Chapter 2
Measures of Noncompactness and Their Applications

Mohammad Mursaleen, Syed M.H. Rizvi and Bessem Samet

Abstract In this chapter, we present a survey of theory and applications of measures of noncompactness. The standard measures of noncompactness are discussed and their properties are compared. Some results concerning standard measures of noncompactness in different spaces including $C([a, b]; \mathbb{R})$, $L^p([a, b]; \mathbb{R})$, Banach spaces with Schauder bases, and paranormed spaces are presented. Moreover, we study different classes of operators, for which we establish fixed point results via an arbitrary measure of noncompactness in the sense of Banaś and Goebel. Finally, we present some applications of the measure of noncompactness concept to functional equations including nonlinear integral equations of fractional orders, implicit fractional integral equations and q-integral equations of fractional orders.

2.1 Introduction

One of the most widely used techniques of proving that certain operator equation has a solution is to reformulate the problem as a fixed point problem and see if the latter can be solved via a fixed point argument. Measures of noncompactness play an important role in fixed point theory and have many applications in various branches of nonlinear analysis, including differential equations, integral and integro-differential equations, optimization, etc. Roughly speaking, a measure of noncompactness is a function defined on the family of all nonempty and bounded subsets of a certain metric space such that it is equal to zero on the whole family of relatively compact sets. The concept of measure of noncompactness was first introduced by Kuratowski [46] in 1930. In 1955, the Italian mathematician Darbo [24] used the Kuratowski measure in

M. Mursaleen (✉) · S.M.H. Rizvi
Department of Mathematics, Aligarh Muslim University, Aligarh, India
e-mail: mursaleenm@gmail.com

S.M.H. Rizvi
e-mail: syedrizvi022@gmail.com

B. Samet
Department of Mathematics, College of Science, King Saud University,
P.O. Box 2455, Riyadh 11451, Saudi Arabia
e-mail: bsamet@ksu.edu.sa

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order to investigate a class of operators (condensing operators) whose properties can be characterized as being intermediate between those of contraction and compact mappings. Darbo's fixed point theorem is useful in establishing existence results for different classes of operator equations.

Other measures of noncompactness have been defined since then. The most important ones are the Hausdorff measure of noncompactness introduced by Goldenstein et al. [35] in 1957 (and later studied by Goldenstein and Markus [36]), the inner Hausdorff measure of noncompactness and the Istrătescu measure introduced by Istrătescu [38] in 1972.

In this chapter, we present a survey of theory and applications of measures of noncompactness. In Sect. 2.2, the classical measures of noncompactness are discussed and their properties are compared. In Sect. 2.3, we discuss some results concerning measures of noncompactness in certain spaces including $C([a, b]; \mathbb{R})$, $L^p([a, b]; \mathbb{R})$, Banach spaces with Schauder bases, and paranormed spaces. In Sect. 2.4, we discuss the axiomatic approach for measure of noncompactness, developed by Banaś and Goebel [17]. In Sect. 2.5, we discuss different classes of operators, for which we study the existence of fixed points via measures of noncompactness. Several generalizations of Darbo's fixed point theorem are presented. In the last section, we present some applications of the measure of noncompactness concept to functional equations including nonlinear integral equations of fractional orders, implicit fractional integral equations and q-integral equations of fractional orders.

2.2 Standard Measures of Noncompactness

In this section, we define the most known measures of noncompactness and recall briefly some of their basic properties.

2.2.1 The Kuratowski Measure of Noncompactness

Definition 2.1 Let (X, d) be a complete metric space. The Kuratowski measure of noncompactness of a nonempty and bounded subsets Q of X, denoted by $\alpha(Q)$, is the infimum of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \varepsilon$, i.e.,

$$\alpha(Q) = \inf \left\{ \varepsilon > 0 \, : \, Q \subset \bigcup_{i=1}^{n} S_i \, : \, S_i \subset X, \, \operatorname{diam}(S_i) < \varepsilon, \, i = 1, 2, \ldots, n, \, n \in \mathbb{N} \right\}.$$

The function α defined on the set of all nonempty and bounded subsets of (X, d), is called Kuratowski's measure of noncompactness.

The following properties are consequences of Definition 2.1.

Proposition 2.1 *Let Q, Q_1, and Q_2 be nonempty and bounded subsets of a complete metric space (X, d). Then*

$\alpha(Q) = 0 \Longleftrightarrow \overline{Q}$ *is compact, where $\overline{Q}$ denotes the closure of Q.*
$\alpha(Q) = \alpha(\overline{Q})$.
$Q_1 \subset Q_2 \Longrightarrow \alpha(Q_1) \leq \alpha(Q_2)$.
$\alpha\,(Q_1 \cup Q_2) = \max\{\alpha(Q_1), \alpha(Q_2)\}$.
$\alpha\,(Q_1 \cap Q_2) \leq \min\{\alpha(Q_1), \alpha(Q_1)\}$.

For the proof of Proposition 2.1, we refer to [18].

The next result is a generalization of the well-known Cantor intersection theorem.

Theorem 2.1 (Kuratowski [46]) *Let (X, d) be a complete metric space. If (F_n) is a decreasing sequence of nonempty, closed and bounded subsets of X such that $\lim_{n\to\infty} \alpha(F_n) = 0$, then the intersection $F_\infty = \bigcap_{n=1}^{\infty} F_n$ is nonempty and compact subset of X.*

Other properties hold if X is a Banach space, the case in which we are more interested.

Proposition 2.2 *Let Q, Q_1 and Q_2 be nonempty and bounded subsets of a Banach space $(X, \|\cdot\|)$ over $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Then*

$\alpha(Q_1 + Q_2) \leq \alpha(Q_1) + \alpha(Q_2)$.
$\alpha(Q + x) = \alpha(Q)$, *for all* $x \in X$.
$\alpha(\lambda Q) = |\lambda|\alpha(Q)$, *for all* $\lambda \in \mathbb{F}$.
$\alpha(Q) = \alpha(conv(Q))$, *where $conv(Q)$ denotes the convex hull of Q.*

We refer to [18] for the proof of Proposition 2.2.

Theorem 2.2 (Furi-Vignoli [34], Nussbaum [52]) *Let $(X, \|\cdot\|)$ be a Banach space. Let B_X be the unit ball in X. Then $\alpha(B_X) = 0$ if X is finite-dimensional, and $\alpha(B_X) = 2$ in the opposite case.*

2.2.2 The Hausdorff Measure of Noncompactness

In general, the computation of the exact value of $\alpha(Q)$ is difficult. Another measure of noncompactness, which seems to be more applicable, is so-called Hausdorff measure of noncompactness (or ball measure of noncompactness). It is defined as follows.

Definition 2.2 Let (X, d) be a complete metric space. The Hausdorff measure of noncompactness of a nonempty and bounded subset Q of X, denoted by $\chi(Q)$, is the infimum of all numbers $\varepsilon > 0$ such that Q can be covered by a finite number of balls with radii $< \varepsilon$, i.e.,

$$\chi(Q) = \inf \left\{ \varepsilon > 0 \,:\, Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X,\ r_i < \varepsilon,\ i = 1, 2, \ldots, n,\ n \in \mathbb{N} \right\}.$$

Here, $B(x_i, r_i)$ denotes the open ball of center x_i and radius r_i. The function χ defined on the set of all nonempty and bounded subsets of (X, d), is called Hausdorff measure of noncompactness.

If $(X, \|\cdot\|)$ is a Banach space, we have the following equivalent definition.

Definition 2.3 Let $(X, \|\cdot\|)$ be a Banach space. The Hausdorff measure of noncompactness of a nonempty and bounded subset Q of X, denoted by $\chi(Q)$, is the infimum of all numbers $\varepsilon > 0$ such that Q has a finite ε-net in X, i.e.,

$$\chi(Q) = \inf \{\varepsilon > 0 \,:\, Q \subset S + \varepsilon B_X,\ S \subset X,\ S \text{ is finite}\}.$$

The following properties follow from Definition 2.2.

Proposition 2.3 *Let Q, Q_1, and Q_2 be nonempty and bounded subsets of a complete metric space (X, d). Then*

$\chi(Q) = 0 \Longleftrightarrow \overline{Q}$ *is compact.*
$\chi(Q) = \alpha(\overline{Q})$.
$Q_1 \subset Q_2 \Longrightarrow \chi(Q_1) \leq \chi(Q_2)$.
$\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$.
$\chi(Q_1 \cap Q_2) \leq \min\{\chi(Q_1), \chi(Q_1)\}$.

Proposition 2.4 *Let Q, Q_1, and Q_2 be nonempty and bounded subsets of a Banach space $(X, \|\cdot\|)$ over $\mathbb{F}$. Then*

$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2)$.
$\chi(Q + x) = \chi(Q)$, *for all* $x \in X$.
$\chi(\lambda Q) = |\lambda|\chi(Q)$, *for all* $\lambda \in \mathbb{F}$.
$\chi(Q) = \chi(conv(Q))$.

The next result shows the equivalence between the Kuratowski's measure of noncompactness and the Hausdorff measure of noncompactness.

Theorem 2.3 *Let (X, d) be a complete metric space and Q be a nonempty and bounded subset of X. Then*

$$\chi(Q) \leq \alpha(Q) \leq 2\chi(Q). \tag{2.1}$$

For the proof of Theorem 2.3, we refer to [18].

Remark 2.1 In the class of all infinite-dimensional spaces inequalities (2.1) are sharp.

Theorem 2.4 *Let $(X, \|\cdot\|)$ be a Banach space. Then $\chi(B_X) = 0$ if X is finite-dimensional, and $\chi(B_X) = 1$ in the opposite case.*

See [18] for the proof.

2.2.3 The Istrătescu Measure of Noncompactness

In this section, we describe briefly another measure of noncompactness which is useful in applications. At first, we need to recall the following concept.

Definition 2.4 Let (X, d) be a complete metric space. Let Q be a nonempty and bounded subset of X. For $\varepsilon > 0$, the subset Q is said to be ε-discrete if the following property holds:

$$x, y \in Q,\ x \neq y \Longrightarrow d(x, y) \geq \varepsilon.$$

Remark 2.2 Let (X, d) be a complete metric space and Q be a nonempty and bounded subset of X. It is not hard to see that the set Q is relatively compact if and only if every ε-discrete subset of Q is finite for all $\varepsilon > 0$.

Definition 2.5 Let (X, d) be a complete metric space and Q be a bounded subset of X. Then the Istrătescu measure of noncompactness of Q, denoted by $\beta(Q)$, is defined by

$$\beta(Q) = \inf\left\{\varepsilon > 0 :\ Q \text{ has no infinite } \varepsilon\text{-discrete subsets}\right\}.$$

The function β defined on the set of all nonempty and bounded subsets of (X, d), is called Istrătescu's measure of noncompactness.

Remark 2.3 The above-mentioned properties of α and χ are also valid for β.

Note that there is no general formula for computing the value of $\beta(B_X)$ (Kottman constant). However, some estimates exist in the literature for some particular spaces. Let us recall some results in this direction.

Theorem 2.5 (Benavides [28]) *Let X be a Hilbert space. Then*

$$\beta(B_X) = \sqrt{2}.$$

Theorem 2.6 (Kottman [45]) *Let $X = l^p$, $1 \leq p < \infty$. Then*

$$\beta(B_X) = 2^{1/p}.$$

Theorem 2.7 (see [22, 53]) *Let $X = L^p$, $p \geq 1$. Then*

$$\beta(B_X) = \begin{cases} 2^{1/p} & \text{if } 1 \leq p \leq 2, \\ 2^{1-1/p} & \text{if } 2 \leq p < \infty. \end{cases}$$

2.2.4 Inner Hausdorff Measure of Noncompactness

Now, we will mention another measure of noncompactness, namely inner Hausdorff measure of noncompactness, denoted as χ_i which is very similar to the Hausdorff measure of noncompactness χ, except that in this case the balls which cover the set, have their center inside the set. It is defined as follows.

Definition 2.6 Let (X, d) be a complete metric space and Q be a nonempty and bounded subset of X. Then the inner Hausdorff measure of noncompactness of Q, denoted by $\chi_i(Q)$, is the infimum of all the numbers $\varepsilon > 0$ such that Q can be covered by a finite number of balls with radii $< \varepsilon$ and centers in Q, that is,

$$\chi_i(Q) = \inf\left\{\varepsilon > 0 :\ Q \subset \bigcup_{i=1}^{n} B(x_i, r_i),\ x_i \in Q,\ r_i < \varepsilon,\ (i = 1, 2, \ldots, n),\ n \in \mathbb{N}\right\}. \tag{2.2}$$

The function χ_i defined on the set of all nonempty and bounded subsets of (X, d), is called inner Hausdorff measure of noncompactness.

Note that the measure χ_i does not have some properties of the measures α and β. More precisely, if Q_1 and Q_2 are nonempty and bounded subsets of (X, d), then

$$Q_1 \subset Q_2 \not\Longrightarrow \chi_i(Q_1) \le \chi_i(Q_2),$$
$$\chi_i\,(Q_1 \cup Q_2) \ne \max\{\chi_i(Q_1), \chi_i(Q_2)\}.$$

Moreover, if X has the structure of a Banach space, and Q is a nonempty and bounded subset of X, then

$$\chi_i(Q) \ne \chi_i(\mathrm{conv}(Q)).$$

For some counter-examples illustrating the above facts, we refer to [18].

The following theorem due to Daneš gives the relationship for the estimate of the four measures of noncompactness discussed so far.

Theorem 2.8 ([Daneš [23]) *Let (X, d) be a complete metric space and Q be a nonempty and bounded subset of X. Then*

$$\chi(Q) \le \chi_i(Q) \le \beta(Q) \le \alpha(Q) \le 2\chi(Q). \tag{2.3}$$

Remark 2.4 It follows immediately from (2.3) that

$$\frac{1}{2}\alpha(Q) \le \beta(Q) \le \alpha(Q) \quad \text{and} \quad \chi(Q) \le \beta(Q) \le 2\chi(Q).$$

2.3 Measures of Noncompactness in Some Spaces

In this section, we will discuss some results concerning well-known measures of noncompactness in certain spaces including $C([a, b]; \mathbb{R})$, $L^p([a, b]; \mathbb{R})$, Banach spaces with Schauder bases, and paranormed spaces.

2.3.1 Hausdorff Measure of Noncompactness in the Space $C([a, b]; \mathbb{R})$

Let $C([a, b]; \mathbb{R})$ be the space of all real valued and continuous functions on the interval $[a, b]$, $-\infty < a < b < +\infty$. It is well known that such space is a Banach space with respect to the norm

$$\|u\|_\infty = \max\{|u(t)| : a \le t \le b\}, \quad u \in C([a, b]; \mathbb{R}).$$

Let $x \in C([a, b]; \mathbb{R})$. For $r \ge 0$ (small enough), we denote by x_r the r-translate of the function x, i.e.,

$$x_r(t) = \begin{cases} x(t + r) & \text{if } a \le t \le b - r, \\ x(b) & \text{if } b - r \le t \le b. \end{cases}$$

The following theorem gives us an explicit formula for the Hausdorff measure of noncompactness in the space $C([a, b]; \mathbb{R})$.

Theorem 2.9 *Let Q be a nonempty and bounded subset of $C([a, b]; \mathbb{R})$. Then*

$$2\chi(Q) = \lim_{\rho \to 0} \left\{ \sup_{x \in Q} \left[\max_{0 \le r \le \delta} \|x - x_r\|_\infty \right] \right\}.$$

For the proof of the above result, we refer to [18].

2.3.2 Hausdorff Measure of Noncompactness in the Space $L^p([a, b]; \mathbb{R})$

Let $L^p([a, b]; \mathbb{R})$, $-\infty < a < b < +\infty$, $1 \le p < \infty$, be the Banach space of equivalence classes x of measurable functions $u : [a, b] \to \mathbb{R}$, which are p-integrable, endowed with the norm

$$\|x\|_p = \left(\int_a^b |x(t)|^p \, dt \right)^{1/p}, \quad x \in L^p([a, b]; \mathbb{R}).$$

Let $x \in L^p([a, b]; \mathbb{R})$. For $h > 0$ (small enough), we denote by x_h the Steklov mean of the function x defined as

$$x_h(t) = \frac{1}{2h} \int_{t-h}^{t+h} x(s) \, ds, \quad t \in [a, b].$$

Here we put $x(t) = 0$ outside of the interval $[a, b]$.

For a nonempty and bounded subset Q of $L^p([a,b];\mathbb{R})$, let

$$\mu(Q) = \lim_{\varepsilon\to 0}\left\{\sup_{x\in Q}\left[\max_{0\le h\le\varepsilon}\|x - x_h\|_p\right]\right\}$$

We have the following result that gives us an estimate of $\chi(Q)$.

Theorem 2.10 *Let Q be a nonempty and bounded subset of $L^p([a,b];\mathbb{R})$. Then*

$$\frac{1}{2}\mu(Q) \le \chi(Q) \le \mu(Q). \tag{2.4}$$

We refer to [18] for the proof of the above result.

Remark 2.5 It may be shown [30] that the estimates (2.4) are sharp.

2.3.3 Hausdorff Measure of Noncompactness in Banach Spaces with Schauder Bases

We start this section with some concepts concerning sequence spaces.

Definition 2.7 Let $(E, \|\cdot\|)$ be a Banach space over $\mathbb{F}$. A sequence $\{e_n\} \subset E$ is said to be a Schauder basis of E if for every element $x \in E$, there exists a unique sequence $\{a_n\}$ of scalars in $\mathbb{F}$ so that

$$x = \sum_{n=0}^{\infty} a_n e_n,$$

where the convergence is understood with respect to the norm topology, i.e.,

$$\lim_{n\to\infty}\left\| x - \sum_{k=0}^{n} a_k e_k \right\| = 0.$$

Definition 2.8 Let $(E, \|\cdot\|)$ be a Banach space with a Schauder basis $\{e_k\} \subset E$. For each $n \in \mathbb{N}$, the projection mapping $P_n : E \to E$ is defined by

$$P_n(x) = P_n\left(\sum_{k=0}^{\infty} a_k e_k\right) = \sum_{k=0}^{n} a_k e_k, \quad x \in E.$$

Remark 2.6 It follows from the Banach-Steinhaus theorem that all operators P_n and $I_E - P_n$ are equibounded, where $I_E : E \to E$ is the identity mapping.

Example 2.1 Let c_0 be the space of all null sequences in $\mathbb{F}$, that is,

$$c_0 = \{x = (x_n) \in \mathbb{F} : x_n \to 0 \text{ as } n \to \infty\}.$$

The space c_0 is a Banach space with respect to the norm

$$\|x\|_\infty = \sup_n |x_n|, \quad x = (x_n) \in c_0.$$

For all $k \in \mathbb{N}$, consider the vector $e_k = (e_{k,n})_n \in c_0$ defined by

$$e_{k,n} = \delta_{k,n}, \tag{2.5}$$

where $\delta_{k,n}$ is the Kronecker delta. Then $\{e_k\}$ is a Schauder basis of c_0.

Example 2.2 Let ℓ_p $(1 \leq p < \infty)$ be the space of all absolutely p-summable series, that is,

$$\ell_p = \left\{ x = (x_n) \in \mathbb{F} : \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}.$$

The space ℓ_p is a Banach space with respect to the norm

$$\|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p}, \quad x = (x_n) \in \ell_p.$$

For all $k \in \mathbb{N}$, consider the vector $e_k = (e_{k,n})_n \in \ell_p$ defined by (2.5). Then $\{e_k\}$ is a Schauder basis of ℓ_p.

Example 2.3 Every orthonormal basis in a separable Hilbert space is a Schauder basis.

Definition 2.9 A sequence space X is called an FK space if it is a complete linear metric space with continuous coordinates $p_k : X \to \mathbb{F}$, $k \in \mathbb{N}$, where

$$p_k(x) = x_k, \quad x = (x_n) \in X,\ k \in \mathbb{N}.$$

A normed FK space is called a BK space, that is, a BK space is a Banach sequence space with continuous coordinates.

Example 2.4 The sequence spaces $(c_0, \|\cdot\|_\infty)$ and $(\ell_p, \|\cdot\|_p)$, $1 \leq p < \infty$, are BK spaces.

Definition 2.10 Let ϕ be the set of all finite sequences in $\mathbb{F}$. A BK space $(X, \|\cdot\|)$ containing ϕ is said to have AK if every sequence $x = (x_k) \in X$ has a unique representation

$$x = \sum_{k=0}^{\infty} x_k e_k,$$

where $\{e_k\}$ is defined by (2.5), that is,

$$\lim_{n\to\infty} \left\| x - \sum_{k=0}^{n} x_k e_k \right\| = 0.$$

Example 2.5 The sequence spaces $(c_0, \|\cdot\|_\infty)$ and $(\ell_p, \|\cdot\|_p)$, $1 \le p < \infty$, have AK.

Now, we have the following important result, which is due to Goldenštein, Gohberg, and Markus [35].

Theorem 2.11 *Let $(X, \|\cdot\|_X)$ be a BK-space with Schauder basis $\{e_n\}$ and $Q \in \mathscr{M}_X$. Then*

$$\begin{aligned}\frac{1}{a}\lim_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|_X\right) &\le \chi(Q) \le \inf_n\left(\sup_{x\in Q}\|(I-P_n)(x)\|_X\right)\\ &\le \limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)(x)\|_X\right),\end{aligned}$$

where $a = \limsup_{n\to\infty}\|(I-P_n)\|$. Here, $\|\cdot\|$ denotes the standard norm on the set of all linear and bounded operators $L : X \to X$.

We now mention a result, which is used to obtain the formula for Hausdorff measure of noncompactness in some of the widely used classical Banach spaces. At first, we need the following definition.

Definition 2.11 Let $(X, \|\cdot\|_X)$ be a sequence space. We say that the norm $\|\cdot\|_X$ is monotone if the following condition is satisfied:

$$x = (x_n),\, y = (y_n) \in X,\ \ |x_n| \le |y_n| \text{ for all } n \in \mathbb{N} \Longrightarrow \|x\|_X \le \|y\|_X.$$

Example 2.6 Obviously, $\|\cdot\|_\infty$ is a monotone norm in the space c_0. Similarly, $\|\cdot\|_p$ is a monotone norm in the space ℓ_p, $1 \le p < \infty$.

Theorem 2.12 (see [18]) *Let $(X, \|\cdot\|_X)$ be a BK-space with AK and monotone norm $\|\cdot\|_X$, $Q \in \mathscr{M}_X$, and $P_n : X \to X$ $(n \in \mathbb{N})$ be the projector operator defined by $P_n(x_1, x_2, \ldots) = x^{[n]} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$, for all $x = (x_1, x_2, \ldots) \in X$. Then*

$$\chi(Q) = \lim_{n\to\infty}\left(\sup_{x\in Q}\|(I-P_n)x\|_X\right).$$

Since c_0 space and ℓ_p space, $1 \leq p < \infty$, are BK-spaces with AK-property and their respective norms are monotone, we have the following results as consequences of Theorem 2.12.

Theorem 2.13 *Let Q be a nonempty and bounded subset of $(\ell_p, \|\cdot\|_p)$, $1 \leq p < \infty$. Then*

$$\chi(Q) = \lim_{n\to\infty} \sup_{x\in Q} \left(\sum_{k\geq n} |x_k|^p \right)^{1/p}.$$

Theorem 2.14 *Let Q be a nonempty and bounded subset of $(c_0, \|\cdot\|_\infty)$. Then*

$$\chi(Q) = \lim_{n\to\infty} \sup_{x\in Q} \left(\max_{k\geq n} |x_k| \right).$$

2.3.4 Inner Measure of Noncompactness in Paranormed Spaces

The relation $\mu(X) = \mu(\text{conv}(X))$, where μ is a certain measure of noncompactness, is of great importance in fixed point theory in normed spaces, or more generally in locally convex spaces. Hadžić [37] studied the inner Hausdorff measure of noncompactness in paranormed spaces. Under certain conditions, she proved the inequality $\chi_i(\text{conv}(Q)) \leq \varphi(\chi_i(Q))$, where $\varphi : [0, \infty) \to [0, \infty)$. Using such inequality, she got some fixed point theorems for multivalued mappings. Before recalling some of the obtained results in [37], we need some concepts.

Definition 2.12 Let E be a vector space over $\mathbb{F}$ and $\|\cdot\|^* : E \to [0, \infty)$ so that the following conditions are satisfied:

(i) $\|0_E\|^* = 0$, where 0_E is the zero vector of E.
(ii) $\|x\|^* = \|-x\|^*$, for every $x \in E$.
(iii) $\|x + y\|^* \leq \|x\|^* + \|y\|^*$, for every $x, y \in E$.
(iv) If $\lim_{n\to\infty} \lambda_n = \lambda$ $(\lambda_n, \lambda \in \mathbb{F})$ and $\lim_{n\to\infty} \|x_n - x\|^* = 0$ $(x_n, x \in E)$, then we have $\lim_{n\to\infty} \|\lambda_n x_n - \lambda x\|^* = 0$.

Then the pair $(E, \|\cdot\|^*)$ is called a paranormed space and $\|\cdot\|^*$ is a paranorm on E.

Remark 2.7 If $(E, \|\cdot\|^*)$ is a paranormed space, then E is a metrizable topological vector space in which the fundamental system of neighborhoods of zero is given by the family $V = \{V_r : r > 0\}$, where

$$V_r = \{x \in E : \|x\|^* < r\}.$$

Let us give some examples of paranormed spaces.

Example 2.7 Obviously, any normed space is a paranormed space.

Example 2.8 Let $E = \mathbb{R}^2$ and $\|\cdot\|^* : E \to [0, \infty)$ be the mapping defined by

$$\|(x, y)\|^* = |x|, \quad (x, y) \in E.$$

Then $\|\cdot\|^*$ is a paranorm on E.

Example 2.9 Let (E, p) be a paranormed space. Define the mapping $\|\cdot\|^* : E \to [0, \infty)$ by

$$\|x\|^* = \frac{p(x)}{1 + p(x)}, \quad x \in E.$$

Then $\|\cdot\|^*$ is a paranorm on E.

Definition 2.13 Let $(E, \|\cdot\|^*)$ be a paranormed space, M a nonempty subset of E and $\varphi : (0, \infty) \to (0, \infty)$. The set M is said to be of Z_φ-type, if for every $r > 0$,

$$\operatorname{conv}(V_r \cap (M - M)) \subset V_{\varphi(r)}.$$

Example 2.10 Let $E = L^1(0, 1)$ and $\|\cdot\|^* : E \to [0, \infty)$ be the mapping defined by

$$\|x\|^* = \int_0^1 \frac{|x(t)|}{1 + |x(t)|}\, dt, \quad x \in E.$$

It is not difficult to observe that $\|\cdot\|^*$ is a paranorm on E. Let $h > 0$ be fixed and consider the nonempty subset of E defined by

$$M_h = \{x \in E : |x(t)| \le h \text{ for a.e. } t \in (0, 1)\}.$$

We claim that

$$\|r(x_1 - x_2)\|^* \le (1 + 2h) r \|x_1 - x_2\|^*, \text{ for all } r > 0,\ x_i \in M_h,\ i = 1, 2. \tag{2.6}$$

In order to prove (2.6), let us take $r > 0$ and $x_1, x_2 \in E$ such that $|x_i(t)| \le h$ for a.e. $t \in (0, 1)$, $i = 1, 2$. For a.e. $t \in (0, 1)$, we have

$$\begin{aligned} 1 + |x_1(t) - x_2(t)| &\le 1 + 2h \\ &\le 1 + 2h + (1 + 2h) r |x_1(t) - x_2(t)| \\ &= (1 + 2h)\left(1 + r|x_1(t) - x_2(t)|\right), \end{aligned}$$

which yields

$$\frac{1}{1 + r|x_1(t) - x_2(t)|} \le \frac{(1 + 2h)}{1 + |x_1(t) - x_2(t)|}, \quad \text{for a.e. } t \in (0, 1).$$

Using the above inequality, we obtain

$$\begin{aligned}\|r(x_1-x_2)\|^* &= \int_0^1 \frac{r|x_1(t)-x_2(t)|}{1+r|x_1(t)-x_2(t)|}\,dt\\ &\le \int_0^1 (1+2h)\frac{r|x_1(t)-x_2(t)|}{1+|x_1(t)-x_2(t)|}\,dt\\ &= (1+2h)r\|x_1-x_2\|^*,\end{aligned}$$

which proves (2.6).

Now, let $z \in \operatorname{conv}(V_r \cap (M_h - M_h))$, for some $r > 0$. By the definition of conv, we can write z as

$$z = \sum_{i=1}^{n} \lambda_i x_i,$$

where $n \in \mathbb{N}$ $(n \geq 1)$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $x_i \in V_r \cap (M_h - M_h)$, for all $i = 1, 2, \ldots, n$. Therefore, using (2.6), we obtain

$$\|z\|^* \le \sum_{i=1}^{n} \|\lambda_i x_i\|^* \le \sum_{i=1}^{n} (1+2h)\lambda_i \|x_i\|^* < (1+2h)r,$$

which proves that $\operatorname{conv}(V_r \cap (M_h - M_h)) \subset V_{\varphi(r)}$, where $\varphi(r) = (1+2h)r$. As a consequence, M_h is of Z_φ-type.

The inner measure of noncompatness in a paranormed space $(E, \|\cdot\|^*)$ is the function χ_i defined by (2.2), where

$$B(x_i, r_i) = \{x \in E : \|x - x_i\|^* < r_i\}.$$

Theorem 2.15 *Let $(E, \|\cdot\|^*)$ be a paranormed space, K a nonempty and bounded subset of E which is of Z_φ-type, where $\varphi : [0, \infty) \to [0, \infty)$ is a right continuous function. Then*

$$\chi_i(conv(Q)) \le \varphi(\chi_i(Q)), \quad Q \subset K,\ Q \neq \emptyset.$$

Proof Let Q be a nonempty subset of K, and let $\varepsilon > 0$. Since φ is right continuous, we have

$$\varphi(\chi_i(Q) + \rho) + \rho \le \varphi(\chi_i(Q)) + \varepsilon, \tag{2.7}$$

for some $\rho > 0$. For such $\rho > 0$, there exists a finite set $\{x_i\}_{i=1}^{n} \subset Q$ such that

$$Q \subset \bigcup_{i=1}^{n} B(x_i, \chi_i(Q) + \rho). \tag{2.8}$$

From the precompactness of the set conv $\left(\{x_i\}_{i=1}^n\right)$, it follows that there exists a finite set $\{u_j\}_{j=1}^p \subset$ conv $\left(\{x_i\}_{i=1}^n\right)$ such that

$$\operatorname{conv}\left(\{x_i\}_{i=1}^n\right) \subset \bigcup_{j=1}^{p} B(u_j, \rho). \tag{2.9}$$

We claim that

$$\operatorname{conv}(Q) \subset \bigcup_{k=1}^{p} B(u_k, \varphi(\chi_i(Q)) + \varepsilon). \tag{2.10}$$

Let y be an arbitrary element of conv(Q). Then y can be written as

$$y = \sum_{i=1}^{m} \lambda_i y_i,$$

where $\lambda_i \geq 0$, $y_i \in Q$, $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} \lambda_i = 1$. It follows from (2.8) that for every $i = 1, 2, \ldots, m$, there exists $n(i) \in \{1, 2, \ldots, m\}$ such that

$$y_i \in B(x_{n(i)}, \chi_i(Q) + \rho),$$

that is,

$$\|y_i - x_{n(i)}\|^* < \chi_i(Q) + \rho, \quad i = 1, 2, \ldots, m. \tag{2.11}$$

Let

$$x = \sum_{i=1}^{m} \lambda_i x_{n(i)}.$$

Then $x \in$ conv $\left(\{x_i\}_{i=1}^n\right)$, and from (2.9), we have $x \in B(u_k, \rho)$, for some $k \in \{1, 2, \ldots, p\}$. On the other hand, we have

$$y - x = \sum_{i=1}^{m} \lambda_i (y_i - x_{n(i)}).$$

Since $\{y_i\}_{i=1}^m \subset Q \subset K$ and $\{x_i\}_{i=1}^m \subset Q \subset K$, using (2.11), we obtain

$$y_i - x_{n(i)} \in V_{\chi_i(Q)+\rho} \cap (K - K), \quad i = 1, 2, \ldots, m.$$

Therefore,

$$y - x \in \operatorname{conv}\left(V_{\chi_i(Q)+\rho} \cap (K - K)\right).$$

But K is of Z_φ-type. Then

$$\operatorname{conv}\left(V_{\chi_i(Q)+\rho} \cap (K-K)\right) \subset V_{\varphi(\chi_i(Q)+\rho)},$$

which yields

$$y - x \in V_{\varphi(\chi_i(Q)+\rho)}.$$

Hence, we have

$$\|y-u_k\|^* = \|(y-x)+(x-u_k)\|^* \leq \|y-x\|^* + \|x-u_k\|^* < \varphi\left(\chi_i(Q)+\rho\right)+\rho,$$

which from (2.7) yields

$$\|y-u_k\|^* < \varphi(\chi_i(Q)) + \varepsilon.$$

As a consequence, (2.10) holds. Therefore, we obtain

$$\varphi(\chi_i(Q)) + \varepsilon \geq \chi_i(\operatorname{conv}(Q)).$$

Note that the above inequality holds for every $\varepsilon > 0$. Then, passing to the limit as $\varepsilon \to 0$, we get

$$\chi_i(\operatorname{conv}(Q)) \leq \varphi(\chi_i(Q)),$$

which is the desired inequality. □

2.4 Constructing Measures of Noncompactness

The notion of measure of noncompactness is defined in many ways. At first, Kuratowski [46] has introduced for the family of all nonempty and bounded subsets of metric space (X, d) the function α defined in the previous section. Similarly, Hausdorff measure of noncompactness was defined by Goldenstein et al. [35]. In a given space most suitable, a useful measure of noncompactness, is one, which satisfies some criterion for relative compactness in the underlying space and can be expressed by some simple formula. The Hausdorff measure of noncompactness satisfies these requirements in certain spaces (Chap. 5, [18]). But it is not an easy task to develop a useful measure of noncompactness in a desired space. In order to overcome this hurdle, an axiomatic approach was developed by several authors to define a general concept of a measure of noncompactness.

We will mention here the axiomatic approach for measure of noncompactness, developed by Banaś and Goebel [17] in 1980. Let $(E, \|\cdot\|)$ be a Banach space. We denote by $\mathscr{M}_E$ the collection of all nonempty and bounded subsets of E. We denote by $\mathscr{N}_E$ the collection of all relatively compact subsets of E.

Definition 2.14 A function $\mu : \mathscr{M}_E \to \mathbb{R}_+$ is said to be measure of noncompactness in the space E if it satisfies the following conditions:

1. The family $ker\mu = \{X \in \mathscr{M}_E : \mu(X) = 0\}$ is nonempty and $ker\mu \subset \mathscr{N}_E$.
2. $X \subset Y \implies \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(\text{conv}(X)) = \mu(X)$.
4. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for all $\lambda \in [0, 1]$.
5. If (X_n) is a sequence of closed sets from $\mathscr{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim\limits_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap\limits_{n=1}^{\infty} X_n \neq \emptyset$.

Remark 2.8 Observe that from the axioms 1, 2, and 5 in the above definition, we have $X_\infty \in ker\mu$, which implies that X_∞ is relatively compact.

2.4.1 Measure of Noncompactness in $C([a, b]; \mathbb{R})$

Given $X \in \mathscr{M}_{C([a,b];\mathbb{R})}$ and $\varepsilon > 0$, let

$$\omega(X, \varepsilon) = \sup\{\omega(x, \varepsilon) : x \in X\},$$

where

$$\omega(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [a, b], |t - s| \leq \varepsilon\}, \quad x \in X.$$

We have the following result, which is due to Banaś and Goebel [17].

Theorem 2.16 *Let $\omega_0 : \mathscr{M}_{C([a,b];\mathbb{R})} \to \mathbb{R}_+$ be the mapping defined by*

$$\omega_0(X) = \lim_{\varepsilon\to 0} \omega(X, \varepsilon), \quad X \in \mathscr{M}_{C([a,b];\mathbb{R})}. \tag{2.12}$$

Then ω_0 is a measure of noncompactness in $C([a, b]; \mathbb{R})$ in the sense of Definition 2.14. Moreover, we have

$$\omega_0(X) = 2\chi(X), \quad X \in \mathscr{M}_{C([a,b];\mathbb{R})}.$$

2.4.2 Some Measures of Noncompactness in $BC(\mathbb{R}_+; \mathbb{R})$

We denote by $BC(\mathbb{R}_+; \mathbb{R})$ the space of all real functions defined, continuous, and bounded on $\mathbb{R}_+$ with the standard supremum norm

$$\|x\|_\infty = \sup\{|x(t)| : t \geq 0\}, \quad x \in BC(\mathbb{R}_+; \mathbb{R}).$$

Let $X \in \mathscr{M}_{BC(\mathbb{R}_+;\mathbb{R})}$. Let $\varepsilon > 0, T > 0$ and $x \in X$ be fixed. Let us define the following quantities:

$$\begin{aligned}
\omega^T(x,\varepsilon) &= \sup\{|x(t)-x(s)| : t,s\in[0,T],\ |t-s|\le\varepsilon\},\\
\omega^T(X,\varepsilon) &= \sup\left\{\omega^T(x,\varepsilon) : x\in X\right\},\\
\omega_0^T(X) &= \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon),\\
\omega_0(X) &= \lim_{T\to\infty}\omega_0^T(X).
\end{aligned}$$

Further, let us define the set functions $a(X)$, $b(X)$, $c(X)$ by putting

$$\begin{aligned}
a(X) &= \lim_{T\to\infty}\left\{\sup_{x\in X}\left[\sup\{|x(t)| : t\ge T\}\right]\right\},\\
b(X) &= \lim_{T\to\infty}\left\{\sup_{x\in X}\left[\sup\{|x(t)-x(s)| : t,s\ge T\}\right]\right\},\\
c(X) &= \limsup_{t\to\infty}\operatorname{diam} X(t),
\end{aligned}$$

where

$$X(t) = \{x(t) : x\in X\},\quad t\ge 0$$

and diam $X(t)$ is the diameter of the set $X(t)$.

Finally, let us consider the functions μ_a, μ_b, μ_c defined on the family $\mathscr{M}_{BC(\mathbb{R}_+;\mathbb{R})}$ by

$$\begin{aligned}
\mu_a(X) &= \omega_0(X) + a(X),\\
\mu_b(X) &= \omega_0(X) + b(X),\\
\mu_c(X) &= \omega_0(X) + c(X). \qquad (2.13)
\end{aligned}$$

In [15], Banaś proved the following result.

Theorem 2.17 *The set functions $\mu_a, \mu_b, \mu_c : \mathscr{M}_{BC(\mathbb{R}_+;\mathbb{R})} \to \mathbb{R}_+$ are measures of noncompactness in $BC(\mathbb{R}_+;\mathbb{R})$ in the sense of Definition 2.14.*

Remark 2.9 Note that the quantity $\omega_0(X)$ is not a measure of noncompactness in the space $BC(\mathbb{R}_+;\mathbb{R})$. A counter-example illustrating this fact was presented in [18].

2.4.3 Measure of Noncompactness with Kernel

Another approach of constructing measures of noncompactness was introduced by Banaś [14]. This approach is based on the computation of a nonempty family $\mathscr{P} \subset \mathscr{N}_E$, which is called the kernel of a measure of noncompactness.

Definition 2.15 A nonempty family $\mathscr{P} \subset \mathscr{N}_E$ is said to be a kernel of a measure of noncompactness if it satisfies following axioms:

1. $X \in \mathscr{P} \implies \bar{X} \in \mathscr{P}$.
2. $X \in \mathscr{P}, \emptyset \neq Y \subset X \implies Y \in \mathscr{P}$.
3. $X, Y \in \mathscr{P} \implies \lambda X + (1-\lambda)Y \in \mathscr{P}$, for all $\lambda \in [0, 1]$.
4. $X \in \mathscr{P} \implies \overline{\text{conv}(X)} \in \mathscr{P}$.
5. $\mathscr{P}^c$ (i.e. collection of all compacts belonging to $\mathscr{P}$) is closed in $\mathscr{M}_E^c$ with respect to Hausdorff topology.

Definition 2.16 A function $\mu : \mathscr{M}_E \to [0, \infty)$ is said to be a measure of noncompactness with kernel $\mathscr{P}$ ($ker\,\mu = \mathscr{P}$) if it satisfies the following conditions:

1. $\mu(X) = 0 \Longleftrightarrow X \in \mathscr{P}$.
2. $X \subset Y \implies \mu(X) \leq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(\text{conv}(X)) = \mu(X)$.
4. $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$, for all $\lambda \in [0, 1]$.
5. If $(X_n) \subset \mathscr{M}_E^c$ is such that $X_{n+1} \subset X_n$ for $n = 1, 2, \ldots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

Remark 2.10 Notice that the Kuratowski's measure α and the Hausdorff measure χ defined previously are measures of noncompactness with kernel $\mathscr{P} = \mathscr{N}_E$. The simplest example of a measure with $\mathscr{P} \neq \mathscr{N}_E$ is the diameter, diam$X$. Its kernel is the family of all one-point sets.

The next result gives us a way to construct a measure of noncompactness from a kernel $\mathscr{P}$.

Theorem 2.18 (Banaś [14]) *For any kernel $\mathscr{P}$, the function $\mu : \mathscr{M}_E \to [0, \infty)$ defined by*

$$\mu(X) = D(X, \mathscr{P}) = \inf\{D(X, Y) : Y \in \mathscr{P}\}, \quad X \in \mathscr{M}_E,$$

where D is the Hausdorff metric, is a measure of noncompactness with kernel $\mathscr{P}$.

2.5 Fixed Point Theorems Involving a Measure of Noncompactness

The measure of noncompactness concept plays an important role in fixed point theory. In 1955, Darbo, using such concept, proved a theorem guaranteing the existence of fixed points of the so-called condensing operators [24]. That theorem found an abundance of applications in proving the existence of solutions for a large class of functional equations including differential and integral equations.

In this section, we present some fixed point theorems involving an arbitrary measure of noncompactness in the sense of Definition 2.14. So, if E is a Banach space, we denote by $\mu : \mathscr{M}_E \to \mathbb{R}_+$ an arbitrary measure of noncompactness in E in the sense of Definition 2.14.

At first, let us recall the well-known Schauder fixed point theorem that will be used later.

Theorem 2.19 (Schauder [56]) *Let C be a nonempty, convex and compact subset of a Banach space E. Then, every continuous mapping $F : C \to C$ has at least one fixed point.*

2.5.1 The Class of μ-Contractive Mappings and Darbo's Fixed Point Theorem

Definition 2.17 Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E. A self-mapping $T : \Omega \to \Omega$ is said to be a μ-contraction if there exists some constant $k \in (0, 1)$ such that

$$\mu(TX) \leq k\,\mu(X),$$

for every nonempty subset X of Ω.

Darbo's fixed point theorem with respect to a measure μ can be stated as follows.

Theorem 2.20 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator. If T is a μ-contraction, then T has at least one fixed point.*

For the proof of Theorem 2.20, we refer to [17].

Remark 2.11 If we denote by $\mathrm{Fix}(T)$ the set of fixed points of T, i.e.,

$$\mathrm{Fix}(T) = \{x \in \Omega : Tx = x\},$$

then from Theorem 2.20 and axiom 1 of Definition 2.14, we have $\mathrm{Fix}(T) \in \mathscr{N}_E$.

Many generalizations of Theorem 2.20 appeared recently. Most of those results are inspirated from metric fixed point theory [2]. Further, we present some of those generalizations.

2.5.2 A Fixed Point Theorem for (ψ, φ)-μ-Contractive Mappings

In [29], the following fixed point theorem was proved.

Theorem 2.21 *Let (E, d) be a complete metric space and let $T : E \to E$ be a self-mapping such that for all $x, y \in E$,*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \varphi(d(x, y)),$$

where

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,
(b) $\varphi : [0, \infty) \to [0, \infty)$ *is a lower semicontinuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$.

Then T has a unique fixed point.

Using the measure of noncompactness concept, Aghajani et al. [6] extended Theorem 2.21 as follows.

Theorem 2.22 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that for every nonempty subset X of Ω,*

$$\psi(\mu(TX)) \leq \psi(\mu(X)) - \varphi(\mu(X)), \tag{2.14}$$

where

(a) $\psi : [0, \infty) \to [0, \infty)$ *is a continuous function,*
(b) $\varphi : [0, \infty) \to [0, \infty)$ *is a lower semicontinuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$.

Then T has at least one fixed point. Moreover, we have $Fix(T) \in \mathscr{N}_E$.

Proof Consider the sequence of sets $\{\Omega_n\} \subset \mathscr{M}_E$ defined by

$$\Omega_0 = \Omega, \ \ \Omega_{n+1} = \overline{\mathrm{conv}(T\Omega_n)}, \ \ n \in \mathbb{N}. \tag{2.15}$$

Observe that

$$\Omega_{n+1} \subset \Omega_n, \quad n \in \mathbb{N}. \tag{2.16}$$

Using axiom 2 of Definition 2.14, we deduce that there exists some $r \geq 0$ such that

$$\lim_{n \to \infty} \mu(\Omega_n) = r. \tag{2.17}$$

On the other hand, using (2.14) and axiom 3 of Definition 2.14, for every $n \in \mathbb{N}$, we have

$$\psi(\mu(\Omega_{n+1})) = \psi(\mu(\overline{\mathrm{conv}(T\Omega_n)})) = \psi(\mu(T\Omega_n)) \leq \psi(\mu(\Omega_n)) - \varphi(\mu(\Omega_n)),$$

that is,

$$\psi(\mu(\Omega_{n+1})) \leq \psi(\mu(\Omega_n)) - \varphi(\mu(\Omega_n)), \quad n \in \mathbb{N}. \tag{2.18}$$

Passing to the limit as $n \to \infty$ in (2.18), we obtain

$$\limsup_{n\to\infty} \psi(\mu(\Omega_{n+1})) \leq \limsup_{n\to\infty} \psi(\mu(\Omega_n)) - \liminf_{n\to\infty} \varphi(\mu(\Omega_n)).$$

Using (2.17) and properties (a) and (b), we deduce that

$$\psi(r) \leq \psi(r) - \varphi(r),$$

which yields $\varphi(r) = 0$. Therefore, by property (b), we have $r = 0$, i.e.,

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

Now, using (2.16), on the base of axiom 5 of Definition 2.14, we derive that the set $\Omega_\infty = \bigcap_{n=0}^{\infty} \Omega_n$ is nonempty, closed, convex and $\Omega_\infty \subset \Omega$. Moreover, the set Ω_∞ is invariant under the operator T and $\Omega_\infty \in \mathscr{N}_E$ (from axiom 1 of Definition 2.14). Therefore, applying Theorem 2.19 to the operator $T : \Omega_\infty \to \Omega_\infty$, we obtain the desired result. □

Remark 2.12 Theorem 2.20 follows immediately from Theorem 2.22 by taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$.

Now, we consider another class of operators.

2.5.3 A Fixed Point Theorem for φ-μ-Contractive Mappings

Let Φ be the class of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(Φ_1) φ is a nondecreasing function.
(Φ_2) For all $t > 0$, $\lim_{n\to\infty} \varphi^n(t) = 0$, where φ^n is the n-th iterate of φ.

We have the following fixed point result [6].

Theorem 2.23 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that for every nonempty subset X of Ω,*

$$\mu(TX) \leq \varphi(\mu(X)),$$

where $\varphi \in \Phi$. Then T has at least one fixed point. Moreover, we have $Fix(T) \in \mathscr{N}_E$.

Proof Consider the sequence of sets $\{\Omega_n\}$ defined by (2.15). Without restriction of the generality, we may assume that $\mu(\Omega) > 0$. Further, taking into account our assumptions, for all $n \in \mathbb{N}$, we have

$$\mu(\Omega_{n+1}) = \mu(\overline{\mathrm{conv}(T\Omega_n)}) = \mu(T\Omega_n) \le \varphi(\mu(\Omega_n)).$$

Therefore, by induction we get

$$\mu(\Omega_n) \le \varphi^n(\mu(\Omega_0)), \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

The rest of the proof is similar to that of Theorem 2.22. □

As a consequence of Theorem 2.23, we have the following fixed point result.

Corollary 2.1 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space $(E, \|\cdot\|_E)$ and let $T : \Omega \to \Omega$ be a continuous operator such that*

$$\|Tx - Ty\|_E \le \varphi(\|x - y\|_E), \quad x, y \in \Omega,$$

where $\varphi \in \Phi$. Then T has a fixed point.

Proof Consider the mapping $\mu : \mathscr{M}_E \to \mathbb{R}_+$ defined by

$$\mu(X) = \mathrm{diam}(X), \quad X \in \mathscr{M}_E,$$

where $\mathrm{diam}(X)$ denotes the diameter of X. It is not difficult to observe that μ is a measure of noncompactness in the sense of Definition 2.14. Therefore, using the considered assumptions, for every nonempty subset X of Ω, we have

$$\begin{aligned}
\mu(TX) &= \mathrm{diam}(TX) \\
&= \sup\{\|Tx - Ty\|_E : x, y \in X\} \\
&\le \sup\{\varphi(\|x - y\|_E) : x, y \in X\} \\
&\le \varphi\left(\sup\{\|x - y\|_E : x, y \in X\}\right) \\
&= \varphi(\mathrm{diam}(X)) = \varphi(\mu(X)).
\end{aligned}$$

The application of Theorem 2.23 completes the proof. □

2.5.4 A Fixed Point Theorem for an Implicit μ-Contraction

Let $\mathscr{F}$ be the class of functions $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathscr{F}_1$) For every $z_i > 0$, $i = 1, 2$, we have $f(z_1, z_2) < z_2 - z_1$.
($\mathscr{F}_2$) If $\{u_n\}$ and $\{v_n\}$ are two sequences in $(0, \infty)$ such that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = \ell > 0,$$

then

$$\limsup_{n\to\infty} f(u_n, v_n) < 0.$$

Observe that $\mathscr{F}$ includes a large class of mappings.

Example 2.11 Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the mapping defined by

$$f(z_1, z_2) = k\, z_2 - z_1, \quad z_1, z_2 \geq 0,$$

where $k \in (0, 1)$ is a some constant. Then $f \in \mathscr{F}$.

Example 2.12 Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the mapping defined by

$$f(z_1, z_2) = z_2 - z_1 - \varphi(z_2), \quad z_1, z_2 \geq 0,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then $f \in \mathscr{F}$.

Example 2.13 Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the mapping defined by

$$f(z_1, z_2) = z_2\varphi(z_2) - z_1, \quad z_1, z_2 \geq 0,$$

where $\varphi : [0, \infty) \to [0, 1)$ is a function satisfying $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$. Then $f \in \mathscr{F}$.

Example 2.14 Let $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be the mapping defined by

$$f(z_1, z_2) = \varphi(z_2) - z_1, \quad z_1, z_2 \geq 0,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function satisfying $\varphi(t) < t$ for all $t > 0$. Then $f \in \mathscr{F}$.

We have the following fixed point result, which was proved in [41].

Theorem 2.24 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that for every nonempty subset X of Ω,*

$$f(\mu(TX), \mu(X)) \geq 0,$$

where $f \in \mathscr{F}$. Then T has at least one fixed point. Moreover, we have $Fix(T) \in \mathscr{N}_E$.

Proof Consider the sequence of sets $\{\Omega_n\}$ defined by (2.15). Without restriction of the generality, we may assume that $\mu(\Omega_n) > 0$, for all $n \in \mathbb{N}$. Further, taking into account our assumptions, we have

$$\mu(\Omega_n) - \mu(\Omega_{n+1}) > f(\mu(\Omega_{n+1}), \mu(\Omega_n)) \geq 0, \quad n \in \mathbb{N}. \tag{2.19}$$

Therefore, we get

$$\mu(\Omega_n) > \mu(\Omega_{n+1}), \quad n \in \mathbb{N}.$$

Then there is some $r \geq 0$ such that $\lim_{n\to\infty} \mu(\Omega_n) = r$. If $r > 0$, it follows from condition $(\mathscr{F}_2)$ that

$$\limsup_{n\to\infty} f(\mu(\Omega_{n+1}), \mu(\Omega_n)) < 0,$$

which contradicts (2.19). As a consequence, we have $\lim_{n\to\infty} \mu(\Omega_n) = 0$. The rest of the proof is similar to that of Theorem 2.22. □

Note that Darbo's fixed point theorem (Theorem 2.20) follows immediately from Theorem 2.24 by taking f the function defined in Example 2.11. Several other consequences follow from Theorem 2.24. Let us present some consequences.

The following result follows from Theorem 2.24 and Example 2.12.

Corollary 2.2 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that for every nonempty subset X of Ω,*

$$\mu(TX) \leq \mu(X) - \varphi(\mu(X)),$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then T has at least one fixed point. Moreover, we have $Fix(T) \in \mathscr{N}_E$.

Remark 2.13 The result given by Corollary 2.2 can be deduced also from Theorem 2.22 by taking $\psi(t) = t$.

The next result follows from Theorem 2.24 and Example 2.13.

Corollary 2.3 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that for every nonempty subset X of Ω,*

$$\mu(TX) \leq \mu(X)\varphi(\mu(X)),$$

where $\varphi : [0, \infty) \to [0, 1)$ *is a function satisfying* $\limsup_{t \to r^+} \varphi(t) < 1$ *for all* $r > 0$. *Then* T *has at least one fixed point. Moreover, we have* $Fix(T) \in \mathscr{N}_E$.

2.5.5 A Fixed Point Theorem for θ-μ-Contractive Mappings

In [43], Jleli and Samet established the following generalization of Banach contraction principle.

Let $\theta : (0, \infty) \to (1, \infty)$ be a function satisfying the following conditions:

- θ is a nondecreasing function.
- For every sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$.
- There exist $r \in (0, 1)$ and $\ell \in (0, \infty)$ such that $\lim_{t\to 0^+} \dfrac{\theta(t) - 1}{t^r} = \ell$.

Then we have the following fixed point result.

Theorem 2.25 *Let* (E, d) *be a complete metric space and let* $T : E \to E$ *be a given mapping such that*

$$x, y \in E,\ \ d(Tx, Ty) \neq 0 \Longrightarrow \theta(d(Tx, Ty)) \leq [\theta(d(x, y)]^k,$$

where $k \in (0, 1)$ *is a some constant. Then* T *has a unique fixed point.*

Observe that Banach contraction principle follows immediately from Theorem 2.25 by taking $\theta(t) = e^{\sqrt{t}}$.

Following the idea in [43], the authors in [41] obtained the following extension of Theorem 2.25.

Let Θ be the class of functions $\theta : (0, \infty) \to (1, \infty)$ satisfying the following condition: For every sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$.

Theorem 2.26 *Let* Ω *be a nonempty, bounded, closed, and convex subset of a Banach space* E *and let* $T : \Omega \to \Omega$ *be a continuous operator such that for every nonempty subset* X *of* Ω,

$$\mu(X)\mu(TX) > 0 \Longrightarrow \theta(\mu(TX)) \leq [\theta(\mu(X))]^k,$$

where $\theta \in \Theta$ *and* $k \in (0, 1)$ *are a some constant. Then* T *has at least one fixed point. Moreover, we have* $Fix(T) \in \mathscr{N}_E$.

Proof Consider the sequence of sets $\{\Omega_n\}$ defined by (2.15). Without restriction of the generality, we may assume that $\mu(\Omega_n) > 0$, for all $n \in \mathbb{N}$. Further, taking into account our assumptions, we obtain

$$1 < \theta(\mu(\Omega_n)) \leq [\theta(\mu(\Omega_0))]^{k^n}, \quad n \in \mathbb{N}.$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \theta(\mu(\Omega_n)) = 1,$$

which yields

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

The rest of the proof is similar to that of Theorem 2.22. □

Remark 2.14 Taking $\theta(t) = e^{\sqrt{t}}$ in Theorem 2.26, we obtain Darbo's fixed point result (see Theorem 2.20).

Corollary 2.4 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator such that for every nonempty subset X of Ω with $\mu(X)\mu(TX) > 0$,*

$$2 - \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{\mu(TX)}}\right) \leq \left[2 - \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{\mu(X)}}\right)\right]^k$$

where $k \in (0, 1)$ is a some constant. Then T has at least one fixed point. Moreover, we have $Fix(T) \in \mathscr{N}_E$.

Proof We have just to observe that the function $\theta : (0, \infty) \to (1, \infty)$ defined by

$$\theta(t) = 2 - \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{t}}\right), \quad t > 0$$

belongs to Θ. Then an application of Theorem 2.26 yields the desired result. □

2.5.6 Meir–Keeler Generalization of Darbo's Theorem

We discuss here an interesting generalization of Banach contraction principle, via Meir–Keeler contraction stated below, proved by Meir and Keeler in 1969 [49].

Definition 2.18 Let (E, d) be a metric space and let $T : E \to E$ be a giving operator. Then T is said to be a Meir–Keeler contraction if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in E, \ \ \varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon.$$

The fixed point theorem of Meir and Keeler can be stated as follows.

Theorem 2.27 *Let (E, d) be a complete metric space. If $T : E \to E$ is a Meir–Keeler contraction, then T has a unique fixed point.*

Aghajani et al. [7] (see also [41]) extended Theorem 2.27 to the class of Meir–Keeler condensing operators.

Definition 2.19 Let Ω be a nonempty and bounded subset of a Banach space E. We say that an operator $T : \Omega \to \Omega$ is a Meir–Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(X) < \varepsilon + \delta \implies \mu(TX) < \varepsilon,$$

for any nonempty subset X of Ω.

Remark 2.15 It is not difficult to observe that any μ-contractive operator is a Meir–Keeler condensing operator.

The characterization of Meir–Keeler contractions in metric spaces was studied by Lim [48] and Suzuki [57] by introducing class of L-functions, defined below.

Definition 2.20 A function $\phi : [0, \infty) \to [0, \infty)$ is said to be an L-function if it satisfies the following conditions:

- $\phi(s) = 0$ if and only if $s = 0$.
- For every $s > 0$, there exists $\delta > 0$ such that

$$s \leq t \leq s + \delta \implies \phi(t) \leq s.$$

Following the idea of Suzuki [57], Aghajani et al. [7] proved the following characterization of Meir–Keeler condensing operators with the help of L-functions.

Theorem 2.28 *Let Ω be a nonempty and bounded subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator. Then T is Meir–Keeler condensing operator if and only if there exists an L-function ϕ such that*

$$\mu(T(X) < \phi(\mu(X)),$$

for every nonempty, closed and bounded subset X of Ω with $\mu(X) > 0$.

Now, we present the following extension of Theorem 2.27 to the class of Meir–Keeler condensing operators [7, 41].

Theorem 2.29 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : \Omega \to \Omega$ be a continuous operator. If T is a Meir–Keeler condensing operator, then T has at least one fixed point. Moreover, we have $Fix(T) \in \mathcal{N}_E$.*

Proof Consider the sequence of sets $\{\Omega_n\}$ defined by (2.15). Without restriction of the generality, we may assume that $\mu(\Omega_n) > 0$, for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed. Since $\mu(\Omega_n) > 0$, there exists $\delta_n > 0$ such that

$$\mu(\Omega_n) \leq \mu(X) < \mu(\Omega_n) + \delta_n \implies \mu(TX) < \mu(\Omega_n),$$

for every nonempty subset X of Ω. Taking $X = \Omega_n$, we obtain $\mu(\Omega_{n+1}) < \mu(\Omega_n)$. Therefore, $\{\mu(\Omega_n)\}$ is a decreasing sequence in $(0, \infty)$, which yields

$$\lim_{n\to\infty} \mu(\Omega_n) = r^+,$$

for some $r \geq 0$. We shall prove that $r = 0$. We argue by contradiction by supposing that $r > 0$. In this case, there exists $\delta_r > 0$ such that

$$r \leq \mu(X) < r + \delta_r \implies \mu(TX) < r,$$

for every nonempty subset X of Ω. On the other hand, for n large enough, we have

$$r \leq \mu(\Omega_n) < r + \delta_r,$$

which yields the following contradiction

$$r \leq \mu(\Omega_{n+1}) < r, \text{ for } n \text{ large enough.}$$

Therefore, $r = 0$, i.e.,

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

The rest of the proof is similar to that of Theorem 2.22. □

Definition 2.21 Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be contractive on M if

$$x, y \in M, \ x \neq y \implies d(Tx, Ty) < d(x, y).$$

Let us recall the following result due to Edelstin (see [2]).

Lemma 2.1 *Let (M, d) be a compact metric space. If $T : M \to M$ is a contractive map on M, then T has a unique fixed point.*

Aghajani et al. [7] introduced the following concept of asymptotic Meir–Keeler condensing operators.

Definition 2.22 Let Ω be a nonempty and bounded subset of a Banach space E. We say that $T : \Omega \to \Omega$ is an asymptotic Meir–Keeler condensing operator if there exists a sequence $\phi_n : [0, \infty) \to [0, \infty)$, $n \in \mathbb{N}$, satisfying the following conditions:

(A1) For every $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that

$$\phi_\nu(t) \leq \varepsilon, \quad \varepsilon \leq t \leq \varepsilon + \delta.$$

(A2) For every $n \in \mathbb{N}$,

$$\mu(T^n \Omega) < \phi_n(\mu(\Omega)).$$

We have the following fixed point result for the class of asymptotic Meir–Keeler condensing operators, where the convexity of Ω is not required [7].

Theorem 2.30 *Let Ω be a nonempty, bounded and closed (not necessarily convex) subset of a Banach space E. Let $T : \Omega \to \Omega$ be contractive and asymptotic Meir–Keeler condensing operator. Then T has a unique fixed point in Ω.*

Proof Consider the sequence of sets $\{\Omega_n\}$ defined by

$$\Omega_n = \overline{T^n \Omega}, \ \ n \in \mathbb{N}.$$

As T is contractive, then it is continuous and $T(\overline{A}) \subset \overline{TA}$. On the other hand, $T^{n+1}\Omega \subset T^n \Omega$, so $\Omega_{n+1} \subset \Omega_n$ and $T\Omega_n \subset \Omega_n$. If there exists $N \in \mathbb{N}$ such that $\mu(\Omega_N) = 0$, then Ω_N will be compact. Thus by Theorem 2.19, T has a fixed point. So, we may assume that $\mu(\Omega_n) > 0$, for every $n \in \mathbb{N}$. On the other hand, from the inclusions $\Omega_{n+1} \subset \Omega_n$, for every $n \in \mathbb{N}$, we deduce that $\{\mu(\Omega_n)\}$ is a decreasing sequence in $(0, \infty)$. Therefore, there exists some $r \geq 0$ such that

$$\lim_{n\to\infty} \mu(\Omega_n) = r^+. \tag{2.20}$$

Suppose that $r > 0$. In this case, it follows from (A1) that there exists $\delta_r > 0$ and $\nu = \nu_r \in \mathbb{N}$ such that

$$\phi_\nu(t) \leq r, \quad r \leq t \leq r + \delta_r.$$

On the other hand, from (2.20), there exists some $n_0 \in \mathbb{N}$ such that

$$r \leq \mu(\Omega_n) \leq r + \delta_r, \quad n \geq n_0.$$

Then

$$r \leq \mu(\Omega_{n_0+\nu}) = \mu(T^\nu(T^{n_0}\Omega)) < \phi_\nu(\mu(T^{n_0}\Omega)) = \phi_\nu(\mu(\Omega_{n_0})) \leq r,$$

which is a contradiction. Therefore, $r = 0$, i.e.,

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

As $\Omega_{n+1} \subset \Omega_n$ and $T\Omega_n \subset \Omega_n$, for all $n \in \mathbb{N}$, then $\Omega_\infty = \bigcap_{n=0}^{\infty} \Omega_n$ is nonempty, compact and invariant under T. Then by Lemma 2.1, T has a unique fixed point in Ω_∞. On the other hand, since $\mathrm{Fix}(T) = \{x \in \Omega : x = Tx\} \subset \Omega_n$, for every $n \in \mathbb{N}$, then $\mathrm{Fix}(T) \subset \Omega_\infty$, and T has a unique fixed point in Ω. □

2.5.7 A Fixed Point Result in a Banach Algebra

In this section, we will discuss the results obtained by Aghajani et al. [4], in relation with measure of noncompactness in Banach algebra.

Let $(E, \cdot\|_E)$ be a Banach algebra. For given subsets X and Y of E, we define the subset XY of E by

$$XY = \{xy : x \in X,\ y \in Y\}.$$

For $X \in \mathscr{M}_E$, we denote by $\|X\|$ the quantity

$$\|X\| = \sup\{\|x\|_E : x \in X\}.$$

Definition 2.23 Let $\mu : \mathscr{M}_E \to \mathbb{R}_+$ be a measure of noncompactness (in the sense of Definition 2.14). We say that μ satisfies the condition (m) if

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X),$$

for every $X, Y \in \mathscr{M}_E$.

Example 2.15 Consider the Bnach algebra $E = BC(\mathbb{R}_+; \mathbb{R})$ and the measure of noncompactness $\mu_c : \mathscr{M}_{BC(\mathbb{R}_+;\mathbb{R})} \to \mathbb{R}_+$ defined in (2.13). Let us prove that the measure μ_c satisfies the condition (m). Let $X, Y \in \mathscr{M}_E$. We claim that

$$\omega_0(XY) \leq \|X\|\omega_0(Y) + \|Y\|\omega_0(X). \tag{2.21}$$

In order to prove our claim, let us consider an arbitrary pair $(x, y) \in X \times Y$ and $t, s \in [0, T]$ with $|t - s| \leq \varepsilon$. We have

$$\begin{aligned} |x(t)y(t) - x(s)y(s)| &\leq |x(t)y(t) - x(t)y(s)| + |x(t)y(s) - x(s)y(s)| \\ &\leq \|X\||y(t) - y(s)| + \|Y\||x(t) - x(s)| \\ &\leq \|X\|\omega^T(y, \varepsilon) + \|Y\|\omega^T(x, \varepsilon) \\ &\leq \|X\|\omega^T(Y, \varepsilon) + \|Y\|\omega^T(X, \varepsilon). \end{aligned}$$

Therefore,

$$\omega^T(xy, \varepsilon) \leq \|X\|\omega^T(Y, \varepsilon) + \|Y\|\omega^T(X, \varepsilon),$$

which implies that

$$\omega^T(XY, \varepsilon) \leq \|X\|\omega^T(Y, \varepsilon) + \|Y\|\omega^T(X, \varepsilon),$$

Passing to the limit as $\varepsilon \to 0$, we get

$$\omega_0^T(XY) \leq \|X\|\omega_0^T(Y) + \|Y\|\omega_0^T(X).$$

Passing to the limit as $T \to \infty$, we get

$$\omega_0(XY) \leq \|X\|\omega_0(Y) + \|Y\|\omega_0(X),$$

which is the desired inequality (2.21).

Now, we claim that

$$c(XY) \leq \|X\|c(Y) + \|Y\|c(X). \tag{2.22}$$

In order to prove the above claim, let us consider two pairs $(x_i, y_i) \in X \times Y, i = 1, 2,$ and $t > 0$. We have

$$\begin{aligned}|x_1(t)y_1(t) - x_2(t)y_2(t)| &\leq |x_1(t)y_1(t) - x_1(t)y_2(t)| + |x_1(t)y_2(t) - x_2(t)y_2(t)|\\ &\leq \|X\|\text{diam}Y(t) + \|Y\|\text{diam}X(t).\end{aligned}$$

Therefore,

$$\text{diam } (XY)(t) \leq \|X\|\text{diam } Y(t) + \|Y\|\text{diam } X(t).$$

Passing to the limit as $t \to \infty$, we obtain

$$c(XY) \leq \|X\|c(Y) + \|Y\|c(X),$$

which is the desired inequality (2.22).

Finally, combining both inequalities (2.21) and (2.22), we deduce that the measure μ_c satisfies the condition (m).

In the sequel, we will use the following notation. Let Ω be a nonempty subset of a Banach algebra E, and let $P, T : \Omega \to E$ be two given operators. We define the operator $PT : \Omega \to E$ as follows:

$$(PT)x = (Px)(Tx), \quad x \in \Omega.$$

Now, we present the following fixed point result involving a measure of noncompactness satisfying the condition (m) in a Banach algebra.

Theorem 2.31 *Let Ω be a nonempty, bounded, closed, and convex subset of the Banach algebra E. Let $P, T : \Omega \to E$ be two given operators. Suppose that the following conditions are satisfied:*

(i) *The operators P and T are continuous.*
(ii) $P\Omega, T\Omega \in \mathscr{M}_E$.
(iii) $(PT)\Omega \subseteq \Omega$.
(iv) *For any nonempty subset X of Ω, we have*

$$\mu(PX) \leq \psi_1(\mu(X)), \ \ \mu(TX) \leq \psi_2(\mu(X)),$$

where μ is an arbitrary measure of noncompactness satisfying condition (m) and $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ are nondecreasing functions such that for all $t > 0$,

$$\lim_{n\to\infty} \psi_1^n(t) = \lim_{n\to\infty} \psi_2^n(t) = 0,$$
$$\lim_{n\to\infty} (\|P\Omega\|\psi_2 + \|T\Omega\|\psi_1)^n (t) = 0.$$

Then $S = PT$ has at least one fixed point in Ω.

Proof Let X be a nonempty subset of Ω. Then in view of the assumption that μ satisfies condition (m), we obtain

$$\begin{aligned}\mu(SX) &= \mu((PX)(TX))\\ &\leq \|PX\|\mu(TX) + \|TX\|\mu(PX)\\ &\leq \|P\Omega\|\mu(TX) + \|T\Omega\|\mu(PX)\\ &\leq \|P\Omega\|\psi_2(\mu(X)) + \|T\Omega\|\psi_1(\mu(X))\\ &= \varphi(\mu(X)),\end{aligned}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is the function defined by

$$\varphi(t) = (\|P\Omega\|\psi_2 + \|T\Omega\|\psi_1)(t), \quad t \geq 0.$$

On the other hand, from the considered assumptions, the function φ satisfies the following conditions:

- φ is nondecreasing,
- $\lim_{n\to\infty} \varphi^n(t) = 0$, for every $t > 0$.

Now, An application of Theorem 2.23 gives us the desired result. □

2.6 Some Applications of the Measure of Noncompactness Concept

One of the most used fixed point theorems in proving existence results for functional equations is Schauder fixed point theorem (see Theorem 2.19), which asserts that every continuous self-mapping on a nonempty, convex and compact subset of a Banach space E has at least one fixed point. The main difficulty in applying this theorem lies in finding a convex and compact subset of E, which is transformed into itself by a continuous operator that depends on the considered equation. In order to overcome these difficulties, one of the possible strategies is the use of techniques associated with the concept of the measure of noncompactness.

Recently, there have been several successful efforts to apply the concept of measure of noncompactness in the study of the existence and behavior of solutions of different kinds of functional equations (see [2–5, 8, 10, 14–21, 25, 26, 41, 42, 50, 51] and the references therein). In this section, we present some applications of the measure of noncompactness concept to the study of the existence of solutions for certain functional equations including nonlinear integral equations of fractional orders, implicit fractional integral equations and q-integral equations of fractional orders.

2.6.1 An Existence Result for a Class of Nonlinear Integral Equations of Fractional Orders

In this section, we present some results obtained recently in [2] concerning the existence of solutions to the nonlinear integral equation

$$\begin{aligned} y(t) &= f(t, y(M(t))) \\ &\quad + g(t, y(N(t))) \int_a^t \frac{h'(\tau)u(t, \tau, y(c_1(\tau)), y(c_2(\tau)), \dots, y(c_n(\tau)))}{(h(t) - h(\tau))^{1-\alpha}}\, d\tau, \end{aligned} \tag{2.23}$$

where $\alpha \in (0, 1)$, $0 \leq a < T$, $f, g : [a, T] \times \mathbb{R} \to \mathbb{R}$, $M, N, c_i : [a, T] \to [a, T]$, $i = 1, \dots, n$, $u : [a, T] \times [a, T] \times \mathbb{R}^n \to \mathbb{R}$, and $h : [a, T] \to \mathbb{R}$. Equation (2.23) can be written in the form

$$y(t) = f(t, y(M(t))) + \Gamma(\alpha) g(t, y(N(t))) I^{\alpha}_{a^+,h}(u(t, \cdot, y(c_1(\cdot)), \dots, y(c_n(\cdot)))(t), \; t \in [a, T],$$

where $I^{\alpha}_{a^+,h}$ is the fractional integral of order α with respect to the function h defined by (see [55])

$$I^{\alpha}_{a^+,h} \psi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h'(\tau)}{(h(t) - h(\tau))^{1-\alpha}} \psi(\tau)\, d\tau, \quad t \in [a, T].$$

In the case $h(\tau) = \tau$, Eq. (2.23) models some problems related to queuing theory and biology (see [27]).

Using a measure of non-compactness argument, we provide sufficient conditions for the existence of at least one solution to Eq. (2.23).

We will investigate Eq. (2.23) under the following assumptions:

(H1) The functions

$$M, N, c_i : [a, T] \to [a, T], \quad i = 1, \dots, n$$

are continuous

(H2) There exist nonnegative constants L and p such that

$$|M(t) - M(s)| \leq L|t - s|^p, \quad (t, s) \in [a, T] \times [a, T].$$

(H3) there exist nonnegative constants D and q such that

$$|N(t) - N(s)| \leq D|t - s|^q, \quad (t, s) \in [a, T] \times [a, T].$$

(H4) The function $f : [a, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(t, u) - f(t, v)| \leq \lambda |u - v|,$$

for all $(t, u, v) \in [a, T] \times \mathbb{R}^2$, where λ is a nonnegative constant.
(H5) The function $g : [a, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|g(t, u) - g(t, v)| \leq \theta |u - v|,$$

for all $(t, u, v) \in [a, T] \times \mathbb{R}^2$, where θ is a nonnegative constant.
(H6) The function $u : [a, T] \times [a, T] \times \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies

$$|u(t, \tau, x_1, x_2, \ldots, x_n)| \leq \varphi\left(\max_{i=1,\ldots,n} |x_i|\right),$$

for all $(t, \tau, x_1, x_2, \ldots, x_n) \in [a, T] \times [a, T] \times \mathbb{R}^n$, where $\varphi : [0, \infty) \to [0, \infty)$ is nondecreasing.
(H7) The function $h : [a, T] \to \mathbb{R}$ is C^1 and nondecreasing.
(H8) There exists $r_0 > 0$ such that

$$\lambda r_0 + A + (\theta r_0 + B)\,\frac{\varphi(r_0)}{\alpha}(h(T) - h(a))^\alpha < r_0,$$

where

$$A = \max\{|f(t, 0)| : t \in [a, T]\}$$

and

$$B = \max\{|g(t, 0)| : t \in [a, T]\}.$$

Now, we are able to formulate our existence result.

Theorem 2.32 *Under assumptions (H1)–(H8), Eq.* (2.23) *has at least one solution* $y^* \in C([a, T]; \mathbb{R})$. *Moreover, such solution satisfies*

$$\|y^*\| \leq r_0.$$

Proof For any $y \in C([a, T]; \mathbb{R})$, let

$$\begin{aligned}(Ty)(t) &= f(t, y(M(t))) \\ &\quad + g(t, y(N(t))) \int_a^t \frac{h'(\tau) u(t, \tau, y(c_1(\tau)), y(c_2(\tau)), \ldots, y(c_n(\tau)))}{(h(t) - h(\tau))^{1-\alpha}}\, d\tau,\end{aligned}$$

for all $t \in [a, T]$. We claim that

$$TC([a, T]; \mathbb{R}) \subseteq C([a, T]; \mathbb{R}). \tag{2.24}$$

In order to prove our claim, we have just to justify that the function

$$\gamma : t \in [a, T] \mapsto \gamma(t) = \int_a^t \frac{h'(\tau)u(t, \tau, y(c_1(\tau)), y(c_2(\tau)), \ldots, y(c_n(\tau)))}{(h(t) - h(\tau))^{1-\alpha}}\, d\tau$$

is continuous in $[a, T]$. Let $\{t_n\}$ be a sequence in $[a, T]$ such that $\{t_n\}$ converges to a certain $t \in [a, T]$. Without restriction of the generality, we may assume that $t_n \geq t$ for n large enough. We have

$$|\gamma(t_n) - \gamma(t)| = \left| \int_a^{t_n} \frac{h'(\tau)U(t_n, \tau)}{(h(t_n) - h(\tau))^{1-\alpha}}\, d\tau - \int_a^t \frac{h'(\tau)U(t, \tau)}{(h(t) - h(\tau))^{1-\alpha}}\, d\tau \right|,$$

where

$$\begin{aligned} U(t_n, \tau) &= u(t_n, \tau, y(c_1(\tau)), y(c_2(\tau)), \ldots, y(c_n(\tau))), \\ U(t, \tau) &= u(t, \tau, y(c_1(\tau)), y(c_2(\tau)), \ldots, y(c_n(\tau))). \end{aligned}$$

Therefore,

$$\begin{aligned} |\gamma(t_n) - \gamma(t)| &\leq \left| \int_a^t \left(\frac{h'(\tau)U(t_n, \tau)}{(h(t_n) - h(\tau))^{1-\alpha}} - \frac{h'(\tau)U(t, \tau)}{(h(t) - h(\tau))^{1-\alpha}} \right) d\tau \right| \\ &\quad + \left| \int_t^{t_n} \frac{h'(\tau)U(t_n, \tau)}{(h(t_n) - h(\tau))^{1-\alpha}}\, d\tau \right| \\ &\leq \left| \int_a^t \frac{h'(\tau)}{(h(t) - h(\tau))^{1-\alpha}} \left(U(t_n, \tau) - U(t, \tau)\right) d\tau \right| \\ &\quad + \left| \int_a^t \left(\frac{h'(\tau)U(t_n, \tau)}{(h(t_n) - h(\tau))^{1-\alpha}} - \frac{h'(\tau)U(t_n, \tau)}{(h(t) - h(\tau))^{1-\alpha}} \right) d\tau \right| \\ &\quad + \int_t^{t_n} \frac{h'(\tau)|U(t_n, \tau)|}{(h(t_n) - h(\tau))^{1-\alpha}}\, d\tau \\ &:= A_n + B_n + C_n. \end{aligned}$$

A simple application of the Dominated Convergence Theorem, yields

$$\lim_{n \to \infty} A_n = 0.$$

On the other hand, we have

$$\begin{aligned}B_n &\leq \varphi(\|y\|) \int_a^t \left(\frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}} - \frac{h'(\tau)}{(h(t_n)-h(\tau))^{1-\alpha}} \right) d\tau \\ &= \frac{\varphi(\|y\|)}{\alpha} \left((h(t)-h(a))^\alpha + (h(t_n)-h(t))^\alpha - (h(t_n)-h(a))^\alpha \right).\end{aligned}$$

Passing to the limit $n \to \infty$, we get

$$\lim_{n\to\infty} B_n = 0.$$

Next, we have

$$\begin{aligned}C_n &\leq \varphi(\|y\|) \int_t^{t_n} \frac{h'(\tau)}{(h(t_n)-h(\tau))^{1-\alpha}}\, d\tau \\ &= \frac{\varphi(\|y\|)}{\alpha} (h(t_n)-h(t))^\alpha.\end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} C_n = 0.$$

As a consequence, we deduce that

$$\lim_{n\to\infty} |\gamma(t_n) - \gamma(t)| = 0,$$

which proves (2.24). Then

$$T : C([a,T];\mathbb{R}) \to C([a,T];\mathbb{R})$$

is well-defined.

For $r > 0$, let B_r be the closed ball of center 0 and radius r, i.e.,

$$B_r = \{y \in C([a,T];\mathbb{R}) : \|y\|_\infty \leq r\}.$$

Let $y \in B_r$, for some $r > 0$. For all $t \in [a,T]$, we have

$$\begin{aligned}&|(Ty)(t)| \leq |f(t, y(M(t))) - f(t,0)| + |f(t,0)| \\ &+ (|g(t, y(N(t))) - g(t,0)| + |g(t,0)|) \\ &\times \int_a^t \frac{h'(\tau)|u(t,\tau,y(c_1(\tau)),y(c_2(\tau)),\ldots,y(c_n(\tau)))|}{(h(t)-h(\tau))^{1-\alpha}}\, d\tau.\end{aligned}$$

Using the considered assumptions, we obtain

$$
\begin{aligned}
|(Ty)(t)| &\le \lambda |y(M(t))| + |f(t,0)| \\
&+ (\theta |y(N(t))| + |g(t,0)|) \int_a^t \frac{h'(\tau)\varphi\left(\max_{i=1,\dots,n} |y(c_i(\tau))|\right)}{(h(t)-h(\tau))^{1-\alpha}}\, d\tau \\
&\le \lambda \|y\| + A + (\theta \|y\| + B)\, \frac{\varphi(\|y\|)}{\alpha} (h(t)-h(a))^{\alpha} \\
&\le \lambda r + A + (\theta r + B)\, \frac{\varphi(r)}{\alpha} (h(T)-h(a))^{\alpha}.
\end{aligned}
$$

Taking $r = r_0$, from (H8), we obtain $\|Ty\|_\infty \le r_0$. As a consequence, we get

$$
TB_{r_0} \subseteq B_{r_0}
$$

and

$$
T : B_{r_0} \to B_{r_0}
$$

is well-defined.

Now, we claim that T is a continuous operator in B_{r_0}. In order to prove our claim, let us take $y, z \in B_{r_0}$ and $\varepsilon > 0$ so that

$$
\|y - z\|_\infty \le \varepsilon.
$$

For all $t \in [a, T]$, we have

$$
\begin{aligned}
&|(Ty)(t) - (Tz)(t)| \le |f(t, y(M(t))) - f(t, z(M(t)))| + |g(t, y(N(t))) - g(t, z(N(t)))| \\
&\times \int_a^t \frac{h'(\tau)|u(t,\tau,y(c_1(\tau)),\dots,y(c_n(\tau)))|}{(h(t)-h(\tau))^{1-\alpha}}\, d\tau \\
&+ (|g(t, z(N(t))) - g(t,0)| + |g(t,0)|) \int_a^t \frac{h'(\tau)\,(|V(t,\tau) - W(t,\tau)|)}{(h(t)-h(\tau))^{1-\alpha}}\, d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
V(t,\tau) &= u(t,\tau,y(c_1(\tau)),\dots,y(c_n(\tau))), \\
W(t,\tau) &= u(t,\tau,z(c_1(\tau)),\dots,z(c_n(\tau))).
\end{aligned}
$$

Further, let us define the quantity

$$
\begin{aligned}
\gamma_\varepsilon = \sup\{&|u(t,\tau,u_1,\dots,u_n) - u(t,\tau,v_1,\dots,v_n)| : \\
&t,\tau \in [a,T],\ u_i, v_i \in [-r_0, r_0],\ |u_i - v_i| \le \varepsilon,\ i = 1,\dots,n\}.
\end{aligned}
$$

Using the considered assumptions, for all $t \in [a, T]$, we obtain

$$
\begin{aligned}
|(Ty)(t) - (Tz)(t)| &\le \lambda |y(M(t)) - z(M(t))| + \theta |y(N(t)) - z(N(t))| \\
&\times \varphi\left(\max_{i=1,\dots,n} |y(c_i(\tau))|\right) \int_a^t \frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}}\, d\tau \\
&+ (\theta |z(N(t))| + B)\, \gamma_\varepsilon \int_a^t \frac{h'(\tau)}{(h(t)-h(\tau))^{1-\alpha}}\, d\tau \\
&\le \lambda \|y - z\|_\infty + \frac{\theta \|y - z\|_\infty \varphi(\|y\|_\infty)}{\alpha} (h(t) - h(a))^\alpha \\
&+ \frac{(\theta \|z\|_\infty + B)\, \gamma_\varepsilon}{\alpha} (h(t) - h(a))^\alpha \\
&\le \lambda \varepsilon + (h(T) - h(a))^\alpha \left(\frac{\theta \varepsilon \varphi(r_0) + (\theta r_0 + B)\gamma_\varepsilon}{\alpha} \right).
\end{aligned}
$$

Note that from the uniform continuity of the function u in $[a, T] \times [a, T] \times [-r_0, r_0]^n$, we observe easily that

$$
\lim_{\varepsilon \to 0^+} \gamma_\varepsilon = 0.
$$

Therefore,

$$
\|Ty - Tz\|_\infty \le \lambda \varepsilon + (h(T) - h(a))^\alpha \left(\frac{\theta \varepsilon \varphi(r_0) + (\theta r_0 + B)\gamma_\varepsilon}{\alpha} \right).
$$

Passing to the limit as $\varepsilon \to 0^+$, we deduce the continuity of the operator T in B_{r_0}.

Further, take a nonempty subset X of B_{r_0}. Next, fix arbitrary $\varepsilon > 0$. Choose a function $z \in X$ and numbers $t_1, t_2 \in [a, T]$ such that $|t_1 - t_2| \le \varepsilon$. Without restriction of the generality, we may assume that $t_1 \ge t_2$. We obtain

$$
\begin{aligned}
&|(Tz)(t_1) - (Tz)(t_2)| \\
&\le |f(t_1, z(M(t_1))) - f(t_1, z(M(t_2)))| + |f(t_1, z(M(t_2))) - f(t_2, z(M(t_2)))| \\
&+ (|g(t_1, z(N(t_1))) - g(t_1, z(N(t_2)))| + |g(t_1, z(N(t_2))) - g(t_2, z(N(t_2)))|) \\
&\times \int_a^{t_1} \frac{h'(\tau)|u(t_1, \tau, z(c_1(\tau)), \dots, z(c_n(\tau)))|}{(h(t_1) - h(\tau))^{1-\alpha}}\, d\tau \\
&+ (|g(t_2, z(N(t_2))) - g(t_2, 0)| + |g(t_2, 0)|) \\
&\times \left(\int_a^{t_1} \frac{h'(\tau) u(t_1, \tau, z(c_1(\tau)), \dots)}{(h(t_1) - h(\tau))^{1-\alpha}}\, d\tau - \int_a^{t_2} \frac{h'(\tau) u(t_2, \tau, z(c_1(\tau)), \dots)}{(h(t_2) - h(\tau))^{1-\alpha}}\, d\tau \right).
\end{aligned}
$$

Let us define the quantities

$$\begin{aligned}
\omega_1(\varepsilon) &= \sup\{|z(M(t)) - z(M(s))| : t, s \in [a, T],\ |t - s| \le \varepsilon\}, \\
\omega_2(\varepsilon) &= \sup\{|z(N(t)) - z(N(s))| : t, s \in [a, T],\ |t - s| \le \varepsilon\}, \\
\omega_f(\varepsilon) &= \sup\{|f(t, u) - f(s, u)| : t, s \in [a, T],\ |t - s| \le \varepsilon,\ u \in [-r_0, r_0]\}, \\
\omega_g(\varepsilon) &= \sup\{|g(t, u) - g(s, u)| : t, s \in [a, T],\ |t - s| \le \varepsilon,\ u \in [-r_0, r_0]\}, \\
\omega_3(\varepsilon) &= \sup\{|u(t_1, s, u_1, \ldots, u_n) - u(t_2, s, u_1, \ldots, u_n)| : t_1, t_2, s \in [0, T], \\
&\qquad\qquad |t_1 - t_2| \le \varepsilon,\ u_i \in [-r_0, r_0],\ i = 1, \ldots, n\}.
\end{aligned}$$

Then, keeping in mind the considered assumptions, we obtain

$$\begin{aligned}
&|(Tz)(t_1) - (Tz)(t_2)| \\
&\le \lambda|z(M(t_1)) - z(M(t_2))| + \omega_f(\varepsilon) + \big(\theta|z(N(t_1)) - z(N(t_2))| + \omega_g(\varepsilon)\big) \\
&\times \frac{\varphi\left(\max\limits_{i=1,\ldots,n} |z(c_i(\tau))|\right)}{\alpha}(h(T) - h(a))^\alpha + (\theta|z(\nu(t_2))| + B) \\
&\times \left(\int_a^{t_1} \frac{h'(\tau)u(t_1, \tau, z(c_1(\tau)), \ldots)}{(h(t_1) - h(\tau))^{1-\alpha}}\, d\tau - \int_a^{t_2} \frac{h'(\tau)u(t_1, \tau, z(c_1(\tau)), \ldots)}{(h(t_1) - h(\tau))^{1-\alpha}}\, d\tau\right. \\
&+ \int_a^{t_2} \left|\frac{h'(\tau)u(t_1, \tau, z(c_1(\tau)), \ldots)}{(h(t_1) - h(\tau))^{1-\alpha}} - \frac{h'(\tau)u(t_1, \tau, z(c_1(\tau)), \ldots)}{(h(t_2) - h(\tau))^{1-\alpha}}\right| d\tau \\
&\left.+ \int_a^{t_2} \frac{h'(\tau)}{(h(t_2) - h(\tau))^{1-\alpha}} |u(t_1, \tau, z(c_1(\tau)), \cdots) - u(t_2, \tau, z(c_1(\tau)), \cdots)|\, d\tau\right) \\
&\le \lambda\omega_1(\varepsilon) + \omega_f(\varepsilon) + \big(\theta\omega_2(\varepsilon) + \omega_g(\varepsilon)\big) \frac{\varphi(r_0)}{\alpha}(h(T) - h(a))^\alpha + (\theta r_0 + B) \\
&\times \left(\frac{\varphi(r_0)}{\alpha}(h(t_1) - h(t_2))^\alpha + \frac{\omega_3(\varepsilon)}{\alpha}(h(t_2) - h(a))^\alpha\right. \\
&\left.+ \frac{\varphi(r_0)}{\alpha}((h(t_2) - h(a))^\alpha + (h(t_1) - h(t_2))^\alpha - (h(t_1) - h(a))^\alpha)\right) \\
&\le \lambda\omega_1(\varepsilon) + \omega_f(\varepsilon) + \big(\theta\omega_2(\varepsilon) + \omega_g(\varepsilon)\big) \frac{\varphi(r_0)}{\alpha}(h(T) - h(a))^\alpha + (\theta r_0 + B) \\
&\times \left(\frac{2\varphi(r_0)}{\alpha}\omega(h, \varepsilon) + \frac{\omega_3(\varepsilon)}{\alpha}(h(T) - h(a))^\alpha\right).
\end{aligned}$$

Observe that

$$\omega_1(\varepsilon) \le \sup\{|z(t) - z(s)| : t, s \in [a, T],\ |t - s| \le L\varepsilon^p\} = \omega(z, L\varepsilon^p).$$

Similarly,

$$\omega_2(\varepsilon) \le \sup\{|z(t) - z(s)| : t, s \in [a, T],\ |t - s| \le D\varepsilon^q\} = \omega(z, D\varepsilon^q).$$

Note also that

$$\lim_{\varepsilon \to 0^+} \omega_f(\varepsilon) = \lim_{\varepsilon \to 0^+} \omega_g(\varepsilon) = \lim_{\varepsilon \to 0^+} \omega_3(\varepsilon) = 0.$$

Therefore,

$$\omega(TX, \varepsilon) \leq \lambda\omega(X, L\varepsilon^p) + \omega_f(\varepsilon) + \big(\theta\omega(X, D\varepsilon^q) + \omega_g(\varepsilon)\big) \frac{\varphi(r_0)}{\alpha}(h(T) - h(a))^\alpha$$
$$+(\theta r_0 + B)\left(\frac{2\varphi(r_0)}{\alpha}\omega(h, \varepsilon) + \frac{\omega_3(\varepsilon)}{\alpha}(h(T) - h(a))^\alpha\right).$$

Passing to the limit as $\varepsilon \to 0^+$, we get

$$\omega_0(TX) \leq \left(\lambda + \theta\frac{\varphi(r_0)}{\alpha}(h(T) - h(a))^\alpha\right)\omega_0(X),$$

where ω_0 is the measure of noncompactness defined by (2.12). Then we proved that for every nonempty subset X of B_{r_0}, we have

$$\omega_0(TX) \leq K\omega_0(X),$$

where

$$K = \lambda + \theta\frac{\varphi(r_0)}{\alpha}(h(T) - h(a))^\alpha.$$

Note that from (H8), we have $K < 1$. Applying Darbo's theorem (see Theorem 2.20), we deduce that the operator T has at least one fixed point $y^* \in B_{r_0}$, which is a solution to Eq. (2.23). □

2.6.1.1 A Functional Equation Involving Riemann–Liouville Fractional Integral

Taking $h(t) = t$ in Eq. (2.23), we obtain the functional equation

$$y(t) = f(t, y(M(t))) + \Gamma(\alpha)g(t, y(N(t)))I^\alpha_{a^+}\big(u(t, \cdot, y(c_1(\cdot)), \ldots, y(c_n(\cdot)))\big)(t), \tag{2.25}$$

where $I^\alpha_{a^+}$ is the Riemann–Liouville fractional integral defined by (see [55])

$$I^\alpha_{a^+}\psi(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{\psi(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau, \quad t \in [a, T].$$

We can rewrite Eq. (2.25) in the form

$$y(t) = f(t, y(M(t))) + g(t, y(N(t)))\int_a^t \frac{u(t, \tau, y(c_1(\tau)), \ldots, y(c_n(\tau)))}{(t-\tau)^{1-\alpha}}\,d\tau.$$

Then from Theorem 2.32, we deduce the following existence result.

Corollary 2.5 *Suppose that assumptions (H1)–(H6) are satisfied. Suppose also that there is some $r_0 > 0$ such that*

$$\lambda r_0 + A + (\theta r_0 + B)\frac{\varphi(r_0)}{\alpha}(T-a)^\alpha < r_0. \tag{2.26}$$

Then Eq. (2.25) *has at least one solution $y^* \in C([a,T];\mathbb{R})$. Moreover, such solution satisfies*

$$\|y^*\|_\infty \le r_0.$$

Now, we present an example illustrating Corollary 2.5.

Example 2.16 Let us consider the integral equation

$$y(t) = \frac{2y(t^2)}{5} + \frac{1+t}{8} + \left(\frac{y(\cos t)+t^2}{36}\right)\int_0^t \frac{\ln(1+|y(\tau)|)}{(1+t+\tau)\sqrt{t-\tau}}\,d\tau, \quad t \in [0,1]. \tag{2.27}$$

Set

$$\begin{aligned} f(t,x) &= \frac{1+t}{8} + \frac{2u}{5}, \quad (t,x) \in [0,1]\times\mathbb{R},\\ g(t,x) &= \frac{u+t^2}{36}, \quad (t,x) \in [0,1]\times\mathbb{R},\\ u(t,s,x) &= \frac{\ln(1+|x|)}{1+t+s}, \quad (t,s,x) \in [0,1]\times[0,1]\times\mathbb{R} \end{aligned}$$

and

$$\alpha = \frac{1}{2},\ M(t) = t^2,\ N(t) = \cos t,\ c_1(t) = t, \quad t \in [0,1].$$

We can rewrite Eq. (2.27) in the form

$$y(t) = f(t, y(M(t))) + g(t, y(N(t)))\int_0^t \frac{u(t,\tau,y(c_1(\tau)))}{(t-\tau)^{1-\alpha}}\,d\tau,\ t \in [0,1].$$

Observe that the functions involved in Eq. (2.27) satisfy assumptions of Corollary 2.5. Indeed, we have $L = 2$, $p = 1$, $D = q = 1$, $\lambda = \frac{2}{5}$, $\theta = \frac{1}{36}$, $\varphi(r) = \ln(1+r)$, $A = \frac{1}{4}$ and $B = \frac{1}{36}$.

Now, let us consider the inequality (2.26) in Corollary 2.5, which has the form

$$\frac{2}{5}r + \frac{1}{4} + \frac{1}{18}(r+1)\ln(r+1) < r.$$

We can check easily that $r_0 = 1$ satisfies the above inequality. Therefore, from Corollary 2.5, we infer that Eq. (2.27) has at least one solution $y^* \in C([0,1];\mathbb{R})$ such that $\|y^*\|_\infty \leq 1$.

2.6.1.2 A Functional Equation Involving Hadamard Fractional Integral

Taking

$$h(t) = \ln t, \quad t \in [a,T],\ 0 < a < T$$

in Eq. (2.23), we obtain the functional equation

$$y(t) = f(t, y(M(t))) + \Gamma(\alpha) g(t, y(N(t))) J^\alpha_{a^+}(u(t, \cdot, y(c_1(\cdot)), \ldots, y(c_n(\cdot))))(t), \tag{2.28}$$

where $J^\alpha_{a^+}$ is the Hadamard fractional integral defined by (see [55])

$$J^\alpha_{a^+}\psi(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{\psi(\tau)}{\tau}\, d\tau.$$

We can rewrite Eq. (2.28) in the form

$$y(t) = f(t, y(M(t))) + g(t, y(N(t))) \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{u(t, \tau, y(c_1(\tau)), \ldots, y(c_n(\tau)))}{\tau}\, d\tau.$$

Therefore, from Theorem 2.32, we deduce the following result.

Corollary 2.6 *Suppose that assumptions (H1)–(H6) are satisfied. Suppose also that there exists some* $r_0 > 0$ *such that*

$$\lambda r_0 + A + (\theta r_0 + B)\frac{\varphi(r_0)}{\alpha}\left(\ln \frac{T}{a}\right)^\alpha < r_0. \tag{2.29}$$

Then Eq. (2.28) *has at least one solution* $y^* \in C([a,T];\mathbb{R})$. *Moreover, we have*

$$\|y^*\|_\infty \leq r_0.$$

We present the following example to illustrate Corollary 2.6.

Example 2.17 Let us consider the integral equation

$$y(t) = \frac{t}{32} + \frac{y(t)}{8} + \left(\frac{t^2}{64} + \frac{y(t)}{16}\right) \int_1^t \left(\ln \frac{t}{\tau}\right)^{-1/2} \frac{y(\tau)}{\tau}\, d\tau, \quad t \in [1,2]. \tag{2.30}$$

Taking $\alpha = \frac{1}{2}$, $M(t) = N(t) = c_1(t) = t$, $u(t,s,x) = x$ and

$$f(t,x)=\frac{t}{32}+\frac{x}{8},$$
$$g(t,x)=\frac{t^2}{64}+\frac{x}{16},$$

we can rewrite Eq. (2.30) in the form

$$y(t)=f(t,y(M(t)))+g(t,y(N(t)))\int_1^t\left(\ln\frac{t}{\tau}\right)^{-1/2}\frac{u(t,\tau,y(\tau))}{\tau}\,d\tau.$$

We can check easily that the above functions satisfy the required conditions by Corollary 2.6 with $\lambda=\frac{1}{8}$, $\theta=\frac{1}{16}$, $\varphi(r)=r$, and $A=B=\frac{1}{16}$.

Now, let us consider the inequality (2.29) in Corollary 2.6, which has the form

$$\frac{1}{8}r+\frac{1}{16}+\frac{\sqrt{\ln 2}}{8}r(r+1)<r.$$

We can check easily that $r_0=1$ satisfies the above inequality. Therefore, from Corollary 2.6, we infer that Eq. (2.30) has at least one solution $y^*\in C([0,1];\mathbb{R})$ such that $\|y^*\|_\infty\le 1$.

2.6.2 Solvability of an Implicit Fractional Integral Equation

In this section, we are concerned with the existence of solutions to the following implicit integral equation:

$$y(t)=F\left(t,y(t),\psi\left(\int_a^t\frac{g'(s)}{(g(t)-g(s))^{1-\alpha}}h(t,s,y(s))\,ds\right)\right),\quad t\in[a,T], \tag{2.31}$$

where $T>0$, $a\ge 0$, $\alpha\in(0,1)$, $F:[a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\psi:\mathbb{R}\to\mathbb{R}$, $g:[a,T]\to\mathbb{R}$ and $h:[a,T]\times[a,T]\times\mathbb{R}\to\mathbb{R}$. Our considerations are based on recent results obtained in [51].

Equation (2.31) can be written in the form

$$y(t)=F\left(t,y(t),\psi\left(\Gamma(\alpha)I^\alpha_{a,g}h(t,\cdot,y(\cdot))(t)\right)\right),\quad t\in[a,T],$$

where $I^\alpha_{a,g}$ is the fractional integral of order α with respect to the function g defined by

$$I^\alpha_{a,g}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\frac{g'(s)f(s)}{(g(t)-g(s))^{1-\alpha}}\,ds,\quad t\in[a,T].$$

Recall that for $g(s)=s$, $I^\alpha_{a,g}$ is the Riemann–Liouville fractional integral of order α defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t \in [a, T].$$

However, if $a > 0$ and $g(s) = \ln s$, $I_{a,g}^\alpha$ is the Hadamard fractional integral of order α defined by

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \quad t \in [a, T].$$

We will study Eq. (2.31) under the following assumptions:
(A1) The function $\psi : \mathbb{R} \to \mathbb{R}$ satisfies

$$|\psi(b) - \psi(c)| \le C_\psi |b - c|^{\ell_\psi}, \quad (b, c) \in \mathbb{R} \times \mathbb{R},$$

for some nonnegative constants C_ψ and ℓ_ψ.
(A2) The function $F : [a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|F(t, x, y) - F(t, u, v)| \le \varphi(|x - u|) + C_F |y - v|, \quad (t, x, y), (t, u, v) \in [a, T] \times \mathbb{R} \times \mathbb{R},$$

for some nonnegative constant C_F, where $\varphi : [0, \infty) \to [0, \infty)$ is a continuous function satisfying the assumptions of Theorem 2.23.
(A3) The function $g : [a, T] \to \mathbb{R}$ is C^1 and nondecreasing.
(A4) The function $h : [a, T] \times [a, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.
(A5) There exists $r_0 > 0$ such that

$$\varphi(r_0) + C_F C_\psi \left(\frac{H}{\alpha}\right)^{\ell_\psi} (g(T) - g(a))^{\alpha \ell_\psi} + M_F + C_F |\psi(0)| \le r_0,$$

where

$$H := \sup\{|h(t, s, y(s))| : t, s \in [a, T], y \in C([a, T]; \mathbb{R})\} < \infty$$

and

$$M_F := \max\{|F(t, 0, 0)| : t \in [a, T]\}.$$

Remark 2.16 Note that if $\varphi : [0, \infty] \to [0, \infty)$ is a function satisfying the conditions of Theorem 2.23, i.e., φ is nondecreasing and for all $t > 0$, $\lim_{n\to\infty} \varphi^n(t) = 0$, then $\varphi(0) = 0$. In fact, if $\varphi(0) = r > 0$, using the monotone property of the function φ, we obtain $\varphi^n(r) \ge r$, for every $n \in \mathbb{N}$. Passing to the limit as $n \to \infty$, we get $0 \ge r$, which is a contradiction with the fact that $r > 0$.

Now, we can state our main result.

Theorem 2.33 *Under assumptions (A1)–(A5), Eq.* (2.31) *has at least one continuous solution* $y : [a, T] \to \mathbb{R}$*. Moreover, such solution satisfies*

$$\|y\|_\infty \le r_0.$$

Proof For $y \in C([a, T]; \mathbb{R})$, let

$$(Dy)(t) = F\left(t, y(t), \psi\left(\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} h(t, s, y(s))\, ds\right)\right), \quad t \in [a, T].$$

Let $t \in [a, T]$ be fixed and $\{t_n\}$ be a sequence in $[a, T]$ such that $t_n \to t$ as $n \to \infty$. Without restriction of the generality, we may assume that $t_n \ge t$. Therefore, we have

$$\begin{aligned}
&\left|\int_a^{t_n} \frac{g'(s)}{(g(t_n)-g(s))^{1-\alpha}} h(t_n, s, y(s))\, ds - \int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} h(t, s, y(s))\, ds\right| \\
&\le \left|\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} (h(t_n, s, y(s)) - h(t, s, y(s)))\, ds\right| \\
&+ \left|\int_a^t \left(\frac{g'(s)}{(g(t_n)-g(s))^{1-\alpha}} h(t_n, s, y(s)) - \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} (h(t_n, s, y(s))\right) ds\right| \\
&+ \int_t^{t_n} \frac{g'(s)}{(g(t_n)-g(s))^{1-\alpha}} |h(t_n, s, y(s))|\, ds \\
&:= U_n + V_n + W_n.
\end{aligned}$$

Using the continuity of the function $(t, s) \mapsto h(t, s, y(s))$ in $[a, T] \times [a, T]$, a simple application of the Dominated Convergence Theorem yields

$$\lim_{n\to\infty} U_n = 0.$$

On the other hand, we have

$$V_n \le \frac{H}{\alpha}\left((g(t)-g(a))^\alpha + (g(t_n)-g(t))^\alpha - (g(t_n)-g(a))^\alpha\right).$$

Passing to the limit as $n \to \infty$ and using the continuity of g, we get

$$\lim_{n\to\infty} V_n = 0.$$

Similarly, we have

$$W_n \le \frac{H}{\alpha}(g(t_n)-g(t))^\alpha |t_n - t| \to 0 \text{ as } n \to \infty,$$

which yields

$$\lim_{n\to\infty} W_n = 0.$$

Therefore, for every $y \in C([a,T];\mathbb{R})$, we have $Dy \in C([a,T];\mathbb{R})$. Then the mapping

$$D : C([a,T];\mathbb{R}) \to C([a,T];\mathbb{R})$$

is well-defined.

Now, let $y \in C([a,T];\mathbb{R})$ be such that $\|y\|_\infty \le r$, for some $r > 0$. For all $t \in [a,T]$, we have

$$|(Dy)(t)| \le \left| F\left(t, y(t), \psi\left(\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} h(t,s,y(s))\,ds\right)\right) - F(t,0,0)\right| + M_F.$$

Using assumption (A2), we get

$$|(Dy)(t)| \le \varphi(|y(t)|) + C_F \left|\psi\left(\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} h(t,s,y(s))\,ds\right)\right| + M_F.$$

Therefore, via assumption (A1), we can write that

$$\begin{aligned} |(Dy)(t)| &\le \varphi(|y(t)|) + C_F \left|\psi\left(\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} h(t,s,y(s))\,ds\right)\right| + M_F \\ &\le \varphi(r) + C_F C_\psi \left(\frac{H}{\alpha}\right)^{\ell_\psi} (g(T)-g(a))^{\alpha \ell_\psi} + M_F + C_F|\psi(0)|. \end{aligned}$$

Next, by assumption (A5), we infer that $D(B_{r_0}) \subseteq B_{r_0}$, where

$$B_{r_0} := \{y \in C([a,T] : \ \|y\|_\infty \le r_0\}.$$

Then the mapping

$$D : B_{r_0} \to B_{r_0}$$

is well-defined.

Let us prove now that $D : B_{r_0} \to B_{r_0}$ is continuous. We take $y, z \in B_{r_0}$ such that $\|y - z\|_\infty \le \varepsilon, \varepsilon > 0$. Taking in mind the considered assumptions, for all $t \in [a,T]$, we have

$$
\begin{aligned}
&|(Dy)(t)-(Dz)(t)| \\
&\le \varphi(|y(t)-z(t)|)+C_F C_\psi \left|\int_a^t \frac{g'(s)h(t,s,y(s))}{(g(t)-g(s))^{1-\alpha}}\,ds-\int_a^t \frac{g'(s)h(t,s,z(s))}{(g(t)-g(s))^{1-\alpha}}\,ds\right|^{\ell_\psi} \\
&\le \varphi(|y(t)-z(t)|)+C_F C_\psi \left(\int_a^t \frac{g'(s)|h(t,s,z(s))-h(t,s,y(s))|}{(g(t)-g(s))^{1-\alpha}}\,ds\right)^{\ell_\psi} \\
&\le \varphi(\|y-z\|)+C_F C_\psi M_\varepsilon^{\ell_\psi}\left(\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}}\,ds\right)^{\ell_\psi} \\
&\le \varphi(\|y-z\|)+C_F C_\psi \left(\frac{M_\varepsilon}{\alpha}\right)^{\ell_\psi}(g(t)-g(a))^{\alpha\ell_\psi} \\
&\le \varphi(\varepsilon)+C_F C_\psi \left(\frac{M_\varepsilon}{\alpha}\right)^{\ell_\psi}(g(T)-g(a))^{\alpha\ell_\psi},
\end{aligned}
$$

where

$$M_\varepsilon := \sup\{|h(t,s,u)-h(t,s,v)| : t,s\in[a,T],\ |u|\le r_0,\ |v|\le r_0,\ |u-v|\le\varepsilon\}.$$

Note that from the uniform continuity of the function

$$(t,s,u)\mapsto h(t,s,u)$$

in $[a,T]\times[a,T]\times[-r_0,r_0]$, it is clear that

$$\lim_{\varepsilon\to 0^+} M_\varepsilon = 0.$$

As a consequence, we have

$$\|Dy-Dz\|_\infty \le \varphi(\varepsilon)+C_F C_\psi \left(\frac{M_\varepsilon}{\alpha}\right)^{\ell_\psi}(g(T)-g(a))^{\alpha\ell_\psi} \to 0 \text{ as } \varepsilon\to 0^+.$$

Therefore, the operator $D : B_{r_0}\to B_{r_0}$ is continuous.

Further, take a nonempty subset X of the ball B_{r_0}. Next, fix an arbitrary $\varepsilon>0$. Choose a function $y\in X$ and real numbers $\tau,t\in[a,T]$ such that $|\tau-t|\le\varepsilon$. Without restriction of the generality, we may assume that $\tau\ge t$. Then, taking in mind our assumptions, we obtain

$$|(Dy)(\tau)-(Dy)(t)|\le (I)+(II),$$

where the quantities (I) and (II) are given by

$$
\begin{aligned}
&(I)= \\
&\left|F\left(\tau,y(\tau),\psi\left(\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds\right)\right)-F\left(t,y(\tau),\psi\left(\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds\right)\right)\right|
\end{aligned}
$$

and

$$(II)=$$
$$\left|F\left(t, y(\tau), \psi\left(\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds\right)\right)-F\left(t, y(t), \psi\left(\int_a^t \frac{g'(s)h(t,s,y(s))}{(g(t)-g(s))^{1-\alpha}}\,ds\right)\right)\right|.$$

• Estimate of (I). At first, we have

$$\begin{aligned}\left|\psi\left(\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds\right)\right| &\le \left|\psi\left(\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds\right)-\psi(0)\right|+|\psi(0)|\\ &\le C_\psi\left(\frac{H}{\alpha}\right)^{\ell_\psi}(g(T)-g(a))^{\alpha\ell_\psi}+|\psi(0)|\\ &:= D.\end{aligned}$$

Put

$$C(F,\varepsilon):=$$
$$\sup\left\{|F(t,x,y)-F(s,x,y)|:\ t,s\in[a,T],\ |t-s|\le\varepsilon,\ x\in[-r_0,r_0],\ y\in[-D,D]\right\}.$$

Clearly, we have

$$(I)\le C(F,\varepsilon).$$

Note that by the uniform continuity of the function

$$(t,x,y)\mapsto F(t,x,y)$$

in $[a,T]\times[-r_0,r_0]\times[-D,D]$, we have

$$\lim_{\varepsilon\to 0^+} C(F,\varepsilon)=0.$$

• Estimate of (II). It is not difficult to see that

$$\begin{aligned}&(II)\\ &\le \varphi(\omega(y,\varepsilon))+C_F C_\psi\left|\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds-\int_a^t \frac{g'(s)h(t,s,y(s))}{(g(t)-g(s))^{1-\alpha}}\,ds\right|^{\ell_\psi}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\left|\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds-\int_a^t \frac{g'(s)h(t,s,y(s))}{(g(t)-g(s))^{1-\alpha}}\,ds\right|\\ &\le\left|\int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds-\int_a^t \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds\right|\end{aligned}$$

$$+\int_a^t \left| \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}} - \frac{g'(s)h(\tau,s,y(s))}{(g(t)-g(s))^{1-\alpha}} \right| ds$$
$$+\int_a^t \left| \frac{g'(s)h(\tau,s,y(s))}{(g(t)-g(s))^{1-\alpha}} - \frac{g'(s)h(t,s,y(s))}{(g(t)-g(s))^{1-\alpha}} \right| ds.$$

Set

$$C(h,\varepsilon) := \sup\{|h(t,s,x)-h(\tau,s,x)| : t,\tau \in [a,T],\ |t-s| \le \varepsilon,\ x \in [-r_0,r_0]\}.$$

Similarly, we have

$$\lim_{\varepsilon \to 0^+} C(h,\varepsilon) = 0.$$

Next, we obtain

$$\begin{aligned}
&\left| \int_a^\tau \frac{g'(s)h(\tau,s,y(s))}{(g(\tau)-g(s))^{1-\alpha}}\,ds - \int_a^t \frac{g'(s)h(t,s,y(s))}{(g(t)-g(s))^{1-\alpha}}\,ds \right| \\
&\le H\int_t^\tau \frac{g'(s)}{(g(\tau)-g(s))^{1-\alpha}}\,ds + H\int_a^t \left(\frac{g'(s)}{(g(t)-g(s))^{1-\alpha}} - \frac{g'(s)}{(g(\tau)-g(s))^{1-\alpha}} \right) ds \\
&+C(h,\varepsilon)\int_a^t \frac{g'(s)}{(g(t)-g(s))^{1-\alpha}}\,ds \\
&\le \frac{H}{\alpha}(g(\tau)-g(t))^\alpha + \frac{H}{\alpha}\left((g(t)-g(a))^\alpha + (g(\tau)-g(t))^\alpha - (g(\tau)-g(a))^\alpha\right) \\
&+\frac{C(h,\varepsilon)}{\alpha}(g(t)-g(a))^\alpha \\
&\le \frac{2H}{\alpha}(g(\tau)-g(t))^\alpha + \frac{C(h,\varepsilon)}{\alpha}(g(t)-g(a))^\alpha \\
&\le \frac{2H}{\alpha}\omega(g,\varepsilon)^\alpha + \frac{C(h,\varepsilon)}{\alpha}(g(T)-g(a))^\alpha.
\end{aligned}$$

Therefore, we get the estimate

$$(II) \le \varphi(\omega(X,\varepsilon)) + C_F C_\psi \left(\frac{2H}{\alpha}\omega(g,\varepsilon)^\alpha + \frac{C(h,\varepsilon)}{\alpha}(g(T)-g(a))^\alpha \right)^{\ell_\psi}.$$

As a consequence, we obtain

$$\omega(DX,\varepsilon) \le C(F,\varepsilon) + \varphi(\omega(X,\varepsilon)) + C_F C_\psi \left(\frac{2H}{\alpha}\omega(g,\varepsilon)^\alpha + \frac{C(h,\varepsilon)}{\alpha}(g(T)-g(a))^\alpha \right)^{\ell_\psi}.$$

Passing to the limit as $\varepsilon \to 0^+$, we get

$$\omega_0(DX) \le \varphi(\omega_0(X)).$$

Finally, applying Theorem 2.23, we obtain the existence of at least one fixed point of the operator D in B_{r_0}, which is a solution to Eq. (2.31) in B_{r_0}. □

2.6.2.1 An Implicit Functional Equation Involving Riemann–Liouville Fractional Integral

Le us consider the following integral equation involving the Riemann–Liouville fractional integral

$$y(t) = F\left(t, y(t), \psi\left(\int_a^t \frac{h(t,s,y(s))}{(t-s)^{1-\alpha}}\,ds\right)\right), \quad t \in [a,T]. \tag{2.32}$$

Obviously, Eq. (2.32) is a particular case of Eq. (2.31) with $g(s) = s$. Therefore, by Theorem 2.33, we deduce the following existence result.

Corollary 2.7 *Suppose that assumptions (A1), (A2) and (A4) are satisfied. If there exists $r_0 > 0$ such that*

$$\varphi(r_0) + C_F C_\psi \left(\frac{H}{\alpha}\right)^{\ell_\psi} (T-a)^{\alpha\ell_\psi} + M_F + C_F|\psi(0)| \le r_0, \tag{2.33}$$

then Eq. (2.32) has at least one continuous solution $y : [a,T] \to \mathbb{R}$. Moreover, such solution satisfies

$$\|y\|_\infty \le r_0.$$

Next, we give an example illustrating Corollary 2.7.

Example 2.18 Let us consider the integral equation

$$y(t) = \left(\frac{t^2}{1+t^2}\right)\ln(1+|y(t)|) + \cos\left(\int_0^t \frac{y^2(s)}{\sqrt{t-s}(1+y^2(s))}\,ds\right), \quad t \in [0,1]. \tag{2.34}$$

Obviously, Eq. (2.34) is a special case of Eq. (2.32) with $\alpha = \frac{1}{2}$, $\psi(u) = u$, $h(t,s,x) = \frac{x^2}{1+x^2}$ and $F(t,x,y) = \frac{t^2}{1+t^2}\ln(1+|x|) + \cos y$. Moreover, we can check easily that assumptions (A1), (A2), and (A4) are satisfied with $C_\psi = \ell_\psi = 1$, $\varphi(u) = \ln(1+u)$, $M_F = C_F = 1$ and $H \le 1$.

Now, let us consider the inequality (2.33) in Corollary 2.7, which has the form

$$\ln(1+r) + 2H + 1 \le r.$$

Since $H \le 1$, we have

$$\ln(1+r) + 2H + 1 \le \ln(1+r) + 3.$$

On the other hand, using the fact that

$$\lim_{r\to+\infty} r - \ln(1+r) = +\infty,$$

we infer that there exists $r_0 > 0$ such that

$$\ln(1+r_0)+3 \leq r_0.$$

Then r_0 is a solution to (2.33). Therefore, by Corollary 2.7, Eq. (2.34) has at least one continuous solution $y : [0, 1] \to \mathbb{R}$ such that $\|y\|_\infty \leq r_0$.

2.6.2.2 An Implicit Functional Equation Involving Hadamard Fractional Integral

Consider now the integral equation involving the Hadamard fractional integral

$$y(t) = F\left(t, y(t), \psi\left(\int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{h(t,s,y(s))}{s}\, ds\right)\right), \quad t \in [a,T],\ a>0. \tag{2.35}$$

Taking $g(s) = \ln s$ in Theorem 2.33, we obtain the following existence result for Eq. (2.35).

Corollary 2.8 *Suppose that assumptions (A1), (A2) and (A4) are satisfied. If there exists $r_0 > 0$ such that*

$$\varphi(r_0) + C_F C_\psi \left(\frac{H}{\alpha}\right)^{\ell_\psi} \left(\ln \frac{T}{a}\right)^{\alpha \ell_\psi} + M_F + C_F|\psi(0)| \leq r_0,$$

then Eq. (2.35) has at least one continuous solution $y : [a, T] \to \mathbb{R}$. Moreover, such solution satisfies

$$\|y\|_\infty \leq r_0.$$

2.6.3 *q-Integral Equations of Fractional Orders*

The concept of q-calculus (quantum calculus) was introduced by Jackson (see [39, 40]). This subject is rich in history and has several applications (see [31, 44]). Fractional q-difference concept was initiated by Agarwal and by Al-Salam (see [1, 11]). Because of the considerable progress in the study of fractional differential equations, a great interest appeared from many authors in studying fractional q-difference equations (see for examples [9, 11, 32, 33, 47] and the references therein).

In this section, we are concerned with the following functional equation

$$x(t) = F\left(t, x(a(t)), \frac{f(t, x(b(t)))}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s, x(s))\, d_q s\right), \quad t \in I, \tag{2.36}$$

where $\alpha > 1$, $q \in (0, 1)$, $I = [0, 1]$, $f, u : [0, 1] \times \mathbb{R} \to \mathbb{R}$, $a, b : I \to I$ and $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Equation (2.36) can be written as

$$x(t) = F\left(t, x(a(t)), f(t, x(b(t)))I_q^\alpha u(\cdot, x(\cdot))(t)\right), \quad t \in I,$$

where I_q^α is the q-fractional integral of order α defined by (see [1])

$$I_q^\alpha h(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} h(s)\, d_q s, \quad t \in [0, 1].$$

Our considerations are based on recent results obtained in [42].

At first, we recall some concepts on fractional q-calculus and present additional properties that will be used later. For more details, we refer to [1, 12, 54].

Let q be a positive real number such that $q \neq 1$. For $x \in \mathbb{R}$, the q-real number $[x]_q$ is defined by

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

The q-shifted factorial of real number x is defined by

$$(x, q)_0 = 1, \quad (x, q)_k = \prod_{i=0}^{k-1}(1 - xq^i), \quad k = 1, 2, \ldots, \infty.$$

For $(x, y) \in \mathbb{R}^2$, the q-analog of $(x - y)^k$ is defined by

$$(x - y)^{(0)} = 1, \ (x - y)^{(k)} = \prod_{i=0}^{k-1}(x - q^i y), \ k = 1, 2, \ldots$$

For $\beta \in \mathbb{R}$, $(x, y) \in \mathbb{R}^2$ and $x \geq 0$,

$$(x - y)^{(\beta)} = x^\beta \prod_{i=0}^{\infty} \frac{x - yq^i}{x - yq^{\beta+i}}.$$

Note that if $y = 0$, then $x^{(\beta)} = x^\beta$.

The following inequality (see [34]) will be used later.

Lemma 2.2 *If $\beta > 0$ and $0 \leq a \leq b \leq t$, then*

$$(t - b)^{(\beta)} \leq (t - a)^{(\beta)}.$$

The q-gamma function is given by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \notin \{0, -1, -2, \dots\}.$$

We have the following property

$$\Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

Let $f : [0, b] \to \mathbb{R}$ ($b > 0$) be a given function. The q-integral of the function f is given by

$$I_q f(t) = \int_0^t f(s)\, d_q s = t(1-q) \sum_{n=0}^{\infty} f(tq^n) q^n, \quad t \in [0, b].$$

If $c \in [0, b]$, we have

$$\int_c^b f(s)\, d_q s = \int_0^b f(s)\, d_q s - \int_0^c f(s)\, d_q s.$$

Lemma 2.3 *If $f : [0, 1] \to \mathbb{R}$ is a continuous function, then*

$$\Big| \int_0^t f(s)\, d_q s \Big| \le \int_0^t |f(s)|\, d_q s, \quad t \in [0, 1].$$

Remark 2.17 Note that in general, if $0 \le t_1 \le t_2 \le 1$ and $f : [0, 1] \to \mathbb{R}$ is a continuous function, the inequality

$$\left| \int_{t_1}^{t_2} f(s)\, d_q s \right| \le \int_{t_1}^{t_2} |f(s)|\, d_q s$$

is not satisfied. We remark that in many papers dealing with q-difference boundary value problems, the use of such inequality yields wrong results. As a counterexample, we refer the reader to [12, p. 12].

Let $f : [0, 1] \to \mathbb{R}$ be a given function. The fractional q-integral of order $\alpha \ge 0$ of the function f is given by $I_q^0 f(t) = f(t)$ and

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(s)\, d_q s, \quad t \in [0, 1],\ \alpha > 0.$$

Note that for $\alpha = 1$, we have

$$I_q^1 f(t) = I_q f(t), \quad t \in [0, 1].$$

If $f \equiv 1$, then

$$I_q^\alpha 1(t) = \frac{1}{\Gamma_q(\alpha+1)} t^\alpha, \quad t \in [0,1].$$

Let Λ be the set of functions $\eta : [0,\infty) \to [0,\infty)$ such that

1. η is a nondecreasing function.
2. η is an upper semicontinuous function.
3. $\eta(s) < s$, for all $s > 0$.

For our purpose, we need the following generalized version of Darbo's theorem (see [5]).

Theorem 2.34 *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space E. Let $T : \Omega \to \Omega$ be a continuous mapping such that*

$$\mu(TX) \leq \eta(\mu(X)), \quad X \subseteq \Omega,$$

where $\eta \in \Lambda$ and μ is a measure of noncompactness in E (in the sense of Definition 2.14). Then T has at least one fixed point.

Remark 2.18 Observe that Theorem 2.34 is a special case of Theorem 2.24 with $f(z_1, z_2) = \eta(z_2) - z_1$.

The following result will be useful later.

Lemma 2.4 *Let $\eta_1, \eta_2 \in \Lambda$ and $\tau \in (0,1)$. Then the function $\gamma : [0,\infty) \to [0,\infty)$ defined by*

$$\gamma(t) = \max\{\eta_1(t), \eta_2(t), \tau t\}, \quad t \geq 0$$

belongs to the set Λ.

Proof Let $(t, s) \in \mathbb{R}^2$ be such that $0 \leq t \leq s$. Since η_1, η_2 are nondecreasing and $\tau \in (0,1)$, we have

$$\eta_i(t) \leq \eta_i(s) \leq \gamma(s), \quad i = 1, 2,$$
$$\tau t \leq \tau s \leq \gamma(s),$$

which yield $\gamma(t) \leq \gamma(s)$. Therefore, γ is a nondecreasing function. Now, for all $s > 0$, we have $\eta_i(s) < s$ (for $i = 1, 2$) and $\tau s < s$. Since $\gamma(s) \in \{\eta_1(s), \eta_2(s), \tau s\}$, we obtain

$$\gamma(s) < s, \quad s > 0.$$

On the other hand, it is well known that the maximum of finitely many upper semicontinuous functions is upper semicontinuous. As a consequence, the function γ belongs to the set Λ. $\square$

Define the operator T on $E = C(I; \mathbb{R})$ by

$$(Tx)(t) = F\Big(t, x(a(t)), \frac{f(t, x(b(t)))}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big), \quad (x,t) \in \mathbb{E} \times I.$$

We consider the following assumption:

(A1) The functions $f, u : [0,1] \times \mathbb{R} \to \mathbb{R}$, $a, b : I \to I$ and $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous.

Under the above condition, we have the following result.

Proposition 2.5 *Under assumption (A1), the operator T maps E into itself.*

Proof From assumption (A1), we have just to show that the operator H defined on E by

$$(Hx)(t) = \int_0^t (t-qs)^{(\alpha-1)} u(s, x(s))\, d_q s, \quad (x,t) \in E \times I \tag{2.37}$$

maps E into itself. To do this, let us fix $x \in E$. For all $t \in I$, we have

$$\begin{aligned} (Hx)(t) &= \int_0^t (t-qs)^{(\alpha-1)} u(s, x(s))\, d_q s \\ &= t(1-q) \sum_{n=0}^{\infty} q^n (t - q^{n+1} t)^{(\alpha-1)} u(tq^n, x(tq^n)) \\ &= t^{\alpha}(1-q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^{(\alpha-1)} u(tq^n, x(tq^n)). \end{aligned}$$

On the other hand, since $0 < q^{n+1} < 1$, using Lemma 2.2, we have

$$(1 - q^{n+1})^{(\alpha-1)} \leq (1-0)^{(\alpha-1)} = 1.$$

Then by the continuity of u and using the Weierstrass convergence theorem, we obtain the desired result. □

Next, we consider the following assumptions:

(A2) There exist a constant $C_F > 0$ and a nondecreasing function $\varphi_F : [0, \infty) \to [0, \infty)$ such that

$$|F(t,x,y) - F(t,z,w)| \leq \varphi_F(|x-z|) + C_F|y-w|, \quad (t,x,y,z,w) \in I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}.$$

(A3) There exists a constant $C_f > 0$ such that

$$|f(t,x) - f(t,y)| \leq C_f|x-y|, \quad (t,x,y) \in I \times \mathbb{R} \times \mathbb{R}.$$

(A4) There exists a nondecreasing and continuous function $\varphi_u : [0,\infty) \to [0,\infty)$ such that

$$|u(t,x) - u(t,y)| \le \varphi_u(|x-y|), \quad (t,x,y) \in I \times \mathbb{R} \times \mathbb{R},\ \varphi_u(t) < t,\ t > 0,$$
$$u(t,0) = 0, \quad t \in I.$$

(A5) There exists $r_0 > 0$ such that

$$\varphi_F(r_0) + F^* + \frac{C_F(C_f r_0 + f^*)\varphi_u(r_0)}{\Gamma_q(\alpha+1)} \le r_0,$$

where

$$F^* = \max\{|F(t,0,0)| : t \in I\} \quad \text{and} \quad f^* = \max\{|f(t,0)| : t \in I\}.$$

Let B_{r_0} be the closed ball of center 0 and radius r_0, i.e.,

$$B_{r_0} = \{x \in E : \|x\|_\infty \le r_0\}.$$

Proposition 2.6 *Under assumptions (A1)–(A5), the operator T maps B_{r_0} into itself.*

Proof Let $x \in B_{r_0}$. Using the considered assumptions, for all $t \in I$, we have

$$\begin{aligned}
&|(Tx)(t)| \\
&\le \left| F\left(t, x(a(t)), \frac{f(t,x(b(t)))}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s,x(s))\, d_q s\right) - F(t,0,0)\right| \\
&\quad + |F(t,0,0)| \\
&\le \varphi_F(|x(a(t))|) + C_F \frac{|f(t,x(b(t)))|}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} |u(s,x(s))|\, d_q s + F^* \\
&\le \varphi_F(\|x\|_\infty) + C_F \frac{|f(t,x(b(t))) - f(t,0)| + |f(t,0)|}{\Gamma_q(\alpha)} \\
&\quad \times \int_0^t (t-qs)^{(\alpha-1)} |u(s,x(s))|\, d_q s + F^* \\
&\le \varphi_F(\|x\|_\infty) + C_F \frac{\left(C_f |x(b(t))| + f^*\right) \varphi_u(\|x\|_\infty)}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)}\, d_q s + F^* \\
&\le \varphi_F(\|x\|_\infty) + C_F \frac{\left(C_f \|x\|_\infty + f^*\right) \varphi_u(\|x\|_\infty)}{\Gamma_q(\alpha+1)} t^\alpha + F^* \\
&\le \varphi_F(r_0) + C_F \frac{\left(C_f r_0 + f^*\right) \varphi_u(r_0)}{\Gamma_q(\alpha+1)} + F^*.
\end{aligned}$$

Therefore,

$$\|Tx\|_\infty \leq \varphi_F(r_0) + C_F \frac{\left(C_f r_0 + f^*\right) \varphi_u(r_0)}{\Gamma_q(\alpha+1)} + F^*, \quad x \in B_{r_0}.$$

Using the above inequality and assumption (A5), we obtain the desired result. □

Proposition 2.7 *Under assumptions (A1)–(A5), the operator T maps continuously B_{r_0} into itself.*

Proof Define the operators γ_1, γ_2 and γ_3 on E by

$$\begin{aligned}
(\gamma_1 x)(t) &= t, \quad (x,t) \in E \times I,\\
(\gamma_2 x)(t) &= x(a(t)), \quad (x,t) \in E \times I,\\
(\gamma_3 x)(t) &= f(t, x(b(t))), \quad (x,t) \in E \times I.
\end{aligned}$$

Obviously, $\gamma_1 : E \to E$ is continuous. Moreover, for all $x, y \in E$, we have

$$|(\gamma_2 x)(t) - (\gamma_2 y)(t)| = |x(a(t)) - y(a(t))| \leq \|x - y\|_\infty, \quad t \in I,$$

which implies that

$$\|\gamma_2 x - \gamma_2 y\|_\infty \leq \|x - y\|_\infty, \quad (x, y) \in E \times E.$$

Therefore, γ_2 is uniformly continuous on E. Similarly, for all $x, y \in E$, for all $t \in I$, we have

$$\begin{aligned}
|(\gamma_3 x)(t) - (\gamma_3 y)(t)| &= |f(t, x(b(t))) - f(t, y(b(t)))|\\
&\leq C_f |x(b(t)) - y(b(t))\| \leq C_f \|x - y\|_\infty,
\end{aligned}$$

which implies

$$\|\gamma_3 x - \gamma_3 y\|_\infty \leq C_f \|x - y\|_\infty, \quad (x, y) \in E \times E.$$

Then γ_3 is also uniformly continuous on E. So, in order to prove that T is continuous on B_{r_0}, we only need to show that the operator H defined by (2.37) is continuous on B_{r_0}. To do this, let us consider $\varepsilon > 0$ and $(x, y) \in B_{r_0} \times B_{r_0}$ such that $\|x - y\|_\infty \leq \varepsilon$. For all $t \in I$, we have

$$\begin{aligned}
(Hx)(t) - (Hy)(t) &= \int_0^t (t - qs)^{(\alpha-1)} u(s, x(s))\, d_q s - \int_0^t (t - qs)^{(\alpha-1)} u(s, y(s))\, d_q s\\
&= \int_0^t (t - qs)^{(\alpha-1)} \left(u(s, x(s)) - u(s, y(s))\right) d_q s.
\end{aligned}$$

Set

$$u_{r_0}(\varepsilon) = \sup\{|u(t,x) - u(t,y)| : t \in I,\ (x,y) \in [-r_0, r_0] \times [-r_0, r_0],\ |x - y| \le \varepsilon\},$$

we obtain

$$|(Hx)(t) - (Hy)(t)| \le \frac{t^\alpha}{[\alpha]_q} u_{r_0}(\varepsilon) \le \frac{u_{r_0}(\varepsilon)}{[\alpha]_q},$$

for all $t \in I$. Therefore,

$$\|Hx - Hy\|_\infty \le \frac{u_{r_0}(\varepsilon)}{[\alpha]_q}.$$

Passing to the limit as $\varepsilon \to 0^+$ and using the uniform continuity of u on the compact set $I \times [-r_0, r_0]$, we obtain

$$\lim_{\varepsilon \to 0^+} \frac{u_{r_0}(\varepsilon)}{[\alpha]_q} = 0,$$

which completes the proof. □

The following additional assumptions are needed later.

(A6) The function $\varphi_F : [0, \infty) \to [0, \infty)$ is continuous and it satisfies $\varphi_F(s) < s$ for $s > 0$.

(A7) The function $a : I \to I$ satisfies

$$|a(t) - a(s)| \le \varphi_a(|t - s|), \quad (t, s) \in I \times I,$$

where $\varphi_a : [0, \infty) \to [0, \infty)$ is nondecreasing and $\lim_{t \to 0^+} \varphi_a(t) = 0$.

(A8) The function $b : I \to I$ satisfies

$$|b(t) - b(s)| \le \varphi_b(|t - s|), \quad (t, s) \in I \times I,$$

where $\varphi_b : [0, \infty) \to [0, \infty)$ is nondecreasing and $\lim_{t \to 0^+} \varphi_b(t) = 0$.

(A9) We suppose that

$$0 < \varphi_u(r_0) < \frac{\Gamma_q(\alpha + 1)}{C_F C_f} \quad \text{and} \quad \frac{C_F}{\Gamma_q(\alpha)}(C_f r_0 + f^*) < 1.$$

Now, we can state our main result.

Theorem 2.35 *Under assumptions (A1)–(A9), Eq. (2.36) has at least one solution $x^* \in C(I; \mathbb{R})$. Moreover, such solution satisfies*

$$\|x^*\|_\infty \le r_0.$$

Proof From Proposition 2.7, the operator $T : B_{r_0} \to B_{r_0}$ is continuous. Now, let us take a nonempty subset X of B_{r_0}. Fix arbitrary $\varepsilon > 0$ and $x \in X$. Next, choose arbitrary real numbers $(t_1, t_2) \in I \times I$ such that $|t_1 - t_2| \le \varepsilon$. Without restriction of the generality, we may assume that $t_1 \ge t_2$. We obtain

$$\begin{aligned}
&|(Tx)(t_1) - (Tx)(t_2)| \\
&= \Big| F\Big(t_1, x(a(t_1)), \frac{f(t_1, x(b(t_1)))}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big) \\
&\quad - F\Big(t_2, x(a(t_2)), \frac{f(t_2, x(b(t_2)))}{\Gamma_q(\alpha)} \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big)\Big| \\
&\le \Big| F\Big(t_1, x(a(t_1)), \frac{f(t_1, x(b(t_1)))}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big) \\
&\quad - F\Big(t_2, x(a(t_1)), \frac{f(t_1, x(b(t_1)))}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big)\Big| \\
&\quad + \Big| F\Big(t_2, x(a(t_1)), \frac{f(t_1, x(b(t_1)))}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big) \\
&\quad - F\Big(t_2, x(a(t_2)), \frac{f(t_2, x(b(t_2)))}{\Gamma_q(\alpha)} \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big)\Big| \\
&:= (I) + (II).
\end{aligned} \tag{2.38}$$

Let us estimate the quantities (I) and (II).

• Estimate of (I). We have

$$\begin{aligned}
&\Big| \frac{f(t_1, x(b(t_1)))}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \Big| \\
&\le \frac{|f(t_1, x(b(t_1)))|}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} |u(s, x(s))|\, d_q s \\
&\le \frac{|f(t_1, x(b(t_1))) - f(t_1, 0)| + |f(t_1, 0)|}{\Gamma_q(\alpha)} \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} \varphi_u(|x(s)|)\, d_q s \\
&\le \frac{(C_f |x(b(t_1))| + f^*)\varphi_u(\|x\|_\infty)}{\Gamma_q(\alpha+1)} t_1^\alpha \\
&\le \frac{(C_f \|x\|_\infty + f^*)\varphi_u(\|x\|_\infty)}{\Gamma_q(\alpha+1)} \\
&\le \frac{(C_f r_0 + f^*)\varphi_u(r_0)}{\Gamma_q(\alpha+1)} := D.
\end{aligned}$$

Setting

$$\begin{aligned}
C(F, \varepsilon) = \sup \Big\{ &|F(t, x, y) - F(s, x, y)| : (t, s) \in I \times I,\ |t - s| \le \varepsilon, \\
&x \in [-r_0, r_0],\ y \in [-D, D] \Big\},
\end{aligned}$$

we obtain

$$(I) \leq C(F, \varepsilon). \tag{2.39}$$

Next, we estimate (II).

• Estimate of (II). We have

$$\begin{aligned}(II) &\leq \varphi_F(|x(a(t_1)) - x(a(t_2))|)\\ &\quad + \frac{C_F}{\Gamma_q(\alpha)}\Big| f(t_1, x(b(t_1))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\\ &\quad - f(t_2, x(b(t_2))) \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big|.\end{aligned}$$

In order to estimate

$$\varphi_F(|x(a(t_1)) - x(a(t_2))|),$$

observe that

$$|x(a(t_1)) - x(a(t_2))| \leq \omega(x \circ a, \varepsilon).$$

Using the monotone property of the function φ_F, we obtain

$$\varphi_F(|x(a(t_1)) - x(a(t_2))|) \leq \varphi_F(\omega(x \circ a, \varepsilon)).$$

Now, we have to estimate

$$\begin{aligned}&\Big| f(t_1, x(b(t_1))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\\ &\quad - f(t_2, x(b(t_2))) \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big|.\end{aligned}$$

We obtain

$$\begin{aligned}&\Big| f(t_1, x(b(t_1))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\\ &\quad - f(t_2, x(b(t_2))) \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big|\\ &\leq \Big| f(t_1, x(b(t_1))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\\ &\quad - f(t_2, x(b(t_2))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big|\\ &\quad + \Big| f(t_2, x(b(t_2))) \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\end{aligned}$$

$$
\begin{aligned}
&- f(t_2, x(b(t_2))) \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \Big| \\
&\leq \frac{|f(t_1, x(b(t_1))) - f(t_2, x(b(t_2)))| \varphi_u(\|x\|_\infty)}{[\alpha]_q} \\
&+ |f(t_2, x(b(t_2)))| \Big| \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \\
&- \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \Big| \\
&:= (III) + (IV).
\end{aligned}
$$

Let us define

$$\omega_f(r_0, \varepsilon) = \sup\{|f(t, x) - f(s, x)| : (t, s) \in I \times I,\ |t - s| \leq \varepsilon,\ x \in [-r_0, r_0]\}.$$

Then

$$
\begin{aligned}
(III) &\leq \frac{\varphi_u(\|x\|_\infty)}{[\alpha]_q} |f(t_1, x(b(t_1))) - f(t_1, x(b(t_2)))| \\
&\quad + \frac{\varphi_u(\|x\|_\infty)}{[\alpha]_q} |f(t_1, x(b(t_2))) - f(t_2, x(b(t_2)))| \\
&\leq \frac{\left[C_f |x(b(t_1)) - x(b(t_2))| + \omega_f(r_0, \varepsilon)\right] \varphi_u(r_0)}{[\alpha]_q} \\
&\leq \frac{\left[C_f \omega(x \circ b, \varepsilon) + \omega_f(r_0, \varepsilon)\right] \varphi_u(r_0)}{[\alpha]_q}.
\end{aligned}
$$

Now, let us estimate (IV). At first, we have

$$
\begin{aligned}
|f(t_2, x(b(t_2)))| &\leq |f(t_2, x(b(t_2))) - f(t_2, 0)| + |f(t_2, 0)| \\
&\leq C_f |x(b(t_2))| + f^* \leq C_f r_0 + f^*.
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
&\Big| \int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s - \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s \Big| \\
&= (1 - q) \sum_{n=0}^{\infty} q^n (1 - q^{n+1})^{(\alpha-1)} \left| t_1^\alpha u(q^n t_1, x(q^n t_1)) - t_2^\alpha u(q^n t_2, x(q^n t_2)) \right|.
\end{aligned}
$$

We can write

$$\begin{aligned}
&\left|t_1^\alpha u(q^n t_1, x(q^n t_1)) - t_2^\alpha u(q^n t_2, x(q^n t_2))\right| \\
&\le t_1^\alpha \left|u(q^n t_1, x(q^n t_1)) - u(q^n t_1, x(q^n t_2))\right| \\
&\quad + \left|t_1^\alpha u(q^n t_1, x(q^n t_2)) - t_2^\alpha u(q^n t_2, x(q^n t_2))\right| \\
&\le \varphi_u(|x(q^n t_1) - x(q^n t_2)|) + A_\varepsilon \\
&\le \varphi_u(\omega(x, \varepsilon)) + A_\varepsilon,
\end{aligned}$$

where

$$\begin{aligned}
A_\varepsilon = \sup\Big\{&|\mathcal{N}(\tau, s, x) - \mathcal{N}(\tau', s', x)| : (\tau, s, \tau', s') \in I^4,\ |\tau - \tau'| \le \varepsilon, \\
&|s - s'| \le \varepsilon,\ x \in [-r_0, r_0]\Big\}
\end{aligned}$$

and

$$\mathcal{N}(\tau, s, x) = \tau^\alpha u(s, x),\ (\tau, s, x) \in I \times I \times \mathbb{R}.$$

Then, we obtain

$$\begin{aligned}
&\Big|\int_0^{t_1} (t_1 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s - \int_0^{t_2} (t_2 - qs)^{(\alpha-1)} u(s, x(s))\, d_q s\Big| \\
&\le \varphi_u(\omega(x, \varepsilon)) + A_\varepsilon.
\end{aligned}$$

As a consequence, we have

$$(IV) \le (C_f r_0 + f^*)(\varphi_u(\omega(x, \varepsilon)) + A_\varepsilon).$$

Using the above inequalities, we obtain

$$\begin{aligned}
(II) \le \varphi_F(\omega(x \circ a, \varepsilon)) + \frac{C_F}{\Gamma_q(\alpha)}\Big(&\frac{[C_f \omega(x \circ b, \varepsilon) + \omega_f(r_0, \varepsilon)]\varphi_u(r_0)}{[\alpha]_q} \\
&+ (C_f r_0 + f^*)(\varphi_u(\omega(x, \varepsilon)) + A_\varepsilon)\Big).
\end{aligned}$$

Now, observe that from assumption (A7), we have

$$\begin{aligned}
\omega(x \circ a, \varepsilon) &= \sup\{|x(a(t)) - x(a(s))| : (t, s) \in I \times I,\ |t - s| \le \varepsilon\} \\
&\le \sup\{|x(\mu) - x(\nu)| : (\mu, \nu) \in I \times I,\ |\mu - \nu| \le \varphi_a(\varepsilon)\} \\
&= \omega(x, \varphi_a(\varepsilon)).
\end{aligned}$$

Similarly, from assumption (A8), we have

$$\omega(x \circ b, \varepsilon) \le \omega(x, \varphi_b(\varepsilon)).$$

Then

$$
\begin{aligned}
(II) \leq\ & \varphi_F(\omega(x,\varphi_a(\varepsilon))) + \frac{C_F}{\Gamma_q(\alpha)}\Big(\frac{[C_f\omega(x,\varphi_b(\varepsilon)) + \omega_f(r_0,\varepsilon)]\varphi_u(r_0)}{[\alpha]_q} \\
& + (C_f r_0 + f^*)(\varphi_u(\omega(x,\varepsilon)) + A_\varepsilon)\Big).
\end{aligned} \tag{2.40}
$$

Next, using (2.38), (2.39), and (2.40), we obtain

$$
\begin{aligned}
\omega(Tx,\varepsilon) \leq C(F,\varepsilon) + \varphi_F(\omega(x,\varphi_a(\varepsilon))) + \frac{C_F}{\Gamma_q(\alpha)}\Big(\frac{[C_f\omega(x,\varphi_b(\varepsilon)) + \omega_f(r_0,\varepsilon)]\varphi_u(r_0)}{[\alpha]_q} \\
+ (C_f r_0 + f^*)(\varphi_u(\omega(x,\varepsilon)) + A_\varepsilon)\Big),
\end{aligned}
$$

which yields

$$
\begin{aligned}
\omega(TX,\varepsilon) \leq\ & C(F,\varepsilon) + \varphi_F(\omega(X,\varphi_a(\varepsilon))) \\
& + \frac{C_F}{\Gamma_q(\alpha)}\Big(\frac{[C_f\omega(X,\varphi_b(\varepsilon)) + \omega_f(r_0,\varepsilon)]\varphi_u(r_0)}{[\alpha]_q} \\
& + (C_f r_0 + f^*)(\varphi_u(\omega(X,\varepsilon)) + A_\varepsilon)\Big).
\end{aligned}
$$

Recall that from assumptions (A7)–(A8), we have

$$
\lim_{t\to 0^+}\varphi_a(t) = \lim_{t\to 0^+}\varphi_b(t) = 0.
$$

Then passing to the limit as $\varepsilon \to 0^+$ in the above inequality, we obtain

$$
\omega_0(TX) \leq \varphi_F(\omega_0(X)) + \frac{C_F}{\Gamma_q(\alpha)}\Big(\frac{C_f\omega_0(X)\varphi_u(r_0)}{[\alpha]_q} + (C_f r_0 + f^*)\varphi_u(\omega_0(X))\Big).
$$

Therefore,

$$
\omega_0(TX) \leq \eta(\omega_0(X)),
$$

where

$$
\eta(t) = \max\{\varphi_F(t), L\varphi_u(t), Nt\}, \quad t \geq 0,
$$

with

$$
L = \frac{C_F}{\Gamma_q(\alpha)}(C_f r_0 + f^*), \quad N = \frac{C_F C_f}{\Gamma_q(\alpha+1)}\varphi_u(r_0).
$$

Moreover, from assumption (A9) and Lemma 2.4, the function η belongs to the set Λ. Finally, applying Theorem 2.34, we obtain the existence of at least one fixed point of the operator T in B_{r_0}, which is a solution to (2.36). □

Now, we present an example that illustrates Theorem 2.35.

Example 2.19 Consider the integral equation

$$x(t) = \frac{t}{32} + \frac{x(t)}{4} + [\alpha]_q \Big(\frac{t}{2} + \frac{x(t)}{4}\Big) \int_0^t (t - qs)^{(\alpha-1)} \frac{x(s)}{(2 + s^2)} \, d_q s, \qquad (2.41)$$

for $t \in I = [0, 1]$, where $\alpha > 1$ and $q \in (0, 1)$. Observe that Eq. (2.41) is a special case of Eq. (2.36) with

$$\begin{aligned}
a(t) &= t, \quad t \in I, \quad b(t) = t, \quad t \in I,\\
F(t, x, y) &= \frac{t}{32} + \frac{x}{4} + \Gamma_q(\alpha + 1)y, \quad (t, x, y) \in I \times \mathbb{R} \times \mathbb{R},\\
f(t, x) &= \frac{t}{2} + \frac{x}{4}, \quad (t, x) \in I \times \mathbb{R},\\
u(t, x) &= \frac{x}{(2 + t^2)}, \quad (t, x) \in I \times \mathbb{R}.
\end{aligned}$$

Now, let us check that the required assumptions by Theorem 2.35 are satisfied. Assumption (A1) is trivial. In order to check assumption (A2), take $(t, x, y, z, w) \in I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, we have

$$\begin{aligned}
|F(t, x, y) - F(t, z, w)| &= \left|\frac{x}{4} + \Gamma_q(\alpha + 1)y - \frac{z}{4} - \Gamma_q(\alpha + 1)w\right|\\
&\leq \frac{|x - z|}{4} + \Gamma_q(\alpha + 1)|y - w|.
\end{aligned}$$

Then assumption (A2) is satisfied with

$$\begin{aligned}
\varphi_F(t) &= \frac{t}{4}, \quad t \geq 0,\\
C_F &= \Gamma_q(\alpha + 1).
\end{aligned}$$

Now, for all $(t, x, y) \in I \times \mathbb{R} \times \mathbb{R}$, we have

$$|f(t, x) - f(t, y)| = \frac{|x - y|}{4}.$$

Then assumption (A3) is satisfied with $C_f = \frac{1}{4}$. Moreover, for all $(t, x, y) \in I \times \mathbb{R} \times \mathbb{R}$, we have

$$|u(t, x) - u(t, y)| = \frac{|x - y|}{2 + t^2} \leq \frac{|x - y|}{2}.$$

Take $\varphi_u(t) = \frac{t}{2}$, $t \geq 0$, assumption (A4) holds. In order to check assumption (A5), observe that in our case, we have $F^* = \frac{1}{32}$ and $f^* = \frac{1}{2}$. Now, the inequality

$$\varphi_F(r_0) + F^* + \frac{C_F(C_f r_0 + f^*)\varphi_u(r_0)}{\Gamma_q(\alpha + 1)} \leq r_0$$

is equivalent to

$$r_0^2 - 4r_0 + \frac{1}{4} \leq 0.$$

Obviously, the above inequality is satisfied for any $r_0 \in [\frac{4-\sqrt{15}}{2}, \frac{4+\sqrt{15}}{2}]$. Assumptions (A6)–(A8) are trivial. Further, the inequality

$$0 < \varphi_u(r_0) < \frac{\Gamma_q(\alpha+1)}{C_F C_f}$$

is equivalent to

$$0 < r_0 < 8.$$

The inequality

$$\frac{C_F}{\Gamma_q(\alpha)}(C_f r_0 + f^*) < 1$$

is equivalent to

$$r_0 < \frac{4}{[\alpha]_q} - 2.$$

A simple computation gives us that

$$[\frac{4-\sqrt{15}}{2}, \frac{4+\sqrt{15}}{2}] \cap \left(0, \frac{4}{[\alpha]_q} - 2\right) \neq \emptyset$$

for $\alpha = 3/2$ and $q = 1/2$. Therefore, all assumptions (A1)–(A9) are satisfied for $\alpha = 3/2$ and $q = 1/2$. By Theorem 2.35, we deduce that Eq. (2.41) has at least one solution $x^* \in C(I; \mathbb{R})$.

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