Chapter 2
Variational Macroscopic Two-Phase Poroelasticity. Derivation of General Medium-Independent Equations and Stress Partitioning Laws

Abstract A macroscopic continuum theory of two-phase saturated porous media is derived by a purely variational deduction based on the least Action principle. The proposed theory proceeds from the consideration of a minimal set of kinematic descriptors and keeps a specific focus on the derivation of most general *medium-independent* governing equations, which have a form independent from the particular constitutive relations and thermodynamic constraints characterizing a specific medium. The kinematics of the microstructured continuum theory herein presented employs an *intrinsic/extrinsic* split of volumetric strains and adopts, as an additional descriptor, the *intrinsic* scalar volumetric strain which corresponds to the ratio between solid true densities before and after deformation. The present theory integrates the framework of the Variational Macroscopic Theory of Porous Media (VMTPM) which, in previous works, was limited to the variational treatment of the momentum balances of the solid phase alone. Herein, the derivation of the complete set momentum balances inclusive of the momentum balance of the fluid phase is attained on a purely variational basis. Attention is also focused on showing that the singular conditions, in which either the solid or the fluid phase are vanishing, are consistently addressed by the present theory, included conditions over free solid-fluid surfaces.

2.1 Introduction

The mechanics of porous media in multiphase physical system has garnered in years a wide range of applications. Traditionally deployed in the field of soil mechanics [14, 30, 80], in the last decades, multiphase continuum poroelasticity has become an indispensable theoretical tool in biomechanics (see for instance [3, 4, 28, 62]), and, more recently, also in impact engineering [57] due to its importance for the understanding and prediction of several complex physical phenomena occurring in solids interacting with other phases [23, 37, 50, 55, 74].

Given the wide range of applications, poroelastic theories face the challenge to deal with a vast array of microstructural features and properties which determine

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the mechanical behavior of the various types of porous media investigated. This variety of mechanical features and applications has produced a body of literature on continuum modelling of multiphase poroelastic problems which is considerably large, to the extent that a comprehensive overview of it may only be gained by the union of several survey works (see for instance [8, 22, 36, 70, 73]).

The many theories so far proposed differ by the axiomatic schemes and/or methodological approaches employed to infer governing equations [7, 21, 40, 85]. Disagreement is also found in the mathematical and physical meaning of some governing equations. In this respect, it is worth to recall the problem of the "*missing equation*", early pointed out by Truesdell and Noll [83] and generally referred to as *closure problem*, as well as the different identifications proposed by several authors for such closure equations (above all, their questioned constitutive and/or thermodynamic nature) [8, 31, 73, 83]. Theories also differ by the structure of the macroscopic governing PDEs [2, 13, 18, 32, 40, 51], as well as by the physical-mathematical, or engineering, definition employed to introduce macroscopic stress measures [7, 12, 15–17, 20, 35, 37, 65, 70, 79]. In this respect, it can be observed that, while the existence of a multiplicity of differentiated approaches for studying a given physical problem can be deemed to be physiological in a mature research field, a widely spread opinion in multiphase poroelasticity research recognizes the lack of unanimous convergence over a set of governing equations, or over a hierarchy of governing equations. This disagreement can be stigmatized by the words that De Boer used in 2005 according to whom *"the necessity to attack the problem of developing a consistent general poroelasticity theory is still existent"* [16], and has been remarked even more recently [51], leaving the impression that multiphase poroelasticity still remains, in some respects, an 'unfinished chapter' of continuum mechanics. Thus, even the simpler two-phase purely-mechanical problem of poroelasticity can be regarded, under some aspects, as a still-open problem.

A key point for organizing and establishing interrelations between existing multiphase poroelasticity theories is the assessment of the availability of *medium-independent* equilibrium equations for multiphase problems. By medium-independent equations, we refer to a set of equations regulating the dynamics of multiphase media standing in a form which is completely independent from the subsequent specification of constitutive relations and thermodynamic constraints characterizing a specific medium. The importance of such an issue can be recognized by tracing a parallel with the hierarchy of equations standardly encountered in single-phase continuum theories of solids mechanics: although the class of linear and nonlinear constitutive responses and microstructural features addressed by the many available theories is also very large, the linear momentum balance equation is ordinarily regarded as an unquestioned *universal* equation which holds irrespective of constitutive and microstructural properties and of thermodynamic constraints. Actually, in single-phase theories, either the mathematical structure of the linear momentum balance is unaffected by microstructural and constitutive features (this is the case, for instance, of standard elastoplasticity where linear momentum balance is an unquestioned equation [39, 54, 78]), or, in the case of theories with a greater microstructural content, the necessity to preserve downward compatibility of equilibrium equations with linear momentum balance is tacitly given for granted [59].

For two-phase continuum poromechanic formulations, a similar consensus over a set of equilibrium equations general enough to describe porous media independent of the specific constitutive, microstructural and thermodynamic properties of media (i.e., compressibility of the constituent phases, porosity, etc.) is not found. For instance, several theories identify the closure equation with the Clausius-Duhem inequality [18, 41, 70, 83], while other theories introduce supplemental equations which have a constitutive character [18, 27, 51]. Also, as observed in [42, 66, 77], the stress partitioning problem in two-phase saturated media does not admit a medium-independent solution.

The reason why linear momentum balance equations in single-phase continuum mechanics attain medium independence is that they stem as the simplest possible least-Action conditions when a purely-macroscopic and purely-variational approach is adopted. These equations are obtained by considering the *minimal* variational mechanical description in which displacements are the sole kinematic descriptor fields, and a simplest general first-gradient dependence of the strain energy is assumed [5, 9, 47].

Continuum variational approaches, based on Hamilton least-Action principle [48], are indeed suitable tools to investigate the medium independence problem in continuum mechanics, for the main reason that in these approaches the least-Action principle is the sole primitive mechanical concept invoked when deriving momentum balance equations. The resulting equations stem univocally and unambiguosly from the kinematic descriptors and from the form of the Action functional adopted. Hence, when natural deformation descriptors and strain measures are employed for the kinematics, and the form of the Action functional is *sufficiently general*, governing equations of maximum generality are expected to be derived [5]. In addition, continuum variational approaches offer several further advantages: boundary conditions are simultaneously derived with bulk field equations [25, 26, 29, 55, 69] without requiring further mechanical considerations or ad-hoc hypotheses.

In light of the above discussion, the objective of this study is the derivation of a *minimal* medium-independent two-phase poroelastic framework which any more complex theory should be downward compatible to. Specifically, following the parallel with the variational derivation of linear momentum balances in single-phase continuum elasticity, the sought poroelastic framework should be endowed with the following features:

- it should have a purely-mechanical, purely-variational and purely-macroscopic character;
- it should proceed from the consideration of a minimum possible number of kinematic descriptors, which should have a clear physical-mechanical meaning, and their experimental characterization should be possible.

Several two-phase and multi-phase continuum poroelasticity theories have been proposed, whose governing equations at a macroscopic level are based, to different extents, on the application of classical variational principles, or on some variants of these principles, or even on the simple application of some variational concepts [1, 6–8, 10, 11, 21, 24, 33, 34, 44, 50–53, 64, 68, 71]. Reviews specifically dedicated to this subject are the one by Bedford and Drumheller [8] and the more recent one in [73]. As remarked in Chap. 1, most multiphase poroelastic formulations adopt additional descriptors such as the volume fractions and the intrinsic strain. However, even remaining in the purely variational literature, no unanimous consensus over a *minimal* set of governing equations is found. Specific debated issues are the well-posedness of the variational statement of the multiphase problem in presence of constraints such as mass balances, and the assessment of the physical meaning of stress quantities defined with the aid of Lagrange multipliers in relation to boundary data and to the macroscopic measurement process [8, 24, 51, 85]. The development of a constraint-free variational statement of the problem, and of a related suitable kinematical description of open porous systems in which the fluid can freely flow through the porous solid matrix, are recognized to be relevant problems, in particular, in [24]. Therein a study is presented on the variational statement of the two-phase poroelastic problem and on the resulting boundary conditions, as determined by the replacement of the fluid macroscopic placement field with another field, defined in the macroscopic solid reference configuration, which maps solid material points into points of the fluid reference configuration, sharing the same current spatial position at the given time instant.

Most recently, a general variational continuum theory with microstructure of two-phase poroelasticity has been proposed [72, 74–77, 81, 82]. Peculiar feature of this theory, henceforth abbreviated in Variational Macroscopic Theory of Porous Media (VMTPM), is the resort to an *extrinsic/intrinsic* split of volumetric strain measures: VMTPM kinematics includes a scalar field termed *intrinsic* volumetric strain of the solid phase, which essentially corresponds to the ratio between 'true' densities of solid before and after deformation; such field is independent from the primary macroscopic volumetric strain measure, which remains instead ordinarily defined as the determinant of the macroscopic deformation gradient, and accordingly termed *extrinsic* volumetric strain. Importantly, in the above mentioned references, the variational deduction of VMTPM field equations was only limited to the derivation of the momentum balances of the solid phase. The purpose of this work is to complete this framework presenting a more general multiphase variational poroelastic theory by also including the derivation of the fluid linear momentum balance on a purely variational basis from the least-Action principle. Thus, we aim to achieve a purely-variational and purely-macroscopic deduction of all momentum balances for the two-phase poroelastic problem in a *minimal* medium-independent setting, so that the derived equations hold irrespective of the constitutive response and compressibility of the solid and fluid phases, as well as of thermodynamic constraints. Furthermore, we intend to obtain, on a purely-variational basis, the following results:

- a comprehensive rational derivation of the general three-dimensional equations which must be applied at the macroscopic boundaries of the mixture;

- a rational derivation of the equations relevant to the discontinuity surfaces between a porous region contained in the mixture and a contiguous entirely-fluid region (free surfaces);
- a rational derivation of the medium-independent stress partitioning laws, with a subsequent discussion on their range of applicability.

To achieve a general theory, the theoretical derivation hereby proposed proceeds from finite-deformations to subsequently obtain small-displacement equations as a special case upon kinematic linearization. Finally, the medium-independent stress-partitioning laws resulting from this theory are examined.

Attention is also focused on showing that the singular conditions, in which either the solid or the fluid phase are vanishing, are consistenly addressed by the present theory, included conditions over free solid-fluid surfaces. Remarks have been included to place greater attention on the discussion of some technical passages which are important for showing the consistency of the theory, and to provide additional insights.

Due their rather technical character, some of the intermediate developments required for the computation of the explicit form of the Euler Lagrange equations have been moved in Appendices A and B where notation conventions and useful identities for differential operations are reported.

The specific assessment of the predictive capabilities of the equations of the VMTPM theory herein derived and the comparison with other theories are addressed in the next Chaps. 3 and 4. Thereby, the field equations herein obtained are specialized for linear and nonlinear isotropic media subjected to a comprehensive variety of loading and drainage conditions. The results provided in this chapter and in the subsequent ones show that VMTPM recovers governing equations and results of consolidated use in poroelasticity, such as Terzaghi's stress partitioning principle and Biot's equation, and also predicts established experimental results of poromechanics.

2.2 Variational Formulation

The variational formulation is hereby derived following a purely macroscopic approach and based on the use of the intrinsic strain among the primary kinematic descriptors. In addition, we employ the hypothesis of complete saturation of space, and proceed from minimal kinematic and constitutive assumptions substantially analogous to those employed in [74]. The theory herein presented enhances the one proposed in [74]: this is a fully variational derivation in that momentum balance equations of both solid and fluid phases are derived on a purely variational basis. To achieve improved generality and clarity in the derivation, the formulation is first derived in a finite-deformation framework, and then specialized to infinitesimal configuration changes by applying kinematic linearization.

2.2.1 Basic Configuration Descriptors

We consider a purely-macroscopic description of the change of configuration, under finite deformations, of a two-phase immiscible mixture made of a porous solid with interconnected cells allowing independent relative motion of an interstitial fluid. Complete saturation conditions are considered for the mixture throughout the deformation process.

The reference configuration of the mixture is defined by the macroscopic smooth reference domain of the mixture, $\Omega_0^{(M)}$, and by two scalar fields defining at the macroscopic level the volume fractions of the solid and of the fluid phase, $\Phi_0^{(s)} : \mathbf{X} \in \Omega_0^{(M)} \to \Phi_0^{(s)}$, $\Phi_0^{(f)} : \mathbf{X} \in \Omega_0^{(M)} \to \Phi_0^{(f)}$. The description herein considered is based on purely macroscopic quantities. The relation between macroscopic quantities and their microscale counterparts is hereby pointed out, for clarity, with reference to a microscale Representative Volume Element (RVE). The reference solid volume fraction, $\Phi_0^{(s)}(\mathbf{X})$ in a point $\mathbf{X}$ represents, with reference to a RVE $\overset{\text{RVE}}{\Omega}_0(\mathbf{X})$ centered in the point $\mathbf{X}$, the ratio $V_{0_{RVE}}^{(s)}(\mathbf{X})/V_{0_{RVE}}(\mathbf{X})$ between the volume $V_{0_{RVE}}^{(s)}(\mathbf{X})$ of the subset $\overset{\text{RVE}(s)}{\Omega_0}(\mathbf{X}) \subset \overset{\text{RVE}}{\Omega}_0(\mathbf{X})$ occupied by the solid phase and the volume $V_{0_{RVE}}(\mathbf{X})$ of $\overset{\text{RVE}}{\Omega}_0(\mathbf{X})$. The reference fluid volume fraction $\Phi_0^{(f)} = V_{0_{RVE}}^{(f)}(\mathbf{X})/V_{0_{RVE}}(\mathbf{X})$ is analogously defined, with $V_{0_{RVE}}^{(f)}$ being the volume of the subset $\overset{\text{RVE}(f)}{\Omega_0}(\mathbf{X}) \subset \overset{\text{RVE}}{\Omega}_0(\mathbf{X})$ containing the fluid. Due to the saturation condition holding also at reference configuration, one has for any $\mathbf{X} \in \Omega_0^{(M)}$

$$\Phi_0^{(s)}(\mathbf{X}) + \Phi_0^{(f)}(\mathbf{X}) = 1. \tag{2.1}$$

The deformed configuration of the mixture is defined by two invertible a-priori independent vector functions, $\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\boldsymbol{\chi}}^{(f)}$, termed solid placement and fluid placement, respectively, $\bar{\boldsymbol{\chi}}^{(s)} : \mathbf{X} \in \Omega_0^{(M)} \to \mathbf{x} \in \mathbb{R}^3$, $\bar{\boldsymbol{\chi}}^{(f)} : \mathbf{X} \in \Omega_0^{(M)} \to \mathbf{x} \in \mathbb{R}^3$, whose codomains $\bar{\boldsymbol{\chi}}^{(s)}\left(\Omega_0^{(M)}\right)$, $\bar{\boldsymbol{\chi}}^{(f)}\left(\Omega_0^{(M)}\right)$ are contained in the ambient space $\mathbb{R}^3$. In this description when $\bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X}) \neq \bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X})$ the positions of the two solid and fluid particles, which are initially macroscopically superimposed at $\mathbf{X}$, are disjoint in the current configuration.

The description of the state of the deformed configuration is completed by the spatial fields of current volume fraction, $\phi_{\mathbf{x}}^{(s)} : \mathbf{x} \in \Omega^{(M)} \to \phi^{(s)}$, $\phi_{\mathbf{x}}^{(f)} : \mathbf{x} \in \Omega^{(M)} \to \phi^{(f)}$, where the lowercase subscripts $\mathbf{x}$ are added to mark the spatial character of these fields. Fields $\phi_{\mathbf{x}}^{(s)}$ and $\phi_{\mathbf{x}}^{(f)}$ are defined over the subset $\Omega^{(M)}$ of the ambient space occupied by the mixture in the deformed configuration, which contains the codomains of $\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\boldsymbol{\chi}}^{(f)}$, viz., $\bar{\boldsymbol{\chi}}^{(s)}\left(\Omega_0^{(M)}\right) \subseteq \Omega^{(M)}$ and $\bar{\boldsymbol{\chi}}^{(f)}\left(\Omega_0^{(M)}\right) \subseteq \Omega^{(M)}$. Field $\phi_{\mathbf{x}}^{(s)}$, while again introduced on the basis of a purely macroscopic description, is such that its value $\phi_{\mathbf{x}}^{(s)}(\mathbf{x})$ in a point $\mathbf{x}$ is related to the small-scale configuration of the microscale RVE, $\overset{\text{RVE}}{\Omega}(\mathbf{x})$, in the spatial configuration, centered in $\mathbf{x}$ by $\phi^{(s)}(\mathbf{x}) = V_{RVE}^{(s)}(\mathbf{x})/V_{RVE}(\mathbf{x})$, where $V_{RVE}^{(s)}(\mathbf{x})$ is the volume of the (microscale) solid subset $\overset{\text{RVE}(s)}{\Omega}(\mathbf{x}) \subset \overset{\text{RVE}}{\Omega}(\mathbf{x})$ and $V_{RVE}(\mathbf{x})$ is the volume of $\overset{\text{RVE}}{\Omega}(\mathbf{x})$. The counterpart relation for the fluid phase is $\phi^{(f)}(\mathbf{x}) = V^{(f)}_{RVE}(\mathbf{x})/V_{RVE}(\mathbf{x})$ with a completely analogous definition. Also, in the current configuration, volume fractions range between 0 and 1 and the relevant saturation condition similar to (2.1) reads for any $\mathbf{x} \in \Omega^{(M)}$:

$$\phi^{(s)}(\mathbf{x}) + \phi^{(f)}(\mathbf{x}) = 1. \tag{2.2}$$

Remark 2.1 Consistency under limit single phase conditions—Configuration descriptions employing placement fields $\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\boldsymbol{\chi}}^{(f)}$, defined on a common domain $\Omega_0^{(M)}$, in combination with saturation relations of type (2.1) and (2.2), are rather standard in purely-macroscale theories of multiphase flow under saturation hypotheses. These configuration descriptions, or suitable variants, allow to ordinarily describe the relative solid-fluid motion in immiscible mixtures and are, as a matter of fact, explicitly or tacitly employed by most of the formulations surveyed in Chap. 1. For instance the configuration descriptions originarily considered by Cowin [21] and by Bedford and Drumheller [6–8] include the volume fractions among the primary descritors. Similarly, the configuration description exploited by dell'Isola, Madeo and Seppecher [24] employs field $\boldsymbol{\chi}_{sf}$, recalled in Eq. (1.7), which introduces a mapping from the solid to the fluid reference configuration, and which can be identified as a variant of the configuration descriptors herein employed, corresponding to $\boldsymbol{\chi}_{sf} = \left(\bar{\boldsymbol{\chi}}^{(f)}\right)^{-1} \circ \bar{\boldsymbol{\chi}}^{(s)}$.

It is convenient, however, to pay special attention to the *Limit Single-Phase* (LSP) conditions attained by such a representation of configurations when either of solid and fluid volume fractions achieve limit zero or unit values.

In the configuration description presently employed, a point $\mathbf{X} \in \Omega_0^{(M)}$ with nonzero volume fractions ($\Phi_0^{(s)}(\mathbf{X}) \neq 0$, $\Phi_0^{(f)}(\mathbf{X}) \neq 0$) corresponds to the condition of—macroscopically—superimposed positions of solid and fluid phases at point $\mathbf{X}$ in the reference configuration. Fields $\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\boldsymbol{\chi}}^{(f)}$ in these points permit to ordinarily locate the current macroscopic physical positions of the solid and fluid particles. Similarly, in the deformed configuration, a point $\mathbf{x} \in \Omega^{(M)}$ with nonzero volume fractions ($\phi^{(s)}(\mathbf{x}) \neq 0$, $\phi^{(f)}(\mathbf{x}) \neq 0$ accounts for the superposition of phases in the current configuration. In these points the inverse maps $\left(\bar{\boldsymbol{\chi}}^{(s)}\right)^{-1}$ and $\left(\bar{\boldsymbol{\chi}}^{(f)}\right)^{-1}$ permit to locate, again in a ordinary way, the originary, possibly disjoint, reference macroscopic physical positions of the solid and fluid particles.

LSP Conditions of flow (LSPCs) are met when either $\Phi_0^{(s)} = 0$ or $\Phi_0^{(f)} = 0$ in some subregions of $\Omega_0^{(M)}$, or, in the current configuration, when either $\phi^{(s)} = 0$ $\phi^{(f)} = 0$ in some subregions of $\Omega^{(M)}$.

These conditions deserve special attention: LSPCs are customarily required for describing, for instance, the existence of reservoir single-phase regions with the fluid alone (hence with $\phi^{(s)} = 0$), or of nonporous regions (with

$\phi^{(f)} = 0$) in contact with the mixture; on the other hand, the absence of one phase apparently leads to a singularity of the above introduced ($\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\phi^{(s)}$, $\phi^{(f)}$)-based kinematic description since the placement field of the phase with vanishing volume fraction looses a physical counterpart.

Some remarks in this chapter discuss how, even in presence of singular LSP conditions, the statement herein considered of the multiphase problem ordinarily maintains full physical and mathematical consistency. Specifically in this subsection, the consistency of the theory in presence of LSPCs is first illustrated from the kinematic point of view (see Remark 2.3) by showing that the domains so far introduced preserve a physical counterpart in presence of LSPCs when arbitrary a-priori independent fields $\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\boldsymbol{\chi}}^{(f)}$ are assigned. In the subsequent sections, Remark 2.4 comments on the consistency of the ($\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\phi^{(s)}$, $\phi^{(f)}$)-based configuration description in relation to boundary conditions and surface conditions. Remark 2.5 comments on the consistency of the adopted kinematic description in relation to the well-posedness of the corresponding Euler-Lagrange equations in presence of LSPCs.

The description of volume changes achieved in terms of purely macroscopic fields is now examined. The *extrinsic* volumetric deformation of the solid phase is introduced as the scalar $\bar{J}^{(s)} = \det \partial\bar{\boldsymbol{\chi}}^{(s)}/\partial\mathbf{X}$. This quantity is purely macroscopic and its use is standard in finite deformation poroelasticity, see for instance [22]. Concerning the interpretation of the relation of $\bar{J}^{(s)}(\mathbf{X})$ with the volume changes of the reference and deformed RVEs associated with a point $\mathbf{X} \in \Omega_0^{(M)}$, as shown in [74], this quantity is the ratio $\bar{J}^{(s)}(\mathbf{X}) = V_{RVE}(\mathbf{x})/V_{0_{RVE}}(\mathbf{X})\big|_{\bar{\boldsymbol{\chi}}^{(s)}}$, where $V_{RVE}(\mathbf{x})$ is the volume of the microscale deformed RVE $\bar{\boldsymbol{\chi}}^{(s)}(\overset{\text{RVE}}{\Omega}_0(\mathbf{X}))$ obtained applying the microscopic deformation $\boldsymbol{\chi}^{(s)}$ to the whole domain $\overset{\text{RVE}}{\Omega}_0(\mathbf{X})$ upon performing an extrapolation of $\boldsymbol{\chi}^{(s)}$ from $\overset{\text{RVE}(s)}{\Omega_0}(\mathbf{X})$ to the whole set $\overset{\text{RVE}}{\Omega}_0(\mathbf{X})$. The extrinsic volumetric deformation of the fluid phase is specularly introduced as the scalar $\bar{J}^{(f)} = \det \partial\bar{\boldsymbol{\chi}}^{(f)}/\partial\mathbf{X}$, and the following relation holds: $\bar{J}^{(f)}(\mathbf{X}) = V_{RVE}(\mathbf{x})/V_{0_{RVE}}(\mathbf{X})\big|_{\bar{\boldsymbol{\chi}}^{(f)}}$. Trivially, for the inverse mappings $\left(\bar{\boldsymbol{\chi}}^{(s)}\right)^{-1}$, $\left(\bar{\boldsymbol{\chi}}^{(f)}\right)^{-1}$, the extrinsic volume ratios are $V_{0_{RVE}}(\mathbf{X})/V_{RVE}(\mathbf{x})\big|_{(\bar{\boldsymbol{\chi}}^{(s)})^{-1}} = 1/\bar{J}^{(s)}(\mathbf{X})$ and $V_{0_{RVE}}(\mathbf{X})/V_{RVE}(\mathbf{x})\big|_{(\bar{\boldsymbol{\chi}}^{(f)})^{-1}} = 1/\bar{J}^{(f)}(\mathbf{X})$.

In [74], the necessity to introduce an additional volumetric deformation measure in order to achieve a complete description of the volume changes of the mixture was highlighted. This is in consideration of the fact that $\bar{J}^{(s)}(\mathbf{X})$ can be different from unity even when the solid porous skeleton is undergoing isochoric deformations (see Remark 2.2). Accordingly, consistent with [74], an additional macroscopic field of *intrinsic volumetric strain* of the solid phase, $\hat{J}^{(s)}$, independent from $\bar{J}^{(s)}$, is introduced to measure the effective volume changes of the solid phase, $\hat{J}^{(s)} : \mathbf{X} \in \Omega_0^{(M)} \to \hat{J}^{(s)} \in \mathbb{R}$. The relation of $\hat{J}^{(s)}$ with the volumes of the microscale RVE in the reference and deformed configurations is $\hat{J}^{(s)}(\mathbf{X}) = V_{RVE}^{(s)}(\mathbf{x})/V_{0_{RVE}}^{(s)}(\mathbf{X})$ with $\mathbf{x} = \bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X})$. The field of intrinsic volumetric strain of the fluid phase is introduced in completely specular form by replacing the (s) scripts with (f) in the relations above. The intrinsic volumetric strain is related to the RVE volumes by $\hat{J}^{(f)} = V^{(f)}_{RVE}(\mathbf{x})/V^{(f)}_{0_{RVE}}(\mathbf{X})$.

According to the macroscopic description for volume changes so far detailed, a homothetic deformation of the solid phase in a point $\mathbf{X}$ is characterized by the condition $\hat{J}^{(s)}(\mathbf{X}) = \bar{J}^{(s)}(\mathbf{X})$.

Remark 2.2 Basic volumetric strain modes of the solid deformation—To exemplify how the $(\bar{J}^{(s)}, \hat{J}^{(s)})$-based description of configuration changes works, a schematic graphical illustration of three basic volumetric strain modes, which can be represented by nonzero values of the only $\bar{J}^{(s)}$ and $\hat{J}^{(s)}$ strain coordinates, is hereby provided by showing the correspondence between three different choices of coordinate pairs $(\bar{J}^{(s)}, \hat{J}^{(s)})$ and three respectively compatible deformations undergone by the solid microscale RVE, $\overset{\mathrm{RVE}(s)}{\Omega_0} \rightarrow \overset{\mathrm{RVE}(s)}{\Omega}$.

These volumetric strain modes are presented in Fig. 2.1 with reference to a sample square (2D) hollow solid unit RVE with a circular cavity which is saturated by the fluid. Despite this choice for the 2D cell may suggest absence of interconnection of the void space, it is specified that this example is only intended to focus on the kinematics of the solid, and thus no assumption is made on the fluid deformation apart from the fulfillment of the saturation hypothesis, as well as no consideration is made on the permeability of the system.

On the left, the figure shows the undeformed configuration $\overset{\mathrm{RVE}(s)}{\Omega_0}$ and, on the right, three deformed configurations of the RVE, $\overset{\mathrm{RVE}(s)}{\Omega}$, corresponding to three distinct (positive) volumetric deformation modes of the solid phase: a *purely-isochoric* volumetric strain mode (a) $(\bar{J}^{(s)} > 1,\ \hat{J}^{(s)} = 1)$, a *purely-intrinsic* strain mode (b) $(\bar{J}^{(s)} = 1,\ \hat{J}^{(s)} > 1)$ and (c) an homogeneous volumetric strain mode $(\bar{J}^{(s)} = \hat{J}^{(s)} > 1)$.

In particular, in the purely isochoric strain mode, (see Fig. 2.1a), the volume of the solid phase remains unchanged being $\hat{J}^{(s)} = 1$ and $V^{(s)}_{RVE} = V^{(s)}_{0_{RVE}}$. Conversely, for the purely intrinsic strain mode, (Fig. 2.1b), although the outer boundaries of the cell are (in this specific example) held fixed, what thus would give to an observer positioned outside of the cell the apparent measure of a cell experiencing a null volume change, nevertheless the domain $\overset{\mathrm{RVE}(s)}{\Omega}(\boldsymbol{x})$ truly experiments a volume increase since $\hat{J}^{(s)} > 1$ implies $V^{(s)}_{RVE} > V^{(s)}_{0_{RVE}}$. The third illustration (Fig. 2.1c) corresponds to an homogeneous deformation describing an homothety with volume increase, being $\hat{J}^{(s)} > 1$ and hence $V^{(s)}_{RVE} > V^{(s)}_{0_{RVE}}$.

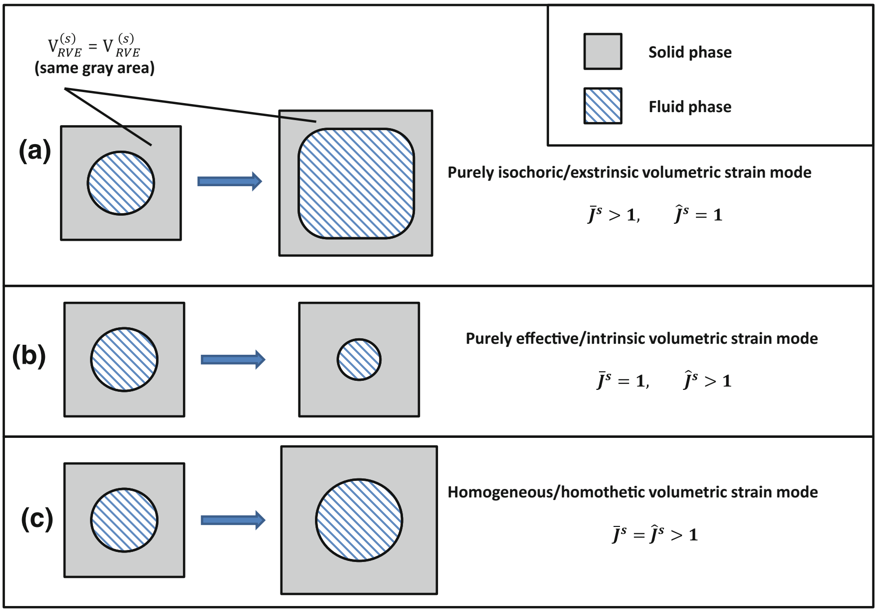


Fig. 2.1 Schematics of the three fundamental volumetric strains modes illustrated for a cell of a RVE. **a** purely isochoric strain, **b** purely intrinsic strain and **c** homogeneous homothetic strain

Current and reference volume fraction fields are related to extrinsic and intrinsic volume deformations by:

$$\phi^{(s)}(\mathbf{x}) = \frac{V^{(s)}_{RVE}(\mathbf{x})}{V_{RVE}(\mathbf{x})} = \frac{V^{(s)}_{RVE}(\mathbf{x})}{V^{(s)}_{0_{RVE}}(\mathbf{X})} \frac{V^{(s)}_{0_{RVE}}(\mathbf{X})}{V_{0_{RVE}}(\mathbf{X})} \frac{V_{0_{RVE}}(\mathbf{X})}{V_{RVE}(\mathbf{x})}\bigg|_{\left(\bar{\chi}^{(s)}\right)^{-1}} = \hat{J}^{(s)}(\mathbf{X})\Phi^{(s)}_0(\mathbf{X})\frac{1}{\bar{J}^{(s)}(\mathbf{X})} \tag{2.3}$$

where the correspondence between $\mathbf{x}$ and $\mathbf{X}$ is defined by the solid placement, viz. $\mathbf{x} = \bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X})$. Analogously, for the fluid phase a relation specular to (2.3) holds:

$$\phi^{(f)}(\mathbf{x}) = \frac{V^{(f)}_{RVE}(\mathbf{x})}{V_{RVE}(\mathbf{x})} = \frac{V^{(f)}_{RVE}(\mathbf{x})}{V^{(f)}_{0_{RVE}}(\mathbf{X})} \frac{V^{(f)}_{0_{RVE}}(\mathbf{X})}{V_{0_{RVE}}(\mathbf{X})} \frac{V_{0_{RVE}}(\mathbf{X})}{V_{RVE}(\mathbf{x})}\bigg|_{\left(\bar{\chi}^{(f)}\right)^{-1}} = \hat{J}^{(f)}(\mathbf{X})\Phi^{(f)}_0(\mathbf{X})\frac{1}{\bar{J}^{(f)}(\mathbf{X})} \tag{2.4}$$

with $\mathbf{x}$ and $\mathbf{X}$ being related by $\mathbf{x} = \bar{\boldsymbol{\chi}}^{(f)}(\mathbf{X})$.

From (2.3) one can also infer a relation for the operative macroscopic measurement of $\hat{J}^{(s)}$ in a point $\mathbf{X}$:

$$\hat{J}^{(s)}(\mathbf{X}) = \frac{\bar{J}^{(s)}(\mathbf{X})}{\Phi^{(s)}_0(\mathbf{X})}\left[1 - \phi^{(f)}(\bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X}))\right]. \tag{2.5}$$

Hence, if the field $\bar{\boldsymbol{\chi}}^{(s)}$ is among the known data, so that $\bar{J}^{(s)}$ is known, the measurement of $\hat{J}^{(s)}$ can be related to the measurement of the porosity $\phi^{(f)}$ which, if the void space is completely interconnected, can also be performed by measuring the fluid volume saturating the void space before and after deformation.

Remark 2.3 Kinematic consistency under LSP conditions—The kinematic consistency of the ($\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\phi^{(s)}$, $\phi^{(f)}$)-based description under LSP conditions is now examined. A LSP condition is exemplified in Fig. 2.2 which shows, on the left, a reference configuration where the domain $\Omega_0^{(M)}$ is partitioned in two macroscopic subregions: a porous saturated region $\Omega_0^{(s)}$, with $\Phi_0^{(s)} \neq 0$, plus a region $\Omega_0^{(f)}$, with $\Phi_0^{(s)} = 0$, which is entirely occupied by the fluid. The condition in this last region is abbreviated as LEFR as the acronym of Limit Entirely Fluid Region, while the opposite situation corresponding to $\Phi_0^{(f)} = 0$, $\phi^{(f)}(\mathbf{x}) = 0$ is denominated Limit of Vanishing Porosity (LVP). Here, and in the subsequent remarks, we comment in detail on the consistency of the ($\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\phi^{(s)}$, $\phi^{(f)}$)-based description in presence of the LEFR, referring to the model problem shown in Fig. 2.2, which does not contemplate the presence of an LVP region. Nevertheless, we specify that the considerations carried out herein by addressing in detail only the presence of LEF regions are straightforwardly translated to the opposite LVP condition, in a specular way, by considering in the relevant formulas a mere swap of indices (s) and (f). LVP conditions are also examined in closer detail in Sect. 3.6 for an isotropic medium which undergoes infinitesimal displacements.

A corresponding deformed configuration, still with a LEFR, is shown on the right with the current entirely-fluid region $\Omega^{(f)} \subset \Omega^{(M)}$ such that $\phi^{(s)}(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega^{(f)}$. Incidentally, it can be shown that $\Omega^{(s)}$ is the image of $\Omega_0^{(s)}$, as determined by the deformation field $\bar{\boldsymbol{\chi}}^{(s)}$, viz., $\Omega^{(s)} = \bar{\boldsymbol{\chi}}^{(s)}\left(\Omega_0^{(s)}\right)$, since, wherever $\Phi_0^{(s)}(\mathbf{X}) \neq 0$, then Eq. (2.3) implies that also in the deformed position one has that $\phi^{(s)}\left((\bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X})\right) \neq 0$, provided that strain $\bar{J}^{(s)}$ remains bounded. Conversely, if $\Phi_0^{(f)}(\mathbf{X}) = 1$, it is recognized from (2.4) that $\phi^{(f)}\left((\bar{\boldsymbol{\chi}}^{(f)}(\mathbf{X})\right)$ must not necessarily maintain a unit value, as a consequence of deformation. Owing to (2.2) this occurs when the moving fluid material 'meets' some solid material in the deformed configuration.

Since in region $\Omega_0^{(f)}$ there are, by hypothesis, no solid particles, the question is raised on the physical meaningfulness of the restriction of fields $\bar{\boldsymbol{\chi}}^{(s)}$ and $\hat{J}^{(s)}$ over $\Omega_0^{(f)}$. It can be easily recognized, however, that a more general definition of the domain of the deformed mixture, $\Omega^{(M)}$, preserves the full *physical traceability* of this set, for arbitrarily assigned functions $\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\Phi_0^{(s)}$ $\Phi_0^{(f)}$ defined over $\Omega_0^{(M)}$. Specifically, the definition of $\Omega^{(M)}$ is generalized as the union of the subset $\bar{\boldsymbol{\chi}}^{(s)}_{\Phi_0^{(s)}\neq 0}\left(\Omega_0^{(M)}\right) \subseteq \bar{\boldsymbol{\chi}}^{(s)}\left(\Omega_0^{(M)}\right)$, of the points of $\bar{\boldsymbol{\chi}}^{(s)}\left(\Omega_0^{(M)}\right)$ fulfilling the further condition $\Phi_0^{(s)}(\mathbf{X}) \neq 0, \mathbf{X} \in \Omega_0^{(M)}$, and of the counterpart fluid subset $\bar{\boldsymbol{\chi}}^{(f)}_{\Phi_0^{(s)}\neq 0} \subseteq \bar{\boldsymbol{\chi}}^{(f)}\left(\Omega_0^{(M)}\right)$ of the points of $\bar{\boldsymbol{\chi}}^{(f)}\left(\Omega_0^{(M)}\right)$ fulfilling $\Phi_0^{(f)} \neq 0$.

This generalization has the effect of making the restriction of $\bar{\boldsymbol{\chi}}^{(s)}$ over $\Omega_0^{(f)}$ *irrelevant* with respect to the definition of the physical deformed domain of the mixture $\Omega^{(M)}$. In regard to such an irrelevancy of the restriction of $\bar{\boldsymbol{\chi}}^{(s)}$ over $\Omega_0^{(f)}$, even if the mechanical problem is specifically addressed in the subsequent sections, it is convenient to anticipate here that, also from the more general mechanical aspect (i.e., with respect to the deduction of physically meaningful evolution equations), the restriction of $\bar{\boldsymbol{\chi}}^{(s)}$ to $\Omega_0^{(f)}$ is necessarily weighted by vanishing density fields of potential and kinetic energies for the solid phase. The null values of these energy density fields has the effect of making the evolution of the restriction of $\bar{\boldsymbol{\chi}}^{(s)}$ to $\Omega_0^{(f)}$ irrelevant also from a mechanical point of view. Actually, as it will be clear from Sect. 2.2.2 (see in particular Remarks 2.5 and 2.7), the value of the restriction of $\bar{\boldsymbol{\chi}}^{(s)}$ to $\Omega_0^{(f)}$ has no effect on the evolution of the physical system (i.e., of the remaining—physically relevant—part of field $\bar{\boldsymbol{\chi}}^{(s)}$ defined over $\Omega_0^{(s)}$, and of the fluid placement field).

To address a variety of relevant boundary conditions (see Remark 2.3), a specific partition, $\Omega_0^{(M)} = \Omega_0^{(s)} \cup \Omega_0^{(f)}$, is considered in the reference configuration of $\Omega_0^{(M)}$ into two subsets: a 'fluid-saturated solid porous' (reference) subset $\Omega_0^{(s)}$ characterized by $\Phi_0^{(s)} \neq 0$, and a (reference) fluid subset $\Omega_0^{(f)}$ with $\Phi_0^{(s)} = 0$ (i.e. where the solid phase is absent and space is completely saturated by the fluid phase alone). A sketch of this partition is shown in Fig. 2.2 with the corresponding partition in the deformed configuration $\Omega^{(M)}$. We assume that both $\Omega_0^{(s)}$ and $\Omega_0^{(f)}$ are simply connected regular domains with piecewise smooth boundaries $\partial\Omega_0^{(s)}$ and $\partial\Omega_0^{(f)}$. Also, in order to address a variety of boundary conditions and interface conditions, we assume non-null the intersection $\partial\Omega_0^{(MU)} = \partial\Omega_0^{(M)} \cap \partial\Omega_0^{(s)}$, as shown in Fig. 2.2. The remaining part of the solid boundary $\partial\Omega_0^{(s)}$ surface, which does not belong to $\partial\Omega_0^{(M)}$, $\mathscr{S}_0^{(sf)} = \partial\Omega_0^{(s)} \backslash \partial\Omega_0^{(MU)}$, is termed *free solid-fluid macroscopic interface* (see dashed line in Fig. 2.2), with its deformed counterpart $\mathscr{S}^{(sf)}$. The term 'interface' is purposefully adopted for $\mathscr{S}_0^{(sf)}$ considering that this surface is not necessarily a boundary of the physical system of the mixture, as it is not necessarily contained entirely in $\partial\Omega_0^{(M)}$, as shown in Fig. 2.2.

It is worth noting that for such surface the denomination 'boundary conditions' is purposefully avoided, since, as discussed in Remark 2.4, the set $\mathscr{S}_0^{(sf)}$ is not properly the boundary of the space domain of the mixture $\Omega_0^{(M)}$. Rather, $\mathscr{S}_0^{(sf)}$ is the macroscopic surface geometrically delimiting the physical subsystem of the solid phase.

Kinematic boundary conditions

Bilateral boundary conditions are considered at $\partial\Omega_0^{(M)}$ (see Remark 2.4 for additional comments on this hypothesis). Accordingly, the solid and fluid macroscopic placements are constrained to be coincident, over $\partial\Omega_0^{(M)}$, with the placement of the exterior environment $\bar{\boldsymbol{\chi}}^{(ext)}$, viz.:

$$\bar{\chi}^{(s)} = \bar{\chi}^{(f)} = \bar{\chi}^{(ext)}, \qquad \text{over } \partial\Omega_0^{(M)}. \tag{2.6}$$

Remark 2.4 Boundary, surface and interface conditions—Even if this section is dedicated to the presentation of the kinematic description, it is worth to anticipate, in view of the subsequently presented variational statement of the problem (see Sect. 2.2.2), a remark concerning the notions of external environment and of external boundaries of $\Omega_0^{(M)}$, $\Omega^{(M)}$, with specific relation to the standard methodology for applicating Hamilton's principle in continuum mechanics and to the hypotheses of full saturation of the physical system.

As observed by Leech and by Bedford and Drumheller, in order to achieve a correct variational statement of the physical problem, a well defined closed-mass system has to be identified as a physical entity consisting of a "*fixed aggregate or control mass*" conceptually disjoint from the external environment (although interacting with it) [8, 49]. This requirement stems from the necessity of preserving agreement with the classical methodology of Lagrange and Hamilton's principles which is characterized by the consideration of systems of fixed mass [48, 60].

In order to combine this condition of conceptually disjoint exterior/interior environments with the fulfillment of the saturation hypothesis, the further condition must be fulfilled that the solid material and the fluid material belonging to the primary traced physical system (i.e. to the system contained in domains $\Omega_0^{(M)}$ and $\Omega^{(M)}$) are never mixed, at the small scale level, with the external material which is located outside of $\Omega_0^{(M)}$ in the reference configuration. Such an hypothesis can be equivalently stated by treating the material outside of $\Omega_0^{(M)}$ as a third phase, say (e), irrespective of its actual solid or fluid nature, whose volume fraction field $\phi^{(e)}$ is either 0, in the points interior to $\Omega^{(M)}$ (with the customary respect of the condition $\phi^{(s)}(\mathbf{x}) + \phi^{(f)}(\mathbf{x}) = 1$), or equal to 1, in the points external to $\Omega^{(M)}$.

At the intersection between the deformed domain $\Omega^{(M)}$ and the deformed exterior environment, this requirement of disjoint exterior/interior environments (yet still traceable by the customary hypothesis in continuum mechanics of regular boundary $\partial\Omega^{(M)}$ endowed with tangent plane, excluded at most a subset of null measure), is contemplated by assuming that the boundary $\partial\Omega^{(M)}$ remains *sharp*, which means that for any point $\mathbf{x} \in \partial\Omega^{(M)}$ (endowed with tangent planes) one can precisely identify an inner side, saturated by the solid and the fluid phase alone, and an outer side, saturated by the exterior environment. In terms of volume fraction fields this hypothesis amounts to the condition $\phi^{(s)}(\mathbf{x}) + \phi^{(f)}(\mathbf{x}) = 1/2$, $\phi^{(e)} = 1/2$ at $\partial\Omega^{(M)}$.

Provided this absence of small-scale mixing of the material inside $\Omega_0^{(M)}$ with the material of the external environment is fulfilled everywhere, included the boundaries, no impermeability constraint is contemplated by the theory presented in this study, in that no constraint between the solid and the fluid kinematics is considered on principle. Hence, from the kinematic point of view, the fluid is completely free to flow through the solid matrix with its placement field $\bar{\boldsymbol{\chi}}^{(f)}$ being a-priori independent from $\bar{\boldsymbol{\chi}}^{(s)}$, both over the interior points of $\Omega_0^{(M)}$ as well as on its boundary $\partial\Omega_0^{(M)}$. This absence of constraints implies in particular at the boundary that $\bar{\boldsymbol{\chi}}^{(f)}$ can be different from $\bar{\boldsymbol{\chi}}^{(s)}$. In particular, the condition $\bar{\boldsymbol{\chi}}^{(f)} \neq \bar{\boldsymbol{\chi}}^{(s)}$ attained in the points of the part $\partial\Omega_0^{(MU)}$ of the boundary $\partial\Omega_0^{(M)}$ with nonvanishing solid volume fraction, (see Fig. 2.2) corresponds to the onset of *segregation* at the boundary between the solid and the fluid phase, and is of significant interest in a wide range of deformation phenomena, primarily in geomechanics.

To properly address such condition, which permits to describe unilateral phenomena such as contact loss, an account of unilateral constraints is required. To proceed incrementally with the exposition of the theory, unilateral boundary conditions are specifically dealt with in Chaps. 3 and 4, see in particular Sects. 3.2 and 4.2, while in the present chapter we focus on the linear theory so that bilateral constraints are considered at the boundary. Accordingly, the kinematic boundary equation employed in this section is $\bar{\boldsymbol{\chi}}^{(s)} = \bar{\boldsymbol{\chi}}^{(f)} = \bar{\boldsymbol{\chi}}^{(ext)}$ where $\bar{\boldsymbol{\chi}}^{(ext)}$ denotes the placement field of the external environment.

It should be duly emphasized that this hypothesis does not prevent the possibility of describing, within the present theory, the ordinary two-phase situations of fluid injection inside porous solids from external gas/liquid reservoirs. Actually, a proper description of this situation by the present variational formulation only requires a careful preliminary identification of a reference domain $\Omega_0^{(M)}$ large enough to contain all the solid and fluid material involved in the poroelastic mixing phenomenon of interest and such that this absence of small-scale mixing/diffusion with the material belonging to the external environment is fulfilled at its boundary $\partial\Omega_0^{(M)}$.

In particular, *free surfaces* delimiting a porous subsystem from an external entirely fluid region acting as a reservoir are ordinarily encompassed by the present theory as *internal* surfaces which are simply characterized by a sudden transition from $\phi^{(s)} \neq 0$ to $\phi^{(s)} = 0$, remarking that these surfaces do not belong to $\partial\Omega_0^{(M)}$. To this end, these surfaces are herein denominated *free solid-fluid macroscopic interfaces* and denoted as $\mathscr{S}_0^{(sf)}$ (see also the illustration in Fig. 2.2).

Relations (2.3) and (2.4) can be straightforwardly linearized and it can be easily recognized that they correspond to standard kinematic relations also derived in [7, 72] for infinitesimal deformations. Specifically, recalling from identity (A.33) reported in the Appendix A.2 that the relation between an infinitesimal increment of the volumetric extrinsic strain measure $d\bar{J}^{(s)}$ and infinitesimal displacement field $d\bar{\boldsymbol{\chi}}^{(s)}$ is

$$d\bar{J}^{(s)} = \bar{J}^{(s)} \frac{\partial d\bar{\chi}_i^{(s)}}{\partial x_i}, \tag{2.7}$$

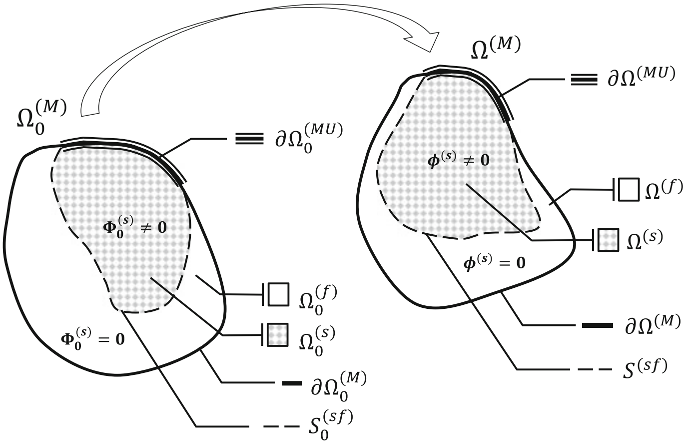


Fig. 2.2 Sketch of the partition of the macroscopic mixture domains into subsets. $\Omega_0^{(M)}$: reference domain of the mixture; $\Omega^{(M)}$: deformed domain of the mixture; $\Omega_0^{(s)}$: macroscopic reference domain of the mixture with nonvanishing volume fraction ($\Phi_0^{(s)} \neq 0$); $\Omega_0^{(f)}$: macroscopic domain of the region entirely occupied by the fluid at reference configuration ($\Phi_0^{(s)} = 0$); $\partial\Omega_0^{(M)}$: boundary o the mixture at reference configuration; $\partial\Omega_0^{(MU)}$: part of the boundary $\partial\Omega_0^{(M)}$ with nonvanishing solid volume fraction; $\mathscr{S}_0^{(sf)}$: free solid-fluid macroscopic interface. Omitted '0' subscripts denote the corresponding current configurations

the extrinsic spatial infinitesimal volumetric strain measures $\bar{e}^{(s)}$ and the intrinsic spatial infinitesimal volumetric strain measures $\hat{e}^{(s)}$ are defined as:

$$\bar{e}^{(s)} = \frac{d\bar{J}^{(s)}}{\bar{J}^{(s)}} = \frac{\partial d\bar{\chi}_i^{(s)}}{\partial x_i}, \qquad \hat{e}^{(s)} = \frac{d\hat{J}^{(s)}}{\hat{J}^{(s)}}. \tag{2.8}$$

According to these definitions, linearization of the porosity-strain relation (2.3) is computed as follows:

$$\begin{aligned} d\phi^{(s)} &= \partial_{(\bar{\chi}^{(s)},\, \hat{J}^{(s)})}\phi^{(s)}\left[d\bar{\chi}^{(s)},\, d\hat{J}^{(s)}\right] \\ &= \Phi_0^{(s)}\left(\frac{1}{\bar{J}^{(s)}}\partial_{\hat{J}^{(s)}}\hat{J}^{(s)}\left[d\hat{J}^{(s)}\right] + \hat{J}^{(s)}\partial_{\bar{\chi}^{(s)}}\left(\frac{1}{\bar{J}^{(s)}}\right)\left[d\bar{\chi}^{(s)}\right]\right) \\ &= \Phi_0^{(s)}\left(\frac{1}{\bar{J}^{(s)}}\partial_{\hat{J}^{(s)}}\hat{J}^{(s)}\left[d\hat{J}^{(s)}\right] - \hat{J}^{(s)}\frac{1}{\left(\bar{J}^{(s)}\right)^2}\partial_{\bar{\chi}^{(s)}}\bar{J}^{(s)}\left[d\bar{\chi}^{(s)}\right]\right) \\ &= \Phi_0^{(s)}\left(\frac{\hat{J}^{(s)}}{\bar{J}^{(s)}}\hat{e}^{(s)} - \frac{\hat{J}^{(s)}}{\bar{J}^{(s)}}\bar{e}^{(s)}\right). \end{aligned} \tag{2.9}$$

When the deformation is infinitesimal, so that $\bar{J}^{(s)} \simeq 1$ and $\hat{J}^{(s)} \simeq 1$, the previous relation yields:

$$d\phi^{(s)} = \Phi_0^{(s)}\left(\hat{e}^{(s)} - \bar{e}^{(s)}\right) \tag{2.10}$$

which corresponds to Eq. (61) derived in [72].

A completely analogous computation of the linearized form of (2.4) yields for the fluid phase:

$$d\phi^{(f)} = \Phi_0^{(f)} \left(\hat{e}^{(f)} - \bar{e}^{(f)}\right), \tag{2.11}$$

with definitions for $\bar{e}^{(s)}$ and $\hat{e}^{(s)}$ analogous to those in (2.8):

$$\bar{e}^{(f)} = \frac{\partial d\bar{\chi}_i^{(f)}}{\partial x_i}, \qquad \hat{e}^{(f)} = \frac{d\hat{J}^{(f)}}{\hat{J}^{(f)}}. \tag{2.12}$$

Equations (2.3) and (2.4) are written in terms of relation between fields as follows:

$$\phi_{\mathbf{x}}^{(s)} \circ \bar{\boldsymbol{\chi}}^{(s)} = \frac{\hat{J}^{(s)}}{\bar{J}^{(s)}} \Phi_0^{(s)}, \qquad \phi_{\mathbf{x}}^{(f)} \circ \bar{\boldsymbol{\chi}}^{(f)} = \frac{\hat{J}^{(f)}}{\bar{J}^{(f)}} \Phi_0^{(f)}, \tag{2.13}$$

where an $\mathbf{x}$ subscript is used to denote spatial fields, all remaining fields with no subscripts are defined in the reference configuration, and symbol '$\circ$' indicates function composition. From (2.13) one infers:

$$\phi_{\mathbf{x}}^{(s)} = \frac{\hat{J}^{(s)}}{\bar{J}^{(s)}} \Phi_0^{(s)} \circ \left(\bar{\boldsymbol{\chi}}^{(s)}\right)^{-1}, \qquad \phi_{\mathbf{x}}^{(f)} = \frac{\hat{J}^{(f)}}{\bar{J}^{(f)}} \Phi_0^{(f)} \circ \left(\bar{\boldsymbol{\chi}}^{(f)}\right)^{-1}, \tag{2.14}$$

whereby the saturation condition (2.2) is expressed in terms of field relations in the following form termed *Finite deformation saturation constraint*:

$$\frac{\hat{J}^{(s)}}{\bar{J}^{(s)}} \Phi_0^{(s)} \circ \left(\bar{\boldsymbol{\chi}}^{(s)}\right)^{-1} + \frac{\hat{J}^{(f)}}{\bar{J}^{(f)}} \Phi_0^{(f)} \circ \left(\bar{\boldsymbol{\chi}}^{(f)}\right)^{-1} = 1. \tag{2.15}$$

The same saturation condition expressed in (2.15) can be also written in terms of field values for a space point $\mathbf{x}$ in the following way:

$$\frac{\hat{J}^{(s)}\left(\mathbf{X}^{(s)}\right)}{\bar{J}^{(s)}\left(\mathbf{X}^{(s)}\right)} \Phi_0^{(s)}\left(\mathbf{X}^{(s)}\right) + \frac{\hat{J}^{(f)}\left(\mathbf{X}^{(f)}\right)}{\bar{J}^{(f)}\left(\mathbf{X}^{(f)}\right)} \Phi_0^{(f)}\left(\mathbf{X}^{(f)}\right) = 1 \tag{2.16}$$

specifying that $\mathbf{x}$ is the common image of the reference points $\mathbf{X}^{(s)}$ and $\mathbf{X}^{(f)}$, respectively via $\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\boldsymbol{\chi}}^{(f)}$, viz.:

$$\mathbf{x} = \bar{\boldsymbol{\chi}}^{(s)}\left(\mathbf{X}^{(s)}\right) = \bar{\boldsymbol{\chi}}^{(f)}\left(\mathbf{X}^{(f)}\right). \tag{2.17}$$

Relationship (2.15) represents the saturation condition in relation to the volumetric deformation measures. Such relation implies that, under the saturation hypothesis and for a given reference configuration defined by $\Omega_0^{(M)}$, $\Phi_0^{(s)}$ and $\Phi_0^{(f)}$, the quantities $\hat{J}^{(s)}$, $\bar{J}^{(s)}$, $\hat{J}^{(f)}$, and $\bar{J}^{(f)}$ are not independent.

2.2.2 Variational Formulation

In this section, the boundary value problem is constructed on a variational basis. Specifically, Sect. 2.2.2.1 details the selection of primary kinematic descriptors, while the integral weak statement of the least-Action condition is reported in Sect. 2.2.3. The general set of strong-form Euler Lagrange equations is subsequently derived in Sect. 2.2.4. For readability, some of the technical passages of Sects. 2.2.3 and 2.2.4 have been moved to Appendices A.3 and B.

In Sect. 2.2.4.1, leaving unmodified the boundary value problem, supplementary strong form conditions are derived, from the general least Action condition, to enucleate the equations applying over free solid-fluid surfaces.

In Sect. 2.2.5, an additional interaction term is included in the mechanical statement of the problem to account for drag forces representing the action phenomenologically described by Darcy and Forchheimer laws.

A kinematic linearization is carried out in Sect. 3.2 and, in this context of infinitesimal displacements, general medium-independent stress partitioning laws are derived in Sect. 2.2.7.1.

2.2.2.1 Selection of Unconstrained Kinematic Descriptors

In this subsection, a variational formulation is developed based on the configuration description detailed in Sect. 2.2.1. To this end, attention is taken in properly selecting a set of independent kinematic descriptor fields which are not constrained to respect further equations of saturation constraints and mass balances. This choice represents a precise element of distinction of the theory presently developed from the variational formulation proposed by Bedford and Drumheller in [7]. In particular, the absence of constraints for the primary descriptors in VMTPM does not require Lagrange multipliers for defining stress quantities. Accordingly, fields $\bar{\boldsymbol{\chi}}^{(s)}$, $\hat{J}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, and $\hat{J}^{(f)}$ cannot be taken altogether as primary kinematic descriptors since quantities $\hat{J}^{(s)}$, $\bar{J}^{(s)}$, $\hat{J}^{(f)}$, and $\bar{J}^{(f)}$ are mutually related by the saturation constraint (2.15). Several options are available for selecting a suitable subset of three kinematically independent descriptors of the change of configuration of the mixture. Hereby, we choose $\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, and $\hat{J}^{(s)}$. Such a choice retains both descriptors of the solid phase deformation while only the macroscopic displacements are included for the fluid phase. Denoting by $\left[t_0,\ t_f\right]$ the time interval of interest, the domain of these fields is the set $\Omega_0^{(M)} \times \left[t_0,\ t_f\right]$. Once these three fields are specified, the intrinsic strain in a point $\mathbf{x} = \bar{\boldsymbol{\chi}}^{(s)}\left(\mathbf{X}^{(s)}\right) = \bar{\boldsymbol{\chi}}^{(f)}\left(\mathbf{X}^{(f)}\right)$, which is the common image of two reference points $\mathbf{X}^{(s)}$ and $\mathbf{X}^{(f)}$ (respectively via $\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\boldsymbol{\chi}}^{(f)}$) can be computed on account of (2.16):

$$
\hat{J}^{(f)}\left(\mathbf{X}^{(f)}\right) = \frac{\bar{J}^{(f)}\left(\mathbf{X}^{(f)}\right)}{\Phi_0^{(f)}\left(\mathbf{X}^{(f)}\right)}\left(1 - \Phi_0^{(s)}\left(\mathbf{X}^{(s)}\right)\frac{\hat{J}^{(s)}\left(\mathbf{X}^{(s)}\right)}{\bar{J}^{(s)}\left(\mathbf{X}^{(s)}\right)}\right). \tag{2.18}
$$

In this way, $\hat{J}^{(f)}$ is treated as a field indirectly related to $\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, and $\hat{J}^{(s)}$ via the saturation constraint. It can be recognized that, provided $\Phi_0^{(f)} \neq 0$, this choice of descriptors is well posed, and the kinematic independence of $\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, and $\hat{J}^{(s)}$ is preserved. The singular condition of null porosity corresponds to a unit value of the solid volume fraction, and is referred to as Limit of Vanishing Porosity (LVP). LVP represents a singular limit of the present theory, since the fluid phase disappears and relation (2.3) yields coincidence of $\bar{J}^{(s)}$ and $\hat{J}^{(s)}$, so that their independence is lost. For this reason, although LVP does not alter the mathematical and physical consistency of the present theory (see in this respect Remarks 2.1 and 2.3), LVP requires additional considerations, and is specifically addressed in Sect. 3.6. Herein LVP is excluded, assuming $\Phi_0^{(f)} \neq 0$ over $\Omega_0^{(M)}$. Conversely, attention is focused on the (specular) issue of the existence with the mixture of an Entirely Fluid Region (EFR) $\Omega_0^{(f)} \subset \Omega_0^{(M)}$ (see Fig. 2.2), where $\Phi_0^{(f)} = 1$.

Accordingly, the argument functions of the considered Lagrange function $L_0^{(M)}$ are the descriptor space-time fields $\bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X}, t)$, $\bar{\boldsymbol{\chi}}^{(f)}(\mathbf{X}, t)$, and $\hat{J}^{(s)}(\mathbf{X}, t)$ of the mixture, viz.:

$$L_0^{(M)} = L_0^{(M)}\left(\bar{\boldsymbol{\chi}}^{(s)},\ \bar{\boldsymbol{\chi}}^{(f)},\ \hat{J}^{(s)}\right). \tag{2.19}$$

The use of primary kinematic fields which are deliberately all based in the reference configuration follows a statement of the least-action problem also considered by Leech [49] and Bedford and Drumheller [7], and expresses the Action functional in relation to the set of possibile evolution histories of a *fixed mass* of mixture. This particular choice, besides being compliant with the originary application of the least-Action principle to Langrangian systems (which have fixed mass), yields the desirable effects of making mass balances uncoupled from the equilibrium equations to be fulfilled. Actually, the densities in the deformed configuration, $\bar{\rho}^{(s)}$, $\bar{\rho}^{(f)}$, are secondary variables which are related to the reference densities, $\bar{\rho}_0^{(s)}$, $\bar{\rho}_0^{(f)}$, by:

$$\bar{\rho}^{(s)} = \frac{\bar{\rho}_0^{(s)}}{\bar{J}^{(s)}}, \qquad \bar{\rho}^{(f)} = \frac{\bar{\rho}_0^{(f)}}{\bar{J}^{(f)}} \tag{2.20}$$

and that can, as such, be post-computed once the least-action boundary value problem is solved.

The condition expressed by (2.18) can be rewritten in terms of a relation of functional dependence between fields. To this end we introduce the definition:

$$\phi_{\bar{\boldsymbol{\chi}}^{(\beta)}}^{(\alpha)} = \phi_{\mathbf{x}}^{(\alpha)} \circ \bar{\boldsymbol{\chi}}^{(\beta)} \tag{2.21}$$

which represents the volume fraction of phase α in the deformed configuration at the space point $\mathbf{x}$ corresponding to point $\mathbf{X}^{(\beta)}$ as a result of the deformation $\bar{\boldsymbol{\chi}}^{(\beta)}$. According to this definition, we have:

$$\phi^{(s)}_{\bar{\chi}^{(s)}} := \phi^{(s)}_{\mathbf{x}} \circ \bar{\chi}^{(s)} = \frac{\hat{J}^{(s)}}{\bar{J}^{(s)}} \Phi^{(s)}_0 \tag{2.22}$$

and

$$\phi^{(s)}_{\bar{\chi}^{(f)}} = \phi^{(s)}_{\mathbf{x}} \circ \bar{\chi}^{(f)} = \phi^{(s)}_{\bar{\chi}^{(s)}} \circ \left(\bar{\chi}^{(s)}\right)^{-1} \circ \bar{\chi}^{(f)} \tag{2.23}$$

whereby (2.18) is expressed as the following relation between fields:

$$\hat{J}^{(f)}_{sat\,\bar{\chi}^{(f)}} := \frac{\bar{J}^{(f)}}{\Phi^{(f)}_0} \left(1 - \phi^{(s)}_{\bar{\chi}^{(f)}}\right). \tag{2.24}$$

2.2.2.2 Mechanical Framework and Form of the Action Functional

The mechanical framework is hereby derived proceeding with a purely variational deduction of the governing equations by following standard methodologies for the application of Hamilton's least Action principle in continuum mechanics [7, 9]. A purely mechanical statement of the problem is first considered, excluding solid-fluid interaction phenomena which require a thermodynamic treatment. Darcy forces, describing solid-fluid interactions, will be subsequently added to the resulting purely mechanical Euler-Lagrange equations (see Sect. 2.2.5) once these will have been derived for the purely mechanical problem.

Constitutive response—Medium independence

From the constitutive point of view, in order to derive governing equations of broadest generality, a completely general functional dependence of the Action functional upon the primary descriptors is considered. This choice allows to retrieve the most general set of equations which apply irrespective of the underlying constitutive responses of the solid phase and of the fluid phase (and hence also irrespective of the degree of anisotropy of the solid phase and from the degree of nonlinearity of the solid and fluid constitutive responses). This feature of the formulation is referred to as *medium independence*.

A purely mechanical theory is herein considered (see Remark 2.13 for a discussion on the extension of the formulation encompassing irreversible processes). Accordingly, the constitutive relations are defined in terms of functions of strain energy densities per unit reference space $\bar{\psi}^{(s)}_0$ and $\bar{\psi}^{(f)}_0$. These are related to the true, or effective, reference densities of strain energy, $\hat{\psi}^{(s)}_0$ and $\hat{\psi}^{(f)}_0$, by:

$$\bar{\psi}^{(s)}_0 = \Phi^{(s)}_0 \hat{\psi}^{(s)}_0, \qquad \bar{\psi}^{(f)}_0 = \Phi^{(f)}_0 \hat{\psi}^{(f)}_0. \tag{2.25}$$

Remark 2.5 Vanishing of solid energy densities over $\Omega^{(f)}_0$—It is immediately inferred from (2.25) that, if $\hat{\psi}^{(s)}_0$ is a bounded function (what is a tacit customary hypothesis), the energy density $\bar{\psi}^{(s)}_0$ is also zero in the region $\Omega^{(f)}_0$ where

$\Phi_0^{(s)} = 0$. A similar consideration holds also for the kinetic energy. This has the consequence of making nil all solid stress measures and inertial properties in the region $\Omega_0^{(f)}$, and of making the restriction of fields $\bar{\boldsymbol{\chi}}^{(s)}$ and $\hat{J}^{(s)}$ to $\Omega_0^{(f)}$ irrelevant from a mechanical point of view. Actually, although field $\bar{\boldsymbol{\chi}}^{(s)}$ is on principle contemplated to be defined, from a kinematic point of view, over $\Omega_0^{(f)}$ the restriction of $\bar{\boldsymbol{\chi}}^{(s)}$ to $\Omega_0^{(f)}$ has no mechanical influence on the evolution of the physical system.

For both solid and fluid phases, standard local constitutive responses of first-gradient type (i.e. depending upon the first gradient of the macroscopic placement) are considered. Accordingly, for the solid phase, a generic dependence of type:

$$\bar{\psi}_0^{(s)}(\mathbf{X}) = \bar{\psi}_0^{(s)}\left(\mathbf{X},\ \left.\frac{\partial \bar{\boldsymbol{\chi}}^{(s)}}{\partial \mathbf{X}}\right|_{\mathbf{X}},\ \hat{J}^{(s)}(\mathbf{X})\right) \tag{2.26}$$

is considered, where dependence upon $\mathbf{X}$ is introduced to address a constitutive response which can be a macroscopically nonhomogeneous function of space. Non-homogeneity of the constitutive response of the solid phase is specifically accounted for, in order to address the possible space nonhomogeneity of $\Phi_0^{(s)}$. In particular, the function $\bar{\psi}_0^{(s)}$ must vanish in the points $\mathbf{X} \in \Omega_0^{(f)} \subset \Omega_0^{(M)}$, where the solid is absent and space is completely saturated by the fluid alone being $\Phi_0^{(s)} = 0$.

For the fluid phase, a generic inviscid behavior is considered and it is assumed that in the reference configuration the fluid is in an homogeneous state, so that the following strain energy density can be adopted:

$$\hat{\psi}_0^{(f)}(\mathbf{X}) = \hat{\psi}_0^{(f)}\left(\hat{J}^{(f)}(\mathbf{X})\right). \tag{2.27}$$

Kinetic energy

For the reference field of density of kinetic energy of the solid phase, $\bar{\kappa}_0^{(s)}$, in addition to the quadratic term of translational kinetic energy associated with the solid velocity, $\bar{\mathbf{v}}^{(s)} = \partial \bar{\boldsymbol{\chi}}^{(s)}(\mathbf{X},\ t)/\partial t$, a further microinertia term $\bar{\kappa}_{0\,add}^{(s)}$ is considered:

$$\bar{\kappa}_0^{(s)}(\mathbf{X}) = \frac{1}{2}\bar{\rho}_0^{(s)}||\bar{\mathbf{v}}^{(s)}(\mathbf{X})||^2 + \bar{\kappa}_{0\,add}^{(s)}. \tag{2.28}$$

Microinertia terms are essentially introduced to retrieve kinetic additional terms in the governing equations which are comparable to those considered in [7, 21, 64] (see also Appendix C of [8, 74]). However, it should be remarked that the addition of $\bar{\kappa}_{0\,add}^{(s)}$-related terms entails some form of constitutive or microscale assumption so that, in some respects, the theory resulting from their introduction lies outside the sought medium-independent treatment. For this reason, while the computation of microinertia related terms is included for completeness, the resulting terms will be subsequently removed from a final summary of the purely medium-independent part of the theory. In particular, the microinertia function is defined according to the expression employed in [74] so as to vanish in presence of a rate of deformation having in $\mathbf{X}$ the shape of an homothety (i.e., when $\dot{\hat{J}}^{(s)} = \dot{\bar{J}}^{(s)}$). To this end, it is set $\bar{\kappa}^{(s)}_{0\,add} = \bar{\kappa}^{(s)}_{0\,add}\left(\dot{\tilde{J}}^{(s)}(\mathbf{X})\right)$ with $\tilde{J}^{(s)}(\mathbf{X}) = \bar{J}^{(s)}(\mathbf{X}) - \hat{J}^{(s)}(\mathbf{X})$. For simplicity, also for $\bar{\kappa}^{(s)}_{0\,add}$ a quadratic expression is chosen:

$$\bar{\kappa}^{(s)}_{0\,add} = \frac{1}{2}\bar{\rho}^{(s)}_{add.0}\left(\dot{\tilde{J}}^{(s)}\right)^2 = \frac{1}{2}\bar{\rho}^{(s)}_{add.0}\left(\dot{\bar{J}}^{(s)} - \dot{\hat{J}}^{(s)}\right)^2, \tag{2.29}$$

where $\bar{\rho}^{(s)}_{add.0}$ is a microinertia density parameter assumed henceforth, for simplicity, a constant reference field.

For the fluid phase, expressions analogous to (2.28) and (2.29) are considered:

$$\bar{\kappa}^{(f)}_{0\,add} = \frac{1}{2}\bar{\rho}^{(f)}_{add.0}\left(\dot{\tilde{J}}^{(f)}\right)^2 = \frac{1}{2}\bar{\rho}^{(f)}_{add.0}\left(\dot{\bar{J}}^{(f)} - \dot{\hat{J}}^{(f)}\right)^2 \tag{2.30}$$

$$\bar{\kappa}^{(f)}_0(\mathbf{X}) = \frac{1}{2}\bar{\rho}^{(f)}_0||\bar{\mathbf{v}}^{(f)}(\mathbf{X})||^2 + \bar{\kappa}^{(f)}_{0\,add}. \tag{2.31}$$

The remaining choices for the definition of the Lagrange function are the standard ones. Accordingly, denoting by $T^{(M)}_0$ the kinetic energy of the mixture and the potential energy by $U^{(M)}_0$, we write:

$$L^{(M)}_0 = T^{(M)}_0 - U^{(M)}_0. \tag{2.32}$$

The potential energy of the mixture $U^{(M)}_0$ is divided into three contributions: the solid phase strain energy $U^{(s)}_0$, the fluid phase strain energy $U^{(f)}_0$, and the potential energy due to external actions U^{ext}_0:

$$U^{(M)}_0 = U^{(s)}_0 + U^{(f)}_0 + U^{ext}_0. \tag{2.33}$$

According to (2.26), the functional dependence of the total potential energy of the solid phase turns out to be:

$$U^{(s)}_0\left(\bar{\chi}^{(s)},\, \hat{J}^{(s)}\right) = \int_{\Omega^{(M)}_0} \bar{\psi}^{(s)}_0\left(\bar{\chi}^{(s)},\, \hat{J}^{(s)}\right) dV_0. \tag{2.34}$$

The functional dependence for the strain energy of the fluid phase $U^{(f)}_0$ is more complex since $\hat{J}^{(f)}$ is not among the primary descriptor fields and this term depends upon all three primary kinematic fields via (2.24). By combining (2.24) and (2.27), $U^{(f)}_0$ is provided by the following integral:

$$U^{(f)}_0\left(\bar{\chi}^{(s)},\, \bar{\chi}^{(f)},\, \hat{J}^{(s)}\right) = \int_{\Omega^{(M)}_0} \Phi^{(f)}_0 \cdot \left[\hat{\psi}^{(f)}_0 \circ \hat{J}^{(f)}_{sat\,\bar{\chi}^{(f)}}\left(\bar{\chi}^{(s)},\, \bar{\chi}^{(f)},\, \hat{J}^{(s)}\right)\right] dV_0. \tag{2.35}$$

In (2.35), $\hat{\psi}_0^{(f)}$ is a predetermined constitutive function which does not depend from the primary fields (although the value $\hat{\psi}_0^{(f)}(\mathbf{X})$ attained by $\hat{\psi}_0^{(f)}$ in a point $\mathbf{X}$ depends from the kinematic descriptors $\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\hat{J}^{(s)}$, via the function combination $\hat{\psi}_0^{(f)} \circ \hat{J}^{(f)}_{sat\bar{\boldsymbol{\chi}}^{(f)}}$). Similarly, $\Phi_0^{(f)}$, as a reference configuration field, does not depend from the kinematic descriptors.

The potential U_0^{ext} is decomposed into the sum of a potential energy term $U_{\Omega 0}^{ext}$, due to external actions associated with the state of the interior points of $\Omega_0^{(M)}$, plus a second term $U_{\partial\Omega 0}^{ext}$ associated with the external actions across the boundary $\partial\Omega_0^{(M)}$, viz.:

$$U_0^{ext} = U_{\Omega 0}^{ext} + U_{\partial\Omega 0}^{ext}, \tag{2.36}$$

where

$$U_{\Omega 0}^{ext} = \int_{\Omega_0^{(M)}} \bar{\psi}_{0ext}^{(s)}(\bar{\boldsymbol{\chi}}^{(s)})\, dV_0 + \int_{\Omega_0^{(M)}} \bar{\psi}_{0ext}^{(f)}(\bar{\boldsymbol{\chi}}^{(f)})\, dV_0 \tag{2.37}$$

and

$$U_{\partial\Omega 0}^{ext} = \int_{\partial\Omega_0^{(M)}} \bar{\psi}_{0ext}^{(\partial)}\left(\bar{\boldsymbol{\chi}}^{(ext)}\right)\, dA_0, \tag{2.38}$$

being $\bar{\boldsymbol{\chi}}^{(ext)}$ the placement field of the external environment at the boundary $\partial\Omega_0^{(M)}$ and $\bar{\psi}_{0ext}^{(\partial)}$ the associated potential energy (of the external environment).

The kinetic energy of the mixture is split into solid ($T^{(s)}$) and fluid ($T_0^{(f)}$) contributions, viz.:

$$T_0^{(M)} = T^{(s)} + T_0^{(f)}, \tag{2.39}$$

with

$$T^{(s)}\left(\dot{\bar{\boldsymbol{\chi}}}^{(s)},\ \dot{\hat{J}}^{(s)}\right) = \int_{\Omega_0^{(M)}} \bar{\kappa}_0^{(s)}\left(\dot{\bar{\boldsymbol{\chi}}}^{(s)},\ \dot{\hat{J}}^{(s)}\right) dV_0, \tag{2.40}$$

$$T_0^{(f)}\left(\dot{\bar{\boldsymbol{\chi}}}^{(f)}\right) = \int_{\Omega_0^{(M)}} \bar{\kappa}_0^{(f)}\left(\dot{\bar{\boldsymbol{\chi}}}^{(f)},\ \hat{J}^{(f)}\right) dV_0, \tag{2.41}$$

where $\bar{\kappa}_0^{(s)}$ and $\bar{\kappa}_0^{(f)}$ are specified by (2.28) and (2.31).

External volume forces

In the reference configuration, the external volume forces, $\bar{\mathbf{b}}_0^{(s,ext)}$ and $\bar{\mathbf{b}}_0^{(f,ext)}$, are related to the external potential energy by:

$$\bar{\mathbf{b}}_0^{(s,ext)} := -\frac{\partial \bar{\psi}_{0ext}^{(s)}}{\partial \bar{\boldsymbol{\chi}}^{(s)}}, \qquad \bar{\mathbf{b}}_0^{(f,ext)} := -\frac{\partial \bar{\psi}_{0ext}^{(f)}}{\partial \bar{\boldsymbol{\chi}}^{(f)}}. \tag{2.42}$$

Spatial fields of external volume forces $\bar{\mathbf{b}}^{(s,ext)}$ and $\bar{\mathbf{b}}^{(f,ext)}$ are provided by the push-forward of the vector fields $\bar{\mathbf{b}}_0^{(s,ext)}$ and $\bar{\mathbf{b}}_0^{(f,ext)}$:

$$\bar{\mathbf{b}}^{(s,ext)} = \frac{1}{\bar{J}^{(s)}}\bar{\mathbf{b}}_0^{(s,ext)}, \qquad \bar{\mathbf{b}}^{(f,ext)} = \frac{1}{\bar{J}^{(f)}}\bar{\mathbf{b}}_0^{(f,ext)}. \tag{2.43}$$

Definition of internal stress measures

The definitions of stress measures are introduced by an extension to the present two-phase framework of the standard mathematical definitions employed in the context of finite-deformation elasticity for single-continuum problems.

Since the theory herein described is purely mechanical, finite stress measures are defined in the usual form in terms of work-association as Lie derivatives [38, 58] of the density of strain energy of the solid and the fluid phase with respect to the associated strain measures. These definitions are hereby given. In the next subsection, their relation with the Cauchy stress tensor of the solid phase and the interstitial fluid pressure p are recalled.

For the solid phase, denoting by $\bar{\mathbf{F}}^{(s)} = \frac{\partial \bar{\boldsymbol{\chi}}^{(s)}}{\partial \mathbf{X}}$ the deformation gradient, the work-conjugate stress measures are:

$$\check{P}_{iK}^{(s)} = \frac{\partial \bar{\psi}_0^{(s)}}{\partial \bar{F}_{iK}^{(s)}}, \qquad \hat{\Pi}^{(s)} = -\frac{\partial \bar{\psi}_0^{(s)}}{\partial \hat{J}^{(s)}}. \tag{2.44}$$

The tensor $\check{\mathbf{P}}^{(s)}$ is a two-point stress tensor termed *extrinsic first Piola-Kirchhoff stress tensor*. The scalar quantity $\hat{\Pi}^{(s)}$ is the material (i.e., reference) stress measure work-associated with $\hat{J}^{(s)}$, denominated (reference) intrinsic solid pressure. The negative sign in the definition of $\hat{\Pi}^{(s)}$ is purposefully introduced to treat the stress measure associated with intrinsic strains as a pressure quantity in a way similar to the treatment employed in [75], limited therein to the context of infinitesimal deformations.

The spatial counterparts of (2.44) are the result of push-forward operations of the quantities in (2.44) along the deformed solid configuration defined by the solid placement $\bar{\boldsymbol{\chi}}^{(s)}$:

$$\check{\sigma}_{ij}^{(s)} = \frac{1}{\bar{J}^{(s)}}\check{P}_{iK}^{(s)}\frac{\partial \bar{\chi}_j^{(s)}}{\partial X_K} = \frac{1}{\bar{J}^{(s)}}\frac{\partial \bar{\psi}_0^{(s)}}{\partial \bar{F}_{iK}^{(s)}}\frac{\partial \bar{\chi}_j^{(s)}}{\partial X_K}, \tag{2.45}$$

$$\hat{p}^{(s)} = \frac{1}{\bar{J}^{(s)}}\hat{\Pi}^{(s)} = -\frac{1}{\bar{J}^{(s)}}\frac{\partial \bar{\psi}_0^{(s)}}{\partial \hat{J}^{(s)}}, \tag{2.46}$$

and are the spatial forms of the extrinsic stress tensor and of the intrinsic solid pressure, respectively.

The scalar stress measure for the fluid phase are similarly defined as the Lie derivatives of the strain energy of the fluid phase with respect to the primary strain measure of the fluid phase $\hat{J}^{(f)}$. Two alternate stress measures of the fluid phase (in pressure form) associated with the reference configuration are the following:

$$\hat{\hat{\Pi}}^{(f)} = -\frac{\partial \hat{\psi}_0^{(f)}}{\partial \hat{J}^{(f)}}, \qquad \hat{\Pi}^{(f)} = -\frac{\partial \bar{\psi}_0^{(f)}}{\partial \hat{J}^{(f)}} \tag{2.47}$$

which, owing to (2.25), are related by $\hat{\Pi}^{(f)} = \Phi_0^{(f)} \hat{\hat{\Pi}}^{(f)}$. The spatial counterparts of (2.47) are again provided by a push forward operation:

$$\hat{\hat{p}}^{(f)} = \frac{1}{\hat{J}^{(f)}} \hat{\hat{\Pi}}^{(f)}, \qquad \hat{p}^{(f)} = \frac{1}{\hat{J}^{(f)}} \hat{\Pi}^{(f)}. \tag{2.48}$$

Remark 2.6 Mechanical interpretation of the stress measures—Within a purely mechanical formulation, the use of definitions (2.45), (2.46) and (2.48), based on work association with primary strain measures and on the concept of Lie derivative, represents a standard method in solid continuum mechanics for defining stress measures. Such definitions, which are self-consistent in that they require no additional mechanical arguments, ensure that all stress measures above introduced are physically founded and mathematically well-posed [58].

It is however convenient to report some additional side-considerations in order to gain physical and engineering insight on the mechanical meaning of the stress quantities above introduced to elucidate their relation with the standard notions of (macroscopic) Cauchy stress tensor for the solid phase $\boldsymbol{\sigma}^{(s)}$, and of interstitial fluid pressure p.

As a general consideration, insights on $\hat{\hat{\Pi}}^{(f)}$, $\hat{\Pi}^{(s)}$, and $\check{\mathbf{P}}$ are obtained by examining the role of these quantities in the momentum balance equations and boundary conditions derived next (see Sect. 2.2.4), and their relation with boundary data (2.79). In particular, in Chap. 4, a thorough mechanical interpretation of internal stress quantities is gained by applying the governing equations and boundary conditions in the analysis of experimental tests including a comprehensive set of loading and drainage conditions at the boundary. These subsequently reported analyses provide explicit relations with the primary measured physical stress and loading quantities during the experiments (i.e., the macroscopic strain, the fluid pressure, and the stress at the load cell).

It is however possible to anticipate that in a point $\mathbf{x}$ the pressure quantity $\hat{\hat{p}}^{(f)}$ corresponds to the interstitial fluid pressure p under the reasonable hypothesis that the state of the interstitial fluid is *microscopically uniform* at such point. The term 'microscopically uniform' in a point $\mathbf{x}$ refers to the condition that the state of the fluid has negligible fluctuations in the interstitial space of the deformed RVE $\overset{\text{RVE}(f)}{\Omega}(\mathbf{x})$ centered in $\mathbf{x}$. In this condition $\psi^{(f)}$ is uniform in $\overset{\text{RVE}(f)}{\Omega}(\mathbf{x})$, and the microscale Jacobian $J^{(f)}$ and microscale strain energy density $\psi_0^{(f)}$ are uniform inside $\overset{\text{RVE}(f)}{\Omega}_0(\mathbf{X})$. As a consequence of such microscale uniformity, macroscopic and microscale quantities coincide inside $\overset{\text{RVE}(f)}{\Omega}_0(\mathbf{X})$:

$$\hat{J}^{(f)} = J^{(f)}, \qquad \hat{\psi}_0^{(f)} = \psi_0^{(f)}, \tag{2.49}$$

where the left hand sides are macroscopic quantities, and the right hand sides are microscale quantities. The direct consequence of (2.49) is that the macroscopic fluid pressure $\hat{\hat{p}}^{(f)}$, defined as:

$$\hat{\hat{p}}^{(f)} = -\frac{1}{\hat{J}^{(f)}} \frac{\partial \hat{\psi}_0^{(f)}}{\partial \hat{J}^{(f)}} = \frac{1}{\hat{J}^{(f)}} \hat{\hat{\Pi}}^{(f)}, \tag{2.50}$$

and the interstitial fluid pressure in the interior points of $\overset{\text{RVE}(f)}{\Omega_0}(\mathbf{X})$, defined instead as:

$$p = -\frac{1}{J^{(f)}} \frac{\partial \psi_0^{(f)}}{\partial J^{(f)}} \tag{2.51}$$

become coincident, viz.:

$$\hat{\hat{p}}^{(f)} \equiv p. \tag{2.52}$$

On account of the consequences of microscopic uniformity, quantities $\hat{\hat{p}}^{(f)}$ and p are both simply referred to as 'fluid pressure'. Also, the relation of the extrinsic stress tensor $\breve{\boldsymbol{\sigma}}^{(s)}$ with the ordinary Cauchy stress tensor $\boldsymbol{\sigma}^{(s)}$ can be elucidated based on considerations analogous to those reported in [77] for the special case of infinitesimal kinematics: within VMTPM, $\breve{\boldsymbol{\sigma}}^{(s)}$ is mechanically defined by (2.45) based on a more general kinematic definition where the deformation state in a point is defined by the pair ($\bar{\mathbf{F}}^{(s)}$, $\hat{J}^{(s)}$). Accordingly, a computation of the partial derivative $\frac{\partial \bar{\psi}_0^{(s)}}{\partial \bar{\mathbf{F}}^{(s)}}$ in (2.45) is carried out, as such, by keeping fixed the other degree of freedom $\hat{J}^{(s)}$. The extrinsic stress tensor $\breve{\boldsymbol{\sigma}}^{(s)}$ is thus recognized to be work-associated with an infinitesimal volumetric deformation mode where no intrinsic volume variation of the solid phase takes place, i.e. with an isochoric strain mode. For this reason, the alternative denomination of *isochoric stress tensor* for identifying $\breve{\boldsymbol{\sigma}}^{(s)}$ is used in [77].

As observed in [77], in absence of interstitial fluid (i.e. when the fluid pressure is null $p = 0$), the extrinsic stress tensor $\breve{\boldsymbol{\sigma}}^{(s)}$ corresponds to the standard notion of macroscopic Cauchy stress tensor $\boldsymbol{\sigma}^{(s)}$ for porous media with voids. This is recognized by conveniently anticipating Eqs. (2.123), (2.125), and (2.126) (rationally deduced in the sequel) which, for $p = 0$, specialize to:

$$p = 0 \qquad \frac{\partial \breve{\sigma}_{ij}^{(s)}}{\partial x_j} + \bar{b}_i^{(s,ext)} = 0, \qquad \hat{p}^{(s)} = 0, \qquad \breve{\boldsymbol{\sigma}}^{(s)} \mathbf{n} = \mathbf{t}^{(ext)}. \tag{2.53}$$

Relation (2.53) states that, when $p = 0$, tensor $\breve{\boldsymbol{\sigma}}^{(s)}$ has to formally satisfy the same interior and boundary equations pertinent to the standard notion of Cauchy stress, and makes these two tensors coincident, viz., $\breve{\boldsymbol{\sigma}}^{(s)} \equiv \boldsymbol{\sigma}^{(s)}$, in conditions of null fluid pressure.

2.2.2.3 Traction Boundary Conditions

Recalling the relation (2.6), the vector field of external tractions over $\partial\Omega_0^{(M)}$ in the reference configuration is standardly defined as:

$$\mathbf{t}_0^{(ext)} := -\frac{\partial\bar{\psi}_{0ext}^{(\partial)}}{\partial\bar{\boldsymbol{\chi}}^{(ext)}}, \tag{2.54}$$

where the push forward of $\mathbf{t}_0^{(ext)}$ is the spatial traction

$$\mathbf{t}^{(ext)} = \left(\frac{dA_0}{dA}\right)\bar{\mathbf{t}}_0^{(s)(ext)}. \tag{2.55}$$

On account of the kinematic bilateral boundary condition (2.6), the virtual variations of placements $\delta\bar{\boldsymbol{\chi}}^{(s)}$ and $\delta\bar{\boldsymbol{\chi}}^{(f)}$ cannot be independently assigned at the boundary but must satisfy the relation:

$$\delta\bar{\boldsymbol{\chi}}^{(s)} = \delta\bar{\boldsymbol{\chi}}^{(f)}, \qquad \text{over } \partial\Omega_0^{(M)}. \tag{2.56}$$

On this basis, boundary conditions of displacement-type and of stress-type are standardly applied. In particular, when the displacements of the external environment $\bar{\boldsymbol{\chi}}^{(ext)}$ are prescribed throughout $\partial\Omega_0^{(M)}$ in all Cartesian directions, these conditions correspond to treating $\bar{\boldsymbol{\chi}}^{(ext)}$ in (2.6) as a fixed vector datum, hence with zero variations, so that the condition for virtual displacements reads accordingly reads:

$$\delta\bar{\boldsymbol{\chi}}^{(s)} = \delta\bar{\boldsymbol{\chi}}^{(f)} = \mathbf{o}, \quad \text{over } \partial\Omega_0^{(M)}. \tag{2.57}$$

Conversely, when boundary conditions of traction type are prescribed in all Cartesian directions, these read:

$$\bar{\boldsymbol{\chi}}^{(s)} = \bar{\boldsymbol{\chi}}^{(f)}, \qquad \delta\bar{\boldsymbol{\chi}}^{(s)} = \delta\bar{\boldsymbol{\chi}}^{(f)} = \delta\bar{\boldsymbol{\chi}}^{(ext)}, \qquad \mathbf{t}_0^{(ext)} = \bar{\mathbf{t}}_0^{ext}, \tag{2.58}$$

where $\delta\bar{\boldsymbol{\chi}}^{(ext)}$ is the virtual variation of the placement of the external environment at the boundary $\partial\Omega_0^{(M)}$ and where $\bar{\mathbf{t}}_0^{ext}$ is the prescribed traction vector field. A more articulated specification of boundary conditions along each Cartesian direction can be also introduced in the customary way by considering three partitions $\partial\Omega_0^{(M)} = {}^d\partial\Omega_{0i}^{(M)} \cup {}^s\partial\Omega_{0i}^{(M)}$ per each i-th Cartesian direction ($i = 1, 2, 3$), where ${}^d\partial\Omega_{0i}^{(M)}$ is the subset where boundary conditions of displacement type are applied along the i-th Cartesian direction:

$$\bar{\chi}_i^{(s)} = \bar{\chi}_i^{(f)} = \bar{\chi}_i^{(e)}, \qquad \delta\bar{\chi}_i^{(s)} = \delta\bar{\chi}_i^{(f)} = 0 \qquad \text{over } {}^d\partial\Omega_{0i}^{(M)}, \tag{2.59}$$

and ${}^s\partial\Omega_{0i}^{(M)}$ is the subset of stress-type boundary conditions for the i-th Cartesian direction:

$$\bar{\chi}_i^{(s)} = \bar{\chi}_i^{(f)}, \qquad \delta\bar{\chi}_i^{(s)} = \delta\bar{\chi}_i^{(f)} = \delta\bar{\chi}_i^{(e)}, \qquad t_{0i}^{(ext)} = \bar{t}_{0i}^{ext} \text{ over } {}^{s}\partial\Omega_{0i}^{(M)}, \tag{2.60}$$

where $\bar{\chi}_i^{(e)}$ and $\bar{t}_{0i}^{ext}$ are the scalar fields of boundary data, defined over ${}^{d}\partial\Omega_{0i}^{(M)}$ and ${}^{s}\partial\Omega_{0i}^{(M)}$, respectively.

2.2.3 Integral Equations

General expression of the least Action condition

Proceeding from the continuum description detailed in Sect. 2.2.2.1, the general condition of Least Action is defined as:

$$\boxed{\begin{array}{c} \delta \displaystyle\int_{t_0}^{t_f} L_0^{(M)}(t)\, dt = 0, \\ \forall \text{ compatible virtual deformation } \left(\delta\bar{\boldsymbol{\chi}}^{(s)},\, \delta\bar{\boldsymbol{\chi}}^{(f)},\, \delta\hat{J}^{(s)}\right) \in \widetilde{\mathscr{D}} \end{array}} \tag{2.61}$$

where $\delta\bar{\boldsymbol{\chi}}^{(s)}$, $\delta\bar{\boldsymbol{\chi}}^{(f)}$, $\delta\hat{J}^{(s)}$ are infinitesimal fields fulfilling kinematic compatibility, i.e., such that the triplets $\left(\delta\bar{\boldsymbol{\chi}}^{(s)},\, \delta\bar{\boldsymbol{\chi}}^{(f)},\, \delta\hat{J}^{(s)}\right)$ belong to the general set of compatible virtual deformations $\widetilde{\mathscr{D}}$ fulfilling the kinematic conditions stated in Sects. 2.2.1 and 2.2.2.3. The integral equation in (2.61) can be written, on account of (A.42) (see Appendix A.3), in the following more explicit form:

$$\begin{aligned} & \frac{d}{dt}\partial_{\dot{\bar{\boldsymbol{\chi}}}^{(s)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(s)}\right] - \partial_{\bar{\boldsymbol{\chi}}^{(s)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(s)}\right] \\ + & \frac{d}{dt}\partial_{\dot{\bar{\boldsymbol{\chi}}}^{(f)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(f)}\right] - \partial_{\bar{\boldsymbol{\chi}}^{(f)}} L_0^{(M)}\left[\delta\bar{\boldsymbol{\chi}}^{(f)}\right] \\ + & \frac{d}{dt}\partial_{\dot{\hat{J}}^{(s)}} L_0^{(M)}\left[\delta\hat{J}^{(s)}\right] - \partial_{\hat{J}^{(s)}} L_0^{(M)}\left[\delta\hat{J}^{(s)}\right] \\ + & \qquad \partial_{\bar{\boldsymbol{\chi}}^{(ext)}} U_{\partial\Omega 0}^{ext}\left[\delta\bar{\boldsymbol{\chi}}^{(ext)}\right] \qquad = 0. \end{aligned} \tag{2.62}$$

In the following, conditions of complete traction-type (2.58) are considered over the whole boundary, viz.:

$$\bar{\boldsymbol{\chi}}^{(s)} = \bar{\boldsymbol{\chi}}^{(f)}, \qquad \delta\bar{\boldsymbol{\chi}}^{(s)} = \delta\bar{\boldsymbol{\chi}}^{(f)} = \delta\bar{\boldsymbol{\chi}}^{(ext)}, \qquad \text{over } \partial\Omega_0^{(M)}. \tag{2.63}$$

This choice implies, in particular, that the general set of compatible virtual deformations $\widetilde{\mathscr{D}}$ is composed of the triplets $\left(\delta\bar{\boldsymbol{\chi}}^{(s)},\, \delta\bar{\boldsymbol{\chi}}^{(f)},\, \delta\hat{J}^{(s)}\right)$ such that:

$$\left(\delta\bar{\boldsymbol{\chi}}^{(s)},\, \delta\bar{\boldsymbol{\chi}}^{(f)},\, \delta\hat{J}^{(s)}\right) : \quad \begin{cases} \delta\bar{\boldsymbol{\chi}}^{(s)} = \delta\bar{\boldsymbol{\chi}}^{(f)} & \text{over } \partial\Omega_0^{(M)} \\ \text{no constraint} & \text{inside } \Omega_0^{(M)} \end{cases} \tag{2.64}$$

Based on the definitions of Sect. 2.2.2.1, the terms in (2.62) are split into:

$$\begin{aligned}
-\partial_{\bar{\chi}^{(s)}} L_0^{(M)} &= \partial_{\bar{\chi}^{(s)}} U_0^{(M)} = \partial_{\bar{\chi}^{(s)}} U_0^{(s)} + \partial_{\bar{\chi}^{(s)}} U_0^{(f)} + \partial_{\bar{\chi}^{(s)}} U_0^{ext}, \\
-\partial_{\bar{\chi}^{(f)}} L_0^{(M)} &= \partial_{\bar{\chi}^{(f)}} U_0^{(M)} = \partial_{\bar{\chi}^{(f)}} U_0^{(s)} + \partial_{\bar{\chi}^{(f)}} U_0^{(f)} + \partial_{\bar{\chi}^{(f)}}, U_0^{ext} \\
-\partial_{\hat{J}^{(s)}} L_0^{(M)} &= \partial_{\hat{J}^{(s)}} U_0^{(M)} = \partial_{\hat{J}^{(s)}} U_0^{(s)} + \partial_{\hat{J}^{(s)}} U_0^{(f)} + \partial_{\hat{J}^{(s)}} U_0^{ext},
\end{aligned} \tag{2.65}$$

$$\begin{aligned}
\frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(s)}} L_0^{(M)}\left[\delta\bar{\chi}_i^{(s)}\right] &= \frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(s)}} T^{(s)}\left[\delta\bar{\chi}_i^{(s)}\right] \\
\frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(f)}} L_0^{(M)}\left[\delta\bar{\chi}_i^{(f)}\right] &= \frac{d}{dt}\partial_{\dot{\bar{\chi}}_i^{(f)}} T_0^{(f)}\left[\delta\bar{\chi}_i^{(f)}\right], \\
\frac{d}{dt}\partial_{\dot{\hat{J}}^{(s)}} L_0^{(M)}\left[\delta\hat{J}^{(s)}\right] &= \frac{d}{dt}\partial_{\dot{\hat{J}}^{(s)}} T^{(s)}\left[\delta\hat{J}^{(s)}\right] + \frac{d}{dt}\partial_{\dot{\hat{J}}^{(s)}} T_0^{(f)}\left[\delta\hat{J}^{(s)}\right].
\end{aligned} \tag{2.66}$$

The computation of each of the terms on the right hand sides of (2.65) and (2.66) is reported in Appendix B. In particular, a key development for obtaining the explicit form of the Euler-Lagrange equations is the computation of the variation $\delta U_0^{(f)}$ and of the associated variation of intrinsic volumetric fluid strain $\delta \hat{J}^{(f)}_{sat\bar{\chi}^{(f)}}$ (see (2.35)). The final computed expressions of these two terms are hereby recalled from Appendix B:

$$\begin{aligned}
\delta\hat{J}^{(f)} &= \partial_{\left(\bar{\chi}^{(s)},\, \bar{\chi}^{(f)},\, \hat{J}^{(s)}\right)} \hat{J}^{(f)}_{sat\bar{\chi}^{(f)}}\left[\delta\bar{\chi}^{(s)},\, \delta\bar{\chi}^{(f)},\, \delta\hat{J}^{(s)}\right] \\
&= -\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\left[-\phi^{(s)}_{\bar{\chi}^{(s)}}\frac{\partial\delta\bar{\chi}_i^{(s)}}{\partial X_J}\frac{\partial\left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i}\circ\left(\bar{\chi}^{(s)}\right)^{-1}\circ\bar{\chi}^{(f)}\right] \\
&\quad -\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\left[-\frac{\partial\phi^{(s)}_{\bar{\chi}^{(s)}}}{\partial X_J}\frac{\partial\left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i}\delta\bar{\chi}_i^{(s)}\circ\bar{\chi}^{(f)}\right] \\
&\quad +\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\left[\phi^{(f)}_{\bar{\chi}^{(f)}}\frac{\partial\left(\bar{\chi}^{(f)}\right)_J^{-1}}{\partial x_i}\frac{\partial\delta\bar{\chi}_i^{(f)}}{\partial X_J}\right] \\
&\quad +\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\left[\frac{\partial\phi^{(f)}_{\bar{\chi}^{(f)}}}{\partial X_J}\frac{\partial\left(\bar{\chi}^{(f)}\right)_J^{-1}}{\partial x_i}\delta\bar{\chi}_i^{(f)}\right] \\
&\quad -\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\frac{\Phi_0^{(s)}}{\bar{J}^{(s)}}\delta\hat{J}^{(s)}.
\end{aligned} \tag{2.67}$$

It should be noted that the expression (2.67) provides a relation written in terms of fields placed in the reference configuration $\Omega_0^{(M)}$. The same relation expressed by (2.67) is more concisely written in terms of spatial variables associated with the space point $\mathbf{x}$, viz.:

$$\delta\hat{J}^{(f)} = \frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\left(\phi^{(s)}\frac{\partial\delta\bar{\chi}_i^{(s)}}{\partial x_i} + \frac{\partial\phi^{(s)}}{\partial x_i}\delta\bar{\chi}_i^{(s)} + \phi^{(f)}\frac{\partial\delta\bar{\chi}_i^{(f)}}{\partial x_i} + \frac{\partial\phi^{(f)}}{\partial x_i}\delta\bar{\chi}_i^{(f)} - \frac{\Phi_0^{(s)}}{\bar{J}^{(s)}}\delta\hat{J}^{(s)}\right). \tag{2.68}$$

Accounting in (2.68) for the product derivatives rule, one obtains the *Linearized Saturation Constraint* for virtual deformations associated with a finite deformation, see [75, 77]. This constraint expresses the relation that virtual infinitesimal variations must fulfill in order to preserve the condition of space saturation (2.15):

$$\Phi_0^{(s)}\frac{\delta\hat{J}^{(s)}}{\hat{J}^{(s)}} + \Phi_0^{(f)}\frac{\delta\hat{J}^{(f)}}{\hat{J}^{(f)}} = \frac{\partial\phi^{(s)}\delta\bar{\chi}_i^{(s)}}{\partial x_i} + \frac{\partial\phi^{(f)}\delta\bar{\chi}_i^{(f)}}{\partial x_i}. \tag{2.69}$$

When linearization is performed in the reference configuration, one can refer to the relation (2.8), written for virtual deformations, and apply $\Phi_0^{(s)} \simeq \phi^{(s)}$ and $\Phi_0^{(f)} \simeq \phi^{(f)}$, obtaining:

$$\phi^{(s)}\delta\hat{e}^{(s)} + \phi^{(f)}\delta\hat{e}^{(f)} = \frac{\partial\phi^{(s)}\delta\bar{\chi}_i^{(s)}}{\partial x_i} + \frac{\partial\phi^{(f)}\delta\bar{\chi}_i^{(f)}}{\partial x_i}. \tag{2.70}$$

In particular, when the fields $\phi^{(s)}$ and $\phi^{(f)}$ are uniform in space, the relation (2.70) specializes to:

$$\phi^{(s)}\delta\hat{e}^{(s)} + \phi^{(f)}\delta\hat{e}^{(f)} = \phi^{(s)}\frac{\partial\delta\bar{\chi}_i^{(s)}}{\partial x_i} + \phi^{(f)}\frac{\partial\delta\bar{\chi}_i^{(f)}}{\partial x_i}. \tag{2.71}$$

Relation (2.71) corresponds to the form of the saturation constraint obtained in [75, 77] for infinitesimal deformations, and has been previously reported in a form combined with mass balances also by Bedford and Drumheller [7]. It is worth remarking that the derivation herein reported for this condition elucidates its pure kinematic significance independent of mass balances. In this respect, it is also worth observing that Eq. (2.71) is also derived by taking the sum of (2.10) and (2.11).

The variation of the fluid potential energy, expressed as:

$$\delta U_0^{(f)} = \partial_{\hat{J}^{(f)}} U_0^{(f)}\left[\delta\hat{J}^{(f)}\right] = \partial_{\left(\bar{\boldsymbol{\chi}}^{(s)},\, \bar{\boldsymbol{\chi}}^{(f)},\, \hat{J}^{(s)}\right)} U_0^{(f)}\left[\delta\bar{\boldsymbol{\chi}}^{(s)},\, \delta\bar{\boldsymbol{\chi}}^{(f)},\, \delta\hat{J}^{(s)}\right] \tag{2.72}$$

is computed differentiating relation (2.35) via the rule for the variation of function composition, see (A.23), (B.4) and (B.5). Its explicit expression is:

$$
\begin{aligned}
\delta U_0^{(f)} &= \int_{\Omega_0^{(M)}} \Phi_0^{(f)} \overbrace{\frac{\partial \hat{\psi}_0^{(f)}}{\partial \hat{J}^{(f)}}}^{-\hat{\hat{\Pi}}^{(f)}} \partial_{\left(\bar{\boldsymbol{\chi}}^{(s)},\, \bar{\boldsymbol{\chi}}^{(f)},\, \hat{J}^{(s)}\right)} \hat{J}^{(f)}_{sat\, \bar{\boldsymbol{\chi}}^{(f)}} \left[\delta \bar{\boldsymbol{\chi}}^{(s)},\, \delta \bar{\boldsymbol{\chi}}^{(f)},\, \delta \hat{J}^{(s)}\right] dV_0 \\
&= -\int_{\Omega_0^{(M)}} \hat{\hat{\Pi}}^{(f)} \bar{J}^{(f)} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \frac{\partial \delta \bar{\chi}_i^{(s)}}{\partial X_J} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} dV_0 \\
&\quad - \int_{\Omega_0^{(M)}} \hat{\hat{\Pi}}^{(f)} \bar{J}^{(f)} \frac{\partial \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}}{\partial X_J} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \delta \bar{\chi}_i^{(s)} dV_0 \\
&\quad - \int_{\Omega_0^{(M)}} \hat{\hat{\Pi}}^{(f)} \bar{J}^{(f)} \phi^{(f)}_{\bar{\boldsymbol{\chi}}^{(f)}} \frac{\partial \left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} \frac{\partial \delta \bar{\chi}_i^{(f)}}{\partial X_J} dV_0 \\
&\quad - \int_{\Omega_0^{(M)}} \hat{\hat{\Pi}}^{(f)} \bar{J}^{(f)} \frac{\partial \phi^{(f)}_{\bar{\boldsymbol{\chi}}^{(f)}}}{\partial X_J} \frac{\partial \left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} \delta \bar{\chi}_i^{(f)} dV_0 \\
&\quad + \int_{\Omega_0^{(M)}} \hat{\hat{\Pi}}^{(f)} \bar{J}^{(f)} \frac{\Phi_0^{(s)}}{\bar{J}^{(s)}} \delta \hat{J}^{(s)} dV_0.
\end{aligned}
\tag{2.73}
$$

Collecting (2.73) with the required expressions reported in Appendix B for $\delta U_0^{(s)}$ and δU_0^{ext}, and for the remaining terms entering (2.66), the following explicit expression of Eq. (2.62) is obtained:

$$
\begin{aligned}
&\int_{\Omega_0^{(M)}} \left[\check{P}_{iJ} - \bar{J}^{(f)} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} + \bar{\rho}^{(s)}_{add.0} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right] \frac{\partial \delta \bar{\chi}_i^{(s)}}{\partial X_J} dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left(-\bar{J}^{(f)} \frac{\partial \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}}}{\partial X_J} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} - \bar{b}_{0i}^{(s,ext)} + \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} \right) \delta \bar{\chi}_i^{(s)} dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left[-\bar{J}^{(f)} \phi^{(f)}_{\bar{\boldsymbol{\chi}}^{(f)}} \frac{\partial \left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} + \bar{\rho}^{(f)}_{add.0} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] \frac{\partial \delta \bar{\chi}_i^{(f)}}{\partial X_J} dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left[-\bar{J}^{(f)} \frac{\partial \phi^{(f)}_{\bar{\boldsymbol{\chi}}^{(f)}}}{\partial X_J} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} - \bar{b}_{0i}^{(f,ext)} + \bar{\rho}_0^{(f)} \ddot{\bar{\chi}}_i^{(f)} \right] \delta \bar{\chi}_i^{(f)} dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left[-\hat{\Pi}^{(s)} + \frac{\bar{J}^{(f)}}{\bar{J}^{(s)}} \Phi_0^{(s)} \hat{\hat{\Pi}}^{(f)} - \bar{\rho}^{(s)}_{add.0} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) + \frac{\bar{J}^{(f)}}{\Phi_0^{(f)}} \frac{\Phi_0^{(s)}}{\bar{J}^{(s)}} \bar{\rho}^{(f)}_{add.0} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] \delta \hat{J}^{(s)} dV_0 \\
&- \int_{\partial \Omega_0^{(M)}} t_{0i}^{(ext)} \delta \bar{\chi}_i^{(e)} dA_0 = 0.
\end{aligned}
\tag{2.74}
$$

Application of the divergence theorem to the terms in (2.74) containing $\frac{\partial \delta \bar{\boldsymbol{\chi}}^{(s)}}{\partial \mathbf{X}}$ and $\frac{\partial \delta \bar{\boldsymbol{\chi}}^{(f)}}{\partial \mathbf{X}}$ yields (see Appendix B for details):

$$
\begin{aligned}
&\int_{\Omega_0^{(M)}} \left[-\frac{\partial \check{P}_{iJ}}{\partial X_J} + \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \frac{\partial}{\partial X_J} \left(\bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} \right) \right] \delta \bar{\chi}_i^{(s)} \, dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left[-\bar{b}_{0i}^{(s,ext)} + \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} - \frac{\partial}{\partial X_J^{(s)}} \left(\bar{\rho}^{(s)}_{add.0} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right) \right] \delta \bar{\chi}_i^{(s)} \, dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left[\phi^{(f)}_{\bar{\boldsymbol{\chi}}^{(f)}} \frac{\partial}{\partial X_J} \left(\bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} \right) - \frac{\partial}{\partial X_J} \left(\bar{\rho}^{(f)}_{add.0} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right) \right] \delta \bar{\chi}_i^{(f)} \, dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left[-\bar{b}_{0i}^{(f,ext)} + \bar{\rho}_0^{(f)} \ddot{\bar{\chi}}_i^{(f)} \right] \delta \bar{\chi}_i^{(f)} \, dV_0 \\
&+ \int_{\Omega_0^{(M)}} \left[-\hat{\Pi}^{(s)} + \frac{\bar{J}^{(f)}}{\bar{J}^{(s)}} \Phi_0^{(s)} \hat{\hat{\Pi}}^{(f)} \delta \hat{J}^{(s)} - \bar{\rho}^{(s)}_{add.0} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) + \frac{\bar{J}^{(f)}}{\Phi_0^{(f)}} \frac{\Phi_0^{(s)}}{\bar{J}^{(s)}} \bar{\rho}^{(f)}_{add.0} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] \delta \hat{J}^{(s)} \, dV_0 \\
&+ \int_{\partial\Omega_0^{(M)}} \left[\check{P}_{iJ} - \bar{J}^{(f)} \phi^{(s)}_{\bar{\boldsymbol{\chi}}^{(s)}} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} + \bar{\rho}^{(s)}_{add.0} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right] N_J \delta \bar{\chi}_i^{(s)} \, dV_0 \\
&+ \int_{\partial\Omega_0^{(M)}} \left[-\phi^{(f)}_{\bar{\boldsymbol{\chi}}^{(f)}} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} + \bar{\rho}^{(f)}_{add.0} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] \delta \bar{\chi}_i^{(f)} N_J \, dV_0 \\
&- \int_{\partial\Omega_0^{(M)}} t_{0i}^{(ext)} \delta \bar{\chi}_i^{(e)} \, dA_0 = 0.
\end{aligned}
\tag{2.75}
$$

where the boundary integrals have been collected in the last three rows of (2.75).

2.2.4 Strong Form Equations

Strong form equations are inferred from (2.75) via application of the fundamental lemma of calculus of variations. Field equations are obtained considering the these integral equations must hold for any triplet $\left(\delta\bar{\boldsymbol{\chi}}^{(s)}, \delta\bar{\boldsymbol{\chi}}^{(f)}, \delta\hat{J}^{(s)}\right) \in \widetilde{\mathscr{D}}$ of virtual deformation functions belonging to the general class of compatible virtual displacements $\widetilde{\mathscr{D}}$, defined by (2.64), with virtual displacements vanishing at the boundary (viz.: such that $\delta\bar{\boldsymbol{\chi}}^{(s)} = \mathbf{o}$, $\delta\bar{\boldsymbol{\chi}}^{(f)} = \mathbf{o}$ over $\partial\Omega_0^{(M)}$). In particular, as shown in Appendix B, by selecting virtual variations of type $\delta\bar{\boldsymbol{\chi}}^{(s)} \neq \mathbf{o}$, $\delta\bar{\boldsymbol{\chi}}^{(f)} = \mathbf{o}$, and $\delta\hat{J}^{(s)} = 0$ over $\Omega_0^{(M)}$ one obtains the *linear momentum balance of the solid phase*:

$$
\begin{aligned}
&\frac{\partial \check{P}_{iJ}}{\partial X_J^{(s)}} - \phi^{(s)} \frac{\partial}{\partial X_J} \left(\bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \hat{\hat{\Pi}}^{(f)} \right) + \bar{b}_{0i}^{(s,ext)} = \\
&= \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} - \frac{\partial}{\partial X_J^{(s)}} \left(\bar{\rho}^{(s)}_{add.0} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right).
\end{aligned}
\tag{2.76}
$$

Similarly, by selecting virtual variations vanishing at the boundary, such that $\delta\bar{\boldsymbol{\chi}}^{(s)} = \mathbf{o}$, $\delta\bar{\boldsymbol{\chi}}^{(f)} \neq \mathbf{o}$, and $\delta\hat{J}^{(s)} = 0$ over $\Omega_0^{(M)}$, one obtains the *linear momentum balance of the fluid phase*:

$$
\begin{aligned}
-\phi^{(f)}\frac{\partial}{\partial X_J}\left(\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\hat{\hat{\Pi}}^{(f)}\right)+\bar{b}_{0i}^{(f,ext)} = \\
=\bar{\rho}_0^{(f)}\ddot{\bar{\chi}}_i^{(f)}-\frac{\partial}{\partial X_J}\left[\bar{\rho}^{(f)}_{add.0}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(f)}-\ddot{\hat{J}}^{(f)}\right)\right].
\end{aligned}
\tag{2.77}
$$

An analogous selection with $\delta\bar{\boldsymbol{\chi}}^{(s)}=\mathbf{o}$, $\delta\bar{\boldsymbol{\chi}}^{(f)}=\mathbf{o}$ and $\delta\hat{J}^{(s)}\neq 0$ yields the *intrinsic momentum balance*:

$$
\hat{\Pi}^{(s)}-\frac{\bar{J}^{(f)}}{\bar{J}^{(s)}}\Phi_0^{(s)}\hat{\hat{\Pi}}^{(f)}=-\bar{\rho}^{(s)}_{add.0}\left(\ddot{\bar{J}}^{(s)}-\ddot{\hat{J}}^{(s)}\right)+\frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\frac{\Phi_0^{(s)}}{\bar{J}^{(s)}}\bar{\rho}^{(f)}_{add.0}\left(\ddot{\bar{J}}^{(f)}-\ddot{\hat{J}}^{(f)}\right)
\tag{2.78}
$$

For the boundary integrals in (2.75) considering variation fields at the boundary $\partial\Omega_0^{(M)}$ such that $\delta\bar{\boldsymbol{\chi}}^{(s)}=\delta\bar{\boldsymbol{\chi}}^{(f)}=\delta\bar{\boldsymbol{\chi}}^{(ext)}$ one obtains:

$$
\begin{aligned}
&\Bigg[\check{P}_{iJ}-\bar{J}^{(f)}\phi^{(s)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\hat{\Pi}^{(f)}-\phi^{(f)}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\hat{\hat{\Pi}}^{(f)}+\\
&+\bar{\rho}^{(s)}_{add.0}\bar{J}^{(s)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(s)}-\ddot{\hat{J}}^{(s)}\right)+\bar{\rho}^{(f)}_{add.0}\bar{J}^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\left(\ddot{\bar{J}}^{(f)}-\ddot{\hat{J}}^{(f)}\right)\Bigg]N_J=t_{0i}^{(ext)}.
\end{aligned}
\tag{2.79}
$$

2.2.4.1 Strong-Form Equations Associated with Virtual Isochoric Deformations

Equations (2.76)–(2.79) provide a complete statement of the boundary value problem of the dynamic evolution of the mixture as dictated by the least Action principle. These equations are complete since they allow to determine, upon integrating in space-time the boundary value problem, the updated physical state of the mixture, which is defined by the updated primary kinematic fields $\bar{\boldsymbol{\chi}}^{(s)}$, $\bar{\boldsymbol{\chi}}^{(f)}$, $\hat{J}^{(s)}$ over $\Omega_0^{(M)}$. It can be also verified that, over the singular LSP regions, where one phase is vanishing, the momentum balance equations associated with the vanishing phase are always automatically satisfied in a trivial way. This has been anticipated in Remarks 2.1, 2.3 and 2.5, and is shown in particular in Remark 2.7, with reference to LSP entirely-fluid region $\Omega_0^{(f)}$ considered in the present model problem.

Remark 2.7 Trivial satisfaction of solid momentum balances over $\Omega_0^{(f)}$— Concerning the evolution of the primary kinematic fields $\bar{\boldsymbol{\chi}}^{(s)}$, $\hat{J}^{(s)}$ over $\Omega_0^{(M)}$ determined the PDEs (2.76)–(2.79), it is worth to remark that, in practice, the integration of these fields has to be carried out only over the region $\Omega_0^{(s)}$ where the solid volume fraction is nonzero. The restriction of fields $\bar{\boldsymbol{\chi}}^{(s)}$ and $\hat{J}^{(s)}$ over $\Omega_0^{(f)}$, which is not mechanically relevant and exemplifies the singularity attained in LSP regions, is excluded from integration as anticipated in Remarks 2.3 and 2.5. This can be recognized by oberving that, over domain $\Omega_0^{(f)}$ where $\Phi_0^{(s)} = 0$, since all energy densities are zero (and hence also all stress measures, included $\check{P}$) all terms of the linear momentum balance of the solid phase (2.76) are zero as well. The second term in the first row of (2.76) containing $\phi^{(s)}$, which is instead related to the strain energy of the fluid, also vanishes since, due to (2.3), one has $\Phi_0^{(s)} = 0 \ \rightarrow \ \phi^{(s)} = 0$. This implies that relation (2.76) collapses to the identity $0 = 0$, and is always trivially satisfied.

An analogous consideration holds for the intrinsic momentum balance (2.78) which also collapses in $\Omega_0^{(f)}$ to the trivial identity $0 = 0$ as a consequence of the vanishing of fields $\Phi_0^{(s)}$, $\hat{\Pi}^{(s)}$ and $\bar{\rho}_{add.0}^{(s)}$ over $\Omega_0^{(f)}$.

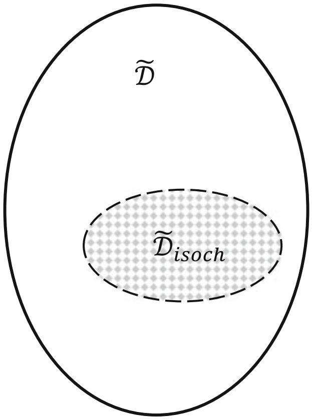
Hence, for the integration of Eqs. (2.76)–(2.79), no additional equations need to be supplemented, besides optional switch to the desired displacement-type or stress-type boundary constraints with the aid of (2.59) and (2.60) pertinent to the specific mechanical boundary problem at hand. Actually, in the statement of the evolution problem given by (2.76)–(2.79), mass balances are not required among the governing equations because updated densities can be post-computed from the macroscopic displacement fields by (2.20).

It is however convenient to derive from the general stationarity condition (2.61) supplementary strong form conditions that the extrinsic Piola stress tensor of the solid phase $\check{\mathbf{P}}$ must comply with over the free solid-fluid macroscopic interface $\mathscr{S}_0^{(sf)}$ (dashed line in Fig. 2.2). Actually, an equation associated with surface $\mathscr{S}_0^{(sf)}$ has not been derived yet since, as remarked in Sect. 2.2.1, this surface is not necessarily a boundary of the primary physical system; this is remarked also in Fig. 2.2 which shows that $\mathscr{S}_0^{(sf)} \not\subset \partial\Omega_0^{(M)}$, although part of $\mathscr{S}_0^{(sf)}$ may be superposed with the boundary.

In order to derive a convenient strong-form relation at the free solid-fluid macroscopic interface $\mathscr{S}_0^{(sf)}$ for $\check{\mathbf{P}}$, we now compute the strong form equations which are inferred from (2.62) when *isochoric* virtual deformations are selected in particular, i.e., when one localizes (2.62) by considering virtual compatible deformations which are additionally characterized by null intrinsic virtual deformations $\delta\hat{J}^{(s)} = 0$ and $\delta\hat{J}^{(f)} = 0$. In order to preserve compatibility, these deformations, referred to as the class of *compatible virtual isochoric deformations* and hereby indicated by the symbol $\widetilde{\mathscr{D}}_{isoch}$, must also keep the fulfillment of Eq. (2.69). Hence, summarizing, the conditions characterizing an element $\left(\delta\bar{\boldsymbol{\chi}}^{(s)}, \delta\bar{\boldsymbol{\chi}}^{(f)}, \delta\hat{J}^{(s)}\right) \in \widetilde{\mathscr{D}}_{isoch}$ are displayed below:

$$\left(\delta\bar{\boldsymbol{\chi}}^{(s)}, \delta\bar{\boldsymbol{\chi}}^{(f)}, \delta\hat{J}^{(s)}\right): \quad \begin{cases} \delta\bar{\boldsymbol{\chi}}^{(s)} = \delta\bar{\boldsymbol{\chi}}^{(f)} & \text{over } \partial\Omega_0^{(M)} \\ \delta\hat{J}^{(s)} = 0, \ \delta\hat{J}^{(f)} = 0 & \text{over } \Omega_0^{(M)} \\ \text{respect of saturation condition (2.69)} & \text{over } \Omega_0^{(M)} \end{cases} \tag{2.80}$$

Fig. 2.3 Venn diagram illustrating the relation between $\widetilde{\mathscr{D}}_{isoch}$ and $\widetilde{\mathscr{D}}$



By definition, $\widetilde{\mathscr{D}}_{isoch}$ is strictly contained in $\widetilde{\mathscr{D}}$, viz., $\widetilde{\mathscr{D}}_{isoch} \subset \widetilde{\mathscr{D}}$ (see Remark 2.8 for additional comments on this evident property) so that the deformations belonging to $\widetilde{\mathscr{D}}_{isoch}$ are compatible. $\widetilde{\mathscr{D}}_{isoch}$ is accordingly denominated *subclass of isochoric compatible virtual deformations*.

Remark 2.8 Strict containment relation $\widetilde{\mathscr{D}}_{isoch} \subset \widetilde{\mathscr{D}}$—It is almost trivial to check that the containment relation $\widetilde{\mathscr{D}}_{isoch} \subset \widetilde{\mathscr{D}}$ holds. Any element $\left(\delta\bar{\boldsymbol{\chi}}^{(s)},\ \delta\bar{\boldsymbol{\chi}}^{(f)},\ \delta\hat{J}^{(s)}\right) \in \widetilde{\mathscr{D}}_{isoch}$ respecting the stronger set of conditions (2.80) also automatically fulfills (2.64), so that $\left(\delta\bar{\boldsymbol{\chi}}^{(s)},\ \delta\bar{\boldsymbol{\chi}}^{(f)},\ \delta\hat{J}^{(s)}\right) \in \widetilde{\mathscr{D}}$. Since the converse is not true this is a strict containment relation, represented in terms of a Venn diagram in Fig. 2.3. For this reason $\widetilde{\mathscr{D}}_{isoch}$ is referred to as the subclass of virtual compatible isochoric deformations.

A strong form equation for surface $\mathscr{S}_0^{(sf)}$ is now derived from (2.61) and (2.62) by constructing a family of elements of $\widetilde{\mathscr{D}}_{isoch}$ which fulfills the saturation hypothesis. In doing this it is convenient to anticipate that the final result obtained is Equation (2.88) and that in the derivation reported below the statement of the boundary value problem is not modified, (i.e., no further ad-hoc mechanical assumptions are added, see in particular Remark 2.9).

Let us observe that, in order to preserve the saturation condition (2.69), the macroscopic displacement fields in a virtual isochoric deformation must fulfill the condition:

$$\frac{\partial \phi^{(s)} \delta\bar{\chi}_i^{(s)}}{\partial x_i} + \frac{\partial \phi^{(f)} \delta\bar{\chi}_i^{(f)}}{\partial x_i} = 0. \tag{2.81}$$

A virtual isochoric deformation compatible with boundary equation (2.56) can be constructed by choosing an arbitrary vector field $\delta\tilde{\mathbf{u}}^{(s)}$ such that $\delta\tilde{\mathbf{u}}^{(s)} = \mathbf{o}$ at the boundary $\partial \Omega_0^{(M)}$, and by setting the variations of the primary fields to:

$$\delta \bar{\boldsymbol{\chi}}^{(s)} = \delta \tilde{\mathbf{u}}^{(s)}, \quad \delta \bar{\boldsymbol{\chi}}^{(f)} = \delta \tilde{\mathbf{u}}^{(f)} = -\frac{\phi^{(s)}}{\phi^{(f)}} \delta \tilde{\mathbf{u}}^{(s)}, \qquad \delta \hat{J}^{(s)} = 0. \tag{2.82}$$

As it can be checked, the virtual deformation field constructed in this way fulfills all conditions in (2.80).

When virtual deformations in (2.62) are set to the form specified in (2.82), the corresponding virtual intrinsic strain of the fluid is null (i.e., $\delta \hat{J}^{(f)} = 0$). Then (2.67) vanishes with the sum of the terms on its right hand side being zero. It stems from (2.72) and (2.73) that also $\delta U_0^{(f)}$ is null, and the terms contained in the right hand side of (2.73) cancel each other.

The effects determined by setting virtual deformations contained in the explicit integral Eqs. (2.74) and (2.75) to the form specified in (2.82) are listed below:

- since $\delta \hat{J}^{(f)} = 0$, all terms provided by (2.73) cancel each other, since their sum is null;
- since $\delta \hat{J}^{(s)} = 0$, all terms multiplying $\delta \hat{J}^{(s)}$ are cancelled;
- considering the partition $\Omega_0^{(M)} = \Omega_0^{(s)} \cup \Omega_0^{(f)}$ sketched by Fig. 2.2 (with $\Phi_0^{(s)} = 0$ over $\Omega_0^{(f)}$), and the associated additive split of the integrals

$$\int_{\Omega_0^{(M)}} (\cdot)\, dV_0 = \int_{\Omega_0^{(s)}} (\cdot)\, dV_0 + \int_{\Omega_0^{(f)}} (\cdot)\, dV_0, \tag{2.83}$$

  it is recognized that all the integrals over $\Omega_0^{(M)}$ in (2.74) and (2.75) containing $\delta \bar{\boldsymbol{\chi}}^{(f)}$ reduce to the only integral over the subset $\Omega_0^{(s)}$ since, as one can infer from (2.82), $\delta \bar{\boldsymbol{\chi}}^{(f)} = \mathbf{o}$ over $\Omega_0^{(f)}$;
- the boundary integrals in (2.75) all vanish due to (2.82) and to the property that $\delta \tilde{\mathbf{u}}^{(s)} = \mathbf{o}$ over $\partial \Omega_0^{(M)}$;

In addition to the simplifications in the bullet list, as previously observed (see Remark 2.5), energy densities for the solid phase are null functions in the points of $\Omega_0^{(f)}$ where $\Phi_0^{(s)} = 0$. Hence, the domain integrals of the solid phase vanish throughout $\Omega_0^{(f)}$, and reduce to only integrals over $\Omega_0^{(s)}$.

Accounting for all of the above listed simplifications, when virtual isochoric deformations are considered into the integral equation (2.75), this specializes to:

$$
\begin{aligned}
&\int_{\Omega_0^{(s)}} \left[-\frac{\partial \check{P}_{iJ}}{\partial X_J} \right] \delta \tilde{u}_i^{(s)} \, dV_0 \\
&+ \int_{\Omega_0^{(s)}} \left[-\bar{b}_{0i}^{(s,ext)} + \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} - \frac{\partial}{\partial X_J^{(s)}} \left(\bar{\rho}_{add.0}^{(s)} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right) \right] \delta \tilde{u}_i^{(s)} \, dV_0 \\
&+ \int_{\Omega_0^{(s)}} \left[-\frac{\partial}{\partial X_J} \left(\bar{\rho}_{add.0}^{(f)} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right) \right] \left(-\frac{\phi^{(s)}}{\phi^{(f)}} \delta \tilde{u}_i^{(s)} \right) dV_0 \\
&+ \int_{\Omega_0^{(s)}} \left[-\bar{b}_{0i}^{(f,ext)} + \bar{\rho}_0^{(f)} \ddot{\bar{\chi}}_i^{(f)} \right] \left(-\frac{\phi^{(s)}}{\phi^{(f)}} \delta \tilde{u}_i^{(s)} \right) dV_0 \\
&+ \int_{\partial \Omega_0^{(s)}} \left[\check{P}_{iJ} + \bar{\rho}_{add.0}^{(s)} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right] N_J \delta \tilde{u}_i^{(s)} \, dV_0 \\
&+ \int_{\partial \Omega_0^{(s)}} \left[\bar{\rho}_{add.0}^{(f)} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] \left(-\frac{\phi^{(s)}}{\phi^{(f)}} \delta \tilde{u}_i^{(s)} \right) N_J \, dV_0 = 0.
\end{aligned}
\tag{2.84}
$$

The previous relation must hold for any $\delta \tilde{\mathbf{u}}^{(s)}$ and can be used to infer strong form equations, again on account of the fundamental lemma of calculus of variations, by a standard localization technique. In particular, a domain strong form equation is first obtained by considering in (2.84) the subclass of virtual isochoric deformations vanishing at the boundary of the solid macroscopic domain, i.e., such that $\delta \tilde{\mathbf{u}}^{(s)} = \mathbf{o}$ over $\partial \Omega_0^{(s)}$. By this choice, the last two rows in (2.84) are cancelled and localization by the fundamental lemma yields the following equation holding in the interior points of $\Omega_0^{(s)}$:

$$
\begin{aligned}
&-\frac{\partial \check{P}_{iJ}}{\partial X_J} - \bar{b}_{0i}^{(s,ext)} + \frac{\phi^{(s)}}{\phi^{(f)}} \bar{b}_{0i}^{(f,ext)} + \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} - \frac{\phi^{(s)}}{\phi^{(f)}} \bar{\rho}_0^{(f)} \ddot{\bar{\chi}}_i^{(f)} \\
&- \frac{\partial}{\partial X_J^{(s)}} \left(\bar{\rho}_{add.0}^{(s)} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right) \\
&+ \frac{\phi^{(s)}}{\phi^{(f)}} \frac{\partial}{\partial X_J} \left(\bar{\rho}_{add.0}^{(f)} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right) = 0.
\end{aligned}
\tag{2.85}
$$

As a second step, substitution of (2.85) into (2.84) yields:

$$
\begin{aligned}
&\int_{\partial \Omega_0^{(s)}} \left[\check{P}_{iJ} + \bar{\rho}_{add.0}^{(s)} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right] N_J \delta \tilde{u}_i^{(s)} \, dV_0 \\
&+ \int_{\partial \Omega_0^{(s)}} \left[\bar{\rho}_{add.0}^{(f)} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)} \right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] \left(-\frac{\phi^{(s)}}{\phi^{(f)}} \delta \tilde{u}_i^{(s)} \right) N_J \, dV_0 = 0.
\end{aligned}
\tag{2.86}
$$

The previous relation must hold true for any field $\delta \tilde{\mathbf{u}}^{(s)}$ simultaneously fulfilling (2.63) and (2.82). Since the simultaneous fulfillment of these two equations necessarily implies $\delta \tilde{\mathbf{u}}^{(s)} = \mathbf{o}$ over $\partial \Omega_0^{(MU)}$, it is inferred from (2.84) the following second integal equation which differs from the previous one since integration is carried out only on the subset $\mathscr{S}_0^{(sf)} = \partial\Omega_0^{(s)} \backslash \partial\Omega_0^{(MU)}$, which is termed *free solid-fluid macroscopic interface*:

$$\begin{aligned} &\int_{\mathscr{S}_0^{(sf)}} \left[\check{P}_{iJ} + \bar{\rho}_{add.0}^{(s)} \bar{J}^{(s)} \frac{\partial \left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) \right] N_J \delta\tilde{u}_i^{(s)} \, dV_0 \\ + &\int_{\mathscr{S}_0^{(sf)}} \left[\bar{\rho}_{add.0}^{(f)} \bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)}\right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] \left(-\frac{\phi^{(s)}}{\phi^{(f)}} \delta\tilde{u}_i^{(s)} \right) N_J \, dV_0 = 0. \end{aligned} \tag{2.87}$$

Since this last integral equation holds for any arbitrary field $\delta\tilde{\mathbf{u}}^{(s)}$ on $\mathscr{S}_0^{(sf)}$, it can be straightforwardly localized into the following strong form equation holding on $\mathscr{S}_0^{(sf)}$:

$$\left[\check{P}_{iJ} + \bar{\rho}_{add.0}^{(s)} \bar{J}^{(s)} \frac{\partial\left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(s)} - \ddot{\hat{J}}^{(s)} \right) - \frac{\phi^{(s)}}{\phi^{(f)}} \bar{\rho}_{add.0}^{(f)} \bar{J}^{(f)} \frac{\partial\left(\bar{\chi}^{(f)}\right)_J^{-1}}{\partial x_i} \left(\ddot{\bar{J}}^{(f)} - \ddot{\hat{J}}^{(f)} \right) \right] N_J = 0. \tag{2.88}$$

Remark 2.9 Significance of Eq. (2.88)—For the readers not acquainted with variational methods, it is convenient to remark the legitimacy of the procedure employed for the derivation of Eqs. (2.85) and (2.88) and the operative significance of Eq. (2.88).

The derivation of Eqs. (2.85) and (2.88) exploits a common technique in variational methods which does not alter the underlying statement of the variational problem, as no further mechanical assumption have been added in such derivation.

Since Eqs. (2.85) and (2.88) have been derived considering the smaller subclass of virtual compatible isochoric deformations $\widetilde{\mathscr{D}}_{isoch} \subset \widetilde{\mathscr{D}}$ strictly contained in $\widetilde{\mathscr{D}}$ (see Remark 2.8), they represent a condition, to be respected by the solution of the boundary value problem, which is less strong than Eqs. (2.76)–(2.79); as such, Eqs. (2.85) and (2.88) are already implicitly contemplated by the primary general Eqs. (2.76)–(2.79). Also, Eq. (2.88) does not 'replace' the primary boundary conditions (2.79) and, as a matter of fact, the former holds on the surface $\mathscr{S}_0^{(sf)}$ which is, in general, a subset different from $\partial\Omega_0^{(s)}$.

The enucleation of the surface Eq. (2.88) is of significant practical relevance when the physical domain of interest is $\Omega_0^{(s)}$ and the objective of the analysis is the direct determination of the stress state inside the solid subdomain $\Omega_0^{(s)}$. In this case Eq. (2.88) represents the boundary condition for the extrinsic stress $\check{P}$ to be applied over free solid-fluid interfaces $\mathscr{S}_0^{(sf)}$. In the present monograph, Eq. (2.88) are employed in Chap. 4 to analyze the linear and nonlinear mechanical response of two-phase media in a variety of compression tests under different loading and drainage conditions.

Surface conditions to be fulfilled over $\partial\Omega_0^{(s)}$ can be specialized for the static case, when inertia terms are negligible. In this condition, relations (2.79) and (2.88) provide:

$$\begin{aligned} &\left(\check{P}_{iJ} - \bar{J}^{(f)}\phi^{(s)}\hat{\hat{\Pi}}^{(f)}\frac{\partial\left(\bar{\chi}^{(s)}\right)^{-1}_J}{\partial x_i} - \bar{J}^{(f)}\hat{\hat{\Pi}}^{(f)}\phi^{(f)}\frac{\partial\left(\bar{\chi}^{(f)}\right)^{-1}_J}{\partial x_i}\right)N_J = 0 \quad \text{over } \partial\Omega_0^{(M)}, \\ &\check{P}_{iJ}N_J = 0 \quad \text{over } \mathscr{S}_0^{(sf)}. \end{aligned} \tag{2.89}$$

Remark 2.10 Parallel with condition of zero Cauchy traction—It is worth observing that the condition (2.89), stating that $\check{\mathbf{P}}\mathbf{N} = \mathbf{o}$ over $\mathscr{S}_0^{(sf)}$, and the formally similar boundary relation for single-phase continuum elasticity $\mathbf{PN} = \mathbf{o}$ (where $\mathbf{P}$ is the classical first Piola tensor) have very different physical meanings. The condition $\mathbf{PN} = \mathbf{o}$ in a point of the boundary surface of the solid domain states that there is no mechanical interaction between the interior solid and the external environment at that point. Conversely, within the present two-phase continuum description, condition $\check{\mathbf{P}}\mathbf{N} = \mathbf{o}$ over $\mathscr{S}_0^{(sf)}$ does not imply that mechanical interaction between the interior solid and the fluid external to $\Omega_0^{(s)}$ is absent in points of $\mathscr{S}_0^{(sf)}$. As a matter of fact, $\mathscr{S}_0^{(sf)}$ is not strictly a boundary of the physical system and, over this surface, coupling between solid and fluid is also mediated by the intrinsic solid stress $\hat{\Pi}^{(s)}$ via the *intrinsic momentum balance* (2.78). Recalling that $\check{\mathbf{P}}$ is work-associated with isochoric strains, it is recognized that vanishing of $\check{\mathbf{P}}$ in a point of $\mathscr{S}_0^{(sf)}$ indicates that virtual isochoric strains applied to the solid in that point produce no strain energy exchange between the solid and the fluid. However, coupling between the solid and the surrounding fluid regions still remains mediated by the intrinsic stress entering (2.78).

2.2.5 Additional Solid-Fluid Interaction

The momentum balance equations derived in Sect. 2.2.4 have been obtained based on the mechanical hypotheses of Sect. 2.2.2.2 where only the individual strain energies $U_0^{(s)}$ and $U_0^{(f)}$ have been specified, and which still do not contemplate a specific solid-fluid interaction apart from the interaction of geometric/volumetric type determined by the saturation constraint (2.15) (see the coupling term in Eq. (2.76) containing the gradient of $\hat{\hat{\Pi}}^{(f)}$).

In multiphase porous media, the interaction between phases is not limited to this volumetric coupling due to saturation. An important role is also played by forces which solid and fluid mutually exchange, which are experimentally found to be dependent on the macroscopic relative solid-fluid flow. This additional interaction, which includes the well-known phenomenological Darcy-Forchheimer laws [56], is herein referred to as *drag interaction*, and is represented by an additional term, $L_0^{(sf)}$, in Lagrange function. Such interaction is examined aiming at its simplest and most general description to preserve the medium independence of the present theoretical approach. Accordingly, in order to achieve a kernel of equations of maximum generality conveniently embracing the widest possible class of two-phase media, minimal assumptions are introduced for $L_0^{(sf)}$. Specifically, for the drag interaction, the only two features that are herein considered are: (1) the *internal force character*; and (2) the local *short-range character*.

Specifically, by *internal force character* we refer to the property of a given term of the Lagrange function, characteristic of internal forces in isolated systems, to fulfill homogeneity of space. Space homogeneity of a Lagrange function L is the invariance of L to parallel translations $\mathbf{v}$ applied to the physical system [48]. This property, which implies the conservation of linear momentum, is fulfilled by the internal strain energies $U_0^{(s)}$ and $U_0^{(f)}$. Actually, definitions (2.35) and (2.34) and the first-gradient nature of $\bar{\psi}_0^{(s)}$ and $\hat{\psi}_0^{(f)}$ imply that the following property holds for any finite uniform translation field $\mathbf{v}$:

$$U_0^{(s)}\left(\bar{\chi}^{(s)},\ \hat{J}^{(s)}\right) = U_0^{(s)}\left(\bar{\chi}^{(s)} + \mathbf{v},\ \hat{J}^{(s)}\right), \tag{2.90}$$

$$U_0^{(f)}\left(\bar{\chi}^{(s)},\ \bar{\chi}^{(f)},\ \hat{J}^{(s)}\right) = U_0^{(f)}\left(\bar{\chi}^{(s)} + \mathbf{v},\ \bar{\chi}^{(f)} + \mathbf{v},\ \hat{J}^{(s)}\right). \tag{2.91}$$

This property is also stated in terms of variations as follows:

$$\partial_{\bar{\chi}^{(s)}} U_0^{(s)}\,[\delta\mathbf{v}] = 0, \qquad \partial_{\bar{\chi}^{(s)}} U_0^{(f)}\,[\delta\mathbf{v}] + \partial_{\bar{\chi}^{(f)}} U_0^{(f)}\,[\delta\mathbf{v}] = 0, \tag{2.92}$$

where $\delta\mathbf{v}$ is a uniform infinitesimal virtual variation field. In particular, accounting for the null gradient of $\delta\mathbf{v}$, the first of (2.92) is directly inferred from (B.1) (see Appendix), while the second of (2.92) is deduced from (2.68) observing that:

$$\partial_{\bar{\chi}^{(s)}} \hat{J}^{(f)}_{sat\,\bar{\chi}^{(f)}}\,[\delta\mathbf{v}] + \partial_{\bar{\chi}^{(f)}} \hat{J}^{(f)}_{sat\,\bar{\chi}^{(f)}}\,[\delta\mathbf{v}] = \frac{\bar{J}^{(f)}}{\Phi_0^{(f)}}\frac{\partial\left(\phi^{(s)} + \phi^{(f)}\right)}{\partial x_i}\delta v_i = 0. \tag{2.93}$$

since $\phi^{(s)} + \phi^{(f)} = 1$ by (2.2).

The requirement that also $L_0^{(sf)}$ must have an *internal force character* is stated in terms analogous to (2.92):

$$\partial_{\bar{\chi}^{(s)}} L_0^{(sf)}\,[\delta\mathbf{v}] + \partial_{\bar{\chi}^{(f)}} L_0^{(sf)}\,[\delta\mathbf{v}] = 0 \tag{2.94}$$

The second hypothesis of *short-range character* implies that, similar to $U_0^{(s)}$ and $U_0^{(f)}$ which are expressed in terms of local strain energy density, also $L_0^{(sf)}$ can be expressed as the space integral of a macroscopic local energy density function $\bar{\psi}_{\mathbf{x}}^{(sf)}$, viz.:

$$L_0^{(sf)} = \int_{\Omega^{(M)}} \bar{\psi}_{\mathbf{x}}^{(sf)} \, dV. \tag{2.95}$$

The integral form provided by (2.94) allows one to write:

$$\int_{\Omega^{(M)}} \left(\frac{\partial \bar{\psi}^{(sf)}}{\partial \bar{\chi}_i^{(s)}} \delta v_i + \frac{\partial \bar{\psi}^{(sf)}}{\partial \bar{\chi}_i^{(f)}} \delta v_i \right) dV = 0. \tag{2.96}$$

The direct consequence of hypotheses (1) and (2) is that the drag interaction can be described by a field of local *drag volume forces* $\bar{\mathbf{b}}^{(sf)}$ and $\bar{\mathbf{b}}^{(fs)}$, defined as:

$$\bar{\mathbf{b}}^{(sf)}(\mathbf{x}) = -\frac{\partial \bar{\psi}^{(sf)}}{\partial \bar{\boldsymbol{\chi}}^{(s)}}, \qquad \bar{\mathbf{b}}^{(fs)}(\mathbf{x}) = -\frac{\partial \bar{\psi}^{(sf)}}{\partial \bar{\boldsymbol{\chi}}^{(f)}}. \tag{2.97}$$

Such forces must have equal magnitude and opposite direction. Actually, since Eq. (2.96) is a spatial relation holding for any subdomain $\Omega \subseteq \Omega^{(M)}$, and since it must also hold for any vector $\delta \mathbf{v}$, one infers that:

$$\frac{\partial \bar{\psi}^{(sf)}}{\partial \bar{\boldsymbol{\chi}}^{(s)}} + \frac{\partial \bar{\psi}^{(sf)}}{\partial \bar{\boldsymbol{\chi}}^{(f)}} = 0, \qquad \text{over } \Omega^{(M)} \tag{2.98}$$

which reads (in field format):

$$\bar{\mathbf{b}}_{\mathbf{x}}^{(fs)} = -\bar{\mathbf{b}}_{\mathbf{x}}^{(sf)}. \tag{2.99}$$

Inclusion of $L_0^{(sf)}$ into the model requires the addition of the terms $\partial_{\bar{\boldsymbol{\chi}}^{(s)}} L_0^{(sf)} \left[\bar{\boldsymbol{\chi}}^{(s)} \right]$ and $\partial_{\bar{\boldsymbol{\chi}}^{(f)}} L_0^{(sf)} \left[\delta \bar{\boldsymbol{\chi}}^{(f)} \right]$ into the general expression of the variation (2.62). These terms, when expressed as integrals over the reference domain of the mixture $\bar{\boldsymbol{\chi}}^{(s)}$, read:

$$\partial_{\bar{\boldsymbol{\chi}}^{(s)}} L_0^{(sf)} \left[\bar{\boldsymbol{\chi}}^{(s)} \right] = -\int_{\Omega_0^{(M)}} \bar{J}^{(s)} \bar{\mathbf{b}}_{\bar{\boldsymbol{\chi}}^{(s)}}^{(sf)} \delta \bar{\boldsymbol{\chi}}^{(s)} \, dV_0, \tag{2.100}$$

$$\partial_{\bar{\boldsymbol{\chi}}^{(f)}} L_0^{(sf)} \left[\bar{\boldsymbol{\chi}}^{(f)} \right] = -\int_{\Omega_0^{(M)}} \bar{J}^{(f)} \bar{\mathbf{b}}_{\bar{\boldsymbol{\chi}}^{(f)}}^{(fs)} \delta \bar{\boldsymbol{\chi}}^{(f)} \, dV_0, \tag{2.101}$$

where, consistent with (2.102), we have:

$$\bar{\mathbf{b}}_{\bar{\boldsymbol{\chi}}^{(s)}}^{(sf)} \circ \left(\bar{\boldsymbol{\chi}}^{(s)} \right)^{-1} + \bar{\mathbf{b}}_{\bar{\boldsymbol{\chi}}^{(f)}}^{(fs)} \circ \left(\bar{\boldsymbol{\chi}}^{(f)} \right)^{-1} = 0. \tag{2.102}$$

The strong form equations resulting from the inclusion of this medium-independent solid-fluid interaction are summarized below. To place emphasis on medium independence, microinertia terms, which have in some respects a constitutive medium-dependent character, are purposefully not included.

Linear momentum balance of the solid phase:

$$\frac{\partial \check{P}_{iJ}}{\partial X_J^{(s)}} - \phi^{(s)} \frac{\partial}{\partial X_J}\left(\bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i} \hat{\hat{\Pi}}^{(f)} \right) + \bar{b}_{0i}^{(s,ext)} + \bar{J}^{(s)} \bar{b}_i^{(sf)} = \bar{\rho}_0^{(s)} \ddot{\bar{\chi}}_i^{(s)} \tag{2.103}$$

Linear momentum balance of the fluid phase:

$$-\phi^{(f)} \frac{\partial}{\partial X_J}\left(\bar{J}^{(f)} \frac{\partial \left(\bar{\chi}^{(f)}\right)_J^{-1}}{\partial x_i} \hat{\hat{\Pi}}^{(f)} \right) + \bar{b}_{0i}^{(f,ext)} + \bar{J}^{(f)} \bar{b}_i^{(fs)} = \bar{\rho}_0^{(f)} \ddot{\bar{\chi}}_i^{(f)} \tag{2.104}$$

Intrinsic momentum balance:

$$\bar{J}^{(s)} \hat{\Pi}^{(s)} - \bar{J}^{(f)} \Phi_0^{(s)} \hat{\hat{\Pi}}^{(f)} = 0 \tag{2.105}$$

The domain equations above are completed by boundary conditions (2.89).

2.2.6 The Kinematically-Linear Medium-Independent Problem

The strong form equations of Sect. 2.2.4 are now specialized for the kinematically linear static boundary value problem, suitable for the description of problems with infinitesimal deformations. This specialization is carried out in a standard way as a first-order Taylor series truncation of deformation and strain measures, and has the primary effect of making the reference and current configurations coincide:

$$\Omega_0^{(M)} \simeq \Omega^{(M)}, \qquad \Omega_0^{(s)} \simeq \Omega^{(s)}, \qquad \Omega_0^{(f)} \simeq \Omega^{(f)}. \tag{2.106}$$

The coincidence of reference and current configurations is reflected in the notation henceforth used by dropping the lowercase/uppercase distinction previously applied to subscripts. Also, the prefix d is used to denote infinitesimal quantities. To achieve a less dense notation, such prefix is omitted for primary descriptor fields, and for symbols ordinarily employed to denote infinitesimal strain measures: $\bar{\boldsymbol{\varepsilon}}^{(s)}$, $\bar{e}^{(s)}$, $\bar{e}^{(f)}$, $\hat{e}^{(s)}$, and $\hat{e}^{(f)}$ (where the prefix is redundant).

Primary descriptors of the linearized formulation are the solid and fluid infinitesimal displacements $\bar{\mathbf{u}}^{(s)} = d\bar{\boldsymbol{\chi}}^{(s)}$ and $\bar{\mathbf{u}}^{(f)} = d\bar{\boldsymbol{\chi}}^{(f)}$ and the infinitesimal intrinsic volumetric strain $\hat{e}^{(s)}$.

Finite volumetric strain measures are replaced by linearized volumetric strain measures $\bar{e}^{(s)}$, $\bar{e}^{(f)}$, $\hat{e}^{(s)}$, and $\hat{e}^{(f)}$, previously defined in (2.8) and (2.12). Their relation is:

$$\begin{gathered} \bar{J}^{(s)} \simeq 1 + \bar{e}^{(s)} = 1 + \frac{\partial d\bar{\chi}_i^{(s)}}{\partial x_i}, \quad \bar{J}^{(f)} \simeq 1 + \bar{e}^{(f)} = 1 + \frac{\partial d\bar{\chi}_i^{(f)}}{\partial x_i}, \\ \hat{J}^{(s)} \simeq 1 + \hat{e}^{(s)}, \quad \hat{J}^{(f)} \simeq 1 + \hat{e}^{(f)}. \end{gathered} \tag{2.107}$$

In the linearized theory, the extrinsic strain of the solid is defined by the infinitesimal extrinsic strain tensor

$$\bar{\boldsymbol{\varepsilon}}^{(s)} = \mathrm{sym}\left(\bar{\mathbf{u}}^{(s)} \otimes \nabla\right) \tag{2.108}$$

and by $\hat{e}^{(s)}$.

For stress measures, the simplifications stemming from the coincidence of reference and current configurations:

$$\begin{gathered} \frac{\partial \bar{\chi}_i^{(s)}}{\partial X_J} \simeq \delta_{ij}, \ \frac{\partial \bar{\chi}_i^{(f)}}{\partial X_J} \simeq \delta_{ij}, \ \frac{\partial \left(\bar{\chi}^{(s)}\right)_J^{-1}}{\partial x_i} = \delta_{ji}, \\ \frac{\partial \left(\bar{\chi}^{(f)}\right)_J^{-1}}{\partial x_i} = \delta_{ji}, \ \bar{J}^{(s)} \simeq 1, \ \bar{J}^{(f)} \simeq 1, \ \hat{J}^{(s)} \simeq 1, \ \hat{J}^{(f)} \simeq 1 \end{gathered} \tag{2.109}$$

are applied to Eqs. (2.45), (2.46), and (2.49)–(2.52), obtaining the following identifications:

$$\check{P}_{iJ}^{(s)} \simeq \check{\sigma}_{ij}^{(s)}, \qquad \hat{\Pi}^{(s)} \simeq \hat{p}^{(s)}, \qquad \hat{\hat{\Pi}}^{(f)} \simeq p, \tag{2.110}$$

so that the fields of spatial extrinsic solid stress tensor, spatial intrinsic solid pressure and fluid pressure provide a complete description of the stress state in the mixture.

Quadratic forms of strain energy

The further choice of a linear constitutive theory determines strain energy densities which are quadratic forms in the relevant infinitesimal strain measures [75]:

$$\bar{\psi}^{(s)} = \frac{1}{2} A_{ijkl} \bar{\varepsilon}_{ij}^{(s)} \bar{\varepsilon}_{kl}^{(s)} + B_{ij} \bar{\varepsilon}_{ij}^{(s)} \hat{e}^{(s)} + \frac{1}{2} C \left(\hat{e}^{(s)}\right)^2, \tag{2.111}$$

$$\bar{\psi}^{(f)} = \phi^{(f)} \frac{1}{2} \hat{k}_f \left(\hat{e}^{(f)}\right)^2, \tag{2.112}$$

where the elastic coefficients of the solid and the fluid phases, A_{ijkl}, B_{ij}, C and $\hat{k}_f$, correspond to the second-order derivatives of the corresponding strain energy densities, and with respect to the relevant primary infinitesimal strain measures. In particular, $\hat{k}_f$ is the fluid intrinsic stiffness. Also, the elastic coefficients in the energy of solid must respect the so-called major and minor symmetries (i.e., $A_{ijkl} = A_{klij} = A_{ijlk}$, $B_{ij} = B_{ji}$) which stem from Schwarz's theorem and from the requirement of objectivity. Stress-strain relations for the solid are:

$$\check{\sigma}_{ij}^{(s)} = \frac{\partial \bar{\psi}^{(s)}}{\partial \bar{\varepsilon}_{ij}^{(s)}} = A_{ijkl} \bar{\varepsilon}_{ij}^{(s)} + B_{ij} \hat{e}^{(s)}, \qquad \hat{p}^{(s)} = -\frac{\partial \bar{\psi}^{(s)}}{\partial \hat{e}^{(s)}} = -B_{ij} \bar{\varepsilon}_{ij}^{(s)} - C \hat{e}^{(s)}, \tag{2.113}$$

and the relation between fluid pressure and intrinsic strain is:

$$\hat{k}_f = \frac{\partial^2 \bar{\psi}^{(f)}}{\partial \hat{e}^{(f)} \partial \hat{e}^{(f)}}, \qquad p = -\hat{k}_f \hat{e}^{(f)}. \tag{2.114}$$

Linearized saturation condition
The saturation condition for the kinematic linear problem can be obtained applying the linearization provided by (2.70) to real deformations, i.e., replacing $\delta \bar{\boldsymbol{\chi}}^{(s)}$ and $\delta \bar{\boldsymbol{\chi}}^{(f)}$ with $\bar{\mathbf{u}}^{(s)}$ and $\bar{\mathbf{u}}^{(f)}$ in (2.70). Accordingly, the following *linearized saturation constraint* is obtained:

$$\phi^{(s)} \hat{e}^{(s)} + \phi^{(f)} \hat{e}^{(f)} = \frac{\partial \phi^{(s)} \bar{u}_i^{(s)}}{\partial x_i} + \frac{\partial \phi^{(f)} \bar{u}_i^{(f)}}{\partial x_i}. \tag{2.115}$$

Equation (2.115) is reported also by Bedford and Drumheller ([7] (see Eq. (42) therein) in a form combined with the mass balances, see also [77]. It is interesting to observe that, herein, this condition is derived independently from any consideration on mass balances. It is also worth noting that, under the hypothesis that the fields $\phi^{(s)}$ and $\phi^{(f)}$ are uniform, the same relation can be directly obtained by combining (2.10) and (2.11) with the linearization of (2.2):

$$d\phi^{(s)} + d\phi^{(f)} = 0. \tag{2.116}$$

Momentum balances with inertia terms
The momentum balances for the kinematic linear problem are obtained specializing (2.76), (2.77) and (2.78), by applying simplifications (2.109). This yields the following kinematically linear momentum balances:

Linear momentum balance of the solid phase

$$\frac{\partial \check{\sigma}_{ij}^{(s)}}{\partial x_j} - \phi^{(s)} \frac{\partial p}{\partial x_i} + \bar{b}_i^{(sf)} + \bar{b}_i^{(s,ext)} = \bar{\rho}^{(s)} \ddot{\bar{u}}_i^{(s)} + \frac{\partial}{\partial x_j} \left[\bar{\rho}_{add.}^{(s)} \left(\frac{\partial \ddot{\bar{u}}_i^{(s)}}{\partial x_i} - \ddot{\hat{e}}^{(s)} \right) \right] \tag{2.117}$$

Linear momentum balance of the fluid phase:

$$-\phi^{(f)} \frac{\partial p}{\partial x_i} + \bar{b}_i^{(fs)} + \bar{b}_i^{(f,ext)} = \bar{\rho}^{(f)} \ddot{\bar{u}}_i^{(f)} - \frac{\partial}{\partial x_j} \left[\bar{\rho}_{add.}^{(f)} \left(\frac{\partial \ddot{\bar{u}}_i^{(f)}}{\partial x_i} - \ddot{\hat{e}}^{(f)} \right) \right] \tag{2.118}$$

Intrinsic momentum balance:

$$\frac{\hat{p}^{(s)}}{\phi^{(s)}} - p = -\frac{1}{\phi^{(s)}} \bar{\rho}_{add.}^{(s)} \left(\frac{\partial \ddot{\bar{u}}_i^{(s)}}{\partial x_i} - \ddot{\hat{e}}^{(s)} \right) + \frac{1}{\phi^{(f)}} \bar{\rho}_{add.}^{(f)} \left(\frac{\partial \ddot{\bar{u}}_i^{(f)}}{\partial x_i} - \ddot{\hat{e}}^{(f)} \right) \tag{2.119}$$

Remark 2.11 Symmetric form of the intrinsic momentum balance—It is interesting to observe that the intrinsic momentum balance (2.119) can be written in a notation symmetric with respect to indexes s and f. Actually, if one introduces the counterpart of $\hat{\hat{p}}^{(f)}$ for the solid phase by the symmetric definition $\hat{\hat{p}}^{(s)} = -\dfrac{\partial \hat{\psi}_0^{(s)}}{\partial \hat{e}^{(s)}} = \dfrac{\hat{p}^{(s)}}{\phi^{(s)}}$, then, recalling also (2.52), Eq. (2.119) achieves the symmetric notation:

$$\hat{\hat{p}}^{(s)} - \hat{\hat{p}}^{(f)} = -\frac{1}{\phi^{(s)}} \bar{\rho}^{(s)}_{add.} \left(\frac{\partial \ddot{u}_i^{(s)}}{\partial x_i} - \ddot{e}^{(s)} \right) + \frac{1}{\phi^{(f)}} \bar{\rho}^{(f)}_{add.} \left(\frac{\partial \ddot{u}_i^{(f)}}{\partial x_i} - \ddot{e}^{(f)} \right). \tag{2.120}$$

We prefer, however, to maintain the unsymmetric notation with stress quantities $\hat{p}^{(s)}$ and p in consideration of their more direct physical identification, and also to avoid proliferation of stress notations.

Boundary conditions
The boundary conditions of the kinematic linear problem are obtained from the specialization of (2.79) provided by (2.109). Recalling that $\phi^{(s)} + \phi^{(f)} = 1$, these turn out to be:

$$\left[\check{\sigma}_{ij}^{(s)} - p\delta_{ij} + \bar{\rho}^{(s)}_{add.} \left(\frac{\partial \ddot{u}_i^{(s)}}{\partial x_i} - \ddot{e}^{(s)} \right) \delta_{ij} + \bar{\rho}^{(f)}_{add.} \left(\frac{\partial \ddot{u}_i^{(f)}}{\partial x_i} - \ddot{e}^{(f)} \right) \delta_{ij} \right] n_j = t_i^{(ext)} \quad \text{over } \partial\Omega^{(M)}. \tag{2.121}$$

A derivation of (2.121) based on simplified arguments in a 1D setting has been also previously reported [75].

Over free solid-fluid interfaces $\mathscr{S}^{(sf)}$ (Fig. 2.2), the relation inferred from (2.88) and (2.109) is:

$$\left[\frac{1}{\phi^{(s)}} \check{\sigma}_{ij}^{(s)} + \frac{1}{\phi^{(s)}} \bar{\rho}^{(s)}_{add.} \left(\frac{\partial \ddot{u}_i^{(s)}}{\partial x_i} - \ddot{e}^{(s)} \right) \delta_{ij} - \frac{1}{\phi^{(f)}} \bar{\rho}^{(f)}_{add.} \left(\frac{\partial \ddot{u}_i^{(f)}}{\partial x_i} - \ddot{e}^{(f)} \right) \delta_{ij} \right] n_j = 0 \quad \text{over } \mathscr{S}^{(sf)}. \tag{2.122}$$

2.2.7 Equations for Static and Quasi-static Problems

The kinematically linear equations are hereby specialized to their form suitable for problems in which inertia terms can be neglected. Accordingly, upon neglecting inertia terms in (2.117), (2.118) and (2.120), one obtains:

Linear momentum balance of the solid phase (static and quasi-static problems)

$$\frac{\partial \check{\sigma}_{ij}^{(s)}}{\partial x_j} - \phi^{(s)} \frac{\partial p}{\partial x_i} + \bar{b}_i^{(sf)} + \bar{b}_i^{(s,ext)} = 0 \tag{2.123}$$

Linear momentum balance of the fluid phase (static and quasi-static problems):

$$-\phi^{(f)}\frac{\partial p}{\partial x_i}+\bar{b}_i^{(fs)}+\bar{b}_i^{(f,ext)}=0 \tag{2.124}$$

Intrinsic momentum balance (static and quasi-static problems):

$$\hat{p}^{(s)}-\phi^{(s)}p=0 \tag{2.125}$$

Observe that Eqs. (2.123)–(2.125) have a medium-independent character since they can be obtained by a direct linearization of (2.103)–(2.105).

The boundary and surface conditions for static and quasi-static problems are obtained from (2.121). For stresses, one obtains a relation which, based on simplified arguments in a 1D setting, has been previously reported [75]:

$$\left(\breve{\sigma}_{ij}^{(s)}-p\delta_{ij}\right)n_j=t_i^{(ext)} \qquad \text{over } \partial\Omega^{(M)}, \tag{2.126}$$

while the boundary condition for displacements is

$$\bar{u}_i^{(s)}=\bar{u}_i^{(f)}=u_i^{(ext)}. \tag{2.127}$$

In particular, over $\partial\Omega^{(M)}\backslash\partial\Omega^{(MU)}$, where $\phi^{(s)}=0$ and, consequently, $\breve{\boldsymbol{\sigma}}^{(s)}=\mathbf{o}$, Eq. (2.126) specializes to:

$$-pn_i=t_i^{(ext)} \qquad \text{over } \partial\Omega^{(M)}\backslash\partial\Omega^{(MU)}. \tag{2.128}$$

Over free solid-fluid interfaces $\mathscr{S}^{(sf)}$ (Fig. 2.2), the relation inferred from (2.122) is:

$$\breve{\sigma}_{ij}^{(s)}n_j=0 \qquad \text{over } \mathscr{S}^{(sf)}. \tag{2.129}$$

It is worth to recall that the considerations previously reported on the physical meaning of a vanishing normal extrinsic stress traction (see Sect. 2.2.4) apply also to (2.129), which is a special case of (2.89).

2.2.7.1 Medium-Independent Stress Partitioning Law

In this subsection, we examine the consequences of Eqs. (2.123)–(2.129) in terms of stress partitioning. We consider those situations in which the stress state of the mixture can be characterized, in a physically meaningful way, by constant stress measures, associated with the entire physical system of the mixture. Two general properties of stress partitioning are specifically examined here: (1) stress partitioning in mixtures undergoing deformation and stress states which are uniform in space, and (2) partitioning in *undrained* mixtures characterized by the property that the macroscopic relative solid-fluid motion induced by deformation is null everywhere. This second condition is also examined in further detail in Chap. 4.

Stress partitioning in mixtures undergoing space-homogeneous deformation states
We introduce the hypothesis that the stress state of the mixture in $\Omega^{(M)}$ is macroscopically uniform, i.e., that fields $\check{\boldsymbol{\sigma}}^{(s)}$ and p have constant values, say $\check{\boldsymbol{\sigma}}_h^{(s)}$, p_h. In this situation, Eq. (2.126) specializes to

$$\mathbf{t}^{(ext)}(\mathbf{x},\ \mathbf{n}) = \check{\boldsymbol{\sigma}}_h^{(s)}\mathbf{n} - p_h\mathbf{n}, \qquad \mathbf{x} \in \partial\Omega^{(M)}, \tag{2.130}$$

so that the external traction field $\mathbf{t}^{(ext)}(\mathbf{x},\ \mathbf{n})$ in (2.130) can be represented by a single constant tensor $\boldsymbol{\sigma}^{(ext)}$:

$$\mathbf{t}^{(ext)}(\mathbf{x},\ \mathbf{n}) = \boldsymbol{\sigma}^{(ext)}\mathbf{n}, \qquad \mathbf{x} \in \partial\Omega^{(M)}, \tag{2.131}$$

where $\boldsymbol{\sigma}^{(ext)}$ is a tensor (associated with the whole domain $\Omega^{(M)}$) defined as

$$\boldsymbol{\sigma}^{(ext)} = \check{\boldsymbol{\sigma}}_h^{(s)} - p_h\mathbf{I}. \tag{2.132}$$

Hence, when a traction is applied over the boundary $\partial\Omega^{(M)}$ so as to produce a uniform stress state $\boldsymbol{\sigma}^{(ext)}$ inside $\Omega^{(M)}$, it is then partitioned in compliance with relation (2.132). Such partitioning is independent from the particular constitutive and microstructural features of the medium considered.

Stress partitioning in regions undergoing undrained flow
A second important consequence of (2.126) is inferred for those regions, $\Omega^{(U)} \subseteq \Omega^{(M)}$, undergoing undrained flow conditions, which are conditions of null relative solid-fluid motion, i.e., $\bar{\mathbf{u}}^{(s)} - \bar{\mathbf{u}}^{(f)} = \mathbf{o}$. This hypothesis implies that any subdomain $\bar{\Omega}^{(U)} \subseteq \Omega^{(U)}$ is a physical system where mass exchanges at the boundary $\partial\bar{\Omega}^{(U)}$ with the mass external to $\bar{\Omega}^{(U)}$ is prevented. This peculiar condition, implies absence of small-scale mixing of the material inside $\bar{\Omega}^{(U)}$ with the material of the environment external to $\bar{\Omega}^{(U)}$ (see Remark 2.4) and allows to construct a boundary value problem for domain $\bar{\Omega}^{(U)}$ in the same way as it has been constructed in this chapter for domain $\Omega_0^{(M)} \simeq \Omega^{(M)}$. For this reason surface $\partial\bar{\Omega}^{(U)}$ is also a boundary of the physical subsystem contained in $\bar{\Omega}^{(U)}$, where relation (2.126) consequently applies, viz.:

$$\left(\check{\sigma}_{ij}^{(s)} - p\delta_{ij}\right)n_j = t_i^{(ext)} \qquad \text{over } \partial\bar{\Omega}^{(U)}. \tag{2.133}$$

Considering (macroscopically) continuous stress fields, due to the arbitrariness of $\bar{\Omega}^{(U)}$, for any point $\mathbf{x} \in \Omega^{(U)}$, a sufficiently small domain $\bar{\Omega}_{\mathbf{x}}^{(U)} \subseteq \Omega^{(U)}$ centered in $\mathbf{x}$ can be found such that the condition of macroscopic homogeneity of the stress state can be recovered in $\bar{\Omega}_{\mathbf{x}}^{(U)}$, as a limit. This condition of uniformity of stresses in $\bar{\Omega}_{\mathbf{x}}^{(U)}$ implies that relations (2.131), (2.132) apply, in an even stronger form, associated with any point $\mathbf{x} \in \Omega^{(U)}$, viz.:

$$\mathbf{t}^{(ext)}(\mathbf{x}, \mathbf{n}) = \boldsymbol{\sigma}^{(ext)}\mathbf{n}, \qquad \forall \mathbf{x} \in \Omega^{(U)}, \ \forall \mathbf{n}, \tag{2.134}$$

where $\boldsymbol{\sigma}^{(ext)}$ is a tensor equal to

$$\boldsymbol{\sigma}^{(ext)} = \check{\boldsymbol{\sigma}}^{(s)} - p\mathbf{I}. \tag{2.135}$$

Since tensor $\boldsymbol{\sigma}^{(ext)}$ is associated, in a physically meaningful way, with any point $\mathbf{x} \in \Omega^{(U)}$ of the undrained region, it is recognized that undrained flow conditions determine the existence of a tensor field $\boldsymbol{\sigma}^{(ext)}$ in $\Omega^{(U)}$.

As observed also in these previous works, relations (2.132) and (2.135) coincide, from a formal point of view, with the classical statement of Terzaghi's principle if one identifies $\check{\boldsymbol{\sigma}}^{(s)}$ with the effective stress tensor. However, it is important to remark that, herein, this condition has been derived in absence of any constitutive hypothesis on the phases, thus representing a stress partitioning law of general validity for homogeneous stress states, not limited to soils.

Remark 2.12 Validity of Terzaghi-like partitioning law beyond the purely-mechanical theory—The Terzaghi-like stress partitioning laws (2.132) and (2.135) stem directly from the general momentum balance equations and boundary conditions in the context of the present general continuum mixture theory where completely general strain potentials have been considered, and in the context of a finite-deformation theory where no ad-hoc constitutive hypotheses have been applied. Hence the stress partitioning laws herein derived should be considered general laws, at least, in so far as linear momentum balances and boundary equations are considered to be general within single-phase nonlinear finite-deformation hyperelasticity [58, 67]. In this respect, it is also worth to anticipate that (2.132) finds an experimental confirmation in Chap. 4, see in particular Sect. 4.5.2.

The only limit to the generality of laws (2.132) and (2.135) is represented by the purely-mechanical character of the theory presented in this chapter, what may raise questions on whether these laws still hold true beyond the purely-mechanical scenario, i.e., for dissipative media. Such a question is significant, for instance, in view of the application of the present theory to geotechnical problem where the rate-independent elastoplastic behavior of saturated soils is of special interest for modeling soil compaction. Another fundamental dissipative mechanism in open-cell porous media is filtration, which is involved in a multiplicity of consolidation and transport mechanisms, and which introduces a rate-dependent behavior in deformation processes.

As the class of nonlinear dissipative behaviors which can be contemplated in a continuum multiphase theory is very large, an answer addressing the problem of encompassing irreversible deformation processes in the present multiphase theory would be desirable. However, this issue represents a theoretical problem of continuum mechanics which finds not an easy solution, since, even for single-phase continuum problems continuum, there exist non-univocal line of thoughts on how a standard theory of *continuum thermodinamics* should be constructed (see, in particular, [45, 63, 84, 86, 87]), and the thermodynamics of continuum multiphase models [11, 19, 63, 83] presents even more controversial issues. Hence, refraining from giving a general answer, we limit here to expose two simple considerations concerning the recovery of the general purely-mechanical partitioning laws (2.132) and (2.135) by two classes of dissipative two-phase models: (1) two-phase models endowed with a rate-dependent seepage law associated with the relative solid-fluid motion, (encompassing drag forces, such as those described by Darcy or Forchheimer laws) and, (2) two-phase models with solid phase endowed with rate-independent elastoplasticity.

For the former class of rate-dependent models, it should be observed that it is no longer possible to find, outside of equilibrium, a straight univocal stress partitioning relation of type (2.132) and (2.135). Actually, outside of equilibrium, as characteristic of rate-dependent models, the transient response to external stresses applied at the boundary of the mixture (and consequently the modalities of stress partitioning) is time-dependent. Moreover, the way stress is partitioned depends on the whole boundary value problem, included on boundary conditions and drainage conditions. Also, as well known, gradients of fluid pressure are originated in the mixture during consolidation, so that the deformation field is macroscopically non-uniform in space, while (2.132) is applicable only when conditions of uniform strain are met. Similarly, (2.135) is not applicable in the transient regime since it only applies when relative sulid-fluid motion is completely prevented. The latter condition is approached, instead, when the characteristic loading time is much higher than the characteristic consolidation time of the mixture. These considerations are further elaborated in Chap. 5 with the transient analysis of a consolidation problem. However, when the transient response is finalized, at equilibrium, the response of the system is expected to match the one predicted by the present general purely-mechanical model and hence relation (2.132) and (2.135), maintain their validity in presence of homogeneous deformation states and prevented relative solid-fluid motion, respectively.

For the second considered class of extended two-phase models, in which the only dissipative feature considered is the elastoplasticity of the solid phase, we limit to observe that if the elastoplastic behavior is included according to the Standard Generalized Model (SGM) [39, 54, 61, 63] by mechanical assumptions which proceed by a straightforward parallel with those employed for the isothermal single-phase elastoplasticity theory, then Eqs. (2.132) and (2.135) are still applicable. Actually, proceeding by steps analogous to those of the single-phase elastoplastic theory (referring to the linear model with infinitesimal displacement, for simplicity in the exposition), the strain energy density of the solid phase in the purely mechanical model, $\bar{\psi}^{(s)}\left(\bar{\boldsymbol{\varepsilon}}^{(s)},\ \hat{e}^{(s)}\right)$, is generalized into a density of global energy of the solid phase (inclusive of thermal energy) $\bar{\psi}^{(s)}_{glob}\left(\bar{\boldsymbol{\varepsilon}}^{(s)},\ \hat{e}^{(s)},\ \bar{\boldsymbol{\varepsilon}}^{(s)}_P,\ \hat{e}^{(s)}_P\right)$ and with the addition of a pseudo-potential $\bar{\varphi}^{(s)}$

providing the complementary constitutive equations. This construction, which is carried out as an exercise in Remark 2.13, has the effect of modifying the equations of the purely mechanical theory only limited to the constitutive equations of the solid phase, without altering the momentum balance equations and the boundary conditions of the purely mechanical theory shown in Sect. 2.2.4. Actually, the derivation of the Euler-Lagrange equations and boundary conditions carried out in Sect. 2.2.3 remains, from a merely formal point of view, the same, with the only replacement of $\bar{\psi}^{(s)}$ with $\bar{\psi}^{(s)}_{glob}$. Specifically, the modification of the constitutive laws is that the simple solid elastic stress-strain law (2.113) are generalized into (2.136)–(2.142). This implies that (2.132) and (2.135), which are a direct consequence of momentum balance equations and boundary equations still continue to be applicable.

Concerning the preservation of the momentum balance equations, it is worth to observe that also in single-continuum elastoplasticity the momentum balance equations do not change when one switches from the hyperelastic theory to the elastoplastic one: linear momentum balances remain the *universal* equations to be applied.

Remark 2.13 Addition of rate-independent plasticity to the solid phase– To support the considerations elaborated in Remark 2.12, and also as a case study, a rate-independent elastoplastic behavior is hereby appended, within the present theory, to the solid phase, proceeding by steps which follow a straightforward parallel with those employed in the framework of a Standard Generalized Model (SGM) [39, 61, 63], for constructing an isothermal single-phase elastoplasticity continuum theory in a standard way. For simplicity, we consider perfect plasticity and infinitesimal kinematics. Accordingly, the steps followed for such an enhancement are listed below.

- A family of thermodynamically-admissibile rate-independent constitutive responses for the solid phase is formulated by adding internal plastic strain variables $\bar{\boldsymbol{\varepsilon}}^{(s)}_P$, $\hat{e}^{(s)}_P$ whose evolution is responsible for dissipation.
- Stress measures and thermodynamic forces of the solid phase are defined employing two non-negative convex functions: a density of global internal energy of the solid phase, $\bar{\psi}^{(s)}_{glob} = \bar{\psi}^{(s)}_{glob}\left(\bar{\boldsymbol{\varepsilon}}^{(s)}, \hat{e}^{(s)}, \bar{\boldsymbol{\varepsilon}}^{(s)}_P, \hat{e}^{(s)}_P\right)$, and a pseudo-potential density function $\bar{\varphi}^{(s)}$ of the solid phase. Primary stress measures are defined by work-association with strain variables in a way similar to (2.44)

$$\check{\sigma}^{(s)}_{ij} = \frac{\partial \bar{\psi}^{(s)}_{glob}}{\partial \bar{\varepsilon}^{(s)}_{ij}}, \qquad \hat{p}^{(s)} = -\frac{\partial \bar{\psi}^{(s)}_{glob}}{\partial \hat{e}^{(s)}}. \tag{2.136}$$

The definition in (2.136) is more general than (2.44) and embraces it when when plastic variables do not evolve in a deformation process, in which case one has that (excluding external volume forces) variations of $\bar{\psi}^{(s)}_{glob}$ coincide with variations of $\bar{\psi}^{(s)}$.

- Inelastic driving forces work-associated with the plastic strain variables are:

$$\check{\sigma}^{(s)}_{Pij} = -\frac{\partial \bar{\psi}^{(s)}_{glob}}{\partial \bar{\varepsilon}^{(s)}_{Pij}}, \qquad \hat{p}^{(s)}_{P} = \frac{\partial \bar{\psi}^{(s)}_{glob}}{\partial \hat{e}^{(s)}_{P}}. \tag{2.137}$$

- In particular, a rate independent elastoplastic model (with no hardening) is obtained by selecting for $\bar{\psi}^{(s)}_{glob}$ a non-negative convex function of variables $\Delta\bar{\varepsilon}^{(s)}_{Pij} = \bar{\varepsilon}^{(s)}_{ij} - \bar{\varepsilon}^{(s)}_{Pij}$ and $\Delta\hat{p}^{(s)}_{P} = \hat{p}^{(s)} - \hat{p}^{(s)}_{P}$

$$\bar{\psi}^{(s)}_{glob} = \bar{\psi}^{(s)}_{glob}(\Delta\bar{\varepsilon}^{(s)}_{Pij}, \ \Delta\hat{p}^{(s)}_{P}). \tag{2.138}$$

This choice implies the coincidence (usual in plasticity) between stress measures and the associated inelastic stresses

$$\check{\sigma}^{(s)}_{Pij} = \check{\sigma}^{(s)}_{ij} = \frac{\partial \bar{\psi}^{(s)}_{glob}}{\partial \Delta\bar{\varepsilon}^{(s)}_{Pij}}, \qquad \hat{p}^{(s)}_{P} = \hat{p}^{(s)} = \frac{\partial \bar{\psi}^{(s)}_{glob}}{\partial \Delta\hat{e}^{(s)}_{P}}. \tag{2.139}$$

- By following the parallel with rate-independent perfect plasticity within the SGM, the pseudopotential $\bar{\varphi}^{(s)}$ is constructed by assigning its dual Legendre-Fenchel transform $\left(\bar{\varphi}^{(s)}\right)^*$, defined in the space $\mathscr{S}_P$ of forces conjugate to internal inelastic variables $(\check{\boldsymbol{\sigma}}^{(s)}_{P}, \ \hat{p}^{(s)}_{P})$ with the aid of a convex domain $\mathscr{P}_P \subset \mathscr{S}_P$, containing the origin. Specifically, $\left(\bar{\varphi}^{(s)}\right)^*$ is defined as:

$$\left(\bar{\varphi}^{(s)}\right)^* = I_{\mathscr{P}_P}\left(\check{\boldsymbol{\sigma}}^{(s)}_{P}, \ \hat{p}^{(s)}_{P}\right), \tag{2.140}$$

where $I_{\mathscr{P}_P}$ is the indicator function of $\mathscr{P}_P$.

- The complementary constitutive equations may then be written as

$$\left(\dot{\boldsymbol{\varepsilon}}^{(s)}{}_P, \ \dot{\hat{e}}^{(s)}{}_P\right) \in \partial\left(\bar{\varphi}^{(s)}\right)^*, \tag{2.141}$$

with $\partial\left(\bar{\varphi}^{(s)}\right)^*$ denoting the subgradient of $\left(\bar{\varphi}^{(s)}\right)^*$. If $\mathscr{P}_P$ is defined by a convex function f as the set of the $(\check{\boldsymbol{\sigma}}^{(s)}_{P}, \ \hat{p}^{(s)}_{P})$ such that $f\left(\check{\boldsymbol{\sigma}}^{(s)}_{P}, \ \hat{p}^{(s)}_{P}\right) \leq 0$, then the complementary constitutive Eq. (2.141) can be written with the aid of a plastic multiplier λ whose rate respects Karush-Kuhn-Tucker conditions [43, 46]:

$$\left(\dot{\boldsymbol{\varepsilon}}^{(s)}{}_P,\ \dot{\hat{e}}^{(s)}{}_P\right) = \dot{\lambda}\frac{\partial f}{\partial(\check{\boldsymbol{\sigma}}^{(s)}_P,\ \hat{p}^{(s)}_P)}, \quad \dot{\lambda} \geq 0, \quad f(\check{\boldsymbol{\sigma}}^{(s)}_P,\ \hat{p}^{(s)}_P) \leq 0, \quad \dot{\lambda} f = 0. \tag{2.142}$$

A linear elastoplastic theory is constructed, in particular, selecting a quadratic form for $\bar{\psi}^{(s)}_{glob}$. It is worth to observe that the only equations of the linear theory of Sect. 2.2.6, which are modified by the 'elastoplastic enhancement' considered in this remark, are the stress-strain relations (2.113) which are replaced by Equations (2.136)–(2.142). The momentum balance equations turn out to be not affected by such an enhancement.

2.3 Discussion and Conclusions

A variational theory of two-phase saturated porous media has been derived proceeding from the adoption of the VMTPM kinematic framework [72, 74, 75, 77], and applying standard concepts of variational continuum mechanics [9, 47, 48] and of continuum theories with microstructure [7, 59].

VMTPM kinematics is based on an *extrinsic/intrinsic* split of volumetric strain measures: the additional descriptor field is the *intrinsic* scalar volumetric strain measure $\hat{J}^{(s)}$, and corresponds to the ratio $\hat{\rho}^{(s)}_0/\hat{\rho}^{(s)}$ between *true* densities of solid before and after deformation. This field is independent from $\bar{J}^{(s)}$, which remains customarily defined in VMTPM as the determinant of the macroscopic deformation gradient. The experimental characterization of $\hat{J}^{(s)}$ is possible by measuring the changes of the porosity field via the relation linking the intrinsic volumetric strain to the porosities before ($\Phi^{(f)}_0$) and after deformation ($\phi^{(f)}$):

$$\hat{J}^{(s)} = \bar{J}^{(s)}\left(1 - \phi^{(f)}\right) / \left(1 - \Phi^{(f)}_0\right). \tag{2.143}$$

The choice of kinematic descriptors in VMTPM is minimal since it consists of the least possible set of fields ensuring the fulfillment of the saturation condition, without adding artificial incompressibility constraints. Previous works have adopted porosity as the additional descriptor [7, 21, 50]. In this work, the use of $\hat{J}^{(s)}$ is preferred alongside with $\bar{J}^{(s)}$ since these fields share analogous properties. In particular, both $\bar{J}^{(s)}$ and $\hat{J}^{(s)}$ are naturally defined in the reference configuration of the solid phase; also they jointly achieve a unit value in presence of rigid deformations, and allow to characterize a homotetic deformation by the condition $\bar{J}^{(s)} = \hat{J}^{(s)}$. These common properties are found to be convenient when introducing the stress measures and stiffness quantities work-associated with $\bar{J}^{(s)}$ and $\hat{J}^{(s)}$, allowing their easier correlation and physical interpretation.

In previous VMTPM derivations [72, 74–77, 81, 82], variational deductions of the governing equations were limited to that of the momentum balances of the solid phase. Hereby, also the fluid momentum balance has been derived on variational basis. Accordingly, the present theory is purely-variational in that no arguments other than Least Action conditions are used to derive all momentum balance equations. Also, similar to background works, the derivation herein reported has a purely-macroscopic character, in that no considerations on the microstructural features of the medium have been introduced. All the equations have been obtained avoiding any assumption on the microscale geometric or constitutive features of the medium. To this end, an Action functional of maximum generality has been employed with kinematic descriptors having a universal character (i.e., the displacements and the intrinsic strain can be ordinarily measured for any deforming medium). Accordingly, this theory meets the sought requirement of *medium independence* and, as such, can be straightfowardly applied, in purely mechanical problems, irrespective of the degree of anisotropy of the medium and of its constitutive linear or nonlinear features, in a way similar to the linear momentum balance for single-phase continua. Furthermore, it has been shown that the momentum balance equations remain the same when a rate-independent elastoplastic behavior is added to the solid phase by constructing an isothermal single-phase elastoplasticity continuum theory in the framework of a Standard Generalized Model (SGM) [39, 61, 63].

This variational formulation has been developed based on the configuration description adopting a set of independent kinematic descriptor fields which are not constrained to respect further equations expressing saturation or mass balances. This choice represents a precise element of distinction of this theory from the variational formulation of Bedford and Drumheller [7], and ensures the well-posedness of the variational statement of the problem. In particular, the absence in VMTPM of constraints for the primary descriptors allows to ordinarily define stress measures based on explicit work-association, as standardly done in the single-continuum elasticity theory, making no recourse to Lagrange multipliers for defining stress quantities.

Importantly, the framework hereby derived is downward compatible with the single continuum Cauchy linear momentum balance equations. More specifically, recalling Eq. (2.53), it is observed that when the fluid pressure is null (i.e., $\check{\boldsymbol{\sigma}}^{(s)} \equiv \boldsymbol{\sigma}^{(s)}$), the solid intrinsic pressure is null, and the extrinsic stress tensor becomes coincident with the standard notion of Cauchy stress tensor. Moreover, as reported in relation (2.53), the extrinsic momentum balance becomes formally coincident with Cauchy linear momentum balances.

The present study also shows that the "*missing equations*" for the *closure* of the two-phase poroelastic boundary value problem are naturally identified when the problem is approached in purely variational terms. The closure equations are represented by the saturation constraint and by the intrinsic momentum balance. Notably, neither of these equations has a constitutive nature or a thermodynamic nature.

In a similar way, a derivation of the general three-dimensional conditions which must be applied at the macroscopic surfaces of the mixture is achieved in purely-variational terms. The reported derivation is general in that it is comprehensive of boundary conditions of stress-type and displacement-type, as well as of free solid-fluid macroscopic interfaces. The derivation proves the variational consistency of the boundary equations obtained, which generalize those reported in [75]. In this respect, the derivation of general surface conditions for free solid-fluid macroscopic interfaces, stemming as the strong-form equations obtained when virtual isochoric deformations are considered, also represents an original result which may be denominated *principle of virtual isochoric deformations*, see Sect. 2.2.4.

Attention has been also placed on showing that the singular conditions, in which some phase is vanishing in some subregion, are ordinarily addressed by the present theory, in a consistent physical and mathematical way, see Remarks 2.1, 2.3, 2.5 and 2.7.

Results relevant to stress partitioning have been also derived. It was shown that, whenever external tractions applied over the (impermeable) boundary induce a deformation in the medium which is (macroscopically) uniform in space, an external stress tensor can be defined such that its partition between the two phases in compliance with relation (2.132), see Sect. 2.2.7. From a formal point of view, such result coincides with the classical statement of Terzaghi's principle, and the extrinsic stress tensor $\breve{\boldsymbol{\sigma}}^{(s)}$ is identified with the largely employed notion of *effective stress* tensor. However, it is important to remark that, in this work, in the context of a purely mechanical theory, such relation has been derived in absence of any constitutive hypothesis on the phases, thus representing a stress partitioning law of general validity for homogeneous stress states in saturated two-phase media not limited to saturated soils. A related result is the proof that in regions of the medium undergoing undrained flow conditions (i.e., with macroscopic relative solid-fluid motion prevented), an *external stress* field can be meaningfully considered with the external stress tensor being partitioned between the two phases in a relation formally coincident with Terzaghi's law, see Eqs. (2.134) to (2.135). A less general proof of this result has been previously reported in [77].

Concerning the validity of this result beyond the purely mechanical theory, as discussed in Remark 2.12, when a rate-dependent drag law is included in the formulation representing, for instance, Darcy or Forchheimer empirical laws, the stress partitioning laws represented by Eqs. (2.132) and (2.135) are expected to maintain their validity, in a restricted sense. Specifically, Eq. (2.135) is expected to be still applicable, in a limit sense, when loading is fast compared to the characteristic consolidation time of the mixture. Conversely, when the purely-mechanical problem is enhanced by adding a rate-independent elastoplastic behavior to the solid phase in the framework of the SGM, the general partitioning laws represented by Eqs. (2.132) and (2.135) always retain their full validity, according to the discussion in Remarks 2.12 and 2.13.

The results obtained on stress partitioning deserve a final consideration. Some experimental results in poromechanics have been interpreted as evidences of deviations from Terzaghi's law for specific classes of two-phase media [65]. Since the general character of the partitioning law herein derived excludes the possibility of such deviations, a dedicated study needs to be conducted to assess the capability of VMTPM to predict such experimental results. Such investigation is reported in Chap. 4 together with a specific assessment of the predictive capabilities of VMTPM theory in relation to the description of tests on biphasic specimens subjected to a comprehensive variety of loading and drainage conditions. The subsequent chapters show that governing equations and results of consolidated use in poroelasticity, such as Terzaghi's stress partitioning principle and Biot's equations, are ordinarily recovered by VMTPM, and present an application of the variational governing equations combined with the employment of a nonlinear constitutive response for the solid phase.

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