

Chapter 2

Bare Parameters at Cutoff Scale

Abstract The discovery of the Higgs boson determines the last parameter in the SM, the Higgs self coupling λ . This allows us to extrapolate the SM up to very high scale such as the string/Planck scale. In this chapter, we calculate the bare parameters as functions of the cutoff scale Λ , and check the theoretical consistency, instability and perturbatively. Especially, we focus on the structure of the Higgs potential. In Sect. 2.1, we present the effective potential of the SM. In Sect. 2.2, we calculate the quadratic divergent part of the bare Higgs mass at the two loop level. In Sect. 2.3, we give the numerical estimation of bare parameters as functions of the cutoff scale. In Sect. 2.4, we give the relation between bare couplings and $\overline{\text{MS}}$ ones. In Sect. 2.5, we discuss the effect of the graviton on bare parameters. In Sect. 2.6, we comment on the metastability issue of the electroweak vacuum.

2.1 The Standard Model Effective Potential

In order to discuss the Higgs potential precisely, we need to calculate the loop correction of the Higgs potential. The general one loop potential [1] in $\overline{\text{MS}}$ scheme is given by [2]

$$V = \sum_n (-1)^{2s_n} (2s_n + 1) \frac{1}{64\pi^2} m_n^4 \left(\log \frac{m_n^2}{\mu^2} + a \right), \quad (2.1)$$

where $a = -3/2$ for spin $s_n = 0, 1/2$ and $a = -5/6$ for spin $s_n = 1$ with m_n being effective mass. The summation is taken over real scalars, two component fermions and vectors.

Then, in the Landau gauge, Coleman-Weinberg potential of the SM Higgs is calculated as

$$V_{\text{tree}} = e^{4\Gamma(\varphi)} \frac{\lambda(\mu)}{4} \varphi^4, \quad (2.2)$$

$$V_{1\text{-loop}} = e^{4\Gamma(\varphi)} \left\{ -\frac{3M_t(\varphi)^4}{16\pi^2} \left(\ln \frac{M_t(\varphi)^2}{\mu^2} - \frac{3}{2} + 2\Gamma(\varphi) \right) \right.$$

$$+ \frac{6M_W(\varphi)^4}{64\pi^2} \left(\ln \frac{M_W(\varphi)^2}{\mu^2} - \frac{5}{6} + 2\Gamma(\varphi) \right) + \frac{3M_Z(\varphi)^4}{64\pi^2} \left(\ln \frac{M_Z(\varphi)^2}{\mu^2} - \frac{5}{6} + 2\Gamma(\varphi) \right) \Bigg\}, \quad (2.3)$$

$$\Gamma(\varphi) = \int_{M_t}^{\varphi} \gamma d \ln \mu, \quad (2.4)$$

$$\gamma = \frac{1}{(4\pi)^2} \left(\frac{9}{4}g_2^2 + \frac{3}{4}g_Y^2 - 3y_t^2 \right), \quad (2.5)$$

where φ is the field value of the Higgs, $H = \varphi/\sqrt{2}$, $M_W(\varphi) = g_2\varphi/2$, $M_Z(\varphi) = \sqrt{g_Y^2 + g_2^2} \varphi/2$, and $M_t(\varphi) = y_t\varphi/\sqrt{2}$.

From above expression, we can define the effective quartic coupling of the Higgs potential as follows.

$$\begin{aligned} \lambda_{\text{eff}}(\varphi, \mu) = & e^{4\Gamma(\varphi)} \lambda(\mu) + e^{4\Gamma(\varphi)} \frac{1}{16\pi^2} \left[-3y_t^4 \left(\ln \frac{y_t^2 \varphi^2}{2\mu^2} - \frac{3}{2} + 2\Gamma(\varphi) \right) \right. \\ & \left. + \frac{3g_2^4}{8} \left(\ln \frac{g_2^2 \varphi^2}{4\mu^2} - \frac{5}{6} + 2\Gamma(\varphi) \right) + \frac{3(g_Y^2 + g_2^2)^2}{16} \left(\ln \frac{(g_Y^2 + g_2^2) \varphi^2}{4\mu^2} - \frac{5}{6} + 2\Gamma(\varphi) \right) \right]. \end{aligned} \quad (2.6)$$

The two loop formula is given in Ref. [3], and we use it for our analysis.

2.1.1 Scheme Independence of the Coleman-Weinberg Potential

In this section, we comment on the scheme independence of the Coleman-Weinberg potential by taking ϕ^4 theory Lagrangian as an example,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \quad (2.7)$$

The Coleman-Weinberg potential is

$$\begin{aligned} -iV_{1\text{-loop}} &= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \log \left(-p^2 + \frac{\lambda\varphi^2}{2} \right) \\ &= -\frac{i}{2} \int dp_E \frac{\Omega_d p_E^{d-1}}{(2\pi)^d} \log \left(p_E^2 + \frac{\lambda\varphi^2}{2} \right), \end{aligned} \quad (2.8)$$

where $\Omega_d = 2\pi^{d/2} / \Gamma(d/2)$. Then, we have

$$\begin{aligned}
 V_{1\text{-loop}} &= \frac{1}{\Gamma(\frac{d}{2})} \frac{1}{2} \int dp_E^2 \frac{(p_E^2)^{d/2-1}}{(4\pi)^{d/2}} \log \left(p_E^2 + \frac{\lambda\varphi^2}{2} \right) \\
 &= \frac{1}{\Gamma(\frac{d}{2})} \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \frac{2}{d} \left((p_E^2)^{d/2} \log \left(p_E^2 + \frac{\lambda\varphi^2}{2} \right) \right) \Big|_0^\infty - \int dp_E^2 \frac{(p_E^2)^{d/2}}{p_E^2 + \frac{\lambda\varphi^2}{2}} \\
 &= -\frac{1}{\Gamma(\frac{d}{2})} \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \frac{2}{d} \left(\frac{\lambda\varphi^2}{2} \right)^{d/2} \int dx \frac{x^{d/2}}{1+x} \\
 &= -\frac{1}{\Gamma(\frac{d}{2})} \frac{1}{2} \frac{1}{(4\pi)^{d/2}} \frac{2}{d} \left(\frac{\lambda\varphi^2}{2} \right)^{d/2} \int dy y^{d/2} (1-y)^{-1-d/2} \\
 &= -\frac{1}{2} \frac{1}{(4\pi)^{2-\epsilon}} \frac{2}{d} \left(\frac{\lambda\varphi^2}{2} \right)^{2-\epsilon} \frac{d}{2} \Gamma \left(-\frac{d}{2} \right) \\
 &= -\frac{1}{2} \frac{1}{(4\pi)^2} \left(\frac{\lambda\varphi^2}{2} \right)^2 \left(1 + \epsilon \log(4\pi) - \epsilon \log \left(\frac{\lambda\varphi^2}{2} \right) \right) \frac{\frac{1}{\epsilon} - \gamma}{(-2+\epsilon)(-1+\epsilon)} \\
 &= \frac{1}{64\pi^2} \left(\frac{\lambda\varphi^2}{2} \right)^2 \left(-\frac{1}{\epsilon} + \gamma - \log(4\pi) + \log \left(\frac{\lambda\varphi^2}{2} \right) - \frac{3}{2} \right). \tag{2.9}
 \end{aligned}$$

In the MS and $\overline{\text{MS}}$ schemes, the divergence is subtracted as follows.

$$\begin{aligned}
 -\frac{1}{\epsilon} &\rightarrow -\log(\mu^2) \quad (\text{MS scheme}) \\
 -\frac{1}{\epsilon} + \gamma - \log(4\pi) &\rightarrow -\log(\mu^2) \quad (\overline{\text{MS}} \text{ scheme}). \tag{2.10}
 \end{aligned}$$

Therefore, Coleman-Weinberg potential becomes

$$\begin{aligned}
 V &= V_{\text{tree}} + V_{1\text{-loop}} \\
 &= \begin{cases} \frac{\varphi^4}{4!} \left\{ \lambda_{\text{MS}} + \frac{3}{32\pi^2} (\lambda_{\text{MS}} \varphi^2)^2 \left(\log \left(\frac{\lambda_{\text{MS}} \varphi^2}{2\mu^2} \right) + \gamma - \log(4\pi) - \frac{3}{2} \right) \right\} & \text{for MS} \\ \frac{\varphi^4}{4!} \left\{ \lambda_{\overline{\text{MS}}} + \frac{3}{32\pi^2} (\lambda_{\overline{\text{MS}}} \varphi^2)^2 \left(\log \left(\frac{\lambda_{\overline{\text{MS}}} \varphi^2}{2\mu^2} \right) - \frac{3}{2} \right) \right\} & \text{for } \overline{\text{MS}} \end{cases} \tag{2.11}
 \end{aligned}$$

In order to compare λ_{MS} and $\lambda_{\overline{\text{MS}}}$, let us calculate four point function with external momentum $(p_1 + p_2)^2 = (p_1 + p_3)^2 = (p_1 + p_4)^2 = P^2$:

$$\begin{aligned}
\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \rangle &= -i\lambda + \frac{(-i\lambda)^2}{2}\mu^\epsilon \int \frac{d^d k}{(2\pi)^d} \frac{i^2}{k^2(P+k)^2} \times 3 \\
&= -i\lambda + \frac{3}{2}i\lambda^2\mu^\epsilon \int \int dx \frac{d^d l_E}{(2\pi)^d} \frac{1}{(l_E^2 - x(1-x)P^2)^2} \\
&= -i\lambda + \frac{3}{32\pi^2}i\lambda^2 \left(\frac{1}{\epsilon} - \gamma + \log(4\pi) + \log\left(\frac{\mu^2}{-x(1-x)P^2} \right) \right).
\end{aligned} \tag{2.12}$$

Then, we obtain

$$\lambda_{\text{MS}} = \lambda_{\overline{\text{MS}}} - \frac{3}{32\pi^2}\lambda_{\overline{\text{MS}}}^2(\gamma - \log(4\pi)). \tag{2.13}$$

Substituting Eq. (2.13) into Eq. (2.11), we can see that effective potential in MS and that in $\overline{\text{MS}}$ are the same.

2.1.2 How to Calculate the Beta Function

This part is based on Refs. [4, 5]. In the previous sections, we have seen that we need the beta function and gamma function in order to evaluate the effective potential. We review the calculation of these functions using the dimensional regularization, namely $d = 4 - 2\epsilon$. Then, requiring that $F_{\mu\nu}F^{\mu\nu}$, $\bar{\psi}i\partial_\mu\gamma^\mu\psi$ and $\bar{\psi}iA_\mu\gamma^\mu\psi$ have same dimension, we can fix the dimension in unit of momentum as $[A_\mu] = 1$, $[\psi] = 3/2$, and overall $1/g_B^2$ factor has dimension $[1/g_B^2] = d - 4 = 2\epsilon$. Hence, we parameterize g_B as

$$g_B = \bar{\mu}^\epsilon \bar{Z}_g g, \tag{2.14}$$

where

$$\bar{\mu} = \frac{\mu}{\sqrt{4\pi}} e^{\gamma_E}. \tag{2.15}$$

Here γ_E is the Euler constant. This corresponds to the usual one loop subtraction of $(1/\epsilon - \gamma_E + \log(4\pi))/(4\pi)^2$, which can be found in the elementary textbook of quantum field theory like Ref. [6].

By using this μ , the renormalization of the coupling constant is

$$g_B = \mu^\epsilon Z_g g, \quad Z_g = 1 + \sum_{i=1}^{\infty} \frac{Z_g^{(i)}(g)}{\epsilon^i}. \tag{2.16}$$

The dimensional analysis tells us that $Z_g^{(i)}(g)$ depends on μ only through $g(\mu)$. We also note that Eq. (2.16) does not contain $\epsilon^0, \epsilon^1, \dots$ terms. This corresponds to

the choice of the counter term in the scheme, and therefore this is nothing but the definition of $\overline{\text{MS}}$ scheme.

The definition of the beta function is

$$\beta_g = \mu \frac{\partial}{\partial \mu} g. \quad (2.17)$$

The partial derivative means that g_B and ϵ are fixed.

From the definition, we have

$$\beta_g = \mu \frac{\partial}{\partial \mu} (\mu^{-\epsilon} Z_g^{-1} g_B) = -\epsilon g - g \beta_g \frac{\partial}{\partial g} \log Z_g, \quad (2.18)$$

which becomes

$$\left(\beta_g + \epsilon g + g \beta_g \frac{\partial}{\partial g} \right) Z_g = 0. \quad (2.19)$$

We substituting Eq. (2.16) into this, and put ansatz on $\beta_g = \sum_{i=0}^{i=N} \beta_g^{(i)} \epsilon^i$, where N is some finite number. Then it turns out that

$$\begin{aligned} \beta_g^{(i)} &= 0, \quad (2 \leq i \leq N) \\ \beta_g^{(1)} &= -g, \\ \beta_g^{(0)} &= -g \beta_g^{(1)} \frac{\partial}{\partial g} Z_g^{(1)}. \end{aligned} \quad (2.20)$$

Hence, in order to calculate the beta function in $\overline{\text{MS}}$ scheme, we take the simple pole of ϵ in Z_g :

$$\lim_{\epsilon \rightarrow 0} \beta_g = g^2 \frac{\partial}{\partial g} Z_g^{(1)}. \quad (2.21)$$

The similar derivation is applicable to the gamma function. From the definition,

$$\gamma_\phi = \frac{1}{2} \mu \frac{\partial}{\partial \mu} \log Z_\phi, \quad (2.22)$$

we can derive

$$\left(\beta_g \frac{\partial}{\partial g} - 2\gamma \right) Z_\phi = 0, \quad (2.23)$$

where Z_ϕ is the wavefunction renormalization of a scale field ϕ , $\phi_B = \sqrt{Z_\phi}\phi$. As a result, we obtain

$$\gamma_\phi = -\frac{1}{4}g\frac{\partial}{\partial g}Z_\phi^{(1)}. \quad (2.24)$$

A few comments on the properties of the beta and gamma functions are in order.

1. The coefficients of $\mathcal{O}(g^3)$, $\mathcal{O}(g^5)$ terms in the beta function and $\mathcal{O}(g^3)$ term in gamma function are independent of the scheme.

This can be seen by starting from the given beta function of some scheme,

$$\beta_g = b_0g^3 + b_1g^5 + \mathcal{O}(g^7). \quad (2.25)$$

Let us calculate the beta function in different scheme, $\beta'_{g'} := \mu\partial_\mu g'$. Here $g' = g + ag^3 + \mathcal{O}(g^5)$ is taken.¹ $\beta'_{g'}$ becomes

$$\begin{aligned} \beta'_{g'} &= \beta_g \frac{dg'}{dg} \\ &= (b_0g^3 + b_1g^5 + \mathcal{O}(g^7))(1 + 3ag^2 + \mathcal{O}(g^4)) \\ &= b_0g^3 + b_1g^5 + \mathcal{O}(g^7). \end{aligned} \quad (2.26)$$

2. The zero of the beta function, $\beta(g_*) = 0$, is independent of the scheme.

If the beta function vanishes at $g = g_*$ in some scheme, $\beta'_{g'}(g' = g_*)$ also does:

$$\beta'_{g'}(g'(g = g_*)) = \beta_g(g_*) \frac{dg'}{dg} = 0. \quad (2.27)$$

Therefore, the existence of a fixed point is scheme independent.

3. $\frac{d\beta}{dg}$ at $g = g_*$ is independent of the scheme.

$$\frac{d\beta'_{g'}}{dg'} = \frac{d\beta_g}{dg} + \frac{\partial g}{\partial g'}\beta_g(g_*)\frac{\partial^2 g'}{\partial g^2} = \frac{d\beta_g}{dg}. \quad (2.28)$$

This independence is importance because $d\beta/dg$ determines the flow around a fixed point. This means that the critical exponent is scheme independent quantity.

4. The leading term of γ_ϕ is independent of the scheme.

We write a relation between the wavefunction renormalization as

$$Z'_\phi(g') = Z_\phi(g)F_\phi(g), \quad F_\phi(g) = 1 + \mathcal{O}(g^2) \quad (2.29)$$

¹In general, scale μ' can also be different from μ . Here we assume that $\mu' \propto \mu$.

For some scheme, $\gamma_\phi(g) = c_0 g^2 + \mathcal{O}(g^4)$, then

$$\begin{aligned}\gamma'_\phi(g') &= \frac{1}{2} \mu \frac{\partial}{\partial \mu} \log(Z_\phi(g) F_\phi(g)) \\ &= \gamma_\phi(g) + \frac{1}{2} \beta_g \frac{\partial}{\partial g} \log F_\phi(g) \\ &= c_0 g^2 + \mathcal{O}(g^4).\end{aligned}\tag{2.30}$$

5. $\gamma_\phi(g) = \gamma'_\phi(g')$ at a fixed point, $\beta_g(g_*) = 0$.

This is obvious from the Eq. (2.30).

The leading contribution in beta function and gamma function corresponds to the summation of the leading logarithm. The renormalization group equation for the one-particle irreducible Green function $\Gamma^{(n)}$ is

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} - n \gamma_\phi \right) \Gamma^{(n)} = 0.\tag{2.31}$$

By introducing $t := \log(\mu/Q)$ where Q is some pivot scale, we have

$$\frac{\partial}{\partial t} \Gamma^{(n)}(t, g) = \left(\beta_g \frac{\partial}{\partial g} - n \gamma_\phi \right) \Gamma^{(n)}.\tag{2.32}$$

Formally we can solve it, and obtain

$$\begin{aligned}\Gamma^{(n)}(t, g) &= \exp \left[t \left(\beta_g \frac{\partial}{\partial g} - n \gamma_\phi \right) \right] \Gamma^{(n)}(0, g) \\ &= \sum_m \frac{1}{m!} \left(\log \frac{\mu}{Q} \right)^m \left(\beta_g \frac{\partial}{\partial g} - n \gamma_\phi \right)^m \Gamma^{(n)}(0, g)\end{aligned}\tag{2.33}$$

The point is that the mass independent scheme such as $\overline{\text{MS}}$, β and γ does not have explicit dependence of μ .

The inclusion of leading term of the beta and gamma function corresponds to the summation of leading log. The next-to-leading, next-to-next-leading, $\dots$ terms correspond to next-leading log, next-to-next-leading log, $\dots$, respectively.

2.2 Bare Higgs Mass at the Two Loop

In the following, we will estimate the bare parameters of the SM, and check the theoretical consistency. To compute the bare parameters is important because these are inputs for numerical consideration of the cutoff scale physics such as string theory. There are two kinds of the bare parameters, dimensionless coupling and dimensionful

one. The dimensionless couplings are the gauge couplings, Yukawa couplings and Higgs self coupling, while dimensionful coupling is the bare Higgs mass. As we will see in Sect. 2.4, the dimensionless couplings are well approximated by $\overline{\text{MS}}$ ones. On the other hand, the bare Higgs mass needs extra computations. In this section, we calculate the bare Higgs mass at the two loop level. See Refs. [7–11] for related topic. The dominant part of the bare Higgs mass is the quadratic divergence, so we focus on this part, and calculate it by a cutoff scheme. We have performed the two loop level calculation, and obtain the bare mass as follows:

$$m_{B, 1\text{-loop}}^2 = -\left(6\lambda_B + \frac{3}{4}g_{YB}^2 + \frac{9}{4}g_{2B}^2 - 6y_{tB}^2\right) I_1, \quad (2.34)$$

$$\begin{aligned} m_{B, 2\text{-loop}}^2 = & -\left\{9y_{tB}^4 + y_{tB}^2\left(-\frac{7}{12}g_{YB}^2 + \frac{9}{4}g_{2B}^2 - 16g_{3B}^2\right) \right. \\ & - \frac{87}{16}g_{YB}^4 - \frac{63}{16}g_{2B}^4 - \frac{15}{8}g_{YB}^2g_{2B}^2 \\ & \left. + \lambda_B(-18y_{tB}^2 + 3g_{YB}^2 + 9g_{2B}^2) - 12\lambda_B^2\right\} I_2. \end{aligned} \quad (2.35)$$

Here I_1 and I_2 are quadratic divergent loop integrals,

$$I_1 := \int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{p_E^2}, \quad (2.36)$$

$$I_2 := \int \frac{d^4 p_E}{(2\pi)^4} \frac{d^4 q_E}{(2\pi)^4} \frac{1}{p_E^2 q_E^2 (p_E + q_E)^2}, \quad (2.37)$$

y_{tB} , λ_B , g_{3B} , g_{2B} and g_{YB} are bare top Yukawa, bare Higgs self coupling, bare $SU(3)_C$ coupling, bare $SU(2)_L$ coupling and bare $U(1)_Y$ coupling, respectively.

The vanishing bare Higgs mass condition at the one loop is nothing but Veltman condition [8]. In the computation, we take the physical Higgs mass to be zero and work in symmetric phase. This is because the dimensional analysis tells us that the quadratic divergent part is same both in symmetric phase and broken phase, and is independent of physical Higgs mass. In addition, Landau gauge is taken. The use of Landau gauge drastically reduces the number of Feynman diagram. In Landau gauge, the following class of diagram vanishes since three point coupling is derivative coupling, and external momentum is zero:


 $\Big|_{k=0} = 0.$

(2.38)

In Figs. 2.1 and 2.2, we show non-vanishing diagrams and their contributions.

	$5g_{YB}^4 + 9g_{2B}^4$
	$-\frac{51}{8}g_{2B}^4$
	$\frac{1}{8}g_{YB}^4 + \frac{3}{8}g_{2B}^4$
	$\frac{5}{16}g_{YB}^4 + \frac{15}{16}g_{2B}^4 + \frac{15}{8}g_{YB}^2g_{2B}^2$
sum	$\frac{87}{16}g_{YB}^4 + \frac{63}{16}g_{2B}^4 + \frac{15}{8}g_{YB}^2g_{2B}^2$

Fig. 2.1 g^4 terms in $m_{B,2\text{-loop}}^2$ in units of I_2

	$12\lambda_B^2$
	$-9y_{tB}^4$
	$y_{tB}^2 \left(\frac{7}{12}g_{YB}^2 - \frac{9}{4}g_{2B}^2 + 16g_{3B}^2 \right)$
	$\lambda_B (18y_{tB}^2 - 3g_{YB} - 9g_{2B}^2)$

Fig. 2.2 The terms other than g^4 in $m_{B,2\text{-loop}}^2$ in units of I_2

In order to fix the ratio of I_1 and I_2 , the cutoff scheme should be specified. In this thesis, we take proper time regularization:

$$\int d^4 k_E \frac{1}{k_E^2} = \int_{\epsilon}^{\infty} d\alpha \int d^4 k_E e^{-\alpha k_E^2}, \quad (2.39)$$

By employing this regularization, the one loop and two loop integrals are calculated as

$$\begin{aligned} I_1 &= \frac{1}{\epsilon} \frac{1}{16\pi^2}, \\ I_2 &= \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} \frac{1}{k^2 p^2 (p+k)^2} \\ &= \int_{\epsilon}^{\infty} d\alpha \int_{\epsilon}^{\infty} d\beta \int_{\epsilon}^{\infty} d\gamma \frac{d^4 k}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} e^{-\alpha p^2 - \beta k^2 - \gamma (p+k)^2} \\ &= \int_{\epsilon}^{\infty} d\alpha \int_{\epsilon}^{\infty} d\beta \int_{\epsilon}^{\infty} d\gamma \frac{\pi^4}{(2\pi)^8} \frac{1}{(\alpha\beta + \beta\gamma + \gamma\alpha)^2} \\ &= \frac{1}{\epsilon} \int_1^{\infty} d\alpha' \int_1^{\infty} d\beta' \int_1^{\infty} d\gamma' \frac{\pi^4}{(2\pi)^8} \frac{1}{(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha')^2} \\ &= \frac{1}{\epsilon} \frac{1}{(16\pi^2)^2} \ln \frac{6}{3^3} \simeq 0.005 I_1. \end{aligned} \quad (2.40)$$

Here $\alpha' = \epsilon \alpha$, $\beta' = \epsilon \beta$ and $\gamma' = \epsilon \gamma$. If we employ the usual momentum cutoff, we obtain

$$I_1 = \frac{\Lambda^2}{16\pi^2}, \quad (2.41)$$

which implies the relation $1/\epsilon = \Lambda^2$. We note that the ratio Eq. (2.40) depends on the cutoff scheme. However, in the next section, we will show that the two loop effect is small, and therefore we can safely neglect the cutoff scheme dependence.

2.3 Bare Parameters at the Cutoff Scale

In this section, we numerically solve the renormalization group equations (RGEs), and evaluate the bare parameters at the cutoff scale. The values of SM couplings at the electroweak scale are given in Ref. [3]:

$$g_2(M_t) = 0.64822 + 0.00004 \left(\frac{M_t}{\text{GeV}} - 173.10 \right) + 0.00011 \left(\frac{M_W - 80.384 \text{ GeV}}{0.014 \text{ GeV}} \right), \quad (2.42)$$

$$g_Y(M_t) = 0.35761 + 0.00011 \left(\frac{M_t}{\text{GeV}} - 173.10 \right) + 0.00021 \left(\frac{M_W - 80.384 \text{ GeV}}{0.014 \text{ GeV}} \right), \quad (2.43)$$

$$g_3(M_t) = 1.1666 + 0.00314 \left(\frac{\alpha_S(M_Z) - 0.1184}{0.0007} \right) - 0.00046 \left(\frac{M_t}{\text{GeV}} - 173.10 \right), \quad (2.44)$$

$$y_t(M_t) = 0.93558 + 0.00550 \left(\frac{M_t}{\text{GeV}} - 173.10 \right) - 0.00042 \left(\frac{\alpha_S(M_Z) - 0.1184}{0.0007} \right) \\ - 0.00042 \left(\frac{M_W - 80.384 \text{ GeV}}{0.014 \text{ GeV}} \right) \pm 0.00050_{\text{th}}, \quad (2.45)$$

$$\lambda(M_t) = 0.12711 + 0.00206 \left(\frac{M_H}{\text{GeV}} - 125.66 \right) - 0.00004 \left(\frac{M_t}{\text{GeV}} - 173.10 \right) \pm 0.00030_{\text{th}}. \quad (2.46)$$

The RGE of the SM at two loop level is in Appendix B. The mass of the Higgs boson is

$$M_H = 125.09 \pm 0.24 \text{ GeV} \quad (2.47)$$

at the 1σ by Particle Data Group [12]. The $SU(3)_C$ gauge coupling is

$$\alpha_S(M_W) = 0.1185 \pm 0.0006 \text{ GeV}. \quad (2.48)$$

We note that the current error of the value of the top mass:

$$M_t^{\text{pole}} = \begin{cases} 171.2 \pm 2.4 \text{ GeV}, & \text{MITP [13],} \\ 176.7^{+4.0}_{-3.4} \text{ GeV}, & \text{PDG [14],} \end{cases} \quad (2.49)$$

$$M_t^{\text{Pythia}} = \begin{cases} 173.21 \pm 0.51 \pm 0.71 \text{ GeV}, & \text{direct measurement, PDG [14],} \\ 174.98 \pm 0.76 \text{ GeV}, & \text{D0 [15],} \\ 174.34 \pm 0.64 \text{ GeV}, & \text{D0+CDF [16],} \\ 173.34 \pm 0.76 \text{ GeV}, & \text{ATLAS [17],} \\ 172.38 \pm 0.10 \pm 0.65 \text{ GeV}, & \text{CMS [18].} \end{cases} \quad (2.50)$$

As we can see, there are two types of top masses, M_t^{pole} and M_t^{Pythia} . M_t^{Pythia} is measured by using Monte Carlo simulation. The experiment observes the momentum of the color singlet decay product of $t\bar{t}$, and compares with the event shape of Monte Carlo simulation. The problem is that we do not know the relation between M_t^{Pythia} and $\overline{\text{MS}}$ parameters, which is important for renormalization group (RG) evolution. Moreover, M_t^{Pythia} inevitably suffers from uncertainty of hadronization.

On the other hand, M_t^{pole} is defined as a pole of two point function of the top field. At least perturbatively, M_t^{pole} is well defined, and can be written by using $\overline{\text{MS}}$ couplings. Therefore, we use M_t^{pole} in the thesis. M_t^{pole} is measured from inclusive cross section of $t\bar{t}$, $\sigma(pp \rightarrow t\bar{t}X)$ at the LHC. We can see that M_t^{pole} still has large

uncertainty. As we will see, in order to discuss the criticality of the SM precisely, the error of M_t should become small.

The way of precision measurement of the top mass is the threshold scan of the toponium state, which can be done by, e.g. international linear collider (ILC) in which initial energy of electron and positron is controlled. In the ILC, the error of top mass is estimated as $\mathcal{O}(10\text{--}100)\text{MeV}$. In the following, we take

$$M_t^{\text{pole}} = 171.2 \pm 2.4 \text{ GeV} \quad (2.51)$$

as a reference value.

Then let us move on numerical calculation. We plot the running of bare parameters in the left panel of Fig. 2.3. The largest uncertainty comes from the error of the top mass Eq. (2.51), we represent it as bands. Surprisingly, we can find triple coincidence if the top quark mass is 170 GeV: the Higgs self coupling λ , its beta function β_λ and bare Higgs mass take zero around the string/Planck scale. Figure 2.4

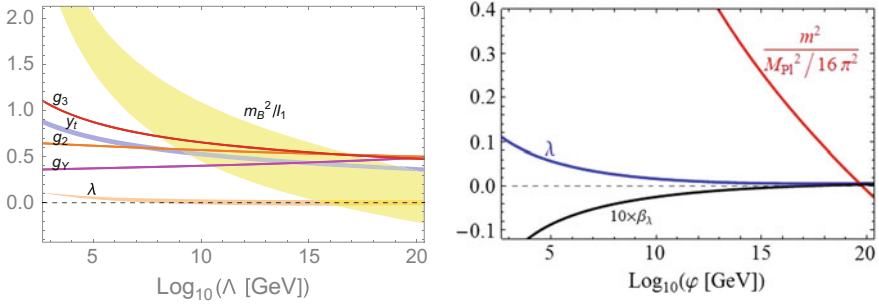
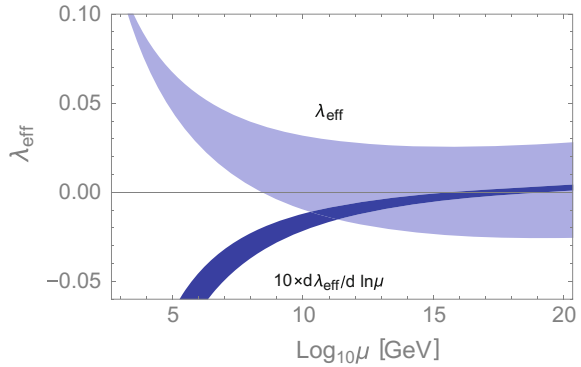


Fig. 2.3 *Left* The bare parameters as functions of the cutoff scale Λ as well as the beta function of λ . The *band* corresponds to the two sigma deviation of the top mass, Eq. (2.51). *Right* The triple coincidence occurs if $M_t = 170 \text{ GeV}$

Fig. 2.4 The *light blue band* is RGE running of $\lambda_{\text{eff}}(\mu)$. $\lambda_{\text{eff}}(\mu)$ is defined by two loop effective potential, and we have used 2 loop RGE. The *dark blue band* is the beta function multiplied by ten, $10 \times d\lambda_{\text{eff}}/d \ln \mu$. $M_H = 125.09 \text{ GeV}$ and $\alpha_s = 0.1185$ are taken. The band corresponds to 95% CL deviation of M_t



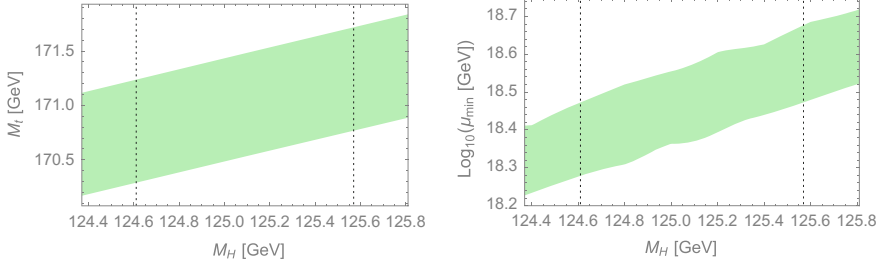


Fig. 2.5 Left: M_t realizing $\lambda_{\min} = 0$ as a function of M_H . We use the two loop RGE and effective potential. The *light green band* and *dotted line* correspond to 95% deviation of α_s and M_H , respectively. Right: The scale $\mu_{\min}$ that realizes $\lambda_{\min} = 0$

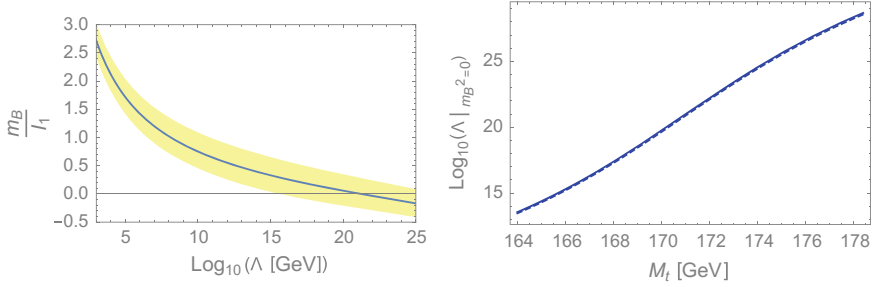
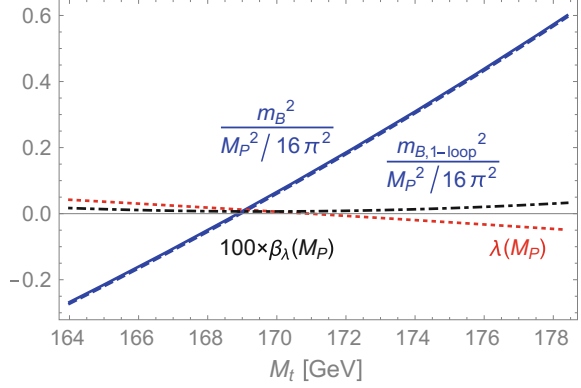


Fig. 2.6 Left: The bare Higgs mass as a function of Λ . The *band* corresponds the deviation of the top mass at 95% C.L. Right: The scale where the bare Higgs mass takes zero as a function of M_t

is the enlarged view of the effective self coupling λ_{eff} and its derivative as functions of the renormalization scale μ . The band is same as Fig. 2.3. In the left panel of the Fig. 2.5, M_t realizing $\lambda_{\min} = 0$ as a function of M_H is plotted. In the right panel, we can see the scale $\mu_{\min}$ that realizes $\lambda_{\min} = 0$. From this we can see that the two loop effect is small. In Fig. 2.6, we show the bare Higgs mass as a function of the cutoff scale Λ at the two loop level. The right panel represents the scale where the bare mass vanishes. The dashed and solid lines correspond to one-loop and two-loop calculations, respectively. We can see that the two loop effect is small, as we noted before. The bare mass and the self coupling at the Planck scale is shown in Fig. 2.7, as well as the beta function of the self coupling. It can be seen that there is three coincidence. The bare mass, self coupling and its beta function simultaneously become zero if the top quark mass is around 170 GeV.

Fig. 2.7 The *blue solid* (*dashed*) line corresponds to the one-plus-two-loop (one-loop) bare mass m_B^2 ($m_{B,1-loop}^2$) in units of $M_P^2/16\pi^2$ for $\Lambda = M_P$. For comparison, we also plot the quartic coupling λ at the Planck scale with the *red dotted line*. The central values $\alpha_s(m_Z) = 0.1185$ and $m_H = 125.9$ GeV are used



2.4 The Relation Between Dimensionless Bare Couplings and $\overline{\text{MS}}$ Couplings

We have approximated the bare couplings by $\overline{\text{MS}}$ ones. In this section, we present the justification of this approximation. The relation between dimensionless bare coupling and $\overline{\text{MS}}$ coupling can be obtained by calculating a physical quantity. Here we present the relation in ϕ^4 theory by performing the one loop integral of the four point function $\langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \rangle$. As there is no wavefunction renormalization at the one loop level, we can take $\phi_B = \phi$. The one loop calculation in the $\overline{\text{MS}}$ scheme can be found in Eq. (2.12). In the momentum cutoff scheme, we get

$$\begin{aligned}
 & -i\lambda_B + \frac{(-i\lambda_B)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i^2}{k^2(P+k)^2} \times 3 \\
 & = -i\lambda_B + \frac{3}{2}i\lambda_B^2 \int_0^{\Lambda^2} \int_0^1 dx \frac{d^4l_E}{(2\pi)^4} \frac{1}{(l_E^2 - x(1-x)P^2)^2}. \quad (2.52)
 \end{aligned}$$

Taking $P^2 = -\Lambda^2$, we obtain

$$\begin{aligned}
 \langle \phi(p_1)\phi(p_2)\phi(p_3)\phi(p_4) \rangle & = -i\lambda_B + \frac{3}{32\pi^2}i\lambda_B^2 \frac{\log(161 + 72\sqrt{5})}{2\sqrt{5}} \\
 & \simeq -i\lambda_B + 0.01 \times i\lambda_B^2. \quad (2.53)
 \end{aligned}$$

Finally we have

$$-i\lambda(\mu) + \frac{3}{32\pi^2}i\lambda(\mu)^2 \left(\log\left(\frac{\mu^2}{\Lambda^2}\right) - 2 \right) = -i\lambda_B + \frac{3}{32\pi^2}i\lambda_B^2 \frac{\log(161 + 72\sqrt{5})}{2\sqrt{5}}, \quad (2.54)$$

and the relation between bare and $\overline{\text{MS}}$ coupling can be derived as

$$\lambda_B = \lambda(\Lambda) - \frac{3}{32\pi^2} i \lambda(\Lambda)^2 \left(2 + \frac{\log(161 + 72\sqrt{5})}{2\sqrt{5}} \right). \quad (2.55)$$

Now we have confirmed that the difference between two couplings are very small, and the approximation $\lambda_B \simeq \lambda(\Lambda)$ is justified.

Now let us evaluate the effect of above approximation on the scale where the bare Higgs mass vanishes. The relation between the bare and $\overline{\text{MS}}$ coupling in the SM is generally written as

$$\lambda_{\overline{\text{MS}}}^i(\mu) = \lambda_B^i + \sum_{jk} c^{ijk}(\mu/\Lambda) \lambda_B^j \lambda_B^k + O(\lambda_B^3), \quad (2.56)$$

$$c^{ijk}(x) := f^{ijk} + b^{ijk} \ln x + O(x^2), \quad (2.57)$$

where b^{ijk} represents the one loop coefficient of the beta function, f^{ijk} represents the finite correction and λ_B^i represent $(\{g_{YB}^2, g_{2B}^2, g_{3B}^2, y_{tB}^2, \lambda_B\})$. Since the all SM couplings are perturbative up to the Planck scale, we have

$$\mu \ll \Lambda, \quad \left| \frac{\lambda_{\overline{\text{MS}}}^i}{16\pi^2} \ln(\mu/\Lambda) \right| \ll 1, \quad (2.58)$$

and therefore

$$\lambda_{\overline{\text{MS}}}^i(\mu) = \lambda_B^i + \sum_{jk} \left(f^{ijk} + b^{ijk} \ln \frac{\mu}{\Lambda} \right) \lambda_B^j \lambda_B^k, \quad (2.59)$$

is the good approximation. On the other hand, the scale dependence of $\overline{\text{MS}}$ coupling can be written by using beta function,

$$\lambda_{\overline{\text{MS}}}^i(\Lambda) = \lambda_{\overline{\text{MS}}}^i(\mu) + \sum_{jk} b^{ijk} \lambda_{\overline{\text{MS}}}^j(\mu) \lambda_{\overline{\text{MS}}}^k(\mu) \ln \frac{\Lambda}{\mu}, \quad (2.60)$$

at one loop level. Combining Eqs. (2.59) and (2.60), we obtain the following formula:

$$\lambda_{\overline{\text{MS}}}^i(\Lambda) = \lambda_B^i + \sum_{jk} f^{ijk} \lambda_B^j \lambda_B^k. \quad (2.61)$$

This is generalization of Eq. (2.55).

As a result, the correction to the bare Higgs mass is given by

$$\Delta m_B^2 = \left(a^i \lambda_{\overline{\text{MS}}}^i(\Lambda) - \sum_{ijk} a^i f^{ijk} \lambda_{\overline{\text{MS}}}^j(\Lambda) \lambda_{\overline{\text{MS}}}^k(\Lambda) \right) I_1, \quad (2.62)$$

Here a_i is the coefficient of the one loop bare mass. We denote the scale where the bare mass without correction vanishes by Λ_0 ,

$$a^i \lambda_{\overline{\text{MS}}}^i(\Lambda_0) = 0. \quad (2.63)$$

Then, the scale where the corrected bare mass takes zero becomes

$$\Lambda|_{m_B^2=0} := e^{\delta t} \Lambda_0, \quad \delta t = \frac{\sum_{ijk} a^i f^{ijk} \lambda_{\overline{\text{MS}}}^j(\Lambda_0) \lambda_{\overline{\text{MS}}}^k(\Lambda_0)}{\sum_{ijk} a^i b^{ijk} \lambda_{\overline{\text{MS}}}^j(\Lambda_0) \lambda_{\overline{\text{MS}}}^k(\Lambda_0)}. \quad (2.64)$$

Since δt is the order of one, we expect that

$$0.1 \lesssim \frac{\Lambda|_{m_B^2=0}}{\Lambda_0} \lesssim 10. \quad (2.65)$$

This is the uncertainty coming from the difference between the bare and $\overline{\text{MS}}$ couplings.

2.5 The Effect of the Graviton

Since we consider the scale close to the Planck scale, the effect of the gravitational interaction may become relevant. Here, we estimate the impact on the bare Higgs mass from gravity. When we take into the gravity, the action becomes

$$S \ni \int d^4x \sqrt{-g} \mathcal{L}_{\text{SM}}. \quad (2.66)$$

We expand the $g_{\mu\nu}$ around the flat space:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_P} h_{\mu\nu}, \quad (2.67)$$

where $\eta_{\mu\nu}$ is the flat space metric, $\text{diag}(+ - - -)$, and $h_{\mu\nu}$ is the fluctuation corresponding to the graviton. It is found that the interaction is

$$\begin{aligned} & \int d^4x \left(\frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-g} \mathcal{L}_{\text{SM}}) \Big|_{g=\eta} h_{\mu\nu} + \frac{1}{2} \frac{\delta}{\delta g_{\rho\sigma}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-g} \mathcal{L}_{\text{SM}}) \Big|_{g=\eta} h_{\mu\nu} h_{\rho\sigma} + \dots \right) \\ &= \int d^4x \sqrt{-g} (T^{\mu\nu} h_{\mu\nu} + \dots). \end{aligned} \quad (2.68)$$

In the second line, we have used the definition of the energy momentum tensor, $T_{\mu\nu} := \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-g} \mathcal{L}_{\text{SM}})$.

The part of $T_{\mu\nu}$ related to the Higgs is given by

$$T_{\mu\nu} \sim (D_\mu H)^\dagger (D^\nu H) + (D_\nu H)^\dagger (D^\mu H) - g_{\mu\nu} \left((D_\mu H)^\dagger (D^\mu H) - m_B^2 H^\dagger H - \lambda_B (H^\dagger H)^2 y_u \bar{Q} \tilde{H} U + y_d \bar{Q} H D + y_l \bar{L} H E + \text{h.c.} \right). \quad (2.69)$$

Among the terms in Eq. (2.69), only the kinetic term is relevant for one loop calculation of the bare mass. However, this contribution vanishes since the momentum of the external line is zero, and there are no correction to the bare mass at the one loop level. As for the second term in Eq. (2.68), the possible interaction is the type of $hh\partial H\partial H$, which gives no contribution at one loop level.

In general, gravitons give the correction to the Higgs mass of the order of $\mathcal{O}(\Lambda^4/M_P^2)$ at the two loop level. If the theory of gravity is weak coupled, this contribution is two loop suppressed. For example, in perturbative string theory, the string coupling, determined by the dilation background, is small and the perturbative expansion is valid. In this sense, our analysis of the bare Higgs mass is valid taking into account the gravitational effect. If the physics around the Planck scale is strong coupled or the cutoff scale becomes much higher than the Planck scale, we can not neglect the effect of gravitons.

2.6 Metastability of the Electroweak Vacuum

We have seen that our electroweak vacuum is in the critical point, the border between stable and unstable vacuum. However, in practice, even if the electroweak vacuum is unstable, there are no problems provided that the lifetime is sufficiently large compared with the age of the universe. In this case, we call the electroweak vacuum is metastable. Therefore, it is phenomenologically important to determine the lifetime of false vacuum. The way of calculation is developed by Coleman [19].

In the actual calculation, the bounce solution of the Euclidean equation of the motion plays the crucial role. By using the bounce action S_0 , the decay rate per unit volume is

$$\frac{\Gamma}{V} \sim \frac{1}{R^4} e^{-S_0}, \quad (2.70)$$

where R is a typical size of the bounce solution.

At the tree level, the equation of the motion is

$$-\frac{d^2}{dr^2}\varphi - \frac{3}{r}\frac{d}{dr}\varphi + \lambda\varphi^3 = 0. \quad (2.71)$$

The boundary condition of the bounce solution is given by

$$\varphi'(0) = 0, \quad \varphi(\infty) = 0, \quad (2.72)$$

The solution of the equation is [20]

$$\varphi(r) = \sqrt{\frac{2}{|\lambda|}} \frac{2R}{r^2 + R^2}, \quad S_{\text{bounce}} = \frac{8\pi^2}{3|\lambda(\mu)|}, \quad (2.73)$$

Note that the action has the scale invariance at the classical level,

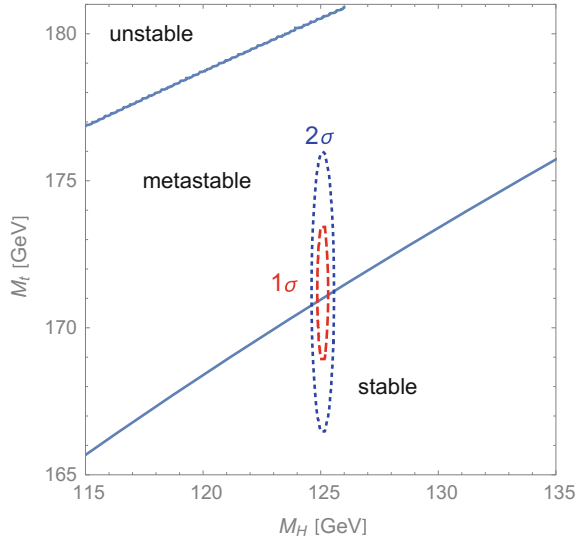
$$\varphi(r) \rightarrow \frac{1}{C} \varphi(Cr). \quad (2.74)$$

Hence, if $\varphi(r)$ is a solution of the equation, $\frac{1}{C} \varphi(Cr)$ is also a solution for arbitrary constant C . This means that, at the classical level, there is huge uncertainty due to the arbitrary choice of scales R and μ . In order to resolve the situation, the one-loop level calculation is needed. The resultant one-loop calculation suggests that the tunneling probability becomes [20]

$$p \sim \text{Max}_R \frac{1}{H_0^4 R^4} \exp\left(-\frac{8\pi^2}{3|\lambda(\mu = 1/R)|}\right), \quad (2.75)$$

where H_0 is the current Hubble parameter. Here we have not included the one-loop correction to the exponent for simplicity. Figure 2.8 shows the phase structure of the SM. We can see that the error of the top mass is larger than that of the Higgs mass, and that we indeed live in the stable or metastable vacuum depending on the mass of the top quark.

Fig. 2.8 The phase diagram of the SM. The *upper*, *middle* and *lower* region correspond to unstable, metastable and stable electroweak vacuum. The *red* and *blue* circles represent the 1σ and 2σ deviations



We note that the above estimate is valid when we consider the zero temperature field theory and vacuum to vacuum transition. The condition of the metastability becomes severe if we take into account the finite temperature effect or dynamical evolution of Higgs field during inflation and preheating, although these estimations highly depend on the assumption of cosmological history. In this sense, Eq. (2.75) gives conservative condition of metastability of electroweak vacuum.

References

1. S.R. Coleman, E.J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking. *Phys. Rev. D* **7**, 1888–1910 (1973)
2. S.P. Martin, A supersymmetry primer (1997). [arXiv:hep-ph/9709356](#). (Adv. Ser. Direct. High Energy Phys. 18(1) (1998))
3. D. Buttazzo, G. Degrandi, P.P. Giardino, G.F. Giudice, F. Sala, A. Salvio, A. Strumia, Investigating the near-criticality of the Higgs boson. *JHEP* **12**, 089 (2013). [arXiv:1307.3536](#)
4. Y. Sumino, *Intensive Lecture*
5. D.J. Gross, *Applications of the Renormalization Group*
6. M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory*
7. K.G. Wilson, The renormalization group and strong interactions. *Phys. Rev. D* **3**, 1818 (1971)
8. M. Veltman, The infrared—ultraviolet connection. *Acta Phys. Polon. B* **12**, 437 (1981)
9. M. Al-sarhi, I. Jack, D. Jones, Quadratic divergences in gauge theories. *Z. Phys. C* **55**, 283–288 (1992)
10. Y. Hamada, H. Kawai, K.-Y. Oda, Bare Higgs mass at Planck scale. *Phys. Rev. D* **87**(5), 053009 (2013). [arXiv:1210.2538](#)
11. D. Jones, *The quadratic divergence in the Higgs mass revisited*. *Phys. Rev. D* **88**, 098301 (2013). [arXiv:1309.7335](#)
12. Particle Data Group, K.A. Olive et al., Review of particle physics. *Chin. Phys. C* **38**, 090001 (2014)
13. S. Moch, S. Weinzierl, S. Alekhin, J. Blumlein, L. de la Cruz, et al., High precision fundamental constants at the TeV scale (2014). [arXiv:1405.4781](#)
14. Particle Data Group, K. Olive et al., Review of particle physics. *Chin. Phys. C* **38**, 090001 (2014)
15. (D0 Collaboration), V.E. Abazov, Precision measurement of the top quark mass in lepton +jets final states. *Phys. Rev. Lett.* **113**, 032002 (2014)
16. Tevatron Electroweak Working Group, T.E.W. Group, Combination of CDF and D0 results on the mass of the top quark using up to 9.7 fb^{-1} at the Tevatron (2014). [arXiv:1407.2682](#)
17. D0 Collaborations, T. ATLAS, CDF, and CMS, First combination of Tevatron and LHC measurements of the top-quark mass (2014). [arXiv:1403.4427](#)
18. CMS Collaboration, C. collaboration, Measurement of the top-quark mass in t t-bar events with all-jets final states in pp collisions at $\sqrt{s}=8 \text{ TeV}$ (2014)
19. S.R. Coleman, *The Uses of Instantons*
20. G. Isidori, G. Ridolfi, A. Strumia, On the metastability of the standard model vacuum. *Nucl. Phys. B* **609**, 387–409 (2001). [arXiv:hep-ph/0104016](#)

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