On the Theory of Acoustic Metamaterials with a Triple-Periodic System of Interior Obstacles

M.A. Sumbatyan and M.Yu. Remizov

Abstract The paper is devoted to the calculation of the reflection and the transmission coefficients, when a plane longitudinal wave is incident on a three-dimensional grating with a periodic array of rectangular cracks in the elastic material. In the one-mode frequency range the problem is reduced to a system of integral equations, which can be solved for various sizes of the cracks to give an explicit representation for the wave field inside the cracked structure, as well as the values of the reflection and the transmission coefficients.

Keywords Acoustic metamaterials · Periodic system of obstacles · Integral equation · Hypersingular kernel · Reflection coefficient · Transmission coefficient

1 Introduction

The study of elastic waves penetration through periodic gratings is an important subject in many practical applications in the field of mechanical, acoustical and electromagnetic sciences. In practice, analytical results can be obtained under assumption of low frequency with a weak interaction regime, where some approximated results can be established in an analytical form [1, 3, 5, 15].

The papers of Scarpetta, Sumbatyan and Tibullo [11–14] provide explicit analytical formulas for reflection and transmission coefficients in the one-mode case for acoustic waves penetrating through a doubly and triple-periodic arrays of arbitrary-shaped apertures and volumetric obstacles in wave propagation through a periodic array of screens in elastic solids.

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In the previous paper we studied the 3-D normal penetration of an elastic wave into a plane screen with an infinite doubly periodic system of cracks [7] with a one-mode frequency assumption. Earlier, the problems of high-frequency diffraction processes by cracks in an elastic material were analyzed in [8–10].

The present work continues to study the 3-D problem for the couple of such plane screens what forms a triple-periodic system. As shown in [7], the qualitative properties of such a geometrical structure are analogous to that in the in-plane problem. Thus, as a particular case, we study here the two arrays with a periodic system of cracks in each of them. The wave process is harmonic in time and all physical quantities contain the factor $e^{-i\omega t}$, which is further omitted, for the sake of brevity. In the same way as in [7] the following assumption is accepted: (a) only one-mode propagation (with normal incidence) is considered, $ak_2 < \pi$, where $2a$ is a period of the grating, k_2 is the transverse wave number; (b) the vertical cracked planes are sufficiently distant from each other so that the ratio D/a is comparatively large, where D is the distance between the two arrays.

The aim of the present work is to generalize the results obtained before extending explicit analytical expressions for the reflection and transmission coefficients for the system of parallel plane screens (using the properties of the kernel of a hyper-singular integral equation) based on the context of the in-plane problem for wave propagation through elastic solids with a periodic array of cracks.

The obtained results are meaningful in the aspect of the so-called "acoustic materials" which become nowadays an important component of the modern technology. The acoustic properties of such materials are prescribed not by the physical substance they are made from, but by their internal structure. Among other helpful properties of these materials, we note the cutoff of the transmitted acoustic energy on certain frequency intervals, i.e. they work as acoustic filters. Typically, this is attained for a certain periodic internal structure, like a triple-periodic system of relatively rigid spheres embedded in the epoxy matrix [4]. The results of the present study show that such a cutoff is an intrinsic feature caused even by simpler kinds of the periodicity. For example, in some sense this can be attained even in the 2D problem with a pair of parallel arrays, each of them containing a periodic linear system of coplanar cracks.

2 Mathematical Formulation of the Problem

Let us consider an unbounded (two-dimensional) elastic medium, which consists of 2 identical periodic systems of cracks, located at $x = 0, D$ with the period $2a$ along axes y, while the size of each crack is $2b$. The distance between the systems of cracks, forming the second period is D. If we study the incidence of a plane longitudinal wave upon the grating along the positive direction of axis x, then the problem is obviously equivalent (due to a symmetry) to a single waveguide of width $2a$ along axis y, see Fig. 1. Hence, if the incident wave of a unit amplitude is assumed to propagate normally to the planes along axis x, then the Lamè potentials in the various regions, satisfying the Helmholtz equation, are:

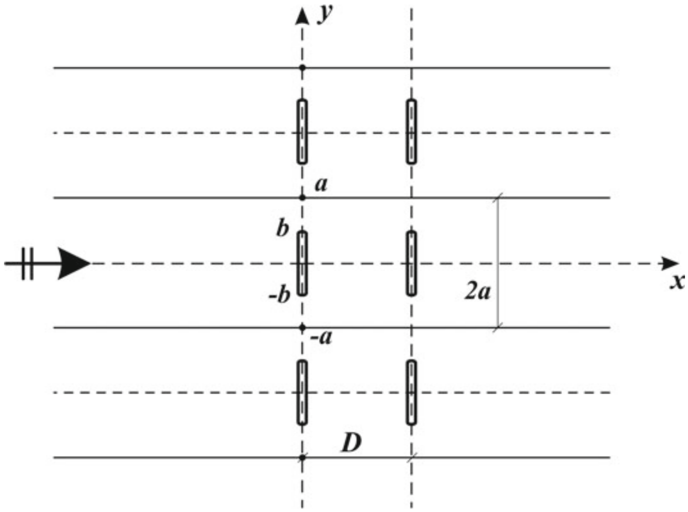

Fig. 1 Propagation of the longitudinal incident wave through a pair of the periodic arrays of cracks

$$\varphi^l = e^{ik_1x} + Re^{-ik_1x} + \sum_{n=1}^{\infty} A_n e^{q_n x} \cos\left(\frac{\pi n y}{a}\right),$$

$$\psi^l = \sum_{n=1}^{\infty} B_n^1 e^{r_n x} \sin\left(\frac{\pi n y}{a}\right), \quad x < 0 \tag{1a}$$

$$\varphi^1 = e^{ik_1x} + F_0^1 \cos[k_1 x] + H_0^1 \cos[k_1(x - D)] +$$

$$+ \sum_{n=1}^{\infty} \left\{F_n^1 ch[q_n x] + H_n^1 ch[q_n(x - D)]\right\} \cos\left(\frac{\pi n y}{a}\right),$$

$$\psi^1 = \sum_{n=1}^{\infty} \left\{G_n^1 ch[r_n x] + P_n^1 ch[q_n(x - D)]\right\} \sin\left(\frac{\pi n y}{a}\right), \quad 0 < x < D, \tag{1b}$$

$$\varphi^r = Te^{ik_1(x-D)} + \sum_{n=1}^{\infty} C_n e^{-q_n(x-D)} \cos\left(\frac{\pi n y}{a}\right),$$

$$\psi^r = \sum_{n=1}^{\infty} D_n e^{-r_n(x-D)} \sin\left(\frac{\pi n y}{a}\right), \quad x > D. \tag{1c}$$

All capital letters are some unknown constants and

$$q_n = \left(a_n^2 - k_1^2\right)^{1/2}, \quad r_n = \left(a_n^2 - k_2^2\right)^{1/2}, \qquad a_n = \frac{\pi n}{a} \tag{2}$$

where k_1, k_2 are the longitudinal and the transverse wave numbers, c_1, c_2—corresponding wave speeds in the elastic material, R and T are the reflection and the transmission coefficients, respectively. Let the consideration be restricted to the one-mode case: $0 < k_2 a < \pi$, then $q_n > 0,\ r_n > 0$ for all $n = 1, 2, \ldots$. Besides, we assume that the cracks arrays are sufficiently distant from each other, this involves $D/a \gg 1$. For $n = 0 \quad q_0 = -ik_1$ and $r_0 = -ik_2$, according to the radiation condition.

The components of the stress tensor can be expressed in terms of the Lamè wave potentials, two of them are represented in the following form:

$$\sigma_{xx} = -c_1^2 k_1^2 \varphi - 2c_2^2 \left(\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \psi}{\partial y \partial x} \right), \quad \sigma_{xy} = c_2^2 \left(2\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right). \tag{3}$$

The displacement field $\mathbf{u} \equiv (u_x, u_y)$ is given by a representation of the Green-Lamè type, as follows:

$$u_x = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}; \qquad u_y = \frac{\partial \varphi}{\partial y} - \frac{\partial \psi}{\partial x}. \tag{4}$$

In the considered structure, a longitudinal plane wave of the form

$$\varphi_0 = e^{ik_1 x}, \qquad \psi_0 = 0 \tag{5}$$

is entering from $x = -\infty$, generating the scattered fields before the first array ($x < 0$) between the two of them ($0 < x < D$), and after the second one ($x > D$).

Due to the natural periodicity in the vertical direction, let us restrict the consideration by the one cell $|y| < a$ only. Accepting the continuity of the displacement field outside the cracks at each vertical periodic system, let us introduce the following unknown functions $g_x^{(s)}(y),\ g_y^{(s)}(y),\ s = 1, 2$ by

$$x = 0: \quad u_x^l - u_x^1 = \begin{cases} g_x^1(y); & |y| < b, \\ 0; & b < |y| < a, \end{cases} \tag{6a}$$

$$x = 0: \quad u_y^l - u_y^1 = \begin{cases} g_y^1(y); & |y| < b, \\ 0; & b < |y| < a, \end{cases} \tag{6b}$$

$$x = D: \quad u_x^1 - u_x^r = \begin{cases} g_x^2(y); & |y| < b, \\ 0; & b < |y| < a, \end{cases} \tag{6c}$$

$$x = D: \quad u_y^1 - u_y^r = \begin{cases} g_y^2(y); & |y| < b, \\ 0; & b < |y| < a. \end{cases} \tag{6d}$$

Now Eqs. (4), (6) can be used to represent expressions for all constants appearing in potentials (1) in terms of $g_x(y), g_y(y)$. By integration of (6) over the domains $|y| < a$, one obtains

$$-ik_1R - H_0^1 k_1 \sin(k_1 D) = \frac{1}{2a}\int_{-b}^{b} g_x^1(\eta)d\eta, \tag{7a}$$

$$-F_0^1 k_1 \sin(k_1 D) + ik_1 e^{ik_1 D} - ik_1 T = \frac{1}{2a}\int_{-b}^{b} g_x^2(\eta)d\eta. \tag{7b}$$

The orthogonality of the trigonometric functions reduces Eq. (6) to the following relations:

$$(A_n + H_n^1 sh(q_n D))q_n + (B_n - P_n^1 ch(r_n D))a_n = \frac{1}{a}\int_{-b}^{b} g_x^1(\eta)\cos(a_n\eta)d\eta, \tag{8a}$$

$$(-A_n + H_n^1 ch(q_n D))a_n - (B_n + P_n^1 sh(r_n D))r_n = \frac{1}{a}\int_{-b}^{b} g_y^1(\eta)\sin(a_n\eta)d\eta, \tag{8b}$$

$$(C_n + F_n^1 sh(q_n D))q_n - (D_n - G_n^1 ch(r_n D))a_n = \frac{1}{a}\int_{-b}^{b} g_x^2(\eta)\cos(a_n\eta)d\eta, \tag{8c}$$

$$(C_n - F_n^1 ch(q_n D))a_n - (D_n + G_n^1 sh(r_n D))r_n = \frac{1}{a}\int_{-b}^{b} g_y^2(\eta)\sin(a_n\eta)d\eta. \tag{8d}$$

When crossing each of the two arrays, $x = 0$ and $x = D$, one can see that the normal stresses on crack's left and right faces are equal due to the boundary conditions, and outside crack's domain—due to continuity of the stress. This implies the continuity of the stress field for arbitrary y:

$$\sigma_{xx}^l = \sigma_{xx}^1, \quad \sigma_{xy}^l = \sigma_{xy}^1, \qquad x = 0, \ \ |y| < a, \tag{9a}$$

$$\sigma_{xx}^1 = \sigma_{xx}^r, \quad \sigma_{xy}^1 = \sigma_{xy}^r, \qquad x = D, \ \ |y| < a. \tag{9b}$$

This leads to the following relations:

$$\begin{gathered}-k_2^2[F_0^1 + H_0^1\cos(k_1 D) - R] - \sum_{n=1}^{\infty}\{k_2^2[H_n^1 ch(q_n D) - A_n] - \\ -2a_n^2[H_n^1 ch(q_n D) - A_n] + 2a_n r_n[P_n^1 sh(r_n D) + B_n]\}\cos(a_n y) = 0,\end{gathered} \tag{10a}$$

$$\begin{gathered}\sum_{n=1}^{\infty}\{2a_n q_n[H_n^1 sh(q_n D) + A_n] - r_n^2[P_n^1 ch(r_n D) - B_n] - \\ -a_n^2[P_n^1 ch(r_n D) - B_n]\}\sin(a_n y) = 0,\end{gathered} \tag{10b}$$

$$-k_2^2[F_0^1\cos(k_1D)+H_0^1+e^{ik_1D}-T]-\sum_{n=1}^{\infty}\{k_2^2[F_n^1ch(q_nD)-C_n]- \\ -2a_n^2[F_n^1ch(q_nD)-C_n]-2a_nr_n[G_n^1sh(r_nD)+D_n]\}\cos(a_ny)=0, \tag{10c}$$

$$-\sum_{n=1}^{\infty}\{2a_nq_n[F_n^1sh(q_nD)+C_n]+r_n^2[G_n^1ch(r_nD)-D_n]+ \\ +a_n^2[G_n^1ch(r_nD)-D_n]\}\sin(a_ny)=0. \tag{10d}$$

Now Eq. (8) together with the relations obtained from (10), with the use of the orthogonality of the trigonometric functions, form the following systems of linear algebraic equations for $x=0,D$:

$$\begin{gathered}\gamma_0E_2-2c_2^2a_nr_nE_3=0,\\ -a_nE_2-r_nE_3=G_y^1,\end{gathered} \tag{11a}$$

$$\begin{gathered}q_nE_1+a_nE_4=G_x^1,\\ -2a_nq_nE_1-(r_n^2+a_n^2)E_4=0,\end{gathered} \tag{11b}$$

$$\begin{gathered}q_nE_5+a_nE_8=G_x^2,\\ -2a_nq_nE_5-(r_n^2+a_n^2)E_8=0,\end{gathered} \tag{11c}$$

$$\begin{gathered}-a_nE_6-r_nE_7=G_y^2,\\ \gamma_0E_6-2c_2^2a_nr_nE_7=0,\end{gathered} \tag{11d}$$

where

$$G_x^{1,2}=\frac{1}{a}\int_{-b}^{b}g_x^{1,2}(\eta)\cos(a_n\eta)d\eta,\quad G_y^{1,2}=\frac{1}{a}\int_{-b}^{b}g_y^{1,2}(\eta)\sin(a_n\eta)d\eta,$$

$$\gamma_0=c_1^2k_1^2-2c_2^2a_n^2$$

and the new unknown quantities $E_m, m=1,\ldots,8$ are defined as follows:

$$\begin{aligned}&E_1=H_n^1sh(q_nD)+A_n, &&E_2=-H_n^1sh(q_nD)+A_n,\\ &E_3=P_n^1sh(r_nD)+B_n, &&E_4=-P_n^1ch(r_nD)+B_n,\\ \\ &E_5=F_n^1sh(q_nD)+C_n, &&E_6=F_n^1ch(q_nD)-C_n,\\ &E_7=G_n^1sh(r_nD)+D_n, &&E_8=G_n^1ch(r_nD)-D_n.\end{aligned} \tag{12}$$

Once the solutions for both the systems ($x=0$ and $x=D$) are constructed, one can easily find the eight unknown constants in the following form:

$$A_n = -\frac{r_n^2 + a_n^2}{2k_2^2 q_n} G_x^1 - \frac{a_n}{k_2^2} G_y^1, \quad B_n = \frac{a_n}{k_2^2} G_x^1 - \frac{\gamma_0}{2k_1^2 c_1^2 r_n} G_y^1,$$

$$H_n^1 = \frac{1}{2sh(q_n D)} \left(-\frac{r_n^2 + a_n^2}{k_2^2 q_n} G_x^1 + \frac{2a_n}{k_2^2} G_y^1 \right),$$

$$P_n^1 = \frac{1}{2sh(r_n D)} \left(-\frac{2a_n}{k_2^2} G_x^1 - \frac{\gamma_0}{k_1^2 c_1^2 r_n} G_y^1 \right),$$

$$F_n^1 = \frac{1}{2sh(q_n D)} \left(-\frac{r_n^2 + a_n^2}{k_2^2 q_n} G_x^2 - \frac{2a_n}{k_2^2} G_y^2 \right),$$

$$G_n^1 = \frac{1}{2sh(r_n D)} \left(\frac{2a_n}{k_2^2} G_x^2 - \frac{\gamma_0}{k_1^2 c_1^2 r_n} G_y^2 \right),$$

$$C_n = -\frac{r_n^2 + a_n^2}{2k_2^2 q_n} G_x^2 + \frac{a_n}{k_2^2} G_y^2, \quad D_n = -\frac{a_n}{k_2^2} G_x^2 - \frac{\gamma_0}{2k_1^2 c_1^2 r_n} G_y^2, \tag{13}$$

By integration of Eq. (10) over domain $|y| < a$, together with relations (7), one obtains the remaining unknown constants F_0^1, H_0^1, R, T from the following algebraic system:

$$\left(\begin{array}{cccc|c} 0 & -k_1 \sin(k_1 D) & -ik_1 & 0 & J_1 \\ -k_1 \sin(k_1 D) & 0 & 0 & -ik_1 & J_2 - ik_1 e^{ik_1 D} \\ 1 & \cos(k_1 D) & -1 & 0 & 0 \\ \cos(k_1 D) & 1 & 0 & -1 & -e^{ik_1 D} \end{array} \right), \tag{14}$$

where we put

$$J_{1,2} = \frac{1}{2a} \int_{-a}^{a} g_x^{1,2}(\eta) d\eta. \tag{15}$$

The solution to system (14) is given as follows:

$$F_0^1 = \frac{J_1 e^{ik_1 D}}{2k_1 \sin(k_1 D)} - \frac{J_2}{2k_1 \sin(k_1 D)}, \quad H_0^1 = -\frac{J_1}{2k_1 \sin(k_1 D)} + \frac{J_2 e^{ik_1 D}}{2k_1 \sin(k_1 D)},$$

$$R = -\frac{J_1}{2ik_1} - \frac{J_2 e^{ik_1 D}}{2ik_1}, \qquad T = -\frac{J_1 e^{ik_1 D}}{2ik_1} - \frac{J_2}{2ik_1} + e^{ik_1 D}. \tag{16}$$

It can easily be shown from the previous equations that the two unknown functions $g_y^{1,2}$ are trivial: $g_y^{1,2}(y) \equiv 0$. Omitting some routine mathematical transformations with the assumptions (a), (b) of Sect. 1, from the equations

$$\sigma_{xx}^{l} + \sigma_{xx}^{1} = 0,\ (x = 0, |y| < b), \qquad \sigma_{xx}^{1} + \sigma_{xx}^{r} = 0,\ (x = D, |y| < b), \tag{17}$$

one finally obtains the following system of two integral equations for the unknown functions $g_x^{1,2}(y)$, holding over the crack $|y| < b$:

$$\frac{1}{2a}\int_{-b}^{b} g_x^1(\eta)\left\{\frac{1}{2ik_1} - \frac{1}{k_2^4}\sum_{n=1}^{\infty}\frac{R_n}{q_n}\cos\left[a_n(y-\eta)\right]\right\}d\eta +$$

$$+\frac{e^{ik_1 D}}{4aik_1}\int_{-b}^{b} g_x^2(\eta)d\eta = 1, \tag{18a}$$

$$\frac{e^{ik_1 D}}{4aik_1}\int_{-b}^{b} g_x^1(\eta)d\eta +$$

$$+\frac{1}{2a}\int_{-b}^{b} g_x^2(\eta)\left\{\frac{1}{2ik_1} - \frac{1}{k_2^4}\sum_{n=1}^{\infty}\frac{R_n}{q_n}\cos\left[a_n(y-\eta)\right]\right\}d\eta = e^{ik_1 D}, \tag{18b}$$

where the numerator in the kernels takes the form of the Rayleigh function

$$R_n = (2a_n^2 - k_2^2)^2 - 4r_n q_n a_n^2. \tag{19}$$

3 The Properties of the Basic Integral Equation

Let us start from the study of the auxiliary integral equation ($|y| < b$):

$$\frac{1}{2ak_2^2}\int_{-b}^{b} h(\eta)K(y-\eta)d\eta = 1, \quad K(y)=\sum_{n=1}^{\infty}L_n\cos(a_n y), \quad L_n=\frac{R_n}{q_n}. \tag{20}$$

Notice that $L_n \approx -2(1 - c_2^2/c_1^2)a_n,\ \ n \to \infty$. Hence, the sum defining the kernel can be transformed as follows:

$$K(y-\eta) = -2\left(1-\frac{c_2^2}{c_1^2}\right)\sum_{n=1}^{\infty} a_n \cos[a_n(y-\eta)] +$$

$$+\sum_{n=1}^{\infty}\left[L_n + 2\left(1-\frac{c_2^2}{c_1^2}\right)a_n\right]\cos[a_n(y-\eta)], \tag{21}$$

$$\sim K(y-\eta) = -2\left(1-\frac{c_2^2}{c_1^2}\right)I(y-\eta) + K_r(y-\eta). \tag{22}$$

Now the second term in the kernel, K_r, is a certain regular function. The first one consists of a regular and a singular part: $I(y) = I_r(y) + I_s(y)$.

Let us introduce the dimensionless variable $\tilde{y} = (y-\eta)/a$, then one rewrites:

$$\frac{a}{\pi}I(\tilde{y}) = \sum_{n=1}^{\infty} n\cos(\pi n\tilde{y}). \tag{23}$$

By using the generalized value of the series in (23), see [6]:

$$\sum_{n=1}^{\infty} n\cos(\pi n\tilde{y}) = \lim_{\varepsilon\to+0}\sum_{n=1}^{\infty} e^{-\varepsilon n} n\cos(\pi n\tilde{y}) =$$

$$= -\frac{1}{4\sin^2(\pi\tilde{y}/2)}, \quad \left(\sim -\frac{1}{\pi^2\tilde{y}^2}, \quad y\to 0\right) \tag{24}$$

one obtains the kernel of the basic integral equation (21) in the following form:

$$K(y-\eta) = K_r(y-\eta) - 2\left(1-\frac{c_2^2}{c_1^2}\right)[I_r(y-\eta) + I_s(y-\eta)], \tag{25}$$

where the singular and the regular parts of $I(\tilde{y})$ are, respectively:

$$I_s = -\frac{a}{\pi(y-\eta)^2}, \quad I_r = \frac{a}{\pi(y-\eta)^2} - \frac{\pi}{4a\sin^2[\pi(y-\eta)/2a]}. \tag{26}$$

This results in the following form of the basic integral equation:

$$\frac{1}{2ak_2^2}\int_{-b}^{b} h(\eta)\left[\Phi_r(y-\eta) + \frac{2a(1-c_2^2/c_1^2)}{\pi(y-\eta)^2}\right]d\eta = 1, \qquad |y| < b,$$

$$\Phi_r(y-\eta) = -2\left(1 - \frac{c_2^2}{c_1^2}\right) I_r(y-\eta) + K_r(y-\eta). \tag{27}$$

The obtained singular behavior of the kernel for small arguments contains a 1D hyper-singular kernel arising in the theory of cracks, well known in the linear elasticity theory in unbounded media [16].

In order to provide the stability of the numerical treatment in the performed numerical experiments, we apply a discrete quadrature formulas for the 1D hyper-singular kernel, known as a "method of discrete vortices" [2]. Transforming the left part of (27) to a discrete form, one obtains

$$\frac{1}{2ak_2^2}\sum_{k=1}^{N}\int_{\eta_{k-1}}^{\eta_k} h(\eta)\left[\Phi_r(y_l-\eta) + \frac{2a(1-c_2^2/c_1^2)}{\pi(y_l-\eta)^2}\right]d\eta =$$
$$= \frac{1}{2ak_2^2}\sum_{k=1}^{N} h(\eta_k)\left[\varepsilon_1\Phi_r(y_l-\eta_k) + \int_{\eta_{k-1}}^{\eta_k}\frac{2a(1-c_2^2/c_1^2)}{\pi(y_l-\eta)^2}d(\eta-y_l)\right] =$$
$$\frac{1}{2ak_2^2}\sum_{k=1}^{N} h(\eta_k)\left[\varepsilon_1\Phi_r(y_l-\eta_k) - \frac{2a(1-c_2^2/c_1^2)}{\pi(\eta_k-y_l)} + \frac{2a(1-c_2^2/c_1^2)}{\pi(\eta_{k-1}-y_l)}\right], \tag{28}$$

where

$$\begin{array}{ll} \eta_k = -b + k\varepsilon_1, & y_l = -b + (l-0.5)\varepsilon_1, \\ l,k = 1,\ldots,N, & \varepsilon_1 = 2b/N. \end{array}$$

Finally we have the system of algebraic equations with respect to the quantities $h(\eta_k)$:

$$\frac{1}{2ak_2^2}\sum_{k=1}^{N} h(\eta_k)\left[\varepsilon_1\Phi_r(y_l-\eta_k) - \frac{2a(1-c_2^2/c_1^2)}{\pi}\left(\frac{1}{\eta_k-y_l} - \frac{1}{\eta_{k-1}-y_l}\right)\right] = 1. \tag{29}$$

Let us give a short description how an efficient treatment of the regular part of the kernel, function K_r in Eq. (21), can be arranged. This is based on the asymptotic estimate that the qualitative behavior of the regular kernel is as follows:

$$K_r(y,z) \sim \sum_{n=1} \frac{\cos(a_n y)}{a_n}, \tag{30}$$

Of course, this series can be calculated explicitly by using the tables [6]. However, some problems arise in this case when integrating over the small sub-intervals (η_{k-1}, η_k) in an explicit form. The simpler alternative way is to apply the explicit integration just to the series (30) itself:

$$\int_{\eta_{k-1}}^{\eta_k} \frac{\cos[a_n(\eta - y_l)]}{a_n} d\eta = \frac{\sin[a_n(\eta_k - y_l)] - \sin[a_n(\eta_{k-1} - y_l)]}{a_n^2}. \tag{31}$$

Note that with such a treatment of the considered particular term of the regular kernel, the factor ε_1 in (29) should be omitted in front of this particular term. It can easily be seen from Eq. (31) that the convergence is rapid enough along the index n. In practice, few hundred terms are sufficient, to guarantee 1% relative error in all diagrams demonstrated below.

4 Calculation of the Wave Characteristics

Let us set

$$H = \int_{-b}^{b} h(t)dt. \tag{32}$$

In terms of the even function $h(y)$, we deduce from system (18):

$$g_x^1(y) = \left\{ [1/(4aik_1)]J_1 + [e^{ik_1D}/[4aik_1]]J_2 - 1 \right\} k_2^2 h(y,z),$$

$$g_x^2(y) = \left\{ [1/(4aik_1)]J_2 + [e^{ik_1D}/[4aik_1]]J_1 - e^{ik_1D} \right\} k_2^2 h(y,z), \tag{33}$$

where the quantities J_1, J_2 are defined in Eq. (15).

The integration of Eq. (33) over the segment $[-b, b]$ leads to the following system of linear algebraic equations for the unknown constants J_1, J_2:

$$\left\{ k_2^{-2} - H/(4aik_1) \right\} J_1 - \left\{ He^{ik_1D}/[4aik_1] \right\} J_2 = -H,$$

$$-\left\{ He^{ik_1D}/[4aik_1] \right\} J_1 + \left\{ k_2^{-2} - H/(4aik_1) \right\} J_2 = -He^{ik_1D}. \tag{34}$$

Therefore, as soon as system (34) is solved, all necessary constants and the wave field can be found. In particular, for the reflection and transmission coefficients we obtain

$$R = -\frac{1}{4aik_1} J_1 - \frac{e^{ik_1D}}{4aik_1} J_2,$$

$$T = -\frac{e^{ik_1D}}{4aik_1} J_1 - \frac{1}{4aik_1} J_2 + e^{ik_1D}. \tag{35}$$

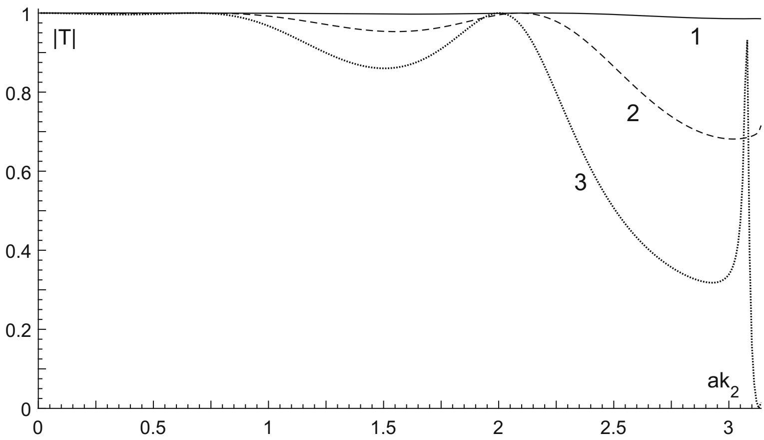

Fig. 2 Transmission coefficient versus frequency parameter: $D = 4.0$, line 1—$b/a = 0.15$, line 2—$b/a = 0.3$, line 3—$b/a = 0.4$

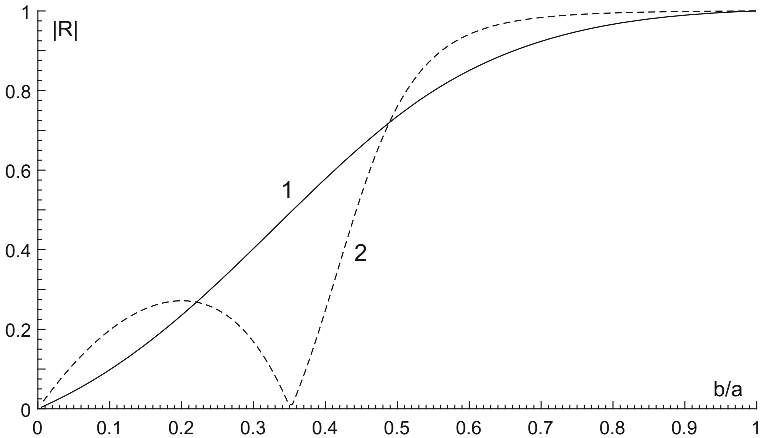

Fig. 3 Reflection coefficient versus crack's size: $D = 4.0$, line 1—$ak_2 = 0.935$, line 2—$ak_2 = 1.870$

The explicit expressions for the scattering parameters developed above complete the semi-analytical approach in the one-mode range. Some examples on the reflection and transmission coefficients versus frequency parameter, crack's size, and the distance between the two vertical arrays, for the elastic media with the wave speeds ratio $c_1/c_2 = 1.870$, are plotted in Figs. 2, 3, 4 and 5.

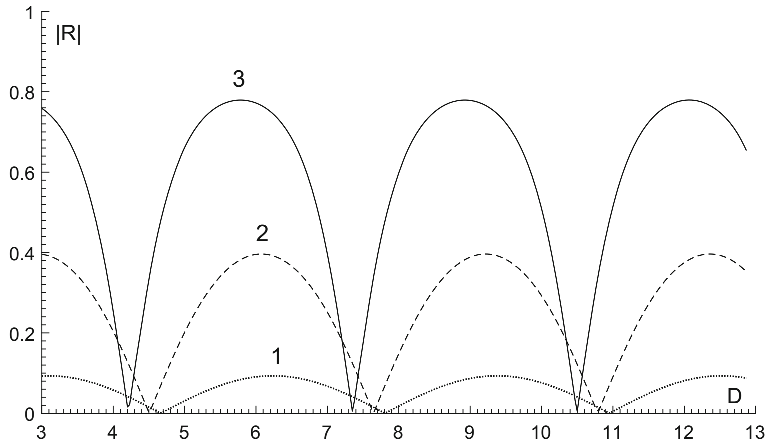

Fig. 4 Reflection coefficient versus distance between the two arrays of periodic systems of cracks: $ak_2 = 1.870$, line 1—$b/a = 0.15$, line 2—$b/a = 0.30$, line 3—$b/a = 0.45$

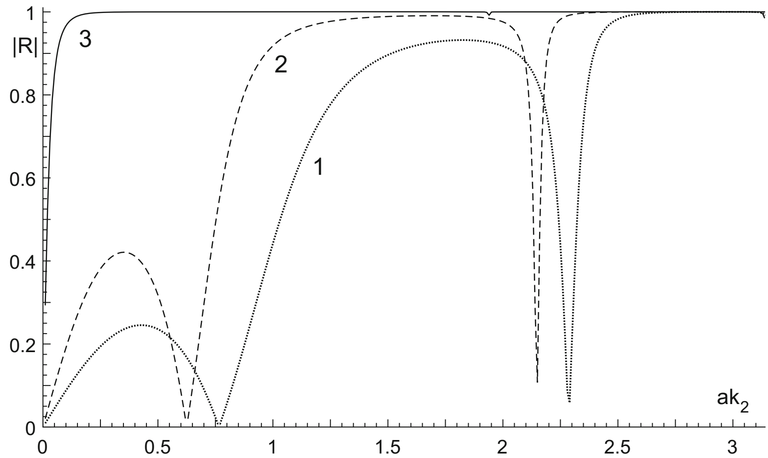

Fig. 5 Reflection coefficient versus frequency parameter: $D = 3.0$, line 1—$b/a = 0.6$, line 2—$b/a = 0.8$, line 3—$b/a = 0.999$

5 Conclusions

The obtained results are analyzed on the subject of the cutoff properties of the acoustic metamaterials possessing an internal periodic geometric structure, as described in the Introduction.

It is obvious from Fig. 2 that the number of frequency intervals with suppressed transmission, which is really a certain cutoff, grows with the increasing of the relative crack's size. If ak_2 and D are fixed, then the reflection coefficient $|R|$ versus relative crack's size b/a may show various behavior. It is a monotonically increasing function for $ak_2 = 0.935$, but when $ak_2 = 1.870$ the monotonic character of the diagram takes place only after the frequency passes a certain critical value, see Fig. 3. If b/a and ak_2 are fixed, then the behavior of function $|R(D)|$ is always wavy. For all that, the higher maxima correspond to the higher crack's length. It means that the strongest reflection takes place for large cracks; this is quite natural from the physical point of view, see Fig. 4.

Figure 5 demonstrates that the property of acoustic filters is attained for not only long cracks, where it is natural, but also for cracks of moderate length. Of course, the extremely long crack with $b/a = 0.999$ shows almost absolute cutoff for almost all frequencies in the considered one-mode range, which is physically natural. However, it is also interesting that the middle-size crack with $b/a = 0.6$ has a pair of relatively long frequency intervals where the reflection coefficient approaches the unit value. And it should also be noted that in the higher frequency part of the one-mode range with $ak_2 \approx \pi$ the cracks of all demonstrated lengths provide the cutoff. With this noting, let us outline that longer cracks make this upper cutoff frequency interval longer too.

It follows from the above discussion that the desired control of the acoustic filtering in the considered grating can be arranged by the appropriate choice of crack's length, respective frequency interval, and finally—but the distance between the two vertical arrays containing periodic systems of cracks.

The method developed in the present work permits efficient treatment of a more complex wave problem, when the number of vertical arrays containing the studied periodic system of cracks may be arbitrary but finite. This only requires to solve an alternative finite-dimensional system of linear algebraic equations, instead of the 2×2 system (34). This case will be studied in the authors' next work.

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