Chapter 2
Spectra of Finite Graphs

2.1 Characteristic Polynomials

Let $G = (V, E)$ be a finite graph on $n = |V|$ vertices. Numbering the vertices, we write down its adjacency matrix in an explicit form of $n \times n$ matrix, say A. The characteristic polynomial of A is defined as usual by

$$\varphi_A(x) = |xI - A| = \det(xI - A),$$

where I is the $n \times n$ identity matrix. It is noted that $\varphi_A(x)$ is determined independently of the numbering. In fact, let A' be the adjacency matrix obtained by a different numbering. It follows from Proposition 1.20 that $A' = S^{-1}AS$ with an $n \times n$ permutation matrix S. Then,

$$\begin{aligned}\varphi_{A'}(x) &= |xI - A'| = |xI - S^{-1}AS| \\ &= |S^{-1}(xI - A)S| = |S|^{-1}|xI - A||S| = \varphi_A(x).\end{aligned}$$

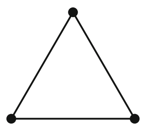
We call $\varphi_A(x)$ the *characteristic polynomial* of G and denote it by $\varphi(x; G)$.

Example 2.1 Here are a few simple examples:

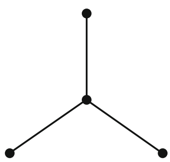
x

$x^2 - 1$

$x^3 - 2x$



$x^3 - 3x - 2 = (x + 1)^2(x - 2)$



$x^4 - 3x^2 = x^2(x^2 - 3)$

N. Obata, *Spectral Analysis of Growing Graphs*,
SpringerBriefs in Mathematical Physics, DOI 10.1007/978-981-10-3506-7_2

By definition the characteristic polynomial of a graph $G = (V, E)$ on $n = |V|$ vertices is of the form:

$$\varphi(x; G) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + c_3 x^{n-3} + \cdots + c_{n-1} x + c_n \,. \tag{2.1}$$

The coefficients represent combinatorial characteristics, see e.g., Bapat [13, Sect. 3.2] and Biggs [17, Chap. 7]. We only mention the following

Theorem 2.2 *For the coefficients of the characteristic polynomial* (2.1) *we have:*

(1) $c_1 = 0$.
(2) $-c_2 = |E|$, *the number of edges.*
(3) $-c_3 = 2\triangle$, *where* $\triangle$ *is the number of triangles in* G.

Proof Let $A = [a_{ij}]$ be the adjacency matrix of G in an explicit form of $n \times n$ matrix after numbering the vertices. Since the diagonal elements of A vanish, the characteristic polynomial of G is given by

$$\varphi(x; G) = |xI - A| = \begin{vmatrix} x & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & x & \cdots & -a_{2n} \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \cdots & \cdots & x \end{vmatrix} .$$

For simplicity, the matrix in the right-hand side is denoted by $B = [b_{ij}]$. Then,

$$\begin{aligned} \varphi(x; G) = |B| &= \sum_{\sigma \in \mathfrak{S}(n)} \operatorname{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)} \\ &= \sum_{k=0}^{n} \sum_{\substack{\sigma \in \mathfrak{S}(n) \\ |\operatorname{supp}\sigma| = k}} \operatorname{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)} \,, \end{aligned}$$

where $\operatorname{supp}\sigma = \{1 \le i \le n \mid \sigma(i) \ne i\}$. Since the indeterminate x appears only in the diagonal of B, comparing with (2.1) we have

$$f_k(x) \equiv \sum_{\substack{\sigma \in \mathfrak{S}(n) \\ |\operatorname{supp}\sigma| = k}} \operatorname{sgn}(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)} = c_k x^{n-k} \,. \tag{2.2}$$

That $c_1 = 0$ is apparent since there is no permutation σ with $|\operatorname{supp}\sigma| = 1$. For c_2 we note that the permutations σ with $|\operatorname{supp}\sigma| = 2$ are parametrized as $\sigma = (i\ j)$ $(1 \le i < j \le n)$. Taking $\operatorname{sgn}(\sigma) = -1$ into account, we obtain

$$f_2(x) = \sum_{1 \le i < j \le n} (-1)(-a_{ij})(-a_{ji}) x^{n-2} = - \sum_{1 \le i < j \le n} a_{ij} x^{n-2}$$

where we used $a_{ij}a_{ji} = a_{ij}^2 = a_{ij}$, and hence $c_2 = -|E|$. Similarly, for c_3 we need to consider permutations σ satisfying $|\mathrm{supp}\,\sigma| = 3$. Those permutations are parametrized as $\sigma = (i\,j\,k)$ and $\sigma = (i\,k\,j)$ for $1 \le i < j < k \le n$. Noting that $\mathrm{sgn}\,(\sigma) = 1$ for such cyclic permutations, we have

$$f_3(x) = -\sum_{1\le i<j<k\le n} (a_{ij}a_{jk}a_{ki} + a_{ik}a_{kj}a_{ji})x^{n-3}.$$

Since $a_{ij}a_{jk}a_{ki} = a_{ik}a_{kj}a_{ji} = 1$ only if i, j, k form a triangle, we get $-c_3 = 2\triangle$. □

Exercise 2.3 Let A be the adjacency matrix of a finite graph $G = (V, E)$. Show that

$$\mathrm{Tr}\ A = 0, \qquad \mathrm{Tr}\ A^2 = 2|E|, \qquad \mathrm{Tr}\ A^3 = 6\triangle.$$

2.2 Spectra

Let $G = (V, E)$ be a finite graph on $n = |V|$ vertices and A the adjacency matrix. The characteristic polynomial of G is factorized as

$$\varphi(x; G) = |xI - A| = \prod_{j=1}^{n}(x - \lambda_j), \tag{2.3}$$

where $\lambda_1, \dots, \lambda_n$ are real numbers since A is a real symmetric matrix. Changing notation, let $\lambda_1 < \lambda_2 < \cdots < \lambda_s$ be the distinct eigenvalues of A. Then (2.3) becomes

$$\varphi(x; G) = |xI - A| = \prod_{i=1}^{s}(x - \lambda_i)^{m_i}, \tag{2.4}$$

where $m_i \ge 1$ is called the *(algebraic) multiplicity* [1] of an eigenvalue λ_i.

Definition 2.4 Let $G = (V, E)$ be a finite graph and $\varphi(x; G)$ its characteristic polynomial in the form (2.4). The *spectrum* of G is by definition the array

$$\mathrm{Spec}\,(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_s \\ m_1 & m_2 & \cdots & m_s \end{pmatrix}. \tag{2.5}$$

Each λ_i is called an *eigenvalue* of G and m_i its *multiplicity*.

Proposition 2.5 *For finite graphs G and G', $G \cong G'$ implies* $\mathrm{Spec}\,(G) = \mathrm{Spec}\,(G')$.

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[1]Let $\varphi_A(x) = |xI - A|$ be the characteristic polynomial of a matrix $A \in M(n, \mathbb{C})$. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\varphi_A(\lambda) = 0$. In that case the *algebraic multiplicity* of the eigenvalue λ is defined to be the multiplicity of λ as a zero of the polynomial $\varphi_A(x)$. While, the *geometric multiplicity* is defined to be the dimension of the eigenspace associated to λ..

Proof Straightforward from Proposition 1.21. □

Example 2.6 The spectra of the five graphs in Example 2.1 are given as follows:

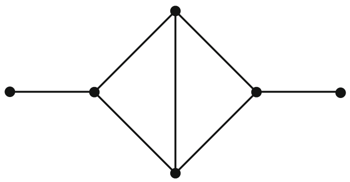
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} -\sqrt{2} & 0 & \sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} -\sqrt{3} & 0 & \sqrt{3} \\ 1 & 2 & 1 \end{pmatrix}.$$

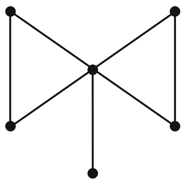
The converse assertion of Proposition 2.5 is not valid, namely, there are non-isomorphic graphs that have the same spectra. The following two graphs have a common characteristic polynomial: $\varphi(x) = x^5 - 4x^3 = x^3(x-2)(x+2)$.

In general, graphs with common spectra are called *cospectral* graphs. It is known that the above is a unique pair of cospectral graphs among 34 non-isomorphic graphs on 5 vertices. For a concise account on small graphs see e.g., Godsil–McKay [54], Harary *et al.* [62], Johnson–Newman [82], and for more recent and detailed results see Cvetković *et al.* [39–41]. There are many attempts to distinguish cospectral graphs by means of another matrices.

Exercise 2.7 (Baker [10]) Show that the following two graphs have a common characteristic polynomial:

$$\begin{aligned}\varphi(x) &= x^6 - 7x^4 - 4x^3 + 7x^2 + 4x - 1 \\ &= (x-1)(x+1)^2(x^3 - x^2 - 5x + 1).\end{aligned}$$

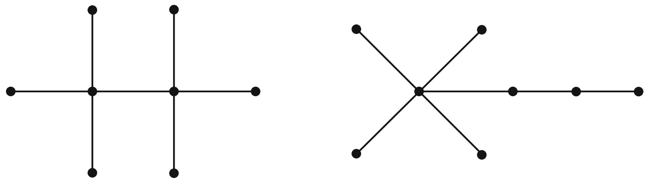




In fact, this is a unique pair of cospectral connected graphs among 112 non-isomorphic connected graphs on 6 vertices.

Exercise 2.8 (Collatz–Sinogowitz [37])
Show that the following two trees have a common characteristic polynomial:

$$\varphi(x) = x^8 - 7x^6 + 9x^4 = x^4(x^4 - 7x^2 + 9).$$



Exercise 2.9 Examine by characteristic polynomials that graphs $G = (V, E)$ with $|V| \le 4$ are uniquely specified by their spectra.

Proposition 2.10 *For a finite graph G let $s(G)$ denote the number of distinct eigenvalues of G. Then we have $s(G) = \dim \mathscr{A}(G)$.*

Proof Let $\lambda_1 < \cdots < \lambda_s$ be the different eigenvalues of the adjacency matrix A, $s = s(G)$. After diagonalization by a suitable orthogonal matrix U we obtain

$$U^{-1}AU = \mathrm{diag}[\overbrace{\lambda_1 \cdots \lambda_1}^{m_1} \;\cdots\; \overbrace{\lambda_s \cdots \lambda_s}^{m_s}] \equiv D.$$

Then $\{I, D, D^2, \dots, D^{s-1}\}$ is linearly independent, while $\{I, D, D^2, \dots, D^{s-1}, D^s\}$ is not since $(D - \lambda_1 I) \cdots (D - \lambda_s I) = O$, where I is the identity matrix. Therefore the algebra $U^{-1}\mathscr{A}(G)U$ is of dimension s, and so is $\mathscr{A}(G)$. □

Corollary 2.11 *For a connected finite graph G we have*

$$s(G) = \dim \mathscr{A}(G) \ge \mathrm{diam}\,(G) + 1.$$

Proof Immediate from Propositions 2.10 and 1.26. □

2.3 Spectra of K_n, C_n and P_n

Lemma 2.12 *Let J be the $n \times n$ matrix with entries being all one. Then J/n is a projection of rank one. Therefore the eigenvalues of J are n with multiplicity one and* 0 *with multiplicity* $n - 1$.

Proof That J/n is a projection follows from $J^2 = nJ$ and $J^* = J$. Let $\{e_k\,;\ 1 \le k \le n\}$ be the canonical basis of $\mathbb{C}^n$ and set $f = \sum_{k=1}^n e_k$. Then we have $Jf = nf$ and $J(e_1 - e_k) = 0$ for $2 \le k \le n$. Since $\{f, e_1 - e_k\,;\ 2 \le k \le n\}$ is a basis of $\mathbb{C}^n$, the eigenvalues of J are n with multiplicity one and 0 with multiplicity $n - 1$. In particular, J/n is a projection onto the one-dimensional subspace spanned by f. □

Theorem 2.13 *For the complete graph K_n with $n \ge 1$ we have*

$$\varphi(x; K_n) = (x+1)^{n-1}(x - n + 1), \qquad \mathrm{Spec}\,(K_n) = \begin{pmatrix} -1 & n-1 \\ n-1 & 1 \end{pmatrix}.$$

Proof Since the adjacency matrix A of K_n is given by $A = J - I$, the assertion is immediate from Lemma 2.12. □

Lemma 2.14 *For $n \geq 2$ let W be the $n \times n$ circulant permutation matrix defined by*

$$W = \begin{bmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}.$$

The eigenvalues of W are $1 = \zeta^0, \zeta, \ldots, \zeta^{n-1}$, where $\zeta = \exp(2\pi i/n)$.

Proof Let $\{e_k\,;\ 1 \leq k \leq n\}$ be the canonical basis of $\mathbb{C}^n$ and set

$$f_l = \sum_{k=1}^{n} \zeta^{-lk} e_k\,, \qquad 0 \leq l \leq n-1.$$

Since $We_k = e_{k+1}$ for $1 \leq k \leq n$ with the understanding that $e_{n+1} = e_1$, we have

$$Wf_l = \sum_{k=1}^{n} \zeta^{-lk} We_k = \sum_{k=1}^{n} \zeta^{-lk} e_{k+1} = \zeta^l \sum_{k=1}^{n} \zeta^{-l(k+1)} e_{k+1} = \zeta^l f_l\,.$$

Namely, ζ^l is an eigenvalue of W with an eigenvector f_l. Since $\{f_l\,;\ 0 \leq l \leq n-1\}$ forms a basis of $\mathbb{C}^n$, $\{\zeta^l\,;\ 0 \leq l \leq n-1\}$ exhaust the eigenvalues of W. □

Theorem 2.15 *The spectrum of the cycle C_{2m+1} for $m \geq 1$ is given by*

$$\mathrm{Spec}\,(C_{2m+1}) = \begin{pmatrix} 2 & \cdots & 2\cos\dfrac{2l\pi}{2m+1} & \cdots & 2\cos\dfrac{2m\pi}{2m+1} \\ 1 & \cdots & 2 & \cdots & 2 \end{pmatrix}, \quad 1 \leq l \leq m. \tag{2.6}$$

The spectrum of the cycle C_{2m} for $m \geq 2$ is given by

$$\mathrm{Spec}\,(C_{2m}) = \begin{pmatrix} 2 & \cdots & 2\cos\dfrac{2l\pi}{2m} & \cdots & -2 \\ 1 & \cdots & 2 & \cdots & 1 \end{pmatrix}, \quad 1 \leq l \leq m-1. \tag{2.7}$$

Proof We keep the same notations as in Lemma 2.14. Note first that the adjacency matrix of C_n is given as $A = W + W^T = W + W^{-1}$. Then for $0 \leq l \leq n-1$ we have

$$Af_l = Wf_l + W^{-1}f_l = (\zeta^l + \zeta^{-l})f_l = \left(2\cos\frac{2l\pi}{n}\right)f_l\,.$$

Therefore, the eigenvalues of A are

$$2\cos\frac{2l\pi}{n}, \qquad 0 \le l \le n-1, \tag{2.8}$$

among which the same values appear at most twice since

$$2\cos\frac{2l\pi}{n} = 2\cos\frac{2(n-l)\pi}{n}.$$

The assertion follows after picking up different values from (2.8). □

Finally, we consider the path P_n ($n \ge 1$). After natural numbering the vertices, we may take the adjacency matrix of the form:

$$A_n = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{bmatrix}.$$

Lemma 2.16 *The characteristic polynomials $\varphi_n(x) = \varphi(x; P_n)$ of the path P_n verify the following recurrence relation:*

$$\varphi_1(x) = x, \quad \varphi_2(x) = x^2 - 1, \tag{2.9}$$

$$\varphi_n(x) = x\varphi_{n-1}(x) - \varphi_{n-2}(x), \quad n \ge 3 \tag{2.10}$$

Proof Relations in (2.9) are immediate. Let $n \ge 3$. We have

$$\varphi_n(x) = \begin{vmatrix} x & -1 & & & & \\ -1 & x & -1 & & & \\ & -1 & x & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & x & -1 \\ & & & & -1 & x \end{vmatrix} = x\varphi_{n-1}(x) + \begin{vmatrix} -1 & -1 & & & & \\ & x & -1 & & & \\ & -1 & x & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & x & -1 \\ & & & & -1 & x \end{vmatrix}$$

$$= x\varphi_{n-1}(x) - \varphi_{n-2}(x),$$

where we applied cofactor expansion twice, first along the first row and then along the first column. □

Chebyshev polynomials of the second kind. Using elementary knowledge of trigonometric functions we see easily that for $n = 0, 1, 2, \ldots$ there exists a polynomial $U_n(x)$ such that

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{2.11}$$

In fact, $U_n(x)$ is determined by the recurrence relation:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0. \tag{2.12}$$

These polynomials $\{U_n(x)\}$ are called the *Chebyshev polynomials of the second kind*. First a few are given by

$$U_0(x) = 1, \qquad U_1(x) = 2x, \qquad U_2(x) = 4x^2 - 1, \qquad U_3(x) = 8x^3 - 4x,$$
$$U_4(x) = 16x^4 - 12x^2 + 1, \qquad U_5(x) = 32x^5 - 32x^3 + 6x.$$

Theorem 2.17 $\varphi(x; P_n) = U_n(x/2)$ *for* $n \geq 1$.

Proof Set $\varphi_n(x) = \varphi(x; P_n)$ for $n \geq 1$ and $\varphi_0(x) = 1$. By Lemma 2.16 we have

$$\varphi_0(x) = 1, \quad \varphi_1(x) = x, \quad \varphi_n(x) = x\varphi_{n-1}(x) - \varphi_{n-2}(x), \quad n \geq 2. \tag{2.13}$$

On the other hand, we see from (2.12) that $\{U_n(x/2)\,;\, n \geq 0\}$ fulfills the same recurrence relation as in (2.13) together with the initial conditions. Therefore $\varphi_n(x) = U_n(x/2)$ for all $n \geq 0$. $\square$

Theorem 2.18 *The spectrum of the path* P_n *is given by*

$$\mathrm{Spec}\,(P_n) = \begin{pmatrix} 2\cos\dfrac{\pi}{n+1} & \cdots & 2\cos\dfrac{k\pi}{n+1} & \cdots & 2\cos\dfrac{n\pi}{n+1} \\ 1 & \cdots & 1 & \cdots & 1 \end{pmatrix}, \quad 1 \leq k \leq n.$$

Proof First we need to find the zeroes of the Chebyshev polynomial of the second kind. It follows from (2.11) that $U_n(\cos\theta) = 0$ for

$$\theta = \theta_k = \frac{k\pi}{n+1}, \qquad k = 1, 2, \ldots, n.$$

Hence, $U_n(x_k) = 0$ for $x_k = \cos\theta_k$, $1 \leq k \leq n$. Since $U_n(x)$ is a polynomial of degree n and $x_1, x_2, \ldots, x_n$ are mutually distinct, they exhaust the zeroes of $U_n(x)$ and each x_k is simple. Then by Theorem 2.17 the zeroes of $\varphi(x; P_n)$ are all simple and given by $2x_1, \ldots, 2x_n$, which gives $\mathrm{Spec}\,(P_n)$ as desired. $\square$

Exercise 2.19 Let $G = (V, E)$ be a graph. Let $a \in V$ be a vertex with $\deg(a) = 1$ and $b \in V$ a unique vertex adjacent to a. Prove that

$$\varphi(x; G) = x\varphi(x; G\backslash\{a\}) - \varphi(x; G\backslash\{a, b\}),$$

where $G\backslash W$ is a short-hand notation for $G[V\backslash W]$, the induced subgraph spanned by $V\backslash W$, where $W \subset V$.

Exercise 2.20 Using Theorem 2.18, prove the following identity:

$$\prod_{k=1}^{m} 2\cos\frac{k\pi}{2m+1} = 1, \qquad m \geq 1.$$

Exercise 2.21 (see Corollary 2.11) Show that $s(G) = \dim \mathscr{A}(G) = \mathrm{diam}\,(G) + 1$ holds for $G = K_n$ with $n \geq 1$ and for $G = P_n$ with $n \geq 1$. Give a small graph G such that $s(G) > \mathrm{diam}\,(G) + 1$.

Exercise 2.22 Show that the characteristic polynomial and the spectrum of the complete bipartite graph $K_{k,l}$ are given by

$$\varphi(x; K_{k,l}) = x^{k+l-2}(x^2 - kl), \qquad \mathrm{Spec}\,(K_{k,l}) = \begin{pmatrix} -\sqrt{kl} & 0 & \sqrt{kl} \\ 1 & k+l-2 & 1 \end{pmatrix}$$

2.4 Bounds of Spectra

We first recall useful results known under the name of Perron–Frobenius theory, for further details see Horn-Johnson [80, Chap. 8] and Seneta [123]. For a matrix $A \in M(n, \mathbb{C})$ the *spectral radius* is defined to be the maximal modulus of its eigenvalue and is denoted by $\rho(A)$.

Proposition 2.23 *Let $A = [a_{ij}] \in M(n, \mathbb{C})$ and assume that $a_{ij} \geq 0$ for all i, j. Then $\rho(A)$ is an eigenvalue of A and there exists a nonzero vector $\xi = [\xi_i] \in \mathbb{C}^n$ such that $A\xi = \rho(A)\xi$ and $\xi_i \geq 0$ for all i.*

A matrix $A \in M(n, \mathbb{C})$ is called *reducible* if there is a permutation matrix S such that $S^T AS$ is partitioned in the form:

$$S^T AS = \begin{bmatrix} B & C \\ O & D \end{bmatrix}, \quad B \in M(r, \mathbb{C}), \quad D \in M(n-r, \mathbb{C}), \quad 1 \leq r \leq n-1.$$

A matrix $A \in M(n, \mathbb{C})$ is called *irreducible* if it is not reducible.

Proposition 2.24 *Let $A = [a_{ij}] \in M(n, \mathbb{C})$ and assume that $a_{ij} \geq 0$ for all i, j. If A is irreducible, $\rho(A)$ is an algebraically simple eigenvalue of A. Moreover, there exists a vector $\xi = [\xi_i] \in \mathbb{C}^n$ such that $A\xi = \rho(A)\xi$ and $\xi_i > 0$ for all i. (This vector is called the Perron vector.)*

Definition 2.25 The *spectral radius* of a finite graph G is defined to be the spectral radius of the adjacency matrix, and is denoted by $\rho(G)$.

As a direct consequence from Propositions 2.23 and 2.24 we have the following

Theorem 2.26 *Let G be a finite graph. Then $\rho(G)$ coincides with the maximal eigenvalue of G. If G is a finite connected graph, $\rho(G)$ is simple and there exists an eigenvector with positive entries.*

Bounds of $\rho(G)$ is interesting. We start with a simple upper bound.

Theorem 2.27 *For a finite graph $G = (V, E)$ it holds that*

$$\rho(G) \le \left\{ \frac{2|E|(|V|-1)}{|V|} \right\}^{1/2}. \tag{2.14}$$

Proof Let A be the adjacency matrix of G and $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ the eigenvalues of A, where $n = |V|$. Then $\rho(G) = \lambda_1$. Since Tr $A = 0$ and Tr $A^2 = 2|E|$, we have

$$\sum_{k=1}^{n} \lambda_k = 0, \qquad \sum_{k=1}^{n} \lambda_k^2 = 2|E|.$$

Then, applying the Schwartz inequality, we obtain

$$\lambda_1^2 = \left\{ \sum_{k=2}^{n} (-\lambda_k) \right\}^2 \le (n-1) \sum_{k=2}^{n} \lambda_k^2 = (n-1)(2|E| - \lambda_1^2),$$

from which we obtain $n\lambda_1^2 \le 2(n-1)|E|$. Then (2.14) follows immediately. □

Some statistics concerning the degrees of vertices play an interesting role. We set

$$d_{\max}(G) = \max\{\deg(x)\,;\, x \in V\}, \qquad d_{\min}(G) = \min\{\deg(x)\,;\, x \in V\},$$

and define the *mean degree* of G by

$$\bar{d}(G) = \frac{1}{|V|} \sum_{x \in V} \deg(x) = \frac{2|E|}{|V|}\,. \tag{2.15}$$

Theorem 2.28 *For a finite graph $G = (V, E)$ it holds that*

$$d_{\min}(G) \le \bar{d}(G) \le \rho(G) \le d_{\max}(G).$$

Proof Obviously, $d_{\min}(G) \le \bar{d}(G) \le d_{\max}(G)$. We first prove that $\bar{d}(G) \le \rho(G)$. Let A be the adjacency matrix of G. From elementary linear algebra we know that

$$\lambda_{\min}(A) \le \frac{\langle f, Af \rangle}{\langle f, f \rangle} \le \lambda_{\max}(A) \quad \text{for all } f \in C(V),\ f \ne 0, \tag{2.16}$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the maximal and minimal eigenvalues of A, respectively. Now we take ψ to be a constant function defined by $\psi(x) = 1$ for all $x \in V$.

Then,

$$\langle \psi, A\psi \rangle = \sum_{x,y\in V} \overline{\psi(x)}\,(A)_{xy}\psi(y) = \sum_{x,y\in V} (A)_{xy} = 2|E|.$$

Since $\langle \psi, \psi \rangle = |V|$, we have

$$\frac{\langle \psi, A\psi \rangle}{\langle \psi, \psi \rangle} = \frac{2|E|}{|V|} = \bar{d}(G). \tag{2.17}$$

Combining (2.16) and (2.17), we obtain $\bar{d}(G) \le \lambda_{\max}(A) = \rho(G)$, as desired.

Next we show that $\rho(G) \le d_{\max}(G)$. We set $\lambda = \rho(G)$ for simplicity. Now we choose an eigenvector $f \in C(V)$ such that $Af = \lambda f$, $f \neq 0$, and $f(x) \in \mathbb{R}$ for all $x \in V$ (here we do not need to refer to a Perron vector). In that case, we may assume without loss of generality that $\alpha \equiv \max\{f(x)\,;\, x \in V\} > 0$. Let $o \in V$ be chosen in such a way that $f(o) = \alpha$. Then,

$$\lambda\alpha = \lambda f(o) = Af(o) = \sum_{y\in V}(A)_{oy}f(y) \le \alpha\ \deg(o).$$

Since $\alpha > 0$, we have $\lambda \le \deg(o)$, which implies that $\rho(G) \le d_{\max}(G)$. □

Corollary 2.29 *If G is a finite regular graph with degree κ, we have $\rho(G) = \kappa$.*

Exercise 2.30 Show that $\min \operatorname{Spec}(G) \ge -2$ for any line graph G of a finite graph.

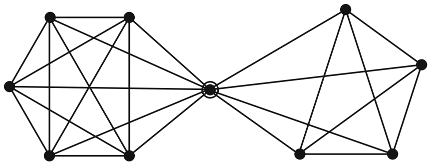
2.5 Star Products

Consider two graphs $G_i = (V_i, E_i)$ with distinguished vertex $o_i \in V_i$ for $i = 1, 2$. The *star product* $G_1 \star G_2 = (G_1, o_1) \star (G_2, o_2)$ is a graph obtaining by gluing G_1 and G_2 at the distinguished vertices (Fig. 2.1), for further study see Sect. 7.7.

Lemma 2.31 *Let $m \ge 1$ and $n \ge 1$ be natural numbers. Let A, B, C, D be $m \times m$-, $m \times n$-, $n \times m$-, and $n \times n$-matrices over the complex number field, respectively. If A is invertible, we have*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A\ \det(D - CA^{-1}B). \tag{2.18}$$

Fig. 2.1 Star product $K_6 \star K_5$



Similarly, if D is invertible, we have

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det D \, \det(A - BD^{-1}C) \tag{2.19}$$

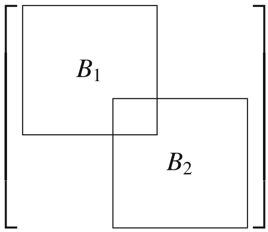
Proof (2.18) follows immediately from the obvious identity:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & O \\ C & I \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ O & D - CA^{-1}B \end{bmatrix},$$

where I stands for the identity matrix. The proof of (2.19) is similar. □

Let B_1, B_2 and A be the adjacency matrices of G_1, G_2 and $G = G_1 \star G_2$, respectively, We number the vertices in such a way that $V_1 = \{1, 2, \dots, m\}$ and $V_2 = \{m, m+1, 2, \dots, m+n-1\}$, where o_1 and o_2 are given the common number m. Then the adjacency matrix A is of the form:

$A =$



$$= \begin{bmatrix} B_1 & F \\ F^T & \tilde{B}_2 \end{bmatrix}, \tag{2.20}$$

where $\tilde{B}_2$ is the adjacency matrix of $G_2 \backslash \{o_2\}$, and F is an $m \times (n-1)$ matrix such that $(F)_{ij} = 0$ for all $1 \le i \le m-1$ and $1 \le j \le n-1$. By Lemma 2.31 we obtain

$$\begin{aligned} \varphi(x) = \det(x - A) &= \det \begin{bmatrix} x - B_1 & -F \\ -F^T & x - \tilde{B}_2 \end{bmatrix} \\ &= \det(x - B_1) \det(x - \tilde{B}_2 - F^T (x - B_1)^{-1} F). \end{aligned} \tag{2.21}$$

We define an $(n-1) \times (n-1)$ matrix $\tilde{F}$ by

$$(\tilde{F})_{ij} = (F)_{mi}(F)_{mj}, \qquad 1 \le i, j \le n-1.$$

By simple calculation, we have

$$F^T (x - B_1)^{-1} F = ((x - B_1)^{-1})_{mm} \tilde{F} = \det(x - B_1)^{-1} \det(x - \tilde{B}_1) \tilde{F}, \tag{2.22}$$

where $\tilde{B}_1$ is a principal submatrix of B_1 obtained by deleting mth row and column, in other words, $\tilde{B}_1$ is the adjacency matrix of $G_1 \backslash \{o_1\}$. Combining (2.21) and (2.22) we obtain

$$\varphi(x) = \det(x - B_1) \det(x - \tilde{B}_2 - \det(x - B_1)^{-1} \det(x - \tilde{B}_1) \tilde{F}).$$

Summing up, we have established the following

Theorem 2.32 *For $i = 1, 2$, let $G_i = (V_i, E_i)$ be a graph with a distinguished vertex $o_i \in V_i$. Let B_i be the adjacency matrix of G_i. Then the characteristic polynomial of the star product $G_1 \star G_2 = (G_1, o_1) \star (G_2, o_2)$ is given by*

$$\varphi(x; G_1 \star G_2) = \det(x - B_1) \det\left(x - \tilde{B}_2 - \frac{\det(x - \tilde{B}_1)}{\det(x - B_1)} \tilde{F} \right),$$

where $\tilde{B}_i$ is the adjacency matrix of $G_i \backslash \{o_i\}$ and $\tilde{F} = [f_i f_j]$ is a matrix indexed by $(V_2 \backslash \{o_2\}) \times (V_2 \backslash \{o_2\})$ defined by $f_i = (B_2)_{o_2 i}$.

Corollary 2.33 *For the star product of the complete graphs we have*

$$\begin{aligned}
&\varphi(x; K_m \star K_n) \\
&\quad = (x+1)^{m+n-4}((x-m-n+2)(x+1)^2 + (m-1)(n-1)(x+2)),
\end{aligned}$$

where $m \geq 1$ and $n \geq 1$.

Proof Noting that $\tilde{B}_i$ is the adjacency matrix of a complete graph and $\tilde{F} = J$, we need only to apply Theorem 2.32. □

Notes. The spectral graph theory has a long history. There are many comprehensive books, some of which are Brouwer–Haemers [28], Cvetković–Doob–Sachs [41], Cvetković–Rowlinson–Simić [39, 40] and van Mieghem [136]. For spectra of the normalized Laplacian, see also Chung [35].

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