),\\
\frac{dz}{dt} &= \hat{b}z + g(x,z) + \mu\tilde{g}(x,z,\mu), \quad (x,z) \notin \Gamma(\mu),\\
\Delta x\ |_{(x,z)\in\Gamma(\mu)} &= B(x,\mu)x,\\
\Delta z\ |_{(x,z)\in\Gamma(\mu)} &= c(z,\mu)z.
\end{aligned} \tag{2.2.67}$$

Assume that the following conditions are satisfied:

(A1) The set $\Gamma(\mu) = \bigcup_{i=1}^{p} \mathscr{S}_i(\mu)$, where $\mathscr{S}_i(\mu) = s_i(\mu) \times \mathbb{R}$ and the equation of $s_i(\mu)$ is given by $s_i(\mu) : \langle a^i, x\rangle + \tau_i(x) + \mu\nu(x,\mu) = 0$, for $i = 1, \ldots, p$;

(A2) There exists a matrix $Q(\mu) \in \mathbb{R}^{2\times 2}$, $Q(0) = Q$, analytic in $(-\mu_0, \mu_0)$, and real numbers γ, χ such that $Q^{-1}(\mu)B(x,\mu)Q(\mu) =$

$$(k + \mu\gamma + \kappa(x)) \begin{bmatrix} \cos(\theta + \mu\chi + \Theta(x)) & -\sin(\theta + \mu\chi + \Theta(x)) \\ \sin(\theta + \mu\chi + \Theta(x)) & \cos(\theta + \mu\chi + \Theta(x)) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $c(z,\mu) = c_0 + \tilde{c}(z) + \mu d(z,\mu)$;

(A3) Associated with (2.2.67) systems

$$\begin{aligned}
&\frac{dx}{dt} = Ax,\\
&\frac{dz}{dt} = \hat{b}z, \quad (x, z) \notin \Gamma_0,\\
&\Delta x \mid_{(x,z)\in\Gamma_0} = B_0 x,\\
&\Delta z \mid_{(x,z)\in\Gamma_0} = c_0 z.
\end{aligned} \tag{2.2.68}$$

and

$$\begin{aligned}
&\frac{dx}{dt} = Ax + f(x, z),\\
&\frac{dz}{dt} = \hat{b}z + g(x, z), \quad (x, z) \notin \Gamma(0),\\
&\Delta x \mid_{(x,z)\in\Gamma(0)} = B(x, 0)x,\\
&\Delta z \mid_{(x,z)\in\Gamma(0)} = c(z, 0)z.
\end{aligned} \tag{2.2.69}$$

are D_0-*system* and D-*system*, respectively;

(A4) Functions $\tilde{f}, \nu : G \times (-\mu_0, \mu_0) \to \mathbb{R}^2$ are analytic in x, z, and μ;

(A5) $\tilde{f}(0, 0, \mu) = 0$, $\nu(0, \mu) = 0$, uniformly for $\mu \in (-\mu_0, \mu_0)$.

Using polar coordinates, one can write system (2.2.67) in the following form:

$$\begin{aligned}
&\frac{dr}{d\phi} = \lambda(\mu)r + P(r, \phi, z, \mu),\\
&\frac{dz}{d\phi} = b(\mu)z + Q(r, \phi, z, \mu), \quad (r, \phi, z) \notin \Gamma(\mu),\\
&\Delta r \mid_{(r,\phi)\in\ell_i(\mu)} = k_i(\mu)r + \omega(r, \phi, \mu),\\
&\Delta \phi \mid_{(r,\phi)\in\ell_i(\mu)} = \theta_i(\mu) + \gamma(r, \phi, \mu),\\
&\Delta z \mid_{(r,\phi)\in\ell_i(\mu)} = c_0(\mu)z + \tilde{c}(z, \mu).
\end{aligned} \tag{2.2.70}$$

Let the system

$$\begin{aligned}
&\frac{d\rho}{d\phi} = \lambda(\mu)\rho + P(\rho, \phi, z, \mu),\\
&\frac{dz}{d\phi} = b(\mu)z + Q(\rho, \phi, z, \mu), \quad \phi \neq \gamma_i(\mu),\\
&\Delta \rho \mid_{\phi=\gamma_i(\mu)} = k_i(\mu)\rho + W_i^1(\rho, z, \mu),\\
&\Delta \phi \mid_{\phi=\gamma_i(\mu)} = \theta_i(\mu),\\
&\Delta z \mid_{\phi=\gamma_i(\mu)} = c_0(\mu)z + W_i^2(\rho, z, \mu),
\end{aligned} \tag{2.2.71}$$

where $\gamma_i(\mu)$, $i = 1, \ldots, p$, are angles of m_i, be B-equivalent to (2.2.70). The functions $W_i^1(\rho, z, \mu)$ and $W_i^2(\rho, z, \mu)$ can be defined in the same manner as in (2.2.51) and (2.2.52), respectively. Applying ψ-substitution to (2.2.71), we get

$$\begin{aligned}
&\frac{d\rho}{d\varphi} = \lambda(\mu)\rho + F(\rho, \varphi, z, \mu),\\
&\frac{dz}{d\varphi} = b(\mu)z + G(\rho, \varphi, z, \mu), \quad \varphi \neq \varphi_i(\mu),\\
&\Delta\rho\ |_{\varphi=\varphi_i(\mu)} = k_i(\mu)\rho + W_i^1(\rho, z, \mu),\\
&\Delta z\ |_{\varphi=\varphi_i(\mu)} = c_0(\mu)z + W_i^2(\rho, z, \mu).
\end{aligned} \tag{2.2.72}$$

Following the methods, as we did to obtain (2.2.57) and (2.2.58), one can see that system (2.2.72) has two integral manifolds whose equations are given by:

$$\begin{aligned}
\Phi_+(\varphi, \rho, \mu) = &\int_{-\infty}^{\varphi} \pi_+(\varphi, s, \mu) G(\rho(s, \varphi, \rho, \mu), s, z(s, \varphi, \rho, \mu), \mu)ds\\
+ &\sum_{\varphi_i(\mu)<\varphi} \pi_+(\varphi, \varphi_i^+, \mu) W_i^2(\rho(\varphi_i^+, \varphi, \rho, \mu), z(\varphi_i^+, \varphi, \rho, \mu)),
\end{aligned} \tag{2.2.73}$$

and

$$\begin{aligned}
\Phi_-(\varphi, z, \mu) = -&\int_{\varphi}^{\infty} \pi_-(\varphi, s, \mu) F(\rho(s, \varphi, z, \mu), s, z(s, \varphi, z, \mu), \mu)ds\\
+ &\sum_{\varphi_i(\mu)<\varphi} \pi_-(\varphi, \varphi_i^+, \mu) W_i^1(\rho(\varphi_i^+, \varphi, z, \mu), z(\varphi_i^+, \varphi, z, \mu)),
\end{aligned} \tag{2.2.74}$$

where

$$\pi_+(\varphi, s, \mu) = e^{b(\varphi-s)} \prod_{s\leq\varphi_j(\mu)<\varphi} (1 + c_0(\mu)),$$

and

$$\pi_-(\varphi, s, \mu) = e^{\lambda(\varphi-s)} \prod_{s\leq\varphi_j(\mu)<\varphi} (1 + k_j(\mu)).$$

In (2.2.73), the pair $(\rho(s, \varphi, \rho, \mu), z(s, \varphi, \rho, \mu))$ denotes a solution of (2.2.72) satisfying $\rho(\varphi, \varphi, \rho, \mu) = \rho$. Similarly, $(\rho(s, \varphi, z, \mu), z(s, \varphi, z, \mu))$, in (2.2.74), is a solution of (2.2.72) with $z(\varphi, \varphi, z, \mu) = z$. Set $S_+(\mu) = \{(\rho, \varphi, z) : z = \Phi_+(\varphi, \rho, \mu)\}$ and $S_-(\mu) = \{(\rho, \varphi, z) : \rho = \Phi_-(\varphi, z, \mu)\}$.

On the local center manifold, $S_+(\mu)$, the first coordinate of the solutions of (2.2.72) satisfies the following system:

$$\begin{aligned}
&\frac{d\rho}{d\varphi} = \lambda(\mu)\rho + F(\rho, \varphi, \Phi_+(\varphi, \rho, \mu)), \quad \varphi \neq \varphi_i(\mu),\\
&\Delta\rho\ |_{\varphi=\varphi_i(\mu)} = k_i(\mu)\rho + W_i^1(\rho, \Phi_+(\varphi, \rho, \mu)).
\end{aligned} \tag{2.2.75}$$

Similar to (2.2.43) and (2.2.44), one can define the functions

$$q_1(\mu) = \exp\left(\lambda(\mu)\left(2\pi - \sum_{i=1}^{p} \theta_i(\mu)\right)\right)\prod_{i=1}^{p}(1 + k_i(\mu)), \tag{2.2.76}$$

and

$$q_2(\mu) = \exp\left(b(\mu)\left(2\pi - \sum_{i=1}^{p} \theta_i(\mu)\right)\right)\prod_{i=1}^{p}(1 + c_0(\mu)). \tag{2.2.77}$$

System (2.2.75) is the system studied in [6], and there it was shown that this system, for sufficiently small μ, has a periodic solution with period T. Here, we will show that if the first coordinate of a solution of (2.2.72) is T-periodic, then so is the second coordinate.

Now, since

$$\pi_+(\varphi + T, s + T, \mu) = \pi_+(\varphi, s, \mu),$$

$$\rho(s + T, \varphi + T, \rho, \mu) = \rho(s, \varphi, \rho, \mu),$$

$$z(s + T, \varphi + T, \rho, \mu) = z(s, \varphi, \rho, \mu),$$

and G is T-periodic in φ, we have,

$$\begin{aligned}
&\Phi_+(\varphi + T, \rho, \mu)\\
&= \int_{-\infty}^{\varphi+T} \pi_+(\varphi + T, s, \mu)G(\rho(s, \varphi + T, \rho, \mu), s, z(s, \varphi + T, \rho, \mu), \mu)ds\\
&+ \sum_{\varphi_i(\mu)<\varphi+T} \pi_+(\varphi + T, \varphi_i^+, \mu)W_i^2(\rho(\varphi_i^+, \varphi + T, \rho, \mu), z(\varphi_i^+, \varphi + T, \rho, \mu))\\
&= \int_{-\infty}^{\varphi} \pi_+(\varphi, t, \mu)G(\rho(t, \varphi, \rho, \mu), t, z(t, \varphi, \rho, \mu), \mu)dt\\
&+ \sum_{\bar{\varphi}_i(\mu)<\varphi} \pi_+(\varphi, \bar{\varphi}_i{}^+, \mu)W_i^2(\rho(\bar{\varphi}_i{}^+, \varphi, \rho, \mu), z(\bar{\varphi}_i{}^+, \varphi, \rho, \mu))\\
&= \Phi_+(\varphi, \rho, \mu)
\end{aligned}$$

Then, we have the following theorem which, in case of two dimension, was shown in Sect. 2.1.

Theorem 2.2.3 *Assume that* $q_1(0) = 1$, $q_1'(0) \neq 0$, $\mid q_2(0) \mid< 1$, *and the origin is a focus for (2.2.69). Then, for sufficiently small* r_0 *and* z_0, *there exists a function* $\mu = \delta(r_0, z_0)$ *such that the solution* $(r(\phi, \delta(r_0, z_0)), z(\phi, \delta(r_0, z_0)))$ *of (2.2.70), with the initial condition* $r(0, \delta(r_0, z_0)) = r_0$, $z(0, \delta(r_0, z_0)) = z_0$, *is periodic with a period,* $T' = \left(2\pi - \sum_{i=1}^{p} \theta_i\right)\beta^{-1} + o(|\mu|)$.

2.2.6 *Examples*

Example 2.2.1 Consider the following dynamical system:

$$
\begin{aligned}
&x_1' = (0.1-\mu)x_1 - 20x_2 + 2x_1x_2,\\
&x_2' = 20x_1 + (0.1-\mu)x_2 + 3x_1^2 z,\\
&z' = (-0.3+\mu)z + \mu^2 x_1 z, \quad (x_1,x_2,z) \notin \mathscr{S},\\
&\Delta x_1\,|_{(x_1,x_2,z)\in\mathscr{S}} = \big((\kappa_1+\mu^3)\cos(\tfrac{\pi}{3})-1\big)\,x_1 - (\kappa_1+\mu^3)\sin(\tfrac{\pi}{3})x_2,\\
&\Delta x_2\,|_{(x_1,x_2,z)\in\mathscr{S}} = (\kappa_1+\mu^3)\sin(\tfrac{\pi}{3})x_1 + \big((\kappa_1+\mu^3)\cos(\tfrac{\pi}{3})-1\big)\,x_2,\\
&\Delta z\,|_{(x_1,x_2,z)\in\mathscr{S}} = (\kappa_2+\mu-1)z,
\end{aligned}
\tag{2.2.78}
$$

where $\kappa_1 = \exp(-\frac{\pi}{120})$, $\kappa_2 = \exp(-\frac{\pi}{400})$, $\mathscr{S} = s \times \mathbb{R}$, and the curve s is given by the equation $x_2 = x_1^2 + \mu x_1^3$, $x_1 > 0$. Using (2.2.76) and (2.2.77), one can define

$$
q_1(\mu) = (\kappa_1+\mu^3)\exp\left((0.1-\mu)\frac{5\pi}{60}\right),
$$

and

$$
q_2(\mu) = (\kappa_2+\mu)\exp\left((-0.3+\mu)\frac{5\pi}{60}\right).
$$

It is easily seen that $q_1(0) = \kappa_1 \exp(\frac{\pi}{120}) = 1$, $q_1'(0) = -\frac{\pi}{12} \neq 0$ and $q_2(0) = \exp(-\frac{11\pi}{200}) < 1$. Therefore, by Theorem 2.2.3, system (2.2.78) has a periodic solution with period $\approx \frac{5\pi}{60}$ if $|\,\mu\,|$ is sufficiently small.

Figure 2.7 shows the trajectory of (2.2.78) with the parameter $\mu = 0.05$ and the initial value $(x_{10}, x_{20}, z_0) = (0.02, 0, 0.05)$. Since there is an asymptotically stable center manifold, no matter which initial condition is taken, the trajectory will get closer and closer to the center manifold as time increases.

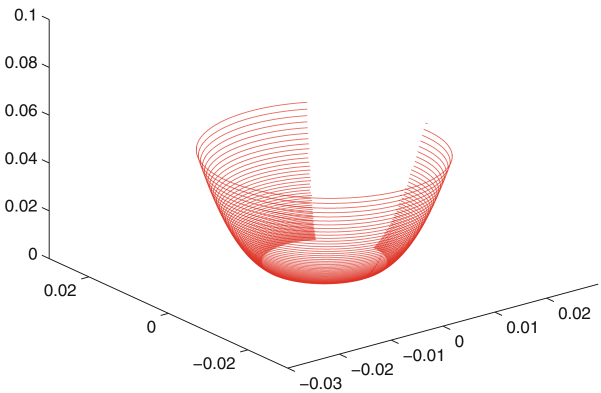


Fig. 2.7 A trajectory of (2.2.78)

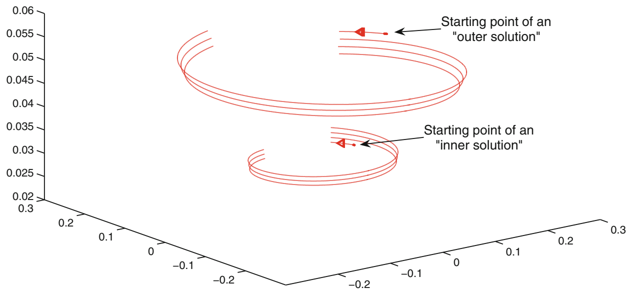


Fig. 2.8 There must exist a discontinuous limit cycle of (2.2.78)

In Fig. 2.8, the existence of a discontinuous limit cycle is illustrated. There an outer and an inner solution are shown which spiral to a trajectory lying between these two. Since the exact value of the initial point for the periodic solution is not known, we have shown two trajectories of (2.2.78).

Example 2.2.2 Consider the following dynamical system:

$$\begin{aligned}
&x_1' = (-2+\mu)x_1 - x_2 + \mu z^2,\\
&x_2' = x_1 + (-2+\mu)x_2,\\
&z' = (-1+\mu)z + \mu^2 x_1 z, \quad (x_1, x_2, z) \notin \mathscr{S},\\
&\Delta x_1 \mid_{(x_1,x_2,z)\in\mathscr{S}} = \big((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\big)\, x_1 - (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3})x_2,\\
&\Delta x_2 \mid_{(x_1,x_2,z)\in\mathscr{S}} = (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3})x_1 + \big((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\big)\, x_2,\\
&\Delta z \mid_{(x_1,x_2,z)\in\mathscr{S}} = (\kappa_2 - 1 - z^2)z,
\end{aligned} \tag{2.2.79}$$

where $\kappa_1 = \exp(\frac{10\pi}{3})$, $\kappa_2 = \exp(\frac{5\pi}{6})$, $\mathscr{S} = s \times \mathbb{R}$, *ands* is a curve given by the equation $x_2 = x_1 + \mu^2 x_1^3$, $x_1 > 0$. Using (2.2.76) and (2.2.77), one can define

$$q_1(\mu) = \kappa_1 \exp\left((-2+\mu)\frac{5\pi}{3}\right),$$

and

$$q_2(\mu) = \kappa_2 \exp\left((-1+\mu)\frac{5\pi}{3}\right).$$

Now, $q_1(0) = \kappa_1 \exp(-\frac{10\pi}{3}) = 1$, $q_1'(0) = \frac{5\pi}{3} \neq 0$, $q_2(0) = \kappa_2 \exp(\frac{5\pi}{3}) = \exp(-\frac{5\pi}{6})$. Moreover, associated D-system is:

$$
\begin{aligned}
&x_1' = -2x_1 - x_2,\\
&x_2' = x_1 - 2x_2,\\
&z' = -z, \quad (x_1, x_2, z) \notin \mathscr{P},\\
&\Delta x_1 \mid_{(x_1,x_2,z)\in\mathscr{P}} = \left((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\right) x_1 - (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3}) x_2,\\
&\Delta x_2 \mid_{(x_1,x_2,z)\in\mathscr{P}} = (\kappa_1 - x_1^2 - x_2^2)\sin(\tfrac{\pi}{3}) x_1 + \left((\kappa_1 - x_1^2 - x_2^2)\cos(\tfrac{\pi}{3}) - 1\right) x_2,\\
&\Delta z \mid_{(x_1,x_2,z)\in\mathscr{P}} = (\kappa_2 - 1 - z^2) z,
\end{aligned}
\tag{2.2.80}
$$

where $\mathscr{P} = \ell \times \mathbb{R}$, ℓ is given by the equation $x_2 = x_1, x_1 > 0$, and the origin is stable focus. Indeed, using cylindrical coordinates, denote the solution of (2.2.80) starting at the angle $\phi = \frac{\pi}{4}$ by $(r(\phi, r_0, z_0), z(\phi, r_0, z_0))$.

We obtain

$$r_n = (\kappa_1 - r_{n-1}^2) r_{n-1} \exp\left(-\frac{10\pi}{3}\right),$$

and

$$z_n = (\kappa_2 - z_{n-1}^2) z_{n-1} \exp\left(-\frac{5\pi}{3}\right),$$

where $r_n = r(\frac{\pi}{4} + 2\pi n, r_0, z_0)$ and $z_n = z(\frac{\pi}{4} + 2\pi n, r_0, z_0)$. It is easily seen that the sequences r_n and z_n are monotonically decreasing for sufficiently small (r_0, z_0), and there exists a limit of (r_n, z_n). Assume that this limit is $(\xi, \eta) \neq (0, 0)$. Then, it implies that there exists a periodic solution of (2.2.80) and $\xi = (\kappa_1 - \xi^2)\xi \exp(-\frac{10\pi}{3})$ and $\eta = (\kappa_2 - \eta^2)\eta \exp(-\frac{5\pi}{3})$ which give us a contradiction. Thus, $(\xi, \eta) = (0, 0)$. Consequently, the origin is a stable focus of (2.2.80), and by Theorem 2.2.3, the system (2.2.79) has a limit cycle with period $\approx \frac{5\pi}{3}$ if $\mid \mu \mid$ is sufficiently small.

2.3 Periodic Solutions of the Van der Pol Equation

In this section, we apply the methods of B-equivalence and ψ-substitution to prove the existence of discontinuous limit cycle for the Van der Pol equation with impacts on surfaces. The result is extended through the center manifold theory for coupled oscillators. The main novelty of the result is that the surfaces, where the jumps occur, are not flat. Examples and simulations are provided to demonstrate the theoretical results as well as application opportunities.

2.3.1 Introduction and Preliminaries

Getting bifurcation in dynamics with impacts relies mainly on collisions near the impact point(s). That is why they are called corner-collision, border-collision, crossing-sliding, grazing-sliding, switching-sliding, etc., bifurcations [64, 66, 86, 112, 116, 133, 145, 181]. That is, the bifurcations are located geometrically. In our present result, we do not have the geometrical source of bifurcation. It is rather reasoned by specifically arranged interaction of continuous and discontinuous stages of the process. To be precise, we use a generalized eigenvalue to evaluate which we apply a characteristic of the impact as well as of the continuous process between moments of discontinuity. This approach when continuous and discontinuous stages are equally participated in creating a certain phenomena is common for the theory of differential equations with impulses [1, 216]. Our results are, rather, close to those, which obtained for systems where continuous flows and surfaces of discontinuity are transversal [1, 6, 39, 109, 150].

The main instruments in this section, except for the Hopf bifurcation technique, are the methods of B-equivalence and ψ-substitution developed in papers [1, 2, 6, 38, 40] for discontinuous limit cycles, and one has to emphasize that the set of all periodic solutions of the nonperturbed system is a proper subset of all solutions near the origin. By a discontinuous cycle, we mean a trajectory of a discontinuous periodic solution.

The Van der Pol equation arises in the study of circuits containing vacuum tubes and is given by

$$y'' + \varepsilon(1 - y^2)y' + y = 0 \tag{2.3.81}$$

where ε is a real parameter. If $\varepsilon = 0$, the equation reduces to the equation of simple harmonic motion $y'' + y = 0$. The term $\varepsilon(1 - y^2)y'$ in (2.3.81) is usually regarded as the friction or resistance. If the coefficient $\varepsilon(1 - y^2)$ is positive, then we have the case of "positive resistance," and when the coefficient $\varepsilon(1 - y^2)$ is negative, then we have the case of "negative resistance." This equation, introduced by Lord Rayleigh (1896), was studied by Van der Pol (1927) [229] both theoretically and experimentally using electric circuits.

Hopf bifurcation is an attractive subject of analysis for mathematicians as well as for mechanics and engineers [6, 39, 63, 66, 84, 110, 112, 133, 148, 160, 171, 190, 228, 229]. Many papers and books have been published about mechanical and electrical systems with impacts [59, 64, 86, 116, 119, 161, 181, 232].

We consider the model with impulses on surfaces which are places in the phase space and are essentially nonlinear while it is known that the Hopf bifurcation is considered either with linear surfaces of discontinuity or with fixed moments of impulses [59, 63, 64, 66, 77, 109, 112, 116, 133, 209]. We have developed a special effective approach to analyze the problem in depth which consists of the method of reduction in equations with variable moments of impacts to systems with fixed moments of impacts [1], a class of equations on variable timescales [6, 40], and a transformation of equations on time scales to systems with impulses [38]. This is all the theoretical basis of the present results.

Specifically, we consider the following system:

$$\begin{aligned} &y'' + 2\alpha y' + (\alpha^2 + \beta^2)y = F(y, y', \mu), \quad (y, y') \notin \Gamma(\mu), \\ &\Delta y'|_{(y,y')\in\Gamma(\mu)} = cy + dy' + J(y, y', \mu), \end{aligned} \tag{2.3.82}$$

where $\alpha,\ \beta \neq 0,\ c,\ d$ are real constants with $c = \alpha d$, and F and J are analytic functions in all variables. $\Gamma(\mu)$ is the set of discontinuity whose equation is given by $m_1 y + m_2 y' + \tau(y, y', \mu) = 0,\ y > 0$, for some real numbers m_1, m_2; the function $\tau(y, y', \mu)$ stands for a small perturbation; and $\Delta y'|_{(y,y')\in\Gamma(\mu)} = y'(\theta^+) - y'(\theta)$ denotes the jump operator in which θ is the time when the solution (y, y') meets the discontinuity set $\Gamma(\mu)$, that is, θ is such that $m_1 y(\theta) + m_2 y'(\theta) + \tau(y(\theta), y'(\theta), \mu) = 0$, and $y'(\theta^+)$ is the right limit of $y'(t)$ at $t = \theta$. After the impact, the phase point $(y(\theta^+), y'(\theta^+))$ will belong to the set $\Gamma'(\mu) = \{(u, v) \in \mathbb{R}^2 : u = y, v = cy + (1 + d)y' + J(y, y', \mu), (y, y') \in \Gamma(\mu)\}$. Here, $y(\theta^+)$ is the right limit of $y(t)$ at $t = \theta$. One can easily see that nonlinearity is inserted into all parts of the model including the surface of discontinuity.

If we choose $\alpha = \varepsilon/2,\ \beta = \sqrt{1 - \alpha^2}$ and $F(y, y', \mu) = \varepsilon y^2 y'$ in the differential equation of the system (2.3.82), then the *Van der Pol equation* will be obtained. Therefore, (2.3.81) is a special case of (2.3.82), if the impulsive condition is not considered. Note that if $F(y, y', \mu) = \varepsilon_2 y^2 y'$ for some nonzero constant ε_2, we still have (2.3.81) after using the linear transformation $y = \sqrt{\varepsilon/\varepsilon_2}\, z$ of the dependent variable.

To explain our application motivations, we consider the oscillator which is subdued to the impacts modeled by the Newton's law of restitution as a concrete mechanical problem. Consider the system

$$\begin{aligned} &y'' + \varepsilon_1 y' + y = \varepsilon_2 y^2 y', \quad (y, y') \notin \Gamma, \\ &\Delta y'|_{(y,y')\in\Gamma} = dy', \end{aligned} \tag{2.3.83}$$

where $\varepsilon_1,\ \varepsilon_2$ are constants, $d = e^{2\pi\varepsilon_1(4-\varepsilon_1^2)^{-1/2}} - 1,\ \Gamma$ is the half-line $y = 0,\ y' > 0$. As it said above, the last system is a generalization of the Van der Pol equation with impacts of Newton's type. If one takes (2.3.83) with $\varepsilon_2 = 0$, then the system is

$$\begin{aligned} &y'' + \varepsilon_1 y' + y = 0, \quad (y, y') \notin \Gamma, \\ &\Delta y'|_{(y,y')\in\Gamma} = dy'. \end{aligned} \tag{2.3.84}$$

Note that the general solution of the differential equation without impulse condition in (2.3.84) is given by

$$y(t) = e^{-\varepsilon_1 t/2}\left(C_1 \cos\left((4 - \varepsilon_1^2)^{1/2} t/2\right) + C_2 \sin\left((4 - \varepsilon_1^2)^{1/2} t/2\right)\right), \tag{2.3.85}$$

where C_1 and C_2 are arbitrary real constants. Let $(0, y_0')$ be any point on the line $\Gamma' = \Gamma$. That is, assume that $y_0' > 0$. Then, $y(0) = 0,\ y'(0) = y_0'$ in (2.3.85) gives us $C_1 = 0,\ C_2 = 2y_0'(4 - \varepsilon_1^2)^{-1/2}$. Thus, we obtain

$$y(t) = 2y_0' \left(4 - \varepsilon_1^2\right)^{-1/2} e^{-\varepsilon_1 t/2} \sin\left((4 - \varepsilon_1^2)^{1/2} t/2\right).$$

Now, the first impact action takes place at time $t = T$ where $T > 0$ and $y(T) = 0$, which means $T = 4\pi(4 - \varepsilon_1^2)^{-1/2}$. At that time, we have $y'(T) = e^{-2\pi\varepsilon_1(4-\varepsilon_1)^{-1/2}} y_0'$, and after the impact, we have $y'(T^+) = (1 + d)y(T) = y_0'$. Therefore, *all* solutions starting on Γ' are $T = 4\pi(4 - \varepsilon_1^2)^{-1/2}$ periodic. One such solution with $y_0' = 0.06$ is depicted in Fig. 2.11.

The obtained result for (2.3.84) shows that the origin is the center for the system, and it is analogous of the planar degenerated linear homogeneous system with constant coefficients in the original Hopf bifurcation theorem. This gives us a hint to apply the bifurcation technique to more general systems of type (2.3.82).

Systems of type (2.3.83) has been analyzed in many papers and books [59, 63, 66, 110, 148, 160, 216, 229] and references cited there. Here, we have mentioned just some of them. We will apply the results of current chapter to prove the existence of a stable periodic motion of the model, in the perturbed system corresponding to this model. Moreover, in Example 2.3.2, we will handle a more complicated case of two coupled oscillators where one of the oscillators is subdued to the impacts modeled by the Newton's law of restitution.

We strictly believe that results of the present section can be applied to other mechanical, electrical, as well as biological problems if one adopts the models by special transformations to the considered case. Moreover, in the upcoming researches, we plan to weaken some restrictions on the model. For example, the approach can be extended to equations where surfaces of discontinuity do not intersect at the origin.

Finally, in the present study, we extend the results to the two-oscillator model through the application of center manifold.

The analysis developed in this chapter can be applied to various problems of mechanics, electronics, and biology.

2.3.2 *Theoretical Results*

2.3.2.1 Reduction to Polar Coordinates

Assume that functions F, J and τ are analytic in all variables,

$$F(0, 0, \mu) = J(0, 0, \mu) = \tau(0, 0, \mu) = 0$$

for all μ, and first derivatives of F, J and τ at $(y, y', \mu) = (0, 0, 0)$ vanish. We start the theoretical investigation by writing (2.3.82) in the following form:

$$\begin{aligned} &y'' + 2\alpha(\mu)y' + (\alpha^2(\mu) + \beta^2(\mu))y = G(y, y', \mu), \quad (y, y') \notin \Gamma(\mu), \\ &\Delta y'|_{(y,y')\in\Gamma(\mu)} = cy + dy' + J(y, y', \mu), \end{aligned} \tag{2.3.86}$$

where $\alpha(\mu) = \alpha - \frac{1}{2}\frac{\partial F}{\partial y'}(0, 0, \mu)$, $\beta(\mu) = \left(\alpha^2 + \beta^2 - \alpha^2(\mu) - \frac{\partial F}{\partial y}(0, 0, \mu)\right)^{1/2}$, $G(y, y', \mu) = F(y, y', \mu) - y\frac{\partial F}{\partial y}(0, 0, \mu) - y'\frac{\partial F}{\partial y'}(0, 0, \mu)$. Note that the functions G, $\frac{\partial G}{\partial y}$ and $\frac{\partial G}{\partial y'}$ vanish at $(0, 0, \mu)$ for all μ. $\Gamma(\mu)$ can also be written as

$$\Gamma(\mu) : m_1(\mu)y + m_2(\mu)y' + \tau_2(y, y', \mu) = 0,$$

where $m_1(\mu) = m_1 + \partial\tau/\partial y(0, 0, \mu)$, $m_2(\mu) = m_1 + \partial\tau/\partial y'(0, 0, \mu)$, and $\tau_2(y, y', \mu) = \tau(y, y', \mu) - y\partial\tau/\partial y(0, 0, \mu) - y'\partial\tau/\partial y'(0, 0, \mu)$.

We write (2.3.86) as a system of first-order equations in x_1x_2-plane so that the linear part has the coefficient matrix in Jordan form. For this purpose, we let $x_1 = (\alpha(\mu)y + y')/\beta(\mu)$ and $x_2 = y$. Then, (2.3.86) is written as

$$\begin{aligned} &x_1' = -\alpha(\mu)x_1 - \beta(\mu)x_2 + H(x_1, x_2, \mu), \\ &x_2' = \beta(\mu)x_1 - \alpha(\mu)x_2, \quad (x_1, x_2) \notin \Gamma(\mu), \\ &\Delta x_1|_{(x_1,x_2)\in\Gamma(\mu)} = Ix_1 + K(x_1, x_2, \mu), \end{aligned} \tag{2.3.87}$$

where $I = d$, functions H and K are analytic in all their variables and they carry all the properties of G and J, respectively. The discontinuity surface $\Gamma(\mu)$ is given by

$$\Gamma(\mu) : m_2(\mu)\beta(\mu)x_1 + (m_1(\mu) - \alpha(\mu)m_2(\mu))x_2 + \tau_3(x_1, x_2, \mu) = 0.$$

Note that system (2.3.87) is more convenient to use the polar coordinates.

We shall now introduce the polar coordinates, but first, consider the set of discontinuity points $\Gamma(\mu)$ in polar coordinates. Using the change of variables $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, the curve $\Gamma(\mu)$ is represented as

$$\Gamma(\mu) : \phi = \phi_0(\mu) + \nu(r, \phi, \mu),$$

where $\phi_0(\mu) = \arctan(m_2(\mu)\beta(\mu)/(\alpha(\mu)m_2(\mu) - m_1(\mu)))$, ν is analytic in all variables, 2π-periodic in ϕ, and $\nu = \mathcal{O}(r)$. Thus, using the polar coordinates, (2.3.87) is transformed into the system

$$\begin{aligned} &r' = -\alpha(\mu)r + R_1(r, \phi, \mu), \\ &\phi' = \beta(\mu) + R_2(r, \phi, \mu), \quad (r, \phi) \notin \Gamma(\mu), \\ &\Delta r|_{(r,\phi)\in\Gamma(\mu)} = k(\mu)r + R(r, \phi, \mu), \\ &\Delta\phi|_{(r,\phi)\in\Gamma(\mu)} = -\theta(\mu) + \Theta(r, \phi, \mu), \end{aligned} \tag{2.3.88}$$

where R_1, R_2, R, Θ are all 2π-periodic in ϕ, $R_1 = \mathcal{O}(r^2)$, $R_2 = \mathcal{O}(r)$, $R = \mathcal{O}(r^2)$, $\Theta = \mathcal{O}(r)$ and

$$k(\mu) = \sqrt{(I^2 + 2I)\cos^2(\phi_0(\mu)) + 1} - 1.$$

Eliminating the time variable, and considering ϕ as the independent variable, we can write (2.3.88) as

$$\begin{aligned}
&\tfrac{dr}{d\phi} = \lambda(\mu)r + \mathscr{R}(r,\phi,\mu), \quad \phi \neq \phi_0(\mu) + \xi_{2j}(r,\phi,\mu),\\
&\Delta r|_{\phi=\phi_0(\mu)+\xi_{2j}(r,\phi,\mu)} = k(\mu)r + R(r,\phi,\mu),\\
&\Delta\phi|_{\phi=\phi_0(\mu)+\xi_{2j}(r,\phi,\mu)} = -\theta(\mu) + \Theta(r,\phi,\mu),
\end{aligned} \tag{2.3.89}$$

where $\lambda(\mu) = -\alpha(\mu)/\beta(\mu)$, $\xi_{2j}(r,\phi,\mu) = \nu(r,\phi,\mu)$. In a neighborhood of the origin, it is easily seen that all solutions of (2.3.89), except for the trivial solution, rotate around the origin. Note that the impacts occur once in every two meetings of the trajectory with the discontinuity set $\Gamma(\mu)$. To indicate this notion, we use $2j$ in the subscript in (2.3.89).

During one rotation around the origin, if a solution $r(\phi)$ of (2.3.89) performs the first impact at the moment when $\phi = \eta_{2j}$, that is, $\eta_{2j} = \phi_0 + \xi_{2j}(r(\eta_{2j}), \eta_{2j}, \mu)$, and if it jumps to the point $(r(\gamma_{2j}), \gamma_{2j})$ after the impact, where $\gamma_{2j} = \eta_{2j} - \theta(\mu) + \Theta(r(\eta_{2j}), \eta_{2j}, \mu)$, then this solution is defined on the variable timescale $\cup_{j\in\mathbb{Z}}(\gamma_{2j} + 2j\pi, \eta_{2j} + 2(j+1)\pi]$. The variable timescale depends on the initial data, and the timescale is different for different solutions. Thus, (2.3.89) is considered as an impulsive differential equation on variable timescale [40].

2.3.2.2 The *B*-Equivalent System

In this part, we shall reduce the system in polar coordinates on variable timescale (2.3.89) to the system on the nonvariable timescale with transition condition [38], using the method of B-equivalence [1, 6, 40].

Let $r(\phi)$ be a solution of (2.3.89) with initial condition $r(\phi_0(\mu)) = r$, and assume that $\phi = \eta_{2j}$ is the first from left solution of $\phi = \phi_0(\mu) + \xi_{2j}(r,\phi,\mu)$. That is, assume that $r(\phi)$ performs the first impact at the moment when $\phi = \eta_{2j}$. Let the solution $r(\phi)$ jump to the point $(r(\gamma_{2j}), \gamma_{2j})$ after the impact. Then, we have

$$\begin{aligned}
\gamma_{2j} &= \eta_{2j} - \theta(\mu) + \Theta(r(\eta_{2j}), \eta_{2j}, \mu),\\
r(\gamma_{2j}) &= (1 + k(\mu))r(\eta_{2j}) + R(r(\eta_{2j}), \eta_{2j}, \mu).
\end{aligned}$$

Throughout this section, $\widehat{[a,b]}$ denotes the oriented interval for any $a, b \in \mathbb{R}$. That is, it denotes $[a,b]$ when $a < b$, and it denotes $[b,a]$ otherwise.

Denote by $r_1(\phi)$ the solution of

$$\frac{dr}{d\phi} = \lambda(\mu)r + \mathscr{R}(r,\phi,\mu) \tag{2.3.90}$$

with the initial condition $r_1(\gamma_{2j}) = r(\gamma_{2j})$.

For $\phi \in \widehat{[\phi_0(\mu), \eta_{2j}]}$, we have

$$r(\phi) = r + \int_{\phi_0(\mu)}^{\phi} [\lambda(\mu)r(s) + \mathscr{R}(r(s), s, \mu)]ds,$$

and on the interval $[\gamma_{2j}, \widehat{\phi_0(\mu) -} \theta(\mu)]$, we have

$$r_1(\phi) = r(\gamma_{2j}) + \int_{\gamma_{2j}}^{\phi} [\lambda(\mu) r_1(s) + \mathscr{R}(r_1(s), s, \mu)] ds.$$

Thus,

$$\begin{aligned} r_1(\phi_0(\mu) - \theta(\mu)) &= r(\gamma_{2j}) + \int_{\gamma_{2j}}^{\phi_0(\mu)-\theta(\mu)} [\lambda(\mu) r_1(s) + \mathscr{R}(r_1(s), s, \mu)] ds \\ &= (1 + k(\mu)) r(\eta_{2j}) + \mathscr{R}(r(\eta_{2j}), \eta_{2j}, \mu) \\ &\quad + \int_{\gamma_{2j}}^{\phi_0(\mu)-\theta(\mu)} [\lambda(\mu) r_1(s) + \mathscr{R}(r_1(s), s, \mu)] ds \\ &= (1 + k(\mu)) \left[r + \int_{\phi_0(\mu)}^{\eta_{2j}} [\lambda(\mu) r(s) + \mathscr{R}(r(s), s, \mu)] ds \right] \\ &\quad + \mathscr{R}(r(\eta_{2j}), \eta_{2j}, \mu) \\ &\quad + \int_{\gamma_{2j}}^{\phi_0(\mu)-\theta(\mu)} [\lambda(\mu) r_1(s) + \mathscr{R}(r_1(s), s, \mu)] ds. \end{aligned}$$

We now let

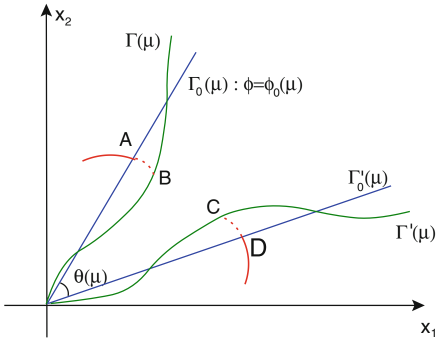
$$\begin{aligned} W(r, \mu) &= r_1(\phi_0(\mu) - \theta(\mu)) - (1 + k(\mu)) r \\ &= (1 + k(\mu)) \int_{\phi_0(\mu)}^{\eta_{2j}} [\lambda(\mu) r(s) + \mathscr{R}(r(s), s, \mu)] ds \\ &\quad + \mathscr{R}(r(\eta_{2j}), \eta_{2j}, \mu) \\ &\quad + \int_{\gamma_{2j}}^{\phi_0(\mu)-\theta(\mu)} [\lambda(\mu) r_1(s) + \mathscr{R}(r_1(s), s, \mu)] ds. \end{aligned}$$

Defining $W(u, \mu)$ and $W(v, \mu)$, using the smallness of the right side function $\mathscr{R}$ and continuous dependence on initial data, we can show that there exists a Lipschitz constant ℓ and a bounded function $m(\ell)$ such that

$$|W(u, \mu) - W(v, \mu)| \leq \ell m(\ell) |u - v| \tag{2.3.91}$$

for all $u, v \in \mathbb{R}$. In Fig. 2.9, the construction of W is demonstrated. There, the point A is $(r, \phi_0(\mu))$, B is the point where the first impact occurs. That is, B is the point $(r(\eta_{2j}), \eta_{2j})$. The phase point jumps to C after the impact. That is, C is the point $(r(\gamma_{2j}), \gamma_{2j})$. Finally, D is the point $(r_1(\phi_0(\mu) - \theta(\mu)), \phi_0(\mu) - \theta(\mu))$.

Fig. 2.9 B-equivalence and the map W



Now, we define the following system

$$\begin{aligned}&\tfrac{d\rho}{d\phi} = \lambda(\mu)\rho + \mathscr{R}(\rho, \phi, \mu), \quad \phi \neq \phi_0(\mu),\\&\Delta\rho|_{\phi=\phi_0(\mu)} = k(\mu)\rho + W(\rho, \mu),\\&\Delta\phi|_{\phi=\phi_0(\mu)} = -\theta(\mu).\end{aligned} \tag{2.3.92}$$

It can be seen that in a neighborhood of the origin, all solutions of (2.3.92), except for the trivial solution, rotate around the origin, as in the case of (2.3.89). The variable ϕ ranges over the timescale $\cup_{n\in\mathbb{Z}}(\phi_0(\mu) - \theta(\mu) + 2n\pi, \phi_0(\mu) + 2(n+1)\pi]$. It can be seen that the timescale is a union of overlapping intervals. Indeed, (2.3.92) is an example of differential equation on a timescale with transition condition (DETCV) [38]. Nevertheless, the timescale is of a new type, since the intervals are overlapping.

Definition 2.3.1 ([6]) We say that (2.3.89) and (2.3.92) are B-equivalent in a neighborhood of the origin if corresponding to each solution $r(\phi)$ of (2.3.89), and there is a solution $\rho(\phi)$ of (2.3.92) such that $\rho(\phi) = r(\phi)$ for all ϕ except possibly on the intervals $[\widehat{\phi_0(\mu), \eta_{2j}}]$ and $[\widehat{\gamma_{2j}, \phi_0(\mu)} - \theta(\mu)]$, where $\eta_{2j} = \eta_{2j}(\mu)$ is the angle when $r(\phi)$ meets $\Gamma(\mu)$ and $\gamma_{2j} = \gamma_{2j}(\mu)$ is the angle where $r(\phi)$ is after the impact.

From the construction made above for W, one can easily see that the following lemma is valid.

Lemma 2.3.1 *Systems (2.3.89) and (2.3.92) are B-equivalent in a neighborhood of the origin.*

2.3.2.3 ψ-Substitution

The independent variable ϕ in (2.3.92) ranges over the domain $\cup_{n\in\mathbb{Z}} I_n$ where

$$I_n = (\phi_0(\mu) - \theta(\mu) + 2n\pi, \phi_0(\mu) + 2(n+1)\pi].$$

Note that $\cup_{n\in\mathbb{Z}} I_n$ is the union of overlapping closed intervals. That is,

$$I_{n,n+1} := I_n \cap I_{n+1} = (\phi_0(\mu) - \theta(\mu) + 2(n+1)\pi, \phi_0(\mu) + 2(n+1)\pi].$$

Therefore, we use a generalization of the so-called ψ-substitution [6, 38]. In our case, the ψ-substitution is defined to be the shifting of the intervals. More precisely, we redefine the intervals I_n as $I'_n := I_n + n\theta(\mu)$. The piece of graph of a solution is shifted accordingly. After the ψ-substitution, we obtain $I'_n \cap I'_{n+1} = \{\}$, and hence, the graph of any trajectory is a single-valued function.

Thus, using the ψ-substitution, we write (2.3.92) as

$$\begin{aligned} &\tfrac{d\rho}{d\varphi} = \lambda(\mu)\rho + \tilde{\mathscr{R}}(\rho,\varphi,\mu), \quad \varphi \neq \phi_0(\mu),\\ &\Delta\rho|_{\varphi=\phi_0(\mu)} = k(\mu)\rho + \tilde{W}(\rho,\mu), \end{aligned} \tag{2.3.93}$$

where $\varphi = \psi(\phi)$. To investigate the Hopf bifurcation in (2.3.93), we follow the classical method, and assume that the nonperturbed system has a family of periodic solutions and the origin in the perturbed system corresponding to $\mu = 0$ is asymptotically stable. Consider the case $\mu = 0$:

$$\begin{aligned} &\tfrac{d\rho}{d\varphi} = \lambda\rho + \tilde{\mathscr{R}}(\rho,\varphi), \quad \varphi \neq \phi_0,\\ &\Delta\rho|_{\varphi=\phi_0} = k\rho + \tilde{W}(\rho), \end{aligned} \tag{2.3.94}$$

where λ, k, ϕ_0, $\tilde{\mathscr{R}}(\rho,\varphi)$ and $\tilde{W}(\rho)$ are the values of $\lambda(\mu)$, $k(\mu)$, $\phi_0(\mu)$, $\tilde{\mathscr{R}}(\rho,\varphi,\mu)$ and $\tilde{W}(\rho,\mu)$ at $\mu = 0$, respectively. The nonperturbed system corresponding to (2.3.94) is

$$\begin{aligned} &\tfrac{d\rho}{d\varphi} = \lambda\rho, \quad \varphi \neq \phi_0,\\ &\Delta\rho|_{\varphi=\phi_0} = k\rho. \end{aligned} \tag{2.3.95}$$

The impacts in (2.3.95) occur on the line $\Gamma : \phi = \phi_0$, and after the impact, the trajectory is on the line $\Gamma' : \phi = \phi_0 - \theta$. The solution $r(\phi) = r(\phi, \phi_0 - \theta, r_0)$, $r_0 > 0$, of (2.3.95) with the initial condition $r(\phi_0 - \theta) = r_0$, where the point $(r_0, \phi_0 - \theta)$ is on the line Γ' (see Fig. 2.10), is given by $r(\phi) = r_0 e^{\lambda(\phi-\phi_0+\theta)}$ for $\phi_0 - \theta \le \phi \le \phi_0 + 2\pi$. Therefore, before the first impact, we have $r(\phi_0 + 2\pi) = r_0 e^{-\alpha T}$, where $T = (2\pi + \theta)/\beta$, and after the impact, the state position is $r(\phi_0 + 2\pi^+) = (1 + k)r(\phi_0 + 2\pi) = (1 + k)e^{-\alpha T} r_0$.

We construct the Poincaré map on the line Γ' and denote

$$q := \frac{r(\phi_0 + 2\pi^+)}{r(\phi_0 - \theta)} = (1 + k)e^{-\alpha T}, \tag{2.3.96}$$

from which we easily see that the following theorem holds.

Theorem 2.3.1 *If*

(a) $q = 1$, *then all solutions of (2.3.95) with the initial conditions on Γ' are T-periodic;*
(b) $q < 1$, *then all solutions of (2.3.95) with the initial conditions on Γ' spiral in toward the origin;*
(c) $q > 1$, *then all solutions of (2.3.95) with the initial conditions on Γ' move away from the origin.*

Remark 2.3.1 In this study, for given α, β, and θ, we fix the number I as $I = (e^{\alpha T} - \cos\theta)/(\cos\theta - e^{-\alpha T}) - 1$ so that $q = 1$ and hence part (a) of the Theorem 2.3.1 holds (see Fig. 2.10). Since $q \neq 1$ is the noncritical case, when the phase portrait is persistent under perturbations, our present interest is only with the case (a) of the Theorem 2.3.1.

Remark 2.3.2 If q, which is defined in (2.3.96), is less than 1, then any solution of (2.3.94) with the initial condition on Γ' spirals in toward the origin, and if $q > 1$, then the solutions move away from the origin. On the other hand, when $q = 1$ we have the critical case. That is, a solution of (2.3.94) with the initial condition on Γ' may be periodic, and it may spiral in toward the origin or it may move away from the origin. In this study, as mentioned before, the case $q \neq 1$ is not of our interest and the critical case is investigated below.

2.3.2.4 Hopf Bifurcation

In this section, we consider (2.3.93) again:

$$\begin{aligned} \frac{d\rho}{d\varphi} &= \lambda(\mu)\rho + \tilde{\mathscr{R}}(\rho, \varphi, \mu), \quad \varphi \neq \phi_0(\mu), \\ \Delta\rho|_{\varphi=\phi_0(\mu)} &= k(\mu)\rho + \tilde{W}(\rho, \mu). \end{aligned} \tag{2.3.97}$$

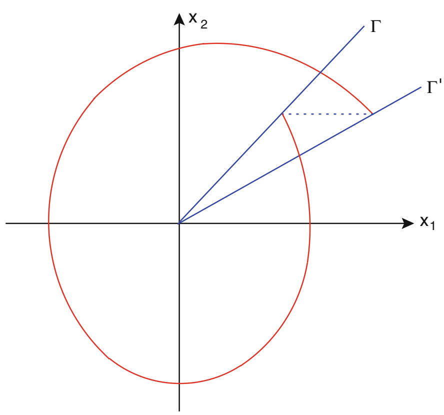


Fig. 2.10 All solutions of (2.3.95) with the initial condition on Γ' are periodic

We consider the corresponding linearized system around the trivial solution:

$$\begin{array}{l} \frac{d\rho}{d\varphi} = \lambda(\mu)\rho, \quad \varphi \neq \phi_0(\mu), \\ \Delta\rho|_{\varphi=\phi_0(\mu)} = k(\mu)\rho. \end{array} \tag{2.3.98}$$

We construct the Poincaré map on the line $\Gamma_0'(\mu)$ and denote

$$q(\mu) = (1 + k(\mu))e^{-\alpha(\mu)T(\mu)}, \tag{2.3.99}$$

in the same way as we defined (2.3.96).

Theorem 2.3.2 *Assume that $q(0) = 1$, $q'(0) \neq 0$. Then for sufficiently small r_0, there exists a function $\mu = \delta(r_0)$ with $\delta(0) = 0$ such that the solution $r(\phi, r_0, \delta(r_0))$ of (2.3.89) is periodic with a period $T = (2\pi + \theta)/\beta + o(|\mu|)$. Furthermore, if solutions of (2.3.94) spiral in toward the origin, then this periodic solution is a discontinuous limit cycle.*

Proof Let $\rho(\varphi, r_0, \mu)$ be a solution of (2.3.97). Because of the analyticity of solutions [1], we have

$$\rho(2\pi, r_0, \mu) = \sum_{i=1}^{\infty} a_i(\mu) r_0^i$$

where $a_i(\mu) = \sum_{j=0}^{\infty} a_{ij}\mu^j$, $a_{10} = q(0) = 1$, $a_{11} = q'(0) \neq 0$. Define

$$\begin{aligned} \mathscr{V}(r_0, \mu) &= \rho(2\pi, r_0, \mu) - r_0 \\ &= q'(0)\mu r_0 + \sum_{i=2}^{\infty} a_{i0} r_0^i + r_0\mu^2 M_1(r_0, \mu) + r_0^2 \mu M_2(r_0, \mu) \end{aligned}$$

where M_1 and M_2 are analytic functions of r_0, μ in a small neighborhood of the $(0, 0)$. When the bifurcation equation, $\mathscr{V}(r_0, \mu) = 0$, is simplified by r_0, one can write

$$\mathscr{H}(r_0, \mu) = 0 \tag{2.3.100}$$

where

$$\mathscr{H}(r_0, \mu) = q'(0)\mu + \sum_{i=2}^{\infty} a_{i0} r_0^{i-1} + \mu^2 M_1(r_0, \mu) + r_0 \mu M_2(r_0, \mu).$$

By the implicit function theorem, since

$$\mathscr{H}(0, 0) = 0, \quad \frac{\partial \mathscr{H}(r_0, \mu)}{\partial \mu} = q'(0) \neq 0,$$

for sufficiently small r_0, there exists a function $\mu = \delta(r_0)$ with $\delta(0) = 0$ such that $r(\phi, r_0, \delta(r_0))$ is a periodic solution. If we assume that $a_{i0} = 0$ for $i = 2, \ldots, \ell - 1$ and $a_{\ell 0} \neq 0$, then from (2.3.100) one can obtain that

$$\delta(r_0) = -\frac{a_{\ell 0}}{q'(0)} r_0^{\ell-1} + \sum_{i=\ell}^{\infty} \delta_i r_0^i. \tag{2.3.101}$$

Analyzing the last expression, one can conclude that the bifurcation of periodic solution exists if a stable focus for $\mu = 0$ is unstable for $\mu \neq 0$ and vice versa. Let $\rho(\varphi) = \rho(\varphi, \bar{r}_0, \bar{\mu})$ be a periodic solution of (2.3.97). It is known that the trajectory is limit cycle if

$$\frac{\partial \mathscr{V}(\bar{r}_0, \bar{\mu})}{\partial r_0} < 0. \tag{2.3.102}$$

Now,

$$\frac{\partial \mathscr{V}(r_0, \mu)}{\partial r_0} = q'(0)\mu + \sum_{i=2}^{\infty} i a_{i0} r_0^{i-1} + \mu^2 N_1(r_0, \mu) + r_0 \mu N_2(r_0, \mu).$$

If $a_{\ell 0}$ is the first nonzero element among a_{i0} and $a_{\ell 0} < 0$, then using (2.3.101) one can obtain

$$\frac{\partial \mathscr{V}(\bar{r}_0, \bar{\mu})}{\partial r_0} = (\ell - 1) a_{\ell 0} \bar{r}_0^{\ell-1} + Q(\bar{r}_0),$$

where Q starts with a member whose order is not less than ℓ. Hence, (2.3.102) is valid. From the ψ-substitution and B-equivalence of (2.3.89) and (2.3.92), one can conclude that the theorem is proved.

Since the change of variables $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, and $y = x_2$, $y' = \beta x_1 - \alpha x_2$ are one-to-one for $\beta \neq 0$, we see that the following theorem is valid.

Theorem 2.3.3 *Assume that $q(0) = 1$, $q'(0) \neq 0$. Then for sufficiently small initial condition $y_0 := (y(0), y'(0))$ there exists a function $\mu = \delta(y_0)$ with $\delta(0) = 0$ such that the solution $y(t, 0, y_0, \mu)$ of (2.3.86) is periodic with a period $T = (2\pi + \theta)/\beta + o(|\mu|)$. Furthermore, if solutions of (2.3.86) with the initial point on $\Gamma_0'(0)$ spiral in toward the origin, then this periodic solution is a discontinuous limit cycle.*

Example 2.3.1 In the following example, we shall consider the model which is studied in many papers and books [59, 63, 66, 110, 148, 160, 216, 229]. Here, we insert the impulse condition and consider

$$\begin{aligned} &y'' + \varepsilon_1 y' + y = \varepsilon_2 y^2 y', \quad (y, y') \notin \Gamma, \\ &\Delta y'|_{(y,y') \in \Gamma} = d y', \end{aligned} \tag{2.3.103}$$

where ε_1 and ε_2 are some nonzero real numbers and Γ is the discontinuity set which is defined, in yy'-plane, by $y = 0,\ y' > 0$. The nonperturbed system is written as

$$\begin{aligned} &y'' + 2\alpha y' + (\alpha^2 + \beta^2)y = 0, \quad (y, y') \notin \Gamma, \\ &\Delta y'|_{(y,y')\in\Gamma} = dy'. \end{aligned} \tag{2.3.104}$$

where $\alpha = \varepsilon_1/2,\ \beta = \sqrt{1-\alpha^2},\ d = e^{2\pi\alpha/\beta} - 1$. Note that the general solution of the differential equation without impulse condition in (2.3.104) is given by

$$y(t) = e^{-\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t)), \tag{2.3.105}$$

where C_1 and C_2 are arbitrary real constants. Let $(0, y_0')$ be any point on the line $\Gamma' = \Gamma$. That is, assume that $y_0' > 0$. Then, $y(0) = 0,\ y'(0) = y_0'$ in (2.3.105) gives us $C_1 = 0,\ C_2 = y_0'/\beta$. Thus, we obtain

$$y(t) = y_0' e^{-\alpha t} \sin(\beta t)/\beta.$$

Now, the first impact action takes place at time $t = T$ where $T > 0$ and $y(T) = 0$, which means $T = 2\pi/\beta$. At that time, we have $y'(T) = e^{-2\pi\alpha/\beta} y_0'$, and after the impact, we have $y'(T^+) = (1+d)y(T) = y_0'$. Therefore, all solutions starting on Γ' are $T = 2\pi/\beta$ periodic. One such solution with $y_0' = 0.06$ is depicted in Fig. 2.11.

The fact that for (2.3.104), the origin is a center makes it suitable for application of our theoretical results. For this reason, let us perturb (2.3.103) and bring it to the notation of system (2.3.82). That is, let us consider the model

$$\begin{aligned} &y'' + 2\alpha y' + (\alpha^2 + \beta^2)y = F(y, y', \mu), \quad (y, y') \notin \Gamma(\mu), \\ &\Delta y'|_{(y,y')\in\Gamma(\mu)} = cy + dy' + J(y, y', \mu), \end{aligned} \tag{2.3.106}$$

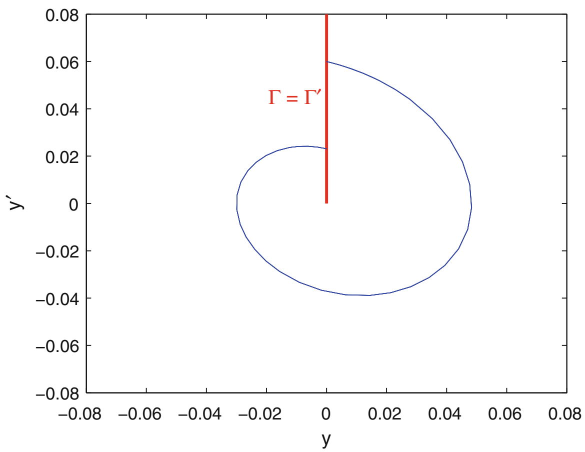


Fig. 2.11 A solution of the nonperturbed system (2.3.104) with the initial condition $y(0) = 0$, $y'(0) = 0.06$

where $\alpha = 0.15$, $\beta = \sqrt{1-\alpha^2}$, $F(y, y', \mu) = 0.02\alpha y^2 y' - \mu y(2 + y + y')$, $\Gamma(\mu)$ is the curve $\Gamma(\mu) = \{(y, y') \in \mathbb{R}^2 : y + 30\mu(y')^2 = 0, y' > 0\}$, $c = \alpha d$, $d = e^{2\pi\alpha/\beta} - 1$, and $J(y, y', \mu) = -(2 + \mu)(y')^2$. Note that (2.3.106) is a special case of (2.3.82) with $m_1 = 1$, $m_2 = 0$, $\tau(y, y', \mu) = 30\mu(y')^2$. The term cy in (2.3.106) has to be considered now as a small perturbation because of smallness of y as the first coordinate of points $\Gamma(\mu)$.

To prove the existence of a periodic solution for (2.3.106), we find that the generalized eigenvalue is

$$q(\mu) = \exp\left(2\pi\alpha\left(\frac{1}{\beta} - \frac{1}{\sqrt{\beta^2 + 2\mu}}\right)\right).$$

Then, one can easily find that $q(0) = 1$ and $q'(0) \neq 0$. Therefore, by Theorem 2.3.3, for sufficiently small μ, there exists a periodic solution with period $\approx 40\pi/\sqrt{391}$.

When $\mu = 0$ in (2.3.106), the origin is a stable focus. This can be seen in simulations and the solution of (2.3.106) corresponding to $\mu = 0$ with the initial condition $y(0) = 0$, $y'(0) = 0.06$ is drawn in Fig. 2.12.

To see an application of Theorem 2.3.3, we take $\mu = 0.08$. By Theorem 2.3.3, we know that there exists a periodic solution corresponding to an initial value $(y(0), y'(0))$ in a sufficiently small neighborhood of the origin. Two solutions of (2.3.106) are drawn in Fig. 2.13. An "inner" solution with the initial condition $y(0) = 0$, $y'(0) = 0.06$ is drawn in red curve and an "outer" solution with the initial condition $y(0) = 0$, $y'(0) = 0.12$ is drawn in blue curve. These two solutions approach to the limit cycle from inside and outside, respectively. Moreover, by Theorem 2.3.3, the period of the discontinuous limit cycle is approximately $40\pi/\sqrt{391}$.

Note that the differential equation of (2.3.106) for $\mu = 0$ becomes (2.3.81) with $\varepsilon = 0.3$. This example shows the application importance of our results.

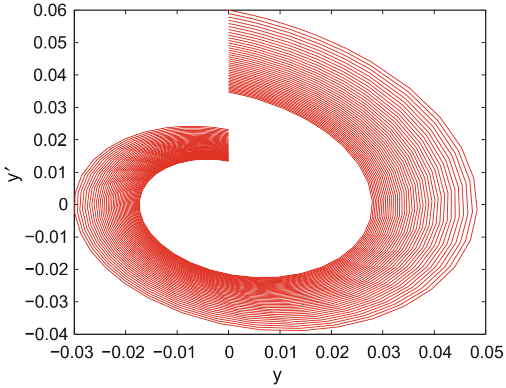


Fig. 2.12 A solution of the perturbed system with the initial condition $y(0) = 0$, $y'(0) = 0.06$

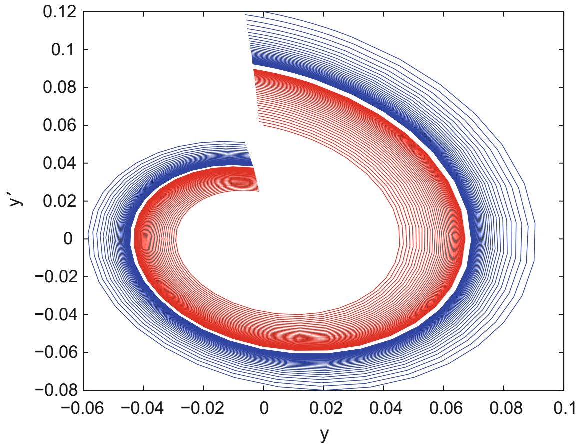


Fig. 2.13 An "inner" and an "outer" solution of (2.3.106). The inner solution is shown in *red* and it corresponds to the initial condition $y(0) = 0,\ y'(0) = 0.06$. The outer solution is shown in *blue* and it corresponds to the initial condition $y(0) = 0$, $y'(0) = 0.12$

System (2.3.106) for $\mu = 0$ is one of the widely investigated models of the mechanisms with impacts determined by the Newton's law of restitution. These models have been studied in many books [66, 216] and papers cited there. One should mention that surfaces of discontinuity in these results are flat. However, it is the first time, we consider the surface of discontinuity as perturbed nonlinearly.

2.3.3 Center Manifold

In this section, we begin our development of the techniques and extend our results to coupled oscillators. We show the existence of a center manifold. Consider

$$\begin{aligned} &y'' + 2\alpha y' + (\alpha^2 + \beta^2)y = F_1(y, y', z, z', \mu), \\ &z'' + 2\gamma z' + (\gamma^2 + \sigma^2)z = F_2(y, y', z, z', \mu), \quad (y, y') \notin \Gamma(\mu), \\ &\Delta y'|_{(y,y')\in\Gamma(\mu)} = cy + dy' + J(y, y', \mu), \end{aligned} \tag{2.3.107}$$

where $y,\ y'$ are the components of one oscillator, say (A), and $z,\ z'$ are the components of another oscillator, say (B).

By using the means of the center manifold theorem [194] and its role for the Hopf bifurcation in multidimensional systems, one can predict that the system of oscillators (A) and (B) admits a limit cycle. Indeed, looking at the results of our simulations given in Figs. 2.14, 2.15, and 2.16, one can see that for the oscillator (A) we have a discontinuous limit cycle, as in Example 2.3.1, and for the oscillator (B) we have a continuous limit cycle. That is, for the whole system, the periodic trajectory (y, z) is a discontinuous limit cycle.

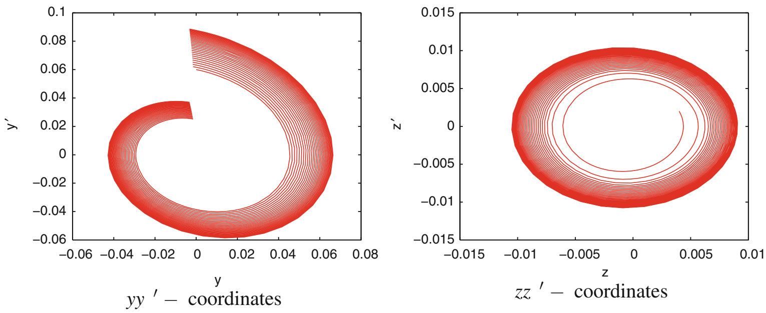


Fig. 2.14 An "inner" solution of (2.3.117) for $\mu = 0.08$ corresponding to the initial condition $y(0) = 0$, $y'(0) = 0.06$, $z(0) = 0.004$, $z'(0) = 0.002$

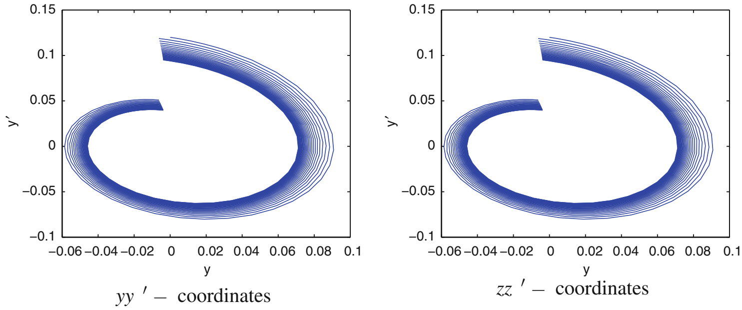


Fig. 2.15 An "outer" solution of (2.3.117) for $\mu = 0.08$ corresponding to the initial condition $y(0) = 0$, $y'(0) = 0.12$, $z(0) = 0.04$, $z'(0) = 0.02$

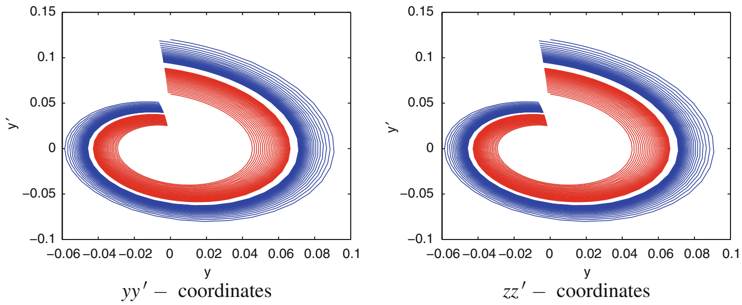


Fig. 2.16 The "inner" and "outer" solutions of (2.3.117) given in Figs. 2.14 and 2.15 are shown in the same picture.

As we did to obtain (2.3.86), we write (2.3.107) in the form

$$
\begin{aligned}
&y'' + 2\alpha(\mu)y' + (\alpha^2(\mu) + \beta^2(\mu))y = G_1(y, y', z, z', \mu),\\
&z'' + 2\gamma(\mu)z' + (\gamma^2(\mu) + \sigma^2(\mu))z = G_2(y, y', z, z', \mu), \quad (y, y') \notin \Gamma(\mu),\\
&\Delta y'|_{(y,y')\in\Gamma(\mu)} = cy + dy' + J(y, y', \mu).
\end{aligned}
\tag{2.3.108}
$$

Let $y_1 = (\alpha(\mu)y + y')/\beta(\mu),\ y_2 = y,\ z_1 = (\gamma(\mu)z + z')/\sigma(\mu),\ z_2 = z$. Then, (2.3.108) becomes

$$
\begin{aligned}
&y_1' = -\alpha(\mu)y_1 - \beta(\mu)y_2 + H_1(y_1, y_2, z_1, z_2, \mu),\\
&y_2' = \beta(\mu)y_1 - \alpha(\mu)y_2,\\
&z_1' = -\gamma(\mu)z_1 - \sigma(\mu)z_2 + H_2(y_1, y_2, z_1, z_2, \mu),\\
&z_2' = \sigma(\mu)z_1 - \gamma(\mu)z_2, \quad (y_1, y_2) \notin \Gamma(\mu),\\
&\Delta y_1|_{(y_1,y_2)\in\Gamma(\mu)} = Iy_1 + K(y_1, y_2, \mu).
\end{aligned}
\tag{2.3.109}
$$

We now use the cylindrical coordinates. That is, we use the polar coordinates for the oscillator (A). Then, we eliminate the variable t, and obtain

$$
\begin{aligned}
&\frac{dr}{d\phi} = \lambda(\mu)r + \mathscr{R}(r, \phi, z_1, z_2, \mu),\\
&\frac{dz_1}{d\phi} = -\tilde{\gamma}(\mu)z_1 - \tilde{\sigma}(\mu)z_2 + \mathscr{R}_2(r, \phi, z_1, z_2, \mu),\\
&\frac{dz_2}{d\phi} = \tilde{\sigma}(\mu)z_1 - \tilde{\gamma}(\mu)z_2, \quad \phi \neq \phi_0(\mu) + \xi_{2j}(r, \phi, \mu),\\
&\Delta r|_{\phi=\phi_0(\mu)+\xi_{2j}(r,\phi,\mu)} = k(\mu)r + R(r, \phi, \mu),\\
&\Delta\phi|_{\phi=\phi_0(\mu)+\xi_{2j}(r,\phi,\mu)} = -\theta(\mu)r + \Theta(r, \phi, \mu),
\end{aligned}
\tag{2.3.110}
$$

where $\tilde{\gamma}(\mu) = \gamma(\mu)/\beta(\mu)$, $\tilde{\sigma}(\mu) = \sigma(\mu)/\beta(\mu)$, and all other elements except for $\mathscr{R}_2$ are the same as in (2.3.89). Using B-equivalence and ψ-substitution, we get the following system:

$$
\begin{aligned}
&\frac{d\rho}{d\varphi} = \lambda(\mu)\rho + \tilde{\mathscr{R}}(\rho, \varphi, z_1, z_2, \mu),\\
&\frac{dz_1}{d\varphi} = -\tilde{\gamma}(\mu)z_1 - \tilde{\sigma}(\mu)z_2 + \tilde{\mathscr{R}}_2(\rho, \phi, z_1, z_2, \mu),\\
&\frac{dz_2}{d\varphi} = \tilde{\sigma}(\mu)z_1 - \tilde{\gamma}(\mu)z_2, \quad \varphi \neq \phi_0(\mu),\\
&\Delta\rho|_{\varphi=\phi_0(\mu)} = k(\mu)\rho + \tilde{W}(\rho, \mu).
\end{aligned}
\tag{2.3.111}
$$

The corresponding linearized system around the origin is

$$
\begin{aligned}
&\frac{d\rho}{d\varphi} = \lambda(\mu)\rho,\\
&\frac{dz_1}{d\varphi} = -\tilde{\gamma}(\mu)z_1 - \tilde{\sigma}(\mu)z_2,\\
&\frac{dz_2}{d\varphi} = \tilde{\sigma}(\mu)z_1 - \tilde{\gamma}(\mu)z_2, \quad \varphi \neq \phi_0(\mu),\\
&\Delta\rho|_{\varphi=\phi_0(\mu)} = k(\mu)\rho.
\end{aligned} \tag{2.3.112}
$$

Like we obtained (2.3.99), we define

$$
\begin{aligned}
q_1(\mu) &= (1+k(\mu))e^{-\alpha(\mu)T(\mu)},\\
q_2(\mu) &= e^{-\tilde{\gamma}(\mu)T(\mu)}.
\end{aligned} \tag{2.3.113}
$$

Note that $q_1(0) = 1$, and hence, we have the critical case for the first oscillator and $q_2(0) < 1$ if and only if $\tilde{\gamma}(0) > 0$. For our system, we assume that $q_1(0) = 1$ and $q_2(0) < 1$.

By using the formulas for the integral manifolds developed in [194], one can see that system (2.3.111) has a center manifold $S_0(\mu) := \{(\rho, \varphi, u) : u = \Phi_0(\varphi, \rho, \mu)\}$ and a stable manifold $S_-(\mu) := \{(\rho, \varphi, u) : \rho = \Phi_-(\varphi, u, \mu)\}$ where

$$
\Phi_0(\varphi, \rho, \mu) = \int_{-\infty}^{\varphi} \pi_0(\varphi, s, \mu)\tilde{\mathscr{R}}_2(\rho(s, \varphi, \rho, \mu), s, u(s, \varphi, \rho, \mu), \mu)ds \tag{2.3.114}
$$

and

$$
\begin{aligned}
\Phi_-(\varphi, u, \mu) = -\int_{\varphi}^{\infty} \pi_-(\varphi, s, \mu)\tilde{\mathscr{R}}_1(\rho(s, \varphi, u, \mu), s, u(s, \varphi, u, \mu), \mu)ds&\\
+ \sum_{\varphi_i<\varphi} \pi_-(\varphi, \varphi_i^+, \mu)\tilde{W}(\rho(s, \varphi, u, \mu), \mu)&
\end{aligned} \tag{2.3.115}
$$

in which $\varphi_i = \varphi + 2\pi i$, $u = (z_1, z_2)$,

$$
\pi_0(\varphi, s, \mu) = e^{-\tilde{\gamma}(\mu)(\varphi-s)} \begin{bmatrix} \cos(\tilde{\sigma}(\mu) - s) & -\sin(\tilde{\sigma}(\mu) - s) \\ \sin(\tilde{\sigma}(\mu) - s) & \cos(\tilde{\sigma}(\mu) - s) \end{bmatrix}
$$

and

$$
\pi_-(\varphi, s, \mu) = e^{\lambda(\mu)(\varphi-s)} \prod_{s\leq\varphi_j(\mu)<\varphi} (1+k(\mu)).
$$

In (2.3.114), the pair $(\rho(s, \phi, \rho, \mu),\ u(s, \phi, \rho, \mu))$ denotes a solution of (2.3.111) satisfying $\rho(s, \phi, \rho, \mu) = \rho$. Similarly, in (2.3.115), the pair $(\rho(s, \phi, u, \mu), u(s, \phi, u, \mu))$ denotes a solution of (2.3.111) satisfying $u(s, \phi, u, \mu) = u$.

On the local center manifold, $S_0(\mu)$, the first coordinate of the solutions of (2.3.111) satisfies the following system

$$\begin{aligned} &\frac{d\rho}{d\varphi} = \lambda(\mu)\rho + \tilde{\mathscr{R}}(\rho, \varphi, \Phi_0(\varphi, \rho, \mu), \mu), \quad \varphi \neq \phi_0(\mu), \\ &\Delta\rho|_{\varphi=\phi_0(\mu)} = k(\mu)\rho + \tilde{W}(\rho, \mu). \end{aligned} \tag{2.3.116}$$

By Theorem 2.3.3, we know that, for sufficiently small μ, system (2.3.116) has a periodic solution with period $T = (2\pi + \theta)/\beta + o(\mu)$. That is, on the local center manifold $S_0(\mu)$, the ρ-coordinate of a solution of (2.3.111) is T-periodic. Because of the T-periodic properties of the right side functions, one can show that z_1 and z_2 components of a solution of (2.3.111) are also T-periodic when the ρ-coordinate is. Thus, we have the following theorem.

Theorem 2.3.4 *Assume that $q_1(0) = 1$, $q_1'(0) \neq 0$, $q_2(0) < 1$. Then for sufficiently small initial condition $(y_0, z_0) := (y(0), y'(0), z(0), z'(0))$ there exists a function $\mu = \delta(y_0, z_0)$ such that the solution $(y(t, 0, y_0, \mu), z(t, 0, z_0, \mu)$ of (2.3.107) is periodic with a period $T = (2\pi + \theta)/\beta + o(|\mu|)$. Furthermore, if solutions of (2.3.107) with the y_0-component of the initial point on $\Gamma'(0)$ spiral in toward the origin, then this periodic solution is a discontinuous limit cycle.*

Example 2.3.2 Let us develop the model studied in Example 2.3.1 further to two coupled oscillators where one of the oscillators is subdued to the impacts. For this reason, consider

$$\begin{aligned} &y'' + 2\alpha y' + (\alpha^2 + \beta^2)y = F_1(y, y', z, z', \mu), \\ &z'' + 2\gamma z' + (\gamma^2 + \sigma^2)z = F_2(y, y', z, z', \mu), \quad (y, y') \notin \Gamma(\mu), \\ &\Delta y'|_{(y,y')\in\Gamma(\mu)} = dy' + J(y, y', \mu), \end{aligned} \tag{2.3.117}$$

where $\gamma = 0.2$, $\sigma = 1$,

$$\begin{aligned} F_1(y, y', z, z', \mu) &= 0.02\alpha y^2 y' - \mu y(2 + y + z + (z')^2), \\ F_2(y, y', z, z', \mu) &= \gamma y z' - \mu y(1 + z^2 + (z')^2), \end{aligned}$$

and all other elements are the same as in Example 2.3.1.

As evaluated before, we have $q_1(0) = 1$, $q_1'(0) \neq 0$, and $\gamma > 0$ implies that $q_2(0) < 1$. Thus, by Theorem 2.3.4, for sufficiently small μ, there exists a periodic solution with period $\approx 40\pi/\sqrt{391}$. Let $\mu = 0.08$. An "inner" solution with the initial condition $y(0) = 0$, $y'(0) = 0.06$, $z(0) = 0.004$, $z'(0) = 0.002$ is drawn in Fig. 2.14 and an "outer" solution with the initial condition $y(0) = 0$, $y'(0) = 0.12$, $z(0) = 0.04$, $z'(0) = 0.02$ is drawn in Fig. 2.15. These two solutions approach to the limit cycle from inside and outside, respectively.

2.4 Notes

The present chapter contains mainly results of papers [6, 39, 42] and is based on the perturbation theory, which was founded by H. Poincaré and A.M. Lyapunov [169, 197], and the bifurcation methods [60, 65, 121, 129, 132, 171, 173, 207, 233]. The main result is the bifurcation of a periodic solution from the equilibrium of the discontinuous dynamical system. After the initial impetus of H. Poincaré [195], A. Andronov [60], and E. Hopf [129], this method of research of periodic motions has been used very successfully for various differential equations by many authors (see [109, 121, 132, 173] and references cited there). There have been two principal obstacles of expansion of this method for discontinuous dynamical systems. While the absence of developed differentiability of solutions has been the first one, the choice of a nonperturbed system convenient to study has been the second. The present investigation utilizes extensively the differentiability and analyticity of discontinuous solutions discussed in Chap. 6 of [1]. The nonperturbed equation is specifically defined. The results of the present chapter can be extended by the dimension enlarging [39] and application to differential equations with discontinuous right side [17]. They are applied to control the population dynamics [19], and can be effectively employed in mechanics, electronics, biology, and medicine [60, 70, 121, 173, 178, 185].

In the second section, we have studied the existence of a center manifold and the Hopf bifurcation for a certain three-dimensional discontinuous dynamical system. The bifurcation of discontinuous cycle is observed by means of the B-equivalence method and its consequences. These results will be extended to arbitrary dimension for a more general type of equations.

Many evolutionary processes are subject to the short-term perturbation whose duration is negligible when compared to that of the whole process. This perturbation results in a change in the state of the process. This change can be at fixed moments or when the state process meets a certain set of discontinuity. These systems model a variety of problems of mechanics, electronics, physics, chemistry, medicine, etc., [59, 63, 64, 66, 77, 84, 109, 112, 116, 130, 145, 150, 179, 209, 228, 232].

In most of the references cited here, the impulse action or the change in the phase space takes place on a flat surface. The theory in which nonlinear surfaces of discontinuity are present has not been investigated fully because of the lack of theoretical results. The problem of discontinuous models where the surfaces of discontinuities are not flat is very actual because of natural possibilities of perturbations. It is natural that one should involve perturbation not only into the differential equation or in the impulse function, but also into the equations for the surfaces of discontinuity.

The discontinuities of the equation which determines the moments of jumps are investigated in many papers and books [216]. In most general form, the results are formulated and expressed in [1], and the present results widely use this information.

In last section, the method of B-equivalence and ψ-substitution [1, 2, 6, 38, 40] is used effectively to observe the Hopf bifurcation of periodic solution. We proved the existence of discontinuous limit cycle for the Van der Pol equation performing impacts on surfaces. We extended the results to two coupled oscillators through the application of the center manifold theory [194]. These theoretical results could be extended to arbitrary dimension and apply them to well-known discontinuous mechanical models.

One should mention that we consider the Hopf bifurcation without reduction in the problem of Hopf bifurcation of the maps as it is usually done in the literature. So, the system which admits the origin as a center in our case is nonlinear one (2.3.95), but its elements are linear. This approach to the investigation is respectively new and promisive.

Based on the present results, one can investigate multioscillatory system where not only one of the oscillators is discontinuous but several of them are discontinuous [63, 114, 118, 190].

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