

Chapter 2

Linear Time-Invariant Descriptor Systems

In this chapter, we recall some basic concepts concerning linear time-invariant descriptor systems for both continuous-time and discrete-time settings, which will be used subsequently. Descriptor systems offer a powerful tool for system modelling since they allow to describe a system by both dynamic equations and algebraic constraints. We give here a quick reminder of the fundamental definitions and results of descriptor systems, such as regularity, admissibility, equivalent realizations, system decomposition, temporal response, controllability, observability, and duality. Most of the results presented in this chapter can be found in [Ros74, Cob84, Lew85, VLK81, YS81, Ail87, CP85, Hou04, Dai89, Lew86, IT02, XY99, XL04, Mar03].

2.1 Introduction

Let us recall the first-order DAE discussed in the previous chapter

$$\mathbf{F}(\dot{x}(t), x(t)) = 0, \quad (2.1)$$

where $\mathbf{F}$ and x are vector value functions. Representing the Jacobians as

$$E \triangleq \frac{\partial \mathbf{F}}{\partial \dot{x}(t)}, \quad A \triangleq -\frac{\partial \mathbf{F}}{\partial x(t)}, \quad (2.2)$$

we can write

$$E d\dot{x}(t) = A dx(t) + \left(d\mathbf{F} - \frac{\partial \mathbf{F}}{\partial t} dt \right). \quad (2.3)$$

As mentioned before, if the matrix E is not singular, i.e. $|E| \neq 0$, we can convert this system into a conventional state-space system by left-multiplying the two sides by E^{-1} . On the other hand, if E is singular, this conversion is no longer possible. In the parts to follow, we omit the time index t for continuous-time descriptor systems to simplify the writing.

A linear time-invariant version of (2.1) including a control input $u(t)$ and a measurement output $y(t)$ is written as

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \quad (2.4)$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are the descriptor variable, measurement and control input vector, respectively. The matrix $E \in \mathbb{R}^{n \times n}$ may be singular, i.e. $\text{rank}(E) = r \leq n$.

Note that the form (2.4) can be used without loss of generality. In the case where the feedthrough matrix from u to y is not null, we can introduce an extra descriptor variable to render the D matrix zero. For example, consider the following system

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \quad (2.5)$$

which can be equivalently represented by

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}}_{\mathcal{E}} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix}}_{\mathcal{A}} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x \\ \zeta \end{bmatrix}}_{\mathcal{B}} + \underbrace{\begin{bmatrix} B \\ I \end{bmatrix}}_{\mathcal{B}} u, \quad (2.6)$$

$$y = \underbrace{\begin{bmatrix} C & D \end{bmatrix}}_{\mathcal{C}} \underbrace{\begin{bmatrix} x \\ \zeta \end{bmatrix}}_{\mathcal{B}}.$$

By introducing the auxiliary variable ζ , this system is rewritten as the form of (2.4).

A descriptor system G associated with the system data (E, A, B, C, D) can also be represented by the form of

$$G(s) := \left[\begin{array}{c|c} A - sE & B \\ \hline C & D \end{array} \right]. \quad (2.7)$$

2.2 Regularity

One of the basic notations of descriptor systems is *regularity* or *solvability*. If a descriptor system is regular, then it has a unique solution for any initial condition and any continuous input function [VLK81, Cob83].

Definition 2.1 (*Regularity*) The descriptor system in (2.4) is said to be regular if $\det(sE - A) \neq 0$ for all $s \in \mathbb{C}$.

This definition is the same as the one called solvability used by Yip and Sincovec in [YS81]. To illustrate the physical mean of regularity, let us examine the Laplace transformed version of $E\dot{x} = Ax + Bu$ as follows

$$sE\mathcal{L}[x] - Ex(0) = A\mathcal{L}[x] + B\mathcal{L}[u], \quad (2.8)$$

which can be arranged as

$$(sE - A)\mathcal{L}[x] = B\mathcal{L}[u] + Ex(0). \quad (2.9)$$

It is observed that, if the system is regular, then

$$\mathcal{L}[x] = (sE - A)^{-1}(B\mathcal{L}[u] + Ex(0)), \quad (2.10)$$

which guarantees the existence and uniqueness of $\mathcal{L}[x]$ for any initial condition and any continuous input function. On the other hand, if the system is not regular, or equivalently, the matrix $sE - A$ is of rank deficiency, there exists a nonzero vector $\theta(s)$ such that

$$(sE - A)\theta(s) \equiv 0. \quad (2.11)$$

Consequently, one can state that, if the system has a solution denoted $\mathcal{L}[x]$, then $\mathcal{L}[x] + \alpha\theta(s)$ is also a solution for any α . It is clear that a solution to this system is not unique, and it is also obvious that there may be no solution to this system. The following characterizations of regularity are given in [YS81].

Lemma 2.1 (*Regularity*) The following statements are equivalent.

- (a) (E, A) is regular.
- (b) If X_0 is the null space of A and $X_i = \{x : Ax \in EX_{i-1}\}$ then $\text{Ker}(E) \cap X_i = 0$ for $i = 0, 1, 2, \dots$
- (c) If Y_0 is the null space of A^T and $Y_i = \{x : A^T x \in E^T Y_{i-1}\}$ then $\text{Ker}(E^T) \cap Y_i = 0$ for $i = 0, 1, 2, \dots$
- (d) The matrix

$$G(n) = \left[\begin{array}{cccc} E & 0 & \cdots & 0 \\ A & E & \cdots & 0 \\ 0 & A & \cdots & 0 \\ & & & E \\ 0 & \cdots & A \end{array} \right] \quad n+1 \quad (2.12)$$

has full column rank for $n = 1, 2, \dots$

(e) *The matrix*

$$F(n) = \underbrace{\begin{bmatrix} E & A & 0 & \cdots & 0 \\ 0 & E & A & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & & & E & A \end{bmatrix}}_{n+1} \quad (2.13)$$

has full row rank for $n = 1, 2, \dots$

(f) *There exist nonsingular matrices M and N such that $E\dot{x} = Ax + Bu$ is decomposed into possibly two subsystems: a subsystem with only a state variable, and an algebraic-like subsystem, i.e. $MENN^{-1}\dot{x} = MANN^{-1}x + MBu$ has one of the following forms.*

(i)

$$\begin{aligned} \dot{x}_1 &= E_1 x_1 + B_1 u, \\ E_2 \dot{x}_2 &= x_2 + B_2 u, \quad E_2^k = 0, \quad E_2^{k-1} \neq 0. \end{aligned} \quad (2.14)$$

In this case, both E and A are singular, or A is nonsingular and E is singular but not nilpotent, i.e., $E^k \neq 0$ for all positive integers k .

(ii)

$$\dot{x}_1 = E_1 x_1 + B_1 u. \quad (2.15)$$

In this case, E is nonsingular.

(iii)

$$E_2 \dot{x}_2 = x_2 + B_2 u, \quad E_2^k = 0, \quad E_2^{k-1} \neq 0. \quad (2.16)$$

In this case, A is nonsingular and E is nilpotent.

In all cases,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N^{-1}x, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = MB, \quad (2.17)$$

and the exact solution is

$$\begin{aligned} x_1(t) &= e^{E_1 t} x_{10} + \int_0^t e^{(t-\tau)E_1} B_1 u(\tau) d\tau, \\ x_2(t) &= - \sum_{i=0}^{k-1} E_2^i B_2 u^{(i)}(t), \end{aligned} \quad (2.18)$$

where x_{10} is the transformed initial condition, i.e., $\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = N^{-1}x_0$.

Among these equivalent statements, the easiest one to characterize regularity for a given descriptor system is the condition (d) or its dual version (e). For the sake of avoiding computing a matrix with huge dimension, Luenberger proposed the so-called shuffle algorithm which requires manipulations only on the rows and columns of the matrix $[E \ A]$ [Lue78]. For convenience, we usually check regularity directly from its definition, that is, $sE - A \neq 0$ for all $s \in \mathbb{C}$. In addition, if the descriptor system (2.4) is regular, $(sE - A)^{-1}$ is a rational matrix and we can further define its transfer function as

$$G(s) = C(sE - A)^{-1}B. \quad (2.19)$$

2.3 Equivalent Realizations and System Decomposition

To model a physical system, one has to choose a set of states which are related to the same performance, such as acceleration, velocity, position, temperature and mass. The choice of these states is in general not unique, and this fact leads to a set of different models (realizations) which yield the same input–output relationship for a given system. Consequently, it is of great interest to determine the relation of equivalence among these different representations.

Definition 2.2 (*Restricted System Equivalence*) Reference [Ros74] Consider two descriptor systems G and $\bar{G}$ given by

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \quad (2.20)$$

and

$$\begin{aligned} \bar{E}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u}, \\ \bar{y} &= \bar{C}\bar{x}. \end{aligned} \quad (2.21)$$

The two systems G and $\bar{G}$ are termed restricted system equivalent if there exist nonsingular matrices M and N such that

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c|c} sE - A & B \\ \hline C & 0 \end{array} \right] \begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} s\bar{E} - \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{bmatrix}. \quad (2.22)$$

Definition 2.3 (*Strong Equivalence*) Reference [VLK81] Consider two descriptor systems G and $\bar{G}$ given in (2.20) and (2.21), respectively. The two systems are termed strongly equivalent if there exist nonsingular matrices M , N and two matrices Q , R such that

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix} \left[\begin{array}{c|c} sE - A & B \\ \hline C & 0 \end{array} \right] \begin{bmatrix} N & R \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} s\bar{E} - \bar{A} & \bar{B} \\ \hline \bar{C} & 0 \end{array} \right], \quad (2.23)$$

$$QE = ER = 0.$$

Note that in book the term “equivalence” means “restricted system equivalence.”

Among many equivalent representations, there are two particular realizations of great importance for system analysis and control. They are referred to as the Kronecker-Weierstrass form [Wei67, Kro90] and the singular value decomposition (SVD) [BL87] form.

Lemma 2.2 (Kronecker-Weierstrass Decomposition) *Reference [Dai89] The descriptor system (2.4) is regular if and only if there exist nonsingular matrices M_1 and N_1 such that*

$$M_1 E N_1 = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \mathcal{N} \end{bmatrix}, \quad M_1 A N_1 = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad (2.24)$$

where $n_1 + n_2 = n$ and $\mathcal{N}$ is a nilpotent matrix.

The form (2.24) is referred to as the Kronecker-Weierstrass decomposition. This form can be viewed as an equivalent condition for regularity, and is also referred to by some scholars as *slow-fast decomposition* [Cob84]. The subsystem related to $\mathcal{A}$ is called the slow subsystem, while what is related to $\mathcal{N}$ is called the fast subsystem. Although the Kronecker-Weierstrass decomposition divides the systems into two parts which may bring simplicity for analysis, the use of this decomposition requires that the underlying system is regular. If the regularity of the system is not known, then this form cannot be applied. Moreover, the Kronecker-Weierstrass decomposition is numerically unreliable, especially in the case where the order of the system is relatively large.

Another decomposition that does not depend upon the regularity of systems is called the SVD form. This form can be obtained via a singular value decomposition on E and followed by scaling of the bases. Under the SVD form, the pair (E, A) is decomposed by two nonsingular matrices M_2 and N_2 as

$$M_2 E N_2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 A N_2 = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}. \quad (2.25)$$

The SVD form was discussed by Bender and Laub for using it to examine general system properties and to derive a linear-quadratic regulator for continuous-time descriptor systems [BL87]. Similar to the Kronecker-Weierstrass decomposition, M_2 and N_2 for SVD form are in general not unique.

2.4 Temporal Response

Assume that the descriptor system (2.4) is regular. According to the Kronecker-Weierstrass decomposition, there exist matrices M_1 and N_1 such that

$$\begin{aligned}\dot{x}_1 &= \mathcal{A}x_1 + B_1u, \\ y_1 &= C_1x_1,\end{aligned}\tag{2.26}$$

$$\begin{aligned}\mathcal{N}\dot{x}_2 &= x_2 + B_2u, \\ y_2 &= C_2x_2,\end{aligned}\tag{2.27}$$

where

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = N_1^{-1}x, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M_1B, \quad [C_1 \ C_2] = CN_1.$$

Suppose that h is the degree of the nilpotent matrix $\mathcal{N}$, that is, $\mathcal{N}^{h-1} \neq 0$ and $\mathcal{N}^h = 0$. The subsystem (2.26) is a normal state-space system, whose temporal response for a given input $u(t)$ and initial condition x_{10} can be written as

$$y_1(t) = C_1 e^{\mathcal{A}t} x_{10} + C_1 \int_0^t e^{\mathcal{A}(t-\tau)} B_1 u(\tau) d\tau.\tag{2.28}$$

Then we consider the subsystem (2.27). Suppose that $u(t) \in \mathcal{C}^{h-1}$, where $\mathcal{C}^{h-1}$ stands for the set of $h-1$ times continuously differentiable functions. Then we have the following relations

$$\begin{aligned}\mathcal{N}\dot{x}_2(t) &= x_2(t) + B_2u(t), \\ \mathcal{N}^2\dot{x}_2^{(2)}(t) &= \mathcal{N}\dot{x}_2(t) + \mathcal{N}B_2\dot{u}(t), \\ &\dots, \\ \mathcal{N}^k\dot{x}_2^{(k)}(t) &= \mathcal{N}^{k-1}x_2^{(k-1)}(t) + \mathcal{N}^{k-1}B_2u^{(k-1)}(t), \\ &\dots, \\ \mathcal{N}^{h-1}\dot{x}_2^{(h-1)}(t) &= \mathcal{N}^{h-2}x_2^{(h-2)}(t) + \mathcal{N}^{h-2}B_2u^{(h-2)}(t), \\ 0 &= \mathcal{N}^{h-1}x_2^{(h-1)}(t) + \mathcal{N}^{h-1}B_2u^{(h-1)}(t).\end{aligned}\tag{2.29}$$

Hence, the expression of $x_2(t)$ can be obtained as

$$\begin{aligned}x_2(t) &= \mathcal{N}\dot{x}_2 - B_2u(t), \\ x_2(t) &= \mathcal{N}^2\dot{x}_2^{(2)}(t) - B_2u(t) - \mathcal{N}B_2\dot{u}(t), \\ &\dots, \\ x_2(t) &= -\sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(t),\end{aligned}\tag{2.30}$$

which gives

$$y_2(t) = - \sum_{k=0}^{h-1} C_2 \mathcal{N}^k B_2 u^{(k)}(t). \quad (2.31)$$

Hence, the temporal response $y(t)$ of the descriptor system (2.4) is

$$y(t) = C_1 \left(e^{\mathcal{A}t} x_{10} + \int_0^t e^{\mathcal{A}(t-\tau)} B_1 u(\tau) d\tau \right) - \sum_{k=0}^{h-1} C_2 \mathcal{N}^k B_2 u^{(k)}(t). \quad (2.32)$$

It is observed that the response of the subsystem (2.26) depends on the matrix $\mathcal{A}$, initial condition x_{10} , as well as the input $u(t)$; while the response of the subsystem (2.27) depends only on the derivative of the input $u(t)$ on time t . That is why we also call these two subsystems slow subsystem and fast subsystem, respectively. If $t \rightarrow 0^+$, then we can deduce the following constraint on the initial condition

$$x(0^+) = N_1 \begin{bmatrix} I \\ 0 \end{bmatrix} x_{10} - N_1 \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(0^+). \quad (2.33)$$

Any initial condition satisfying (2.33) is called an *admissible condition*. From this point of view, only one initial condition is allowed and hence only one solution is allowed for each choice of $u(t)$. In [VLK81, Cob83], the authors used the theory of distributions and generalized this viewpoint to allow arbitrary initial conditions. Under this theory, for the fast subsystem, we have

$$x_2(t) = - \sum_{k=1}^{h-1} \delta^{(k-1)} \mathcal{N}^k x_{20} - \sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(t), \quad (2.34)$$

where δ is the Dirac delta. As pointed out in [Cob81], the form of (2.34) suggests that in any conventional sense the dynamics of the overall system are concentrated in the slow subsystem in (2.26). With the theory of distributions, we can represent the systems whose initial conditions are not admissible or those who contain “jump” behaviors. For example, when we switch an electrical circuit on, there will be a jump in the current or voltage at this moment. For these cases, the first term of (2.34) can transform the system into an admissible state.

2.5 Admissibility

Stability is a fundamental concept for state-space systems, which can be characterized, by one of the definitions, that the system has no poles located in the right-hand plane including the imaginary axis. Under the descriptor framework, a similar yet more complicated concept called *admissibility* plays the same role as stability for state-space systems.

Definition 2.4 (*Admissibility*) References [Dai89, Lew86]

- (a) The descriptor system (2.4) is said to be regular if $\det(sE - A)$ is not identically null;
- (b) The descriptor system (2.4) is said to be impulse-free if $\deg(\det(sE - A)) = \text{rank}(E)$;
- (c) The descriptor system (2.4) is said to be stable if all the roots of $\det(sE - A) = 0$ have negative real parts;
- (d) The descriptor system (2.4) is said to be admissible if it is regular, impulse-free, and stable.

From the definition, the admissibility of a descriptor system concerns stability, as well as regularity and impulsiveness. The latter two are intrinsic properties of conventional state-space systems and are not necessarily considered in the state-space case. Furthermore, it can be deduced that if a descriptor system is impulse-free, then it is regular.

Now we give some equivalent conditions for admissibility.

Lemma 2.3 Reference [Dai89] Suppose that the descriptor system (2.4) is regular and there exist nonsingular matrices M_1 and N_1 such that the Kronecker and Weierstrass form (2.24) holds. Then

- (i) this system is said to be impulse-free if and only if $\mathcal{N} = 0$;
- (ii) this system is said to be stable if and only if $\alpha(\mathcal{A}) < 0$;
- (iii) this system is said to be admissible if and only if $\mathcal{N} = 0$ and $\alpha(\mathcal{A}) < 0$.

Lemma 2.4 Reference [Dai89] Consider the descriptor system (2.4) and suppose that there exist nonsingular matrices M_2 and N_2 such that the SVD form (2.25) holds. Then

- (i) this system is said to be impulse-free if and only if $|A_4| \neq 0$;
- (ii) this system is said to be admissible if and only if $|A_4| \neq 0$ and $\alpha(A_1 - A_2 A_4^{-1} A_3) < 0$.

Furthermore, if the descriptor system is regular and the matrices M_1 and N_1 exist to render it Kronecker-Weierstrass form, then the transfer function of this system can be written as

$$G(s) = C_1(sI - \mathcal{A})^{-1}B_1 + C_2(s\mathcal{N} - I)^{-1}B_2. \quad (2.35)$$

For an impulse-free system, that is, $\mathcal{N} = 0$, we have

$$G(s) = C_1(sI - \mathcal{A})^{-1}B_1 - C_2B_2. \quad (2.36)$$

It is noted that the term $C_2(s\mathcal{N} - I)^{-1}B_2$ leads to polynomial terms of s if both B_2 and C_2 are nonzero. Hence the impulse-free assumption guarantees the properness of the transfer function. The converse statement is, however, not true. Clearly, if either B_2 or C_2 vanishes, the transfer function is still proper, even if the system is

impulsive. Hence, given a stable transfer function $G(s)$ and its corresponding system data $(E, A, B, C, (D))$, the admissibility of this system can not be concluded.

Now we discuss briefly the issue of generalized eigenvalues of a matrix pencil. The theory mentioned here has been reported in the literature, for instance see [GvL96, BDD+00].

Consider a matrix pencil $\lambda E - A$, where E and A are both real $n \times n$ matrices, and λ is a scalar. First, we assure that this pencil is regular, that is, $|\lambda E - A| \neq 0$ for all λ . The generalized eigenvalues are defined as those λ for which

$$|\lambda E - A| = 0. \quad (2.37)$$

Definition 2.5 (*Infinite Generalized Eigenvectors*) Reference [BL87]

1. Grade 1 infinite generalized eigenvectors of the pencil $(sE - A)$ satisfy

$$Ev_i^1 = 0. \quad (2.38)$$

2. Grade k ($k \geq 2$) infinite generalized eigenvectors of the pencil $(sE - A)$ corresponding to the i^{th} grade 1 infinite generalized eigenvectors satisfy

$$Ev_i^{k+1} = Av_i^k. \quad (2.39)$$

Moreover, the finite generalized eigenvalues of $sE - A$ are called the finite dynamic modes. The infinite generalized eigenvalues of $sE - A$ with the grade 1 infinite generalized eigenvectors determine the static modes, while the infinite generalized eigenvalues with the grade k ($k \geq 2$) infinite generalized eigenvectors are the impulsive modes.

Let q be the degree of the polynomial $|\lambda E - A|$. One can state that the matrix pencil $\lambda E - A$ has q finite generalized eigenvalues and $n - q$ infinite generalized eigenvalues where the number of static modes is $n - r$ and the number of impulsive modes is $r - q$.

2.6 Controllability

In this section, we introduce controllability for descriptor systems in a way that reduces to the state-space definition when $E = I$. We suppose that the descriptor system (2.4) is regular and it is transformed into the Kronecker-Weierstrass form as follows

$$\begin{aligned} \theta_s : \dot{x}_1 &= \mathcal{A}x_1 + B_1u, & y_1 &= C_1x_1, \\ \theta_f : \mathcal{N}\dot{x}_2 &= x_2 + B_2u, & y_2 &= C_2x_2, \\ y &= y_1 + y_2, \end{aligned} \quad (2.40)$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$ and $\mathcal{N}$ is a nilpotent matrix with degree h .

Let us define

- C_p^i be the i times piecewise continuously differentiable maps on $\mathbb{R}$ with range depending on context;
- $\mathcal{I}$ be the set of admissible initial conditions, that is,

$$\mathcal{I} = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 \in \mathbb{R}^{n_1}, x_2 = - \sum_{k=0}^{h-1} \mathcal{N}^k B_2 u^{(k)}(0), u \in C_m^{h-1} \right\}; \quad (2.41)$$

- $\langle X, Y \rangle = \beta + X\beta + X^2\beta + \dots + X^{n-1}\beta$, where X is a square matrix, n is the order of X , the product XY is well defined and $\beta = \text{Im}(Y)$.

Definition 2.6 (*Reachable State*) Reference [YS81] A state x_r is reachable from a state x_0 if there exists $u(t) \in C_m^{h-1}$ such that $x(t_r) = x_r$ for some $t_r > 0$.

Lemma 2.5 Reference [YS81] Let $\mathcal{R}(0)$ be the set of reachable states from $x_0 = 0$. Then,

$$\mathcal{R}(0) = \langle \mathcal{A}, B_1 \rangle \oplus \langle \mathcal{N}, B_2 \rangle. \quad (2.42)$$

Lemma 2.6 Reference [YS81] Let $\mathcal{R}(x)$ be the set of reachable states from $x \in \mathcal{I}$. Then the complete set of reachable states $\mathcal{R}$ is

$$\mathcal{R} = \bigcup_{x \in \mathcal{I}} \mathcal{R}(x) = \mathbb{R}^{n_1} \oplus \langle \mathcal{N}, B_2 \rangle. \quad (2.43)$$

We can adopt the conventional definition of controllability for descriptor systems.

Definition 2.7 (*C-controllability*) The descriptor system (2.40) is said to be completely controllable (*C-controllable*) if one can reach any state from any initial state.

Within the descriptor framework, we also define two different types of controllability as follows.

Definition 2.8 (*R-controllability*) The descriptor system (2.40) is said to be controllable within the set of reachable states (*R-controllable*) if, from any initial state $x_0 \in \mathcal{I}$, there exists $u(t) \in C_m^{h-1}$ such that $x(t_f) \in \mathcal{R}$ for any $t_f > 0$.

Note that, for state-space systems, *C*-controllability and *R*-controllability are equivalent. This is, however, not the case for descriptor systems.

Definition 2.9 (*Imp-controllability*) Reference [Cob84] The descriptor system (2.40) is said to be impulse controllable (*Imp-controllable*) if for every $w \in \mathbb{R}^{n_2}$ there exists $u(t) \in C_m^{h-1}$ such that the fast subsystem θ_f satisfies

$$x_2(t_f) = \sum_{k=1}^{h-1} \delta_{t_f}^{(k-1)} \mathcal{N}^k w, \quad \forall t_f > 0. \quad (2.44)$$

Theorem 2.1 (Regarding $\mathcal{C}$ -controllability) References [YS81, Cob84, Dai89, Lew86]

(1) The following statements are equivalent.

- (1i) The descriptor system (2.40) is $\mathcal{C}$ -controllable.
- (1ii) θ_s and θ_f are both controllable.
- (1iii) $\langle \mathcal{A}, B_1 \rangle \oplus \langle \mathcal{N}, B_2 \rangle = \mathbb{R}^{n_1+n_2}$.
- (1iv) $\text{rank}([sE - A \ B]) = n$, for a finite $s \in \mathbb{R}$ and $\text{rank}([E \ B]) = n$.
- (1v) $\text{Im}(\lambda E - A) \oplus \text{Im}(B) = \mathbb{R}^n$ and $\text{Im}(E) \oplus \text{Im}(B) = \mathbb{R}^n$.
- (1vi) The matrix $\mathfrak{C}$ is full row rank,

$$\mathfrak{C} = \begin{bmatrix} -A & & & B & & \\ E & -A & & & B & \\ & E & \ddots & & & B \\ & & \ddots & -A & & \ddots \\ & & & E & & B \end{bmatrix}$$

(2) The following statements are equivalent.

- (2i) θ_s is controllable.
- (2ii) The descriptor system (2.40) is $\mathcal{R}$ -controllable.
- (2iii) $\langle \mathcal{A}, B_1 \rangle = \mathbb{R}^{n_1}$.
- (2iv) $\text{rank}([sE - A \ B]) = n$, for a finite $s \in \mathbb{R}$.
- (2v) $\text{Im}(\lambda E - A) \oplus \text{Im}(B) = \mathbb{R}^n$.

(3) The following statements are equivalent.

- (3i) θ_f is controllable.
- (3ii) $\langle \mathcal{N}, B_2 \rangle = \mathbb{R}^{n_2}$.
- (3iii) $\text{rank}([E \ B]) = n$.
- (3iv) $\text{Im}(E) \oplus \text{Im}(B) = \mathbb{R}^n$.
- (3v) $\text{Im}(\mathcal{N}) \oplus \text{Im}(B_2) = \mathbb{R}^{n_2}$.
- (3vi) The rows of B_2 corresponding to the bottom rows of all Jordan blocks of $\mathcal{N}$ are linearly independent.
- (3vii) $v^\top (s\mathcal{N} - I)^{-1} B_2 = 0$ for constant vector v implies that $v = 0$.

Theorem 2.2 (Regarding $\mathcal{R}$ -controllability) References [YS81, Cob84, Dai89] The following statements are equivalent.

1. The descriptor system (2.40) is $\mathcal{R}$ -controllable.
2. θ_s is controllable.
3. $\langle \mathcal{A}, B_1 \rangle = \mathbb{R}^{n_1}$.
4. $\text{rank}([sE - A \ B]) = n$, for a finite $s \in \mathbb{R}$.
5. $\text{Im}(\lambda E - A) \oplus \text{Im}(B) = \mathbb{R}^n$.

Theorem 2.3 (Regarding Imp-controllability) *References [Cob84, Dai89, Lew86]*
The following statements are equivalent.

1. The descriptor system (2.40) is Imp-controllable.
2. θ_f is Imp-controllable.
3. $\text{Ker}(\mathcal{N}) \oplus \langle \mathcal{N}, B_2 \rangle = \mathbb{R}^{n_2}$.
4. $\text{Im}(\mathcal{N}) = \langle \mathcal{N}, B_2 \rangle$.
5. $\text{Im}(\mathcal{N}) \oplus \text{Ker}(\mathcal{N}) \oplus \text{Im}(B_2) = \mathbb{R}^{n_2}$.
6. $\text{rank} \begin{pmatrix} A & E & B \\ E & 0 & 0 \end{pmatrix} = n + r$.
7. The rows of B_2 corresponding to the bottom rows of the nontrivial Jordan blocks of $\mathcal{N}$ are linearly independent.
8. $v^\top \mathcal{N}(s\mathcal{N} - I)^{-1} B_2 = 0$ for constant vector v implies that $v = 0$.

It is observed that the conditions characterizing $\mathcal{R}$ -controllability are only concerned with the slow subsystem θ_s . The response of the fast subsystem depends only on $u(t)$ and its derivatives. Any reachable state of θ_f $w \in \langle \mathcal{A}, B_1 \rangle$ can be written as $w = \sum_{k=0}^{h-1} \eta_k \mathcal{N}^k B_2$. Then it is easy to find an input $u(t)$ satisfying $u^{(k)}(t_f) = \eta_k$, for $k = 0, 1, \dots, h-1$ (for example $u(t) = \sum_{k=0}^{h-1} \eta_k (t - t_f)^k / k!$) which leads to $x_2(t_f) = w$. Hence, the fast subsystem has no impact on $\mathcal{R}$ -controllability.

Imp-controllability guarantees the ability to generate a maximal set of impulses, at each instant, in the following sense: suppose E and A are given but B and u are allowed to vary over all values.

2.7 Observability

In this section, we introduce observability for descriptor systems in a way that allows for a set of results analogous to the last section. Similarly, the concepts, that is, $\mathcal{C}$ -observability, $\mathcal{R}$ -observability, and Imp-observability are defined.

Definition 2.10 (*$\mathcal{C}$ -observability*) The descriptor system (2.40) is said to be completely observable ($\mathcal{C}$ -observability) if knowledge of $u(t)$ and $y(t)$ for $t \in [0, \infty]$ is sufficient to determine the initial condition x_0 .

Definition 2.11 (*$\mathcal{R}$ -observability*) The descriptor system (2.40) is said to be observable within the set of reachable states ($\mathcal{R}$ -observable) if, for $t \geq 0$, $x(t) \in \mathcal{I}$ can be computed from $u(\tau)$ and $y(\tau)$ for any $\tau \in [0, t]$.

Definition 2.12 (*Imp-observability*) The descriptor system (2.40) is said to be impulse observable (Imp-observable) if, for every $w \in \mathbb{R}^{n_2}$, knowledge of $y(t)$ for $t \in [0, \infty]$ to determine $x_2(t)$.

$$x_2(t) = \sum_{k=1}^{h-1} \delta_{t_f}^{(k-1)} \mathcal{N}^k w. \quad (2.45)$$

Theorem 2.4 (Regarding $\mathcal{C}$ -observability) References [YS81, Cob84, Dai89, Lew86]

(1) The following statements are equivalent.

- (1i) The descriptor system (2.40) is $\mathcal{C}$ -observable.
- (1ii) θ_s and θ_f are both observable.
- (1iii) $\langle \mathcal{A}^\top, C_1^\top \rangle \oplus \langle \mathcal{N}^\top, C_2^\top \rangle = \mathbb{R}^{n_1+n_2}$.
- (1iv) $\text{rank}([sE^\top - A^\top \ C^\top]) = n$, for a finite $s \in \mathbb{R}$ and $\text{rank}([E^\top \ C^\top]) = n$.
- (1v) $\text{Ker}(\lambda E - A) \cap \text{Ker}(C) = \{0\}$ and $\text{Ker}(E) \cap \text{Ker}(C) = \{0\}$.
- (1vi) The matrix $\mathfrak{D}$ is full row rank,

$$\mathfrak{D} = \begin{bmatrix} -A^\top & & & C^\top \\ E^\top & -A^\top & & C^\top \\ & E^\top & \ddots & C^\top \\ & & \ddots & -A^\top & & \ddots \\ & & & E^\top & & C^\top \end{bmatrix}.$$

(2) The following statements are equivalent.

- (2i) θ_s is observable.
- (2ii) The descriptor system (2.40) is $\mathcal{R}$ -observable.
- (2iii) $\langle \mathcal{A}^\top, C_1^\top \rangle = \mathbb{R}^{n_1}$.
- (2iv) $\text{rank}([sE^\top - A^\top \ C^\top]) = n$, for a finite $s \in \mathbb{R}$.
- (2v) $\text{Ker}(\lambda E - A) \cap \text{Ker}(C) = \{0\}$.

(3) The following statements are equivalent.

- (3i) θ_f is observable.
- (3ii) $\langle \mathcal{N}^\top, C_2^\top \rangle = \mathbb{R}^{n_2}$.
- (3iii) $\text{rank}([E^\top \ C^\top]) = n$.
- (3iv) $\text{Ker}(E) \cap \text{Ker}(C) = \{0\}$.
- (3v) $\text{Ker}(\mathcal{N}) \cap \text{Ker}(C_2) = \{0\}$.
- (3vi) The rows of $C_2^\top$ corresponding to the bottom rows of all Jordan blocks of $\mathcal{N}^\top$ are linearly independent.
- (3vii) $C_2(s\mathcal{N} - I)^{-1}v = 0$ for constant vector v implies that $v = 0$.

Theorem 2.5 (Regarding $\mathcal{R}$ -observability) References [YS81, Cob84, Dai89] The following statements are equivalent.

1. The descriptor system (2.40) is $\mathcal{R}$ -observable.
2. θ_s is observable.
3. $\langle \mathcal{A}^\top, C_1^\top \rangle = \mathbb{R}^{n_1}$.
4. $\text{rank}([sE^\top - A^\top \ C^\top]) = n$, for a finite $s \in \mathbb{R}$.
5. $\text{Ker}(\lambda E - A) \cap \text{Ker}(C) = \{0\}$.

Theorem 2.6 (Regarding Imp-observability) References [Cob84, Dai89, Lew86] The following statements are equivalent.

1. The descriptor system (2.40) is *Imp-observable*.
2. θ_f is *Imp-observable*.
3. $\text{Im}(\mathcal{N}^\top) \cap \text{Ker}(\langle \mathcal{N}^\top, C_2^\top \rangle) = \{0\}$.
4. $\text{Ker}(\mathcal{N}^\top) = \mathcal{N} \text{Ker}(\langle \mathcal{N}^\top, C_2^\top \rangle)$.
5. $\text{Ker}(\mathcal{N}) \cap \text{Im}(\mathcal{N}) \cap \text{Ker}(C_2) = \{0\}$.
6. $\text{rank} \left(\begin{bmatrix} A^\top & E^\top & C^\top \\ E^\top & 0 & 0 \end{bmatrix} \right) = n + r$.
7. The rows of $C_2^\top$ corresponding to the bottom rows of the nontrivial Jordan blocks of $\mathcal{N}^\top$ are linearly independent.
8. $C_2(s\mathcal{N} - I)^{-1}\mathcal{N}v = 0$ for constant vector v implies that $v = 0$.

Similar to $\mathcal{R}$ -controllability, the characterizations for evaluating $\mathcal{R}$ -observability are only concerned with the slow subsystem θ_s .

2.8 Duality

As known, there is a strong sense of symmetry between controllability and observability for the state-space setting. We now extend this idea to descriptor systems. Corresponding to (2.4), we define the dual system $\bar{\theta}$

$$\begin{aligned} E^\top \dot{x} &= A^\top x + C^\top u, \\ y &= B^\top x. \end{aligned} \tag{2.46}$$

Then we have the following statements.

Theorem 2.7 (Duality)

1. The descriptor system (2.4) is $\mathcal{C}$ -controllable ($\mathcal{C}$ -observable) if and only if the system (2.46) is $\mathcal{C}$ -observable ($\mathcal{C}$ -controllable).
2. The descriptor system (2.4) is $\mathcal{R}$ -controllable ($\mathcal{R}$ -observable) if and only if the system (2.46) is $\mathcal{R}$ -observable ($\mathcal{R}$ -controllable).
3. The descriptor system (2.4) is *Imp-controllable* (*Imp-observable*) if and only if the system (2.46) is *Imp-observable* (*Imp-controllable*).

2.9 Discrete-Time Descriptor Systems

Consider the following linear time-invariant discrete-time descriptor system:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{2.47}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the descriptor variable and control input vector, respectively. The matrix $E \in \mathbb{R}^{n \times n}$ may be singular, i.e., $\text{rank}(E) = r \leq n$.

The aforementioned notations for continuous-time descriptor systems can be adopted directly for its discrete-time counterpart. The only two differences between continuous-time and discrete-time settings are impulsiveness and stability. For discrete-time descriptor systems, we use the term *causality* instead of impulsiveness; while the discrete-time descriptor system (2.47) is said to be stable if $\rho(E, A) < 1$. Moreover, the definition of the transfer function of a regular discrete-time descriptor system is the same as that defined in the continuous-time setting, except for the use of the shift operator z instead of the Laplace operator s . Interested readers may referred to [Dai89, XL06] for a comprehensive discussion of discrete-time descriptor systems.

2.10 Conclusion

This chapter recalls some basic concepts for linear time-invariant descriptor systems. Some fundamental and important results, such as regularity, admissibility, equivalent realizations, system decomposition and temporal response, are reviewed. The definitions of controllability and observability are also presented. Compared with state-space systems, for a descriptor system, three types of controllability are involved, that is, $\mathcal{C}$ -controllability, $\mathcal{R}$ -controllability, and Imp-controllability. This is also the case for observability. In addition, the duality between controllability and observability is stated.

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