

Chapter 2

Sraffa-Okishio-Nakatani's Fixed Capital Theory

2.1 Introduction

It is known that although fixed capital shares the same function in the process of production, if the working age (or age) is different, one should make a distinction based on the different ages. The fixed capital at age 0 becomes the fixed capital at age 1 after 1 year of use. In other words, when fixed capital is used for commodities production in the production process, while products are manufactured as output, fixed capital that is 1 year older than the inputted fixed capital is also produced. That is to say, this production process produces two or more types of commodities and is a joint production process. Therefore, we can analyze and examine the problem of fixed capital by putting it in the joint production system.

Although the idea of joint-production is not novel at all, Sraffa (1960) seems to be the first literature that treated fixed capital from the angle of joint production; fixed capital is distinguished by ages, namely, regarded as different commodities, and hence, aged fixed capital is grasped as jointly produced with the main commodities of the process that employs fixed capital. Okishio and Nakatani (1975) simplified this narrowly defined Marx-Sraffa model; specifically, they simplified the joint production system, which include both newborn and aged fixed capital as a subsystem with only brand-new commodities. We could label this subsystem as the Sraffa-Okishio-Nakatani economy, in short SON (Asada 1982; Li 2012).

2.2 Reduction à la Sraffa-Okishio-Nakatani

In the outset, Sraffa-Okishio-Nakatani's framework will be made clear by way of listing presuppositions.

SON1 Only aged fixed capital is jointly produced by processes.

SON2 There is no process that produces only aged fixed capital.

SON3 Irrespective of ages, technical efficiency of fixed capital remains the same while it is employed in production.

SON4 Formation of processes obeys the generating rule of processes with fixed capital: a combination of fixed capital starts from the one with brand-new fixed capital alone; in the next period the process will be equipped with 1 year old fixed capital; if some of fixed capital reach their durability, they will be replaced by brand-new one.

The generating rule in SON4 excludes the possibility that worn-out fixed capital is replaced by aged fixed capital. This implies that there is no secondhand market of fixed capital. Besides, from this rule, the number of processes of a sector which produces one type of brand-new commodity does not exceed the minimal multiplier of durabilities of fixed capital in an economy. Intuitively, this implies the number of processes of a whole economy is larger than the number of commodities.

Additionally, *no-cost scrapping* and *exogenously given durability* are assumed with respect to fixed capital.

Sraffa established the prototype of reduction. He started from the system of simultaneous equations concerning all processes and all types of commodities, both brand-new ones and aged fixed capital, and then reduced the system to the one with brand-new commodities.

Okishio and Nakatani (1975) extended Sraffa's approach to the case of Marxian models. Okishio-Nakatani showed that prices of brand-new commodities and equilibrium profit rates are determined in the reduced system of brand-new commodities. Their approach seems to be the one to reduce the number of unknowns and the number of equations, and eventually reached to a system with square coefficient matrix.

In this section, Okishio-Nakatani's reduction procedure will be illustrated with simple examples, and some generalised viewpoint will be presented. Its relationship to replacement dynamics will be shown thereafter.

2.2.1 Okishio-Nakatani Reduction

In the following, two simple examples of Okishio-Nakatani's reduction will be shown on the basis of (1.3.2).

This subsection presents a new aspect to the problem. A new point here is that the coefficient matrix $B - \lambda M$ will be decomposed from the angle of nonsingular transformation of matrices.

One fixed capital case Commodities other than fixed capital are disregarded. Suppose there is basically only one class of fixed capital. The durability is assumed to be 3 years. Fixed capitals are classified according to their ages; there are three types of fixed capital, namely, 0-year old, 1-year old and 2-year old fixed capital. In an economic system, it is assumed that the above introduced fixed capitals are reproduced. Formally, there is a system of production with 3 types of commodities.

They are indexed from 1 to 3. Thus, one obtains the following input and output matrices of the system:

$$M = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ k & 0 & 0 \\ 0 & k & 0 \end{pmatrix},$$

and,

$$B - \lambda M = \begin{pmatrix} 1 - \lambda k & 1 & 1 \\ k & -\lambda k & 0 \\ 0 & k & -\lambda k \end{pmatrix}. \quad (2.2.1)$$

Form the diagonal matrix with λ^2 , λ and 1 as its diagonal elements, $A = \begin{pmatrix} \lambda^2 & & \\ & \lambda & \\ & & 1 \end{pmatrix}$ and postmultiply this to $B - \lambda M$, and one obtains:

$$(B - \lambda M)A = \begin{pmatrix} \lambda^2(1 - \lambda k) & \lambda & 1 \\ \lambda^2 k & -\lambda^2 k & 0 \\ 0 & \lambda k & -\lambda k \end{pmatrix}.$$

Then, postmultiply $E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ to the above result, and it follows that

$$(B - \lambda M)AE \sim \begin{pmatrix} \zeta_{11}(\lambda) & \lambda + 1 & 1 \\ 0 & -\lambda^2 & 0 \\ 0 & 0 & -\lambda \end{pmatrix},$$

where $\zeta_{11}(\lambda) = -\lambda^3 k + \lambda^2 + \lambda + 1$.

In the above, if $\zeta_{11}(\lambda) = 0$, then $\text{rank}(B - \lambda M)AE = \text{rank}(B - \lambda M) = 2 < 3$, so that there exists a $p \neq \ominus$ such that $p(B - \lambda M) = \ominus$. The condition $\zeta_{11}(\lambda) = 0$ determines the profit factor as a function of technical coefficient k . The order of λ is related to the durability of fixed capital.

Thus, the first column of $B - \lambda M$ determines the profit rate and non negative price of brandnew fixed capital. The remaining columns determines the relative ratio of prices of aged fixed capital to the price of brand-new fixed capital.

Two types of fixed capital case Since the above example is too simple, a more complicated case will be shown in the next place. Suppose there are two types of fixed capital, durabilities of them are 4 and 2 years, respectively. Commodities other than fixed capital are again ignored. When ages of fixed capital does not matter, two fixed capitals are identified; fixed capital 1 with durability 4, and fixed capital 2 with durability 2. We distinguish two sectors. The first sector produces fixed capital 1, while the second sector produces fixed capital 2: $B = (B_1, B_2)$ and $M = (M_1, M_2)$ with

$$M_j = \begin{pmatrix} k_{1j} & & & \\ & k_{1j} & & \\ & & k_{1j} & \\ k_{2j} & & & k_{1j} \\ & k_{2j} & & \\ & & k_{2j} & \end{pmatrix}, \quad B_j = \begin{pmatrix} \delta_{j1} & \delta_{j1} & \delta_{j1} & \delta_{j1} \\ k_{1j} & & & \\ & k_{1j} & & \\ & & k_{1j} & \\ \delta_{j2} & \delta_{j2} & \delta_{j2} & \delta_{j2} \\ k_{2j} & & k_{2j} & \end{pmatrix},$$

where $j = 1, 2$, and $\delta_{ij} = 1$ for $i = j$ or 0 otherwise.

For each $B_j - \lambda M_j$ block, one can apply the similar procedure: let

$$L = \begin{pmatrix} \lambda^3 & & & \\ & \lambda^2 & & \\ & & \lambda & \\ & & & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & \dots & 0 \\ & 1 & 1 & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix},$$

and one can compute $(B_j - \lambda M_j)LE$. Then, for the sake of brevity, one should collect rows related to brandnew commodities in the top positions of the matrix. That is, the 4th row should be moved to the 2nd row position. This row-wise operation is expressed by pre-multiplication of an elementary matrix to $(B_j - \lambda M_j)LE$. Let F denote such a matrix, and one obtains:

$$F(B_1 - \lambda M_1)LE = \begin{pmatrix} \psi_{11}(\lambda) & \lambda^2 + \lambda + 1 & \lambda + 1 & 1 \\ -k_{21}\lambda^2(\lambda^2 + 1) & -k_{21}\lambda^2 & -k_{21}\lambda^2 & 0 \\ 0 & -k_{11}\lambda^3 & 0 & 0 \\ 0 & 0 & -k_{11}\lambda^2 & 0 \\ 0 & 0 & 0 & -k_{11}\lambda \\ 0 & -k_{21}\lambda^3 & 0 & -k_{21}\lambda \end{pmatrix},$$

$$F(B_2 - \lambda M_2)LE = \begin{pmatrix} -k_{12}\lambda^4 & 0 & 0 & 0 \\ \psi_{25}(\lambda) & \psi_{26}(\lambda) & -k_{22}\lambda^2 + \lambda + 1 & 1 \\ 0 & -k_{12}\lambda^3 & 0 & 0 \\ 0 & 0 & -k_{12}\lambda^4 & 0 \\ 0 & 0 & 0 & -k_{12}\lambda \\ 0 & -k_{22}\lambda^3 & 0 & -k_{22}\lambda \end{pmatrix},$$

where $\psi_{ij}(\lambda)$ s are polynomials of λ , and

$$\begin{aligned} \psi_{11}(\lambda) &= -k_{11}\lambda^6 + \lambda^3 + \lambda^2 + \lambda + 1, \\ \psi_{25}(\lambda) &= -k_{22}\lambda^4 + \lambda^3 + (1 - k_{22})\lambda^2 + \lambda + 1, \\ \psi_{26}(\lambda) &= (1 - k_{22})\lambda^2 + \lambda + 1. \end{aligned}$$

As an operation to the whole system, one should form diagonal matrices

$$\Lambda = \begin{pmatrix} L & \\ & L \end{pmatrix}, \quad \mathcal{E} = \begin{pmatrix} E & O \\ O & E \end{pmatrix},$$

and, one obtains

$$F(B - \lambda M)\Lambda\mathcal{E} = F\left((B_1 - \lambda M_1)LE, (B_2 - \lambda M_2)LE\right).$$

The final step is to collect all of the first column of each block to the leftmost side of the matrix. Let $\mathcal{G}$ denote the matrix that represents those columnwise interchanges, and one obtains $\mathcal{Z} = F(B - \lambda M)\Lambda\mathcal{E}\mathcal{G}$ as follows:

$$\mathcal{Z} = \begin{pmatrix} \psi_{11}(\lambda) & -k_{12}\lambda^4 & \lambda^2 + \lambda + 1 & \lambda + 1 & 1 & \\ -k_{21}\lambda^2(\lambda^2 + 1) & \psi_{22}(\lambda) & -k_{21}\lambda^2 & -k_{21}\lambda^2 & 0 & \\ 0 & 0 & -k_{11}\lambda^3 & 0 & 0 & \\ 0 & 0 & 0 & -k_{11}\lambda^2 & 0 & \\ 0 & 0 & 0 & 0 & -k_{11}\lambda & \\ 0 & 0 & -k_{21}\lambda^3 & 0 & -k_{21}\lambda & \\ \\ 0 & 0 & 0 & & & \\ \psi_{26}(\lambda) & -k_{22}\lambda^2 + \lambda + 1 & 1 & & & \\ -k_{12}\lambda^3 & 0 & 0 & & & \\ 0 & -k_{12}\lambda^4 & 0 & & & \\ 0 & 0 & -k_{12}\lambda & & & \\ -k_{22}\lambda^3 & 0 & -k_{22}\lambda & & & \end{pmatrix}.$$

Since F , Λ , $\mathcal{E}$, $\mathcal{G}$ are all nonsingular matrices, the above transformation is rank-preserving, so that $\text{rank}(\mathcal{Z}) = \text{rank}(B - \lambda M)$.

Take the leftmost two columns of $\mathcal{Z}$, and that block is related to only brand-new fixed capital. If λ is such that $|\Psi(\lambda)| = \begin{vmatrix} \psi_{11}(\lambda) & -k_{12}\lambda^4 \\ -k_{21}\lambda^2(\lambda^2 + 1) & \psi_{25}(\lambda) \end{vmatrix} = 0$, then $\text{rank}(\mathcal{Z}) < m$, and hence there exists a vector $p \neq \mathbf{0}$ which fulfills $p(B - \lambda M) = \mathbf{0}$ or $x \neq \mathbf{0}$ such that $(B - \lambda M)x = \mathbf{0}$. $\Psi(\lambda)$ is nothing but the coefficient matrix of reduced system of only brand-new commodities.

From the dimension of $\mathcal{Z}$, the number of linearly independent columns of $\mathcal{Z}$ never exceeds its number of rows, so that one has only to consider the matrix formed by column 1, 2, 3, 4, 5 and 6 of $\mathcal{Z}$, for instance.

$$\mathcal{Z}_1 = \begin{pmatrix} \psi_{11}(\lambda) & -k_{12}\lambda^4 & \lambda^2 + \lambda + 1 & \lambda + 1 & 1 & 0 \\ -k_{21}\lambda^2(\lambda^2 + 1) & \psi_{25}(\lambda) & -k_{21}\lambda^2 & -k_{21}\lambda^2 & 0 & \psi_{26}(\lambda) \\ 0 & 0 & -k_{11}\lambda^3 & 0 & 0 & -k_{12}\lambda^3 \\ 0 & 0 & 0 & -k_{11}\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -k_{11}\lambda & 0 \\ 0 & 0 & -k_{21}\lambda^3 & 0 & -k_{21}\lambda & -k_{22}\lambda \end{pmatrix}.$$

By applying appropriate Gauss sweeping-out procedures, one obtains:

$$\mathcal{Z}_1 \sim \mathcal{Z}_1^* = \begin{pmatrix} \psi_{11}(\lambda) & -k_{12}\lambda^4 & \lambda^2 + \lambda + 1 & \lambda + 1 & 1 & 0 \\ -k_{21}\lambda^2(\lambda^2 + 1) & \psi_{25}(\lambda) & 0 & 0 & 0 & \lambda \\ 0 & 0 & -\lambda^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & -\lambda^2 \end{pmatrix}.$$

Remark that in view of the dimension of the matrix, the remaining rightmost block of $\mathcal{Z}$ is consist of all zero columns after applying appropriate Gauss sweeping-out procedures to it further: $\mathcal{Z} \sim (\mathcal{Z}_1^*, O)$.

More general case The above cases can be extended to a general case with more than 3 fixed capitals. Inclusion of current goods does not change the formal framework.

The matrix $B - \lambda M$ is transformed into three blocks via nonsingular transformation of the form $P(B - \lambda M)Q$, where P and Q are nonsingular matrices.

Eventually, $B - \lambda M$ will be reduced to the following form:

$$B - \lambda M \sim \left(\begin{array}{c|ccc|c} \Psi(\lambda) & ** & ** & ** & O \\ \hline & * & & & \\ \hline O & & \ddots & & O \\ \hline & & & * & \end{array} \right). \quad (2.2.2)$$

The leftmost columns with $\Psi(\lambda)$ constitute the block in which prices of brandnew commodities and the profit rate are determined. Okishio-Nakatani's goal of reduction was to solve $p\Psi(\lambda) = \mathbf{0}$, or, equivalently, $p = p(I - \Psi(\lambda))$, whilst, in the middle, the ratios of prices of aged fixed capital to brandnew fixed capital are determined. The remaining rightmost block is the zero block.

Remark that in the above reduction to the eigenvalue problem of $\Psi(\lambda)$, the desired λ is determined implicitly. That is, in Sraffa-Okishio-Nakatani, the basic equation of prices of brandnew commodities is of the form $p = p(I - \Psi(\lambda))$, so that a nonnegative matrix $I - \Psi(\lambda)$ should have the Perron-Frobenius root equal to 1. From this, the profit rate is determined.¹

As shown in the above, the matrix pencil is decomposed into three. Therefore, Okishio-Nakatani's reduction should be regarded as a decomposition of a matrix pencil of Marx-Sraffa price equilibrium. It is worth noting that F , $\mathcal{E}$, Λ and $\mathcal{G}$ characterise the Sraffa-Okishio-Nakatani reduction procedure, that is based on SON1-4. If one of SONs, say SON4, is dropped, the reduction encounters difficulty.

Remark that a similar reduction can be applied to the system of processes to the system of quantities, as was done by Fujimori (1982).

¹As for details, refer to *e.g.* Fujimori (1982).

Linear Theory of Fixed Capital and China's Economy

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2017, XIV, 122 p. 27 illus., 7 illus. in color., Hardcover

ISBN: 978-981-10-4064-1