Chapter 2
Knots

2.1 Knots and Diagrams

A **knot** is a submanifold of $\mathbb{R}^3$ that is homeomorphic to a circle. A **link** with μ components means a union $L = K_1 \cup \cdots \cup K_\mu$ of mutually disjoint μ knots $K_1, \ldots, K_\mu$. Each knot K_i is called a **component** of the link.

The knots illustrated in Fig. 2.1 are called a **trivial knot**, a **trefoil knot**, and a **figure-eight knot**.

The links with two components illustrated in Fig. 2.2 are called a **trivial link**, a **Hopf link**, and a **Whitehead link**. The link with three components illustrated in Fig. 2.3 is called the **Borromean rings**.

When knots K and K' are ambient isotopic in $\mathbb{R}^3$, we say that they are **equivalent**, and denote it by $K \cong K'$. The equivalence class of a knot (or a link) is called a **knot type** (or a **link type**).

A smooth knot can be approximated by a PL knot.[1] In what follows in this section, we assume that knots are PL knots. When a smooth knot is depicted in a figure, we regard it as a PL knot with a lot of small edges.[2]

Let K be a (PL) knot. Suppose that the intersection of a 2-simplex $|a_0a_1a_2|$ in $\mathbb{R}^3$ and K is an edge $|a_0a_1|$. Then $K' = (K \setminus |a_0a_1|) \cup (|a_1a_2| \cup |a_2a_0|)$ is a knot. We say that K' is obtained from K by a **Δ-move** along $|a_0a_1a_2|$. The inverse operation is also called a Δ-move.[3]

Two knots K and K' are said to be **combinatorially equivalent** if there exists a finite sequence of Δ-moves transforming K to K'.

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[1]Refer to R.H. Crowell and R.H. Fox [30], Appendix I.

[2]The idea that a smooth-looking knot is PL with a bunch of edges is due to J.W. Alexander in the 1928 paper [3].

[3]A Δ-move is also called an elementary deformation in J.W. Alexander and G.B. Briggs [4]. This combinatorial move is due to K. Reidemeister [145]. Refer also to K. Reidemeister [146–148].

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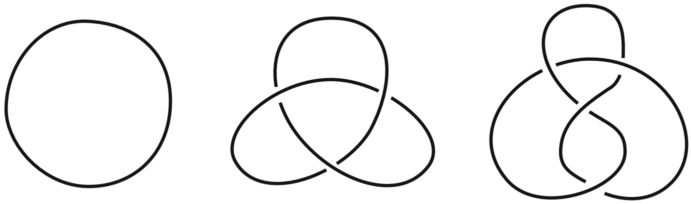


Fig. 2.1 A trivial knot, a trefoil knot, and a figure-eight knot

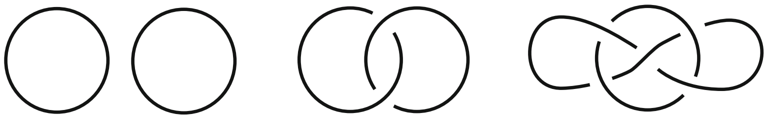
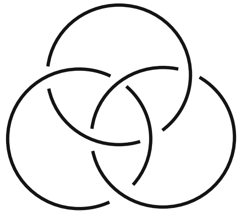


Fig. 2.2 A, trivial link, a Hopf link, and a Whitehead link

Fig. 2.3 Borromean rings



Theorem 2.1.1 *For two (PL) knots K and K', the following conditions are mutually equivalent*[4]*:*

(1) K and K' are equivalent (i.e., ambient isotopic in $\mathbb{R}^3$).
(2) There exists an orientation-preserving PL self-homeomorphism of $\mathbb{R}^3$ carrying K to K'.
(3) There exists an orientation-preserving topological self-homeomorphism of $\mathbb{R}^3$ carrying K to K'.
(4) K and K' are combinatorially equivalent.

An **orientation** of a knot K means an orientation of K as a 1-manifold. A knot with a fixed orientation is called an **oriented knot**. When two oriented knots are ambient isotopic with respect to the orientations, we say that they are **oriented equivalent** or **equivalent**, and we denote it by $K \cong K'$.

For an oriented knot K, we denote by $-K$ the same knot K with the reversed orientation. When K and $-K$ are equivalent, we say that K is **invertible**; otherwise **non-invertible**.

It is easily seen that the knot depicted in Fig. 2.4 is invertible by rotating it along an axis A. A knot is called **strongly invertible** if it is equivalent to a knot, say K, such that K is ambient isotopic to $-K$ by a rotation along an axis.

[4]This theorem is also valid for links. For a proof, refer to G. Burde and H. Zieschang [14] or A. Kawauchi [94].

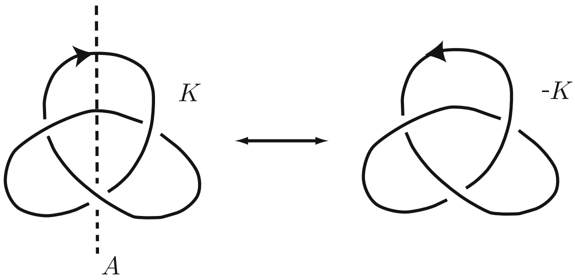


Fig. 2.4 A trefoil knot is strongly invertible

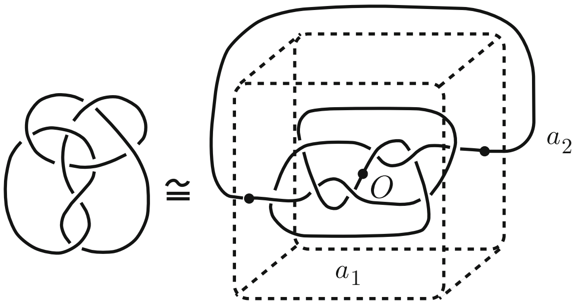


Fig. 2.5 8_{17} is non-invertible and negative amphicheiral

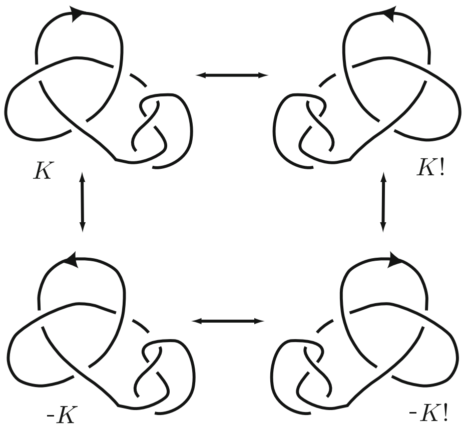


Fig. 2.6 The mirror image and the orientation-reversed of an oriented knot

It is known that the knot depicted in Fig. 2.5 (Left), called 8_{17}, is a non-invertible knot.[5]

The image $r(K)$ of a knot K by an orientation-reversing homeomorphism $r : \mathbb{R}^3 \to \mathbb{R}^3$ is called a **mirror image** of K and is denoted by $K!$. See Fig. 2.6.

[5]Refer to A. Kawauchi [93] and H.F. Trotter [176].

Exercise 2.1.2 Prove that the mirror image of a knot K is uniquely determined up to equivalence; that is, it is independent of choice of reflection r.[6]

Two knots K and K' are said to be **weakly equivalent** if $K \cong K'$ or $K! \cong K'$. Note that two knots K and K' are weakly equivalent if and only if there exists a homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ with $h(K) = K'$.

When a knot K is equivalent to its mirror image $K!$, we say that K is **amphicheiral** or **achiral**. When a knot K is not equivalent to its mirror image $K!$, we say that K is **chiral** or K is not amphicheiral.

The figure-eight knot is amphicheiral, and the trefoil knot is chiral.[7]

A **right-handed trefoil knot** is a knot equivalent to the one depicted in Fig. 2.1, and its mirror image is called a **left-handed trefoil knot**.

It is obvious that $(-K)! = -(K!)$ for any oriented knot K. When $K \cong K!$ as oriented knots, we say that K is **positive amphicheiral**. When $-K \cong K!$, we say that K is **negative amphicheiral**.

Example 2.1.3 The knot 8_{17} (Fig. 2.5) is amphicheiral. Deform the knot as in illustrated in Fig. 2.5 (Right) and denote it by K. The dashed line means a 3-ball B^3. We divide the knot K to two arcs a_1 and a_2 such that a_1 is inside B^3 and a_2 is outside. Let $h_1 : \mathbb{R}^3 \to \mathbb{R}^3$ be the homeomorphism with $h_1(x, y, z) = (-x, -y, -z)$. Then $h_1(a_1) = a_1$. Let $h_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism such that the restriction to B^3 is the identity map and that $h_2(h_1(a_2)) = a_2$. The homeomorphism $h_2 \circ h_1$ revises the orientation of $\mathbb{R}^3$ and $K! = h_2 \circ h_1(K) = K$. Hence K is amphicheiral.

Now we give an orientation to K and regard K as an oriented knot. Since $K! = h_2 \circ h_1(K) = -K$, the knot K is negative amphicheiral. The knot K is not positive amphicheiral. (If K is positive amphicheiral, then it is positive and negative amphicheiral. This implies that K is invertible. However, it is known that K is not invertible.[8])

When two links L_1 and L_2 can be separated by a 2-sphere embedded in $\mathbb{R}^3$, we say that the link $L = L_1 \cup L_2$ is a **split union** or a **split sum** of L_1 and L_2, and we denote it by $L_1 \circ L_2$. A link L is **non-split** if there exists no 2-sphere that separates L into two sub-links. For given links L_1 and L_2, a **split union** $L_1 \circ L_2$ is defined after moving L_1 or L_2 such that they are separated by a 2-sphere in $\mathbb{R}^3$. The split union of L_1 and L_2 is uniquely determined up to equivalence. Note that for links L_1, L_2 and L_3, $(L_1 \circ L_2) \circ L_3 \cong L_1 \circ (L_2 \circ L_3)$.

A split union of μ trivial knots is called a **trivial link** with μ components.

[6]Hint. Let $r' : \mathbb{R}^3 \to \mathbb{R}^3$ be an orientation-reversing homeomorphism. Then $r \circ (r')^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ is an orientation-preserving homeomorphism sending $r'(K)$ to $r(K)$.

[7]The Jones polynomials (Sect. 2.7) of the trefoil knot and its mirror image are different.

[8]Refer to A. Kawauchi [93] and H.F. Trotter [176].

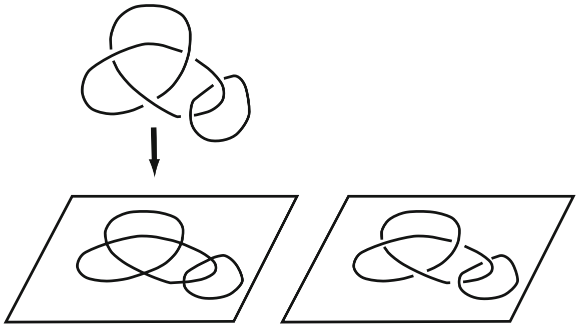


Fig. 2.7 A regular projection (*Left*) and a diagram (*Right*)

Proposition 2.1.4 *For a link L, the following conditions are mutually equivalent:*

(1) *L is a trivial link.*
(2) *L is the boundary of some mutually disjoint 2-disks embedded in* $\mathbb{R}^3$.
(3) *By an ambient isotopy of* $\mathbb{R}^3$, *L can be moved into a plane in* $\mathbb{R}^3$.

Let K be a knot, or a link. Let $f : K \to \mathbb{R}^2$ be the composition of the inclusion map $\iota : K \to \mathbb{R}^3$ and the projection $\mathrm{pr} : \mathbb{R}^3 \to \mathbb{R}^2$; $(x, y, z) \mapsto (x, y)$. When f is an immersion and every multiple point[9] is a transverse double point, we say that K is **in general position** with respect to the projection $\mathrm{pr} : \mathbb{R}^3 \to \mathbb{R}^2$, and we call f or the projection pr a **regular projection** of K. The image $f(K)$ by a regular projection is also called a **regular projection**. See Fig. 2.7 (Left).

A double point of a regular projection is also called a **crossing point** or a **crossing**.

For a crossing p the inverse image $f^{-1}(p)$ consists of two points, say p_1 and p_2. Assume that the z-coordinate of p_1 is greater than that of p_2. We call p_1 the **overcrossing point**, and p_2 the **undercrossing point**. Taking a small neighborhood $N(p) = N(p; \mathbb{R}^2)$ of p in $\mathbb{R}^2$, the inverse image $f^{-1}(N(p))$ is a union of two arcs, say a_1 and a_2, such that $p_1 \in a_1$ and$p_2 \in a_2$. We call the arc a_1 the **over-arc** of K at p, and a_2 the **under-arc**. The image $f(a_1)$ of the over-arc a_1 is called the **over-arc** of $f(K)$ at p, and the image $f(a_2)$ the **under-arc** of $f(K)$.

A **diagram** of a knot K means the regular projection $f(K)$ equipped with the over-arc and under-arc information at each crossing point. A standard method of describing the over-arc and under-arc information at a crossing point is removing a small under-arc from $f(K)$ as in Fig. 2.7 (Right). We use this method in this book.

The local transformations on diagrams depicted in Fig. 2.8 are called **Reidemeister moves**. They are called Reidemeister moves of type I, type II and type III, respectively.[10]

Two diagrams are said to be **Reidemeister equivalent** or **equivalent** if they are related by a finite sequence of Reidemeister moves and ambient isotopies of $\mathbb{R}^2$.

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[9]A point p of $f(K)$ is a multiple point if $f^{-1}(p)$ consists of two points or more.

[10]The move indicated with III$^\#$ in Fig. 2.8 is also a Reidemeister move of type III. For convenience, here we distinguish between the moves of type III and III$^\#$.

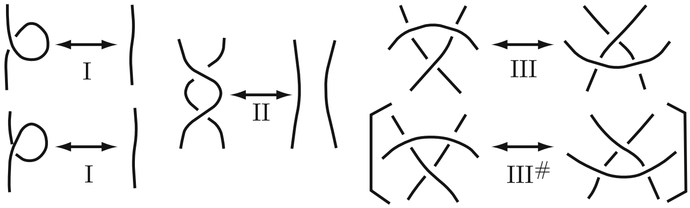

Fig. 2.8 Reidemeister moves

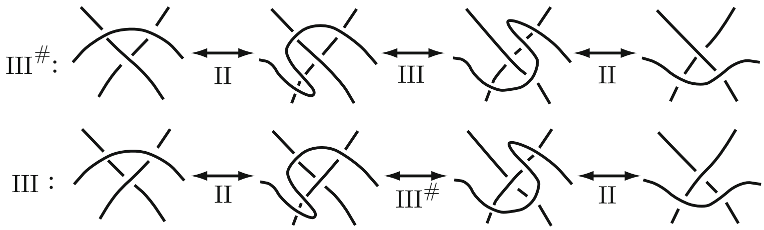

Fig. 2.9 A relationship between a move of type III and a move of type III#

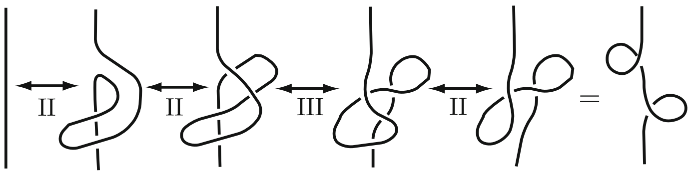

Fig. 2.10 A pair of moves of type I obtained by moves of type II and III

Remark 2.1.5 A move of type III# is obtained by a sequence of moves of type III and of II. Conversely, a move of type III is obtained by a sequence of moves of type III# and II (Fig. 2.9). Thus, for the definition of Reidemeister equivalence, it is sufficient to consider one of the moves of type III or III#.

Remark 2.1.6 Figure 2.10 shows that a pair of moves of type I is obtained by a sequence of moves of type II and III.

Theorem 2.1.7 (Reidemeister's theorem) *Let K and K' be knots (or links), and let D and D' be their diagrams. The knots (or links) K and K' are equivalent if and only if D and D' are Reidemeister equivalent.*[11]

When K is an oriented knot, a diagram D of K is also assigned the orientation. Reidemeister moves are considered on oriented diagrams of knots and links.

[11] This is due to K. Reidemeister in the 1926 article [145]. It is also found in the 1927 article by J.W. Alexander and G.B. Briggs [4]. Refer also to Reidemeister's book [146–148].

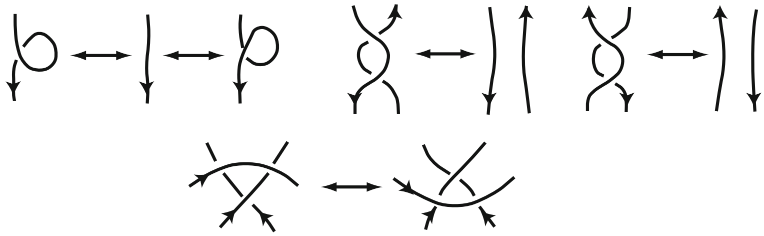

Fig. 2.11 A generating set of Reidemeister moves on oriented diagrams

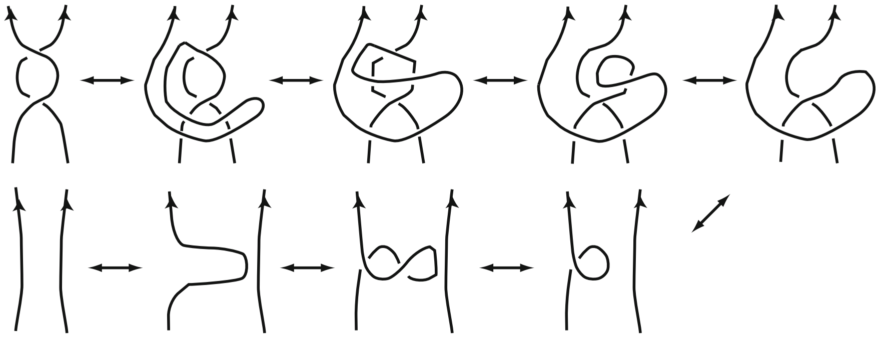

Fig. 2.12 An oriented move of type II with parallel orientations

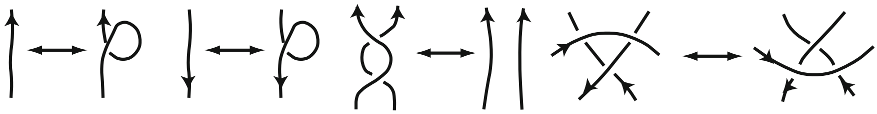

Fig. 2.13 Another generating set of Reidemeister moves on oriented diagrams

Theorem 2.1.8 *Let K and K' be oriented knots (or links), and let D and D' be their diagrams. K and K' are equivalent if and only if D and D' are Reidemeister equivalent.*

It is known that the Reidemeister equivalence on oriented diagrams is accomplished by using the five moves depicted in Fig. 2.11 and ambient isotopies of $\mathbb{R}^2$. For example, an oriented move of type II with parallel orientations can be realized as shown in Fig. 2.12 by using the moves in Fig. 2.11 and ambient isotopies of $\mathbb{R}^2$. The four moves depicted in Fig. 2.13 are another generating set of oriented Reidemeister moves.[12]

Oriented Reidemeister moves of type III are classified into two groups. A move of type III is said to be of **cyclic type** if the arcs around the triangle appearing in the center are oriented cyclically; otherwise, it is said to be of **braid type**.

[12]For details on these generating sets, refer to M. Polyak [136].

Note that the move of type III in Fig. 2.11 is of braid type, while the move of type III in Fig. 2.13 is of cyclic type.

Exercise 2.1.9 Observe that each step in the sequence of diagrams in Fig. 2.12 is a move in Fig. 2.11 or an ambient isotopy of $\mathbb{R}^2$.

The number of crossings of a diagram D is denoted by $c(D)$. For a knot K, the **crossing number** $c(K)$ of K is defined by

$$c(K) := \min\{c(D) \mid D \text{ is a diagram of a knot equivalent to } K\}.$$

2.2 Seifert Surfaces

Let K be an oriented knot. A **Seifert surface** of K means a compact, connected oriented surface S in $\mathbb{R}^3$ with $\partial S = K$.

For an oriented link L, a Seifert surface is defined to be a compact oriented surface S in $\mathbb{R}^3$ with $\partial S = L$ such that each connected component of S has a non-empty boundary.[13]

The following theorem is due to F. Frankl, L. Pontrjagin [41] and H. Seifert [165].

Theorem 2.2.1 *Any oriented knot, or link, has a Seifert surface.*

Proof Let K be an oriented knot or link. Moving K by an ambient isotopy, we assume that K is in general position with respect to the projection $\mathbb{R}^3 \to \mathbb{R}^2$, and consider a diagram, say D. Move K by an ambient isotopy so that most parts of K is contained in the xy-plane, over-arcs of K at the crossings are in the upper half space of $\mathbb{R}^3$ and the under-arcs are in the lower half space of $\mathbb{R}^3$. Let K' be the knot in this way. Obviously K and K' has the identical diagram D. For a crossing of the diagram, replace the over-arc and the under-arc of K' by two arcs as in Fig. 2.14. This operation is called a **smoothing**. Applying smoothing at every crossing of D, we obtain an oriented link, say L. The diagram of L has no crossings, and L is contained in the xy-plane. Thus L is a trivial oriented link. Each circle of L is called a **Seifert circle**. Let $\mathscr{D}$ be a union of mutually disjoint oriented 2-disks embedded in the lower-half space with $\partial \mathscr{D} = L$. See Fig. 2.15. For each crossing of D, attach a twisted band to $\mathscr{D}$ as in Fig. 2.16 and we obtain a compact oriented surface S' with $\partial S' = K'$. By reversing the ambient isotopy carrying K to K', we have an ambient isotopy carrying S' to a surface S such that $\partial S = K$. $\square$

The method of constructing a Seifert surface in the proof above is called the **Seifert algorithm**.[14]

[13] When $\mu \geq 2$, if an oriented link L with μ components has a Seifert surface such that the number of connected components of S is μ, then L is called a boundary link.

[14] This method was introduced by H. Seifert [165]. Note that not every Seifert surface of a knot is obtained by this method.

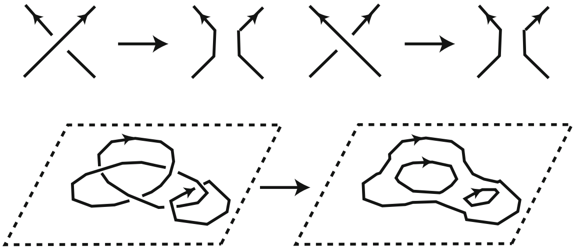

Fig. 2.14 Smoothings of a diagram and Seifert circles

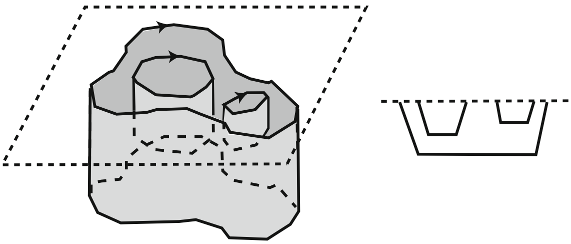

Fig. 2.15 Bounding disks to Seifert circles

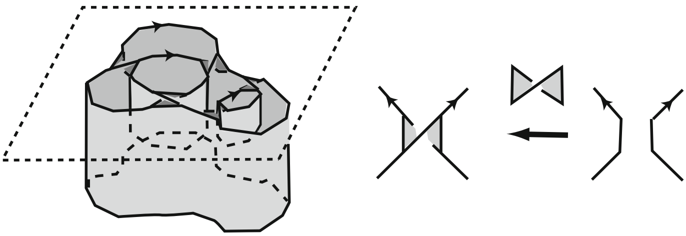

Fig. 2.16 Attaching bands to obtain a Seifert surface

The minimum number among all genera of Seifert surfaces[15] of K is called the **genus** of K.

Exercise 2.2.2 Show that the genus of the trivial knot is 0 while the genus of the figure-eight knot or the trefoil knot is 1.

Let S be a compact oriented surface in $\mathbb{R}^3$. Let $h : D^2 \times D^1 \to \mathbb{R}^3$ be a 3-dimensional 1-handle in $\mathbb{R}^3$ attaching to S (Sect. 1.1). We assume that h is coherent with respect to the orientation of S. We call a 1-handle surgery on S along h a **handle addition** to S.

A 2-handle surgery along a 3-dimensional 2-handle attaching to S is called a **handle reduction** to S.

[15] The genus of a compact, connected, oriented surface means the genus of a closed surface obtained by attaching disks along the boundary components.

Consider an oriented 2-sphere in $\mathbb{R}^3$ that surrounds S, and apply a handle addition to $S \cup S^2$ along a 1-handle connecting S and S^2. This operation is called an **infinity passing move**.

Definition 2.2.3 Let S and S' be compact oriented surfaces in $\mathbb{R}^3$ such that $\partial S = \partial S'$. We say that S and S' are **handle equivalent** if there is a finite sequence of ambient isotopes of $\mathbb{R}^3$ rel ∂S, handle additions, handle reductions and infinity passing moves that transforming S to S'.[16]

Theorem 2.2.4 *Any two Seifert surfaces of an oriented link are handle equivalent.*[17]

When we consider non-orientable surfaces bounding a link, we have the following. Let L be a link and $N(L)$ a regular neighborhood of L.

Theorem 2.2.5 *Let S and S' be compact surfaces in $\mathbb{R}^3$ with $\partial S = \partial S' = L$. If $S \cap N(L) = S' \cap N(L)$, then S and S' are handle equivalent.*[18]

2.3 Meridians and Longitudes

Let K be an oriented knot and $N(K)$ a regular neighborhood of K. There exists a homeomorphism $g : D^2 \times K \to N(K)$ with $g((O, x)) = x$ $(x \in K)$. Here O is the center of the 2-disk D^2. The regular neighborhood $N(K)$ is also called a **tubular neighborhood**.

A **meridian disk** of $N(K)$ means a properly embedded 2-disk D in $N(K)$ such that K intersects D in a point transversely. An oriented loop in $\partial N(K)$ is called a **meridian** of K if it bounds a meridian disk in $N(K)$. Here we assume that the orientation of a meridian should be as in Fig. 2.17 (Left). A meridian with the reversed orientation as in Fig. 2.17 (Right) is called a meridian with the negative orientation.

An oriented simple loop in $\partial N(K)$, say ℓ, is called a **longitude** of K if it is homotopic to K in $N(K)$.

A meridian of K in $\partial N(K)$ is uniquely determined up to ambient isotopy of $\partial N(K)$. However, a longitude ℓ in $\partial N(K)$ is not determined uniquely up to ambient isotopy of $\partial N(K)$. A longitude ℓ with $[\ell] = 0 \in H_1(\mathbb{R}^3 \setminus \mathrm{int}\, N(K))$ is called a **preferred longitude** or a **standard longitude**. A preferred longitude is uniquely determined up to ambient isotopy of $\partial N(K)$.

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[16]When we consider surfaces S and S' in S^3, infinity passing moves are not needed. Handle equivalence is also called tube equivalence in D. Bar-Natan, J. Fulman, and L.H. Kauffman [10].

[17]This theorem is used in J. Levine [109] to prove the uniqueness of the S-equivalence classes of Seifert surfaces (Sect. 2.6). For a proof, refer to D. Bar-Natan, J. Fulman, and L.H. Kauffman [10] or Proposition 7.2.2 of A. Kawauchi [96].

[18]When we consider non-orientable surface S, we do not assume that the 1-handle h is coherent with respect to an orientation of S in the definition of a handle addition. This theorem can be proved by an argument similar to the proof of Proposition 7.2.2 of A. Kawauchi [96].

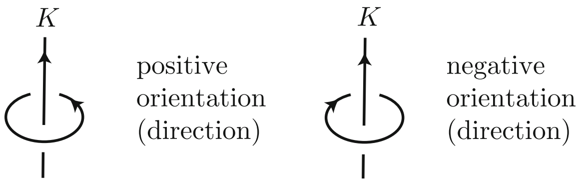

Fig. 2.17 Orientations of a meridian

Let S be a Seifert surface of K. A longitude ℓ with $\ell = \partial N(K) \cap S$ is a preferred longitude. (Since ℓ bounds a compact oriented surface $S \cap \mathbb{R}^3 \setminus \operatorname{int} N(K)$ in $\mathbb{R}^3 \setminus \operatorname{int} N(K)$, we have $[\ell] = 0 \in H_1(\mathbb{R}^3 \setminus \operatorname{int} N(K))$.) Conversely, for any preferred longitude ℓ, there exists a Seifert surface S with $\ell = \partial N(K) \cap S$.

The ambient isotopy class in $\partial N(K)$ of a longitude is called a **framing** of K.

For an oriented link $L = K_1 \cup \cdots \cup K_\mu$, we consider a meridian m_i and a longitude ℓ_i for each component K_i. A meridian m_i of K_i is uniquely determined up to ambient isotopy of $\partial N(K_i)$. However, we may consider two kinds of preferred longitudes of L.

Let $\ell_1, \ldots, \ell_\mu$ be longitudes of $K_1, \ldots, K_\mu$, respectively.

(1) When $[\ell_i] = 0 \in H_1(\mathbb{R}^3 \setminus \operatorname{int} N(K_i))$ for $i = 1, \ldots \mu$, we call $\ell_1, \ldots, \ell_\mu$ **absolutely preferred longitudes** of L. For each i, ℓ_i is a preferred longitude of K_i when we ignore the other components of L.
(2) When $[\ell_1] + \cdots + [\ell_\mu] = 0 \in H_1(\mathbb{R}^3 \setminus \operatorname{int} N(L))$, we call $\ell_1, \ldots, \ell_\mu$ **relatively preferred longitudes** of L.

Let $L = K_1 \cup \cdots \cup K_\mu$ be an oriented link. Longitudes $\ell_1, \ldots, \ell_\mu$ are relatively preferred longitudes of L if and only if there exists a Seifert surface S of L with $S \cap \partial N(L) = \ell_1 \cup \cdots \cup \ell_\mu$.

Figure 2.18 (Left) shows absolutely preferred longitudes of a Hopf link, where bold lines denote the link, and thin lines denote the longitudes. Figure 2.18 (Middle) shows relatively preferred longitudes. When we reverse the orientation of one component K_2 of the link, we have relatively preferred longitudes of $K_1 \cup (-K_2)$ as in Fig. 2.18 (Right). Note that the longitude of K_1 also changes.

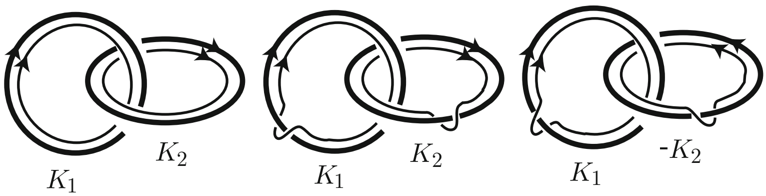

Fig. 2.18 Preferred longitudes

Let K and K' be mutually disjoint oriented knots. The homology group $H_1(\mathbb{R}^3 \setminus K)$ is an infinite cyclic group, and the homology class $[m]$ represented by a meridian m of K is a generator. Then $[K'] = n[m] \in H_1(\mathbb{R}^3 \setminus K)$ for some integer n. We call this integer n the **linking number** of K and K', and we denote it by $\mathrm{Lk}(K, K')$.

Let K and K' be mutually disjoint oriented knots. Let S be a Seifert surface of K. Moving K' slightly, we assume that K' intersects with S transversely in some points. We assign a sign ($\in \{+1, -1\}$) to each intersection of K' and S as in Fig. 2.19. The sum of the signs over all intersections of K' and S is denoted by $\mathrm{Int}(S, K')$, and it is called the **algebraic intersection number** or simply the **intersection number** of S and K'.

Similarly, let S' be a Seifert surface of K'. The algebraic intersection number $\mathrm{Int}(S', K)$ is also considered.

Proposition 2.3.1 *The equality* $\mathrm{Lk}(K, K') = \mathrm{Int}(S, K') = \mathrm{Int}(S', K)$ *holds.*

The linking number and the intersection number are also defined for oriented links. Let $L = K_1 \cup \cdots \cup K_\mu$ and $L' = K'_1 \cup \cdots \cup K'_\nu$ be mutually disjoint links. Let $m_1, \ldots, m_\mu$ be meridians of L. The homology group $H_1(\mathbb{R}^3 \setminus L)$ is a rank μ free abelian group, with basis $\{[m_1], \ldots, [m_\mu]\}$. When $[L'] = n_1[m_1] + \cdots + n_\mu[m_\mu]$, we define the linking number by $\mathrm{Lk}(L, L') = \sum_{i=1}^{\mu} n_i$.

Let S be a Seifert surface of L, and let $\mathrm{Int}(S, L')$ be the algebraic intersection number of S and L'. Then $\mathrm{Lk}(L, L') = \mathrm{Int}(S, L')$.

Exercise 2.3.2 Prove that when we move $L \cup L'$ by an ambient isotopy of $\mathbb{R}^3$, the linking number $\mathrm{Lk}(L, L')$ does not change.

Linking numbers can be calculated from diagrams. For a diagram of an oriented knot or link, each crossing looks as in Fig. 2.20. We call a crossing a **positive crossing** or a **negative crossing**, respectively. The **sign** of a crossing v is $+1$ or -1, respectively, and we denote it by $\mathrm{sign}(v)$.

Let L and L' be mutually disjoint links, and let D be a diagram of $L \cup L'$. Let $X(L > L')$ be the set of crossings of D such that the over-arc is in L and the under-arc is in L'. Let $X(L, L')$ be the set of crossings of D such that (i) the over-arc is in L and the under-arc is in L' or (ii) the over-arc is in L' and the under-arc is in L.

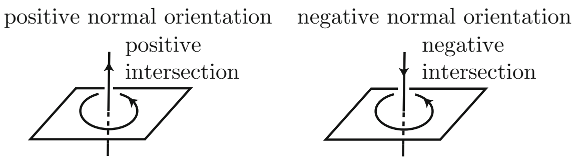

Fig. 2.19 Signs of intersections and normal orientations

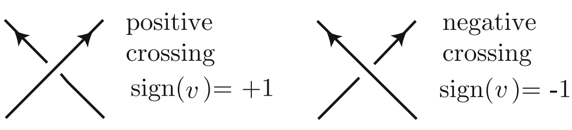

Fig. 2.20 A positive crossing and a negative crossing

Proposition 2.3.3 $\mathrm{Lk}(L, L') = \sum_{v \in X(L>L')} \mathrm{sign}(v) = \frac{1}{2} \sum_{v \in X(L,L')} \mathrm{sign}(v)$.

Proof We first prove the left equality. Let v be a crossing of D belonging to $X(L > L')$. Replace the diagram so that the over/under information at v is switched. (This operation is called a crossing change, which is an unknotting operation (Sect. 7.3).) Let D' be the diagram after the crossing change, and let $L \cup L'_{(1)}$ be a corresponding oriented link. Here we change L' by $L'_{(1)}$ without changing L. The difference $[L'] - [L'_{(1)}]$ in the homology group $H_1(\mathbb{R}^3 \setminus L)$ is the class of a meridian of L. Thus the linking number decreases by $\mathrm{sign}(v)$. Applying crossing changes at all crossings belonging to $X(L > L')$, we obtain a link $L \cup L''$ such that $\mathrm{Lk}(L, L') = \mathrm{Lk}(L, L'') + \sum_{v \in X(L>L')} \mathrm{sign}(v)$. Since L and L'' are split, we have $\mathrm{Lk}(L, L'') = 0$. Thus we have the left equality. We omit the proof for the right equality. □

Linking numbers have the following properties.

Proposition 2.3.4 *For mutually disjoint links L and L', the following holds:*

(1) $\mathrm{Lk}(L, L') = \mathrm{Lk}(L', L)$.
(2) $\mathrm{Lk}(-L, L') = \mathrm{Lk}(L, -L') = -\mathrm{Lk}(L, L')$.
(3) *When $L = L_1 \cup \cdots \cup L_\mu$ and $L' = L'_1 \cup \cdots \cup L'_\nu$,*

$$\mathrm{Lk}(L, L') = \sum_{i=1}^{\mu} \sum_{j=1}^{\nu} \mathrm{Lk}(L_i, L'_j).$$

2.4 Band Surgeries and Connected Sums

Let L be a link. An embedded 2-disk B in $\mathbb{R}^3$ is called a **band** attaching to L if $B \cap L = \partial B \cap L$ and this is a union of two disjoint arcs. The two arcs are called the **attaching arcs** of the band.

A band B attaching to a link L is a 2-dimensional 1-handle in $\mathbb{R}^3$ attaching to L in the sense introduced in Sect. 1.1. An embedding $h : D^1 \times D^1 \to \mathbb{R}^3$ with $L \cap h(D^1 \times D^1) = h(D^1 \times \partial D^1)$ is a 2-dimensional 1-handle, and the image $h(D^1 \times D^1)$ is a band B attaching to L. The attaching arcs $\alpha \cup \alpha'$ are the attaching region. The core $h(O \times D^1)$ of the 1-handle h is called a **core** of the band B.

Put $\mathrm{h}(L; B) := \mathrm{h}(L; h) = L \cup \partial B \setminus (\mathrm{int}\, \alpha \cup \mathrm{int}\, \alpha')$, which is a link. We say that $\mathrm{h}(L; B)$ is obtained from L by a **band surgery** along B. A band surgery is nothing more than a 1-handle surgery. It is also called a **hyperbolic transformation**.

When mutually disjoint bands $B_1, \ldots, B_n$ are attaching to a link L, we denote by $\mathrm{h}(L; B_1, \ldots, B_n)$ the link obtained from L by applying band surgeries along them simultaneously. It is also denoted by $\mathrm{h}(L; \{B_1, \ldots, B_n\})$ or by $\mathrm{h}(L; B_1 \cup \cdots \cup B_n)$. See Fig. 2.21.

Let L be an oriented link and let B be a band attaching to L. When a 1-handle $h : D^1 \times D^1 \to \mathbb{R}^3$ with $h(D^1 \times D^1) = B$ is coherent to the orientation of L, we say

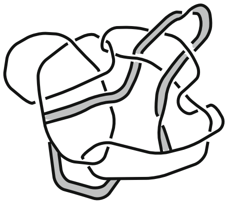

Fig. 2.21 Band surgery

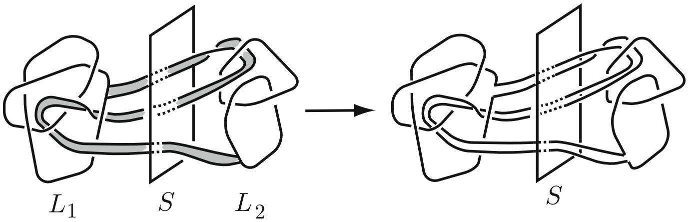

Fig. 2.22 A band sum

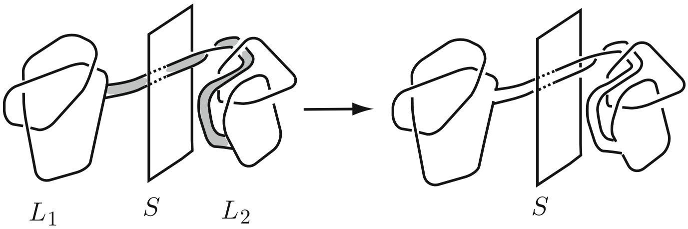

Fig. 2.23 A connected sum

that the band B is **coherent** to L. Then the link $\mathrm{h}(L; B)$ is regarded as an oriented link with the orientation induced from the orientation of L.

Let L_1 and L_2 be two links that are separated by an embedded 2-sphere S in $\mathbb{R}^3$. Let B be a band attached to $L_1 \cup L_2$ such that one of the attaching arcs is on L_1 and the other is on L_2. The link $\mathrm{h}(L_1 \cup L_2; B)$ is called a **band sum** of L_1 and L_2 along B (Fig. 2.22). Moreover, if the core of the band B intersects with S on a point, then the link $\mathrm{h}(L_1 \cup L_2; B)$ is called a **connected sum** of L_1 and L_2, that is denoted by $L_1 \# L_2$ (Fig. 2.23).

Proposition 2.4.1 *When K_1 and K_2 are oriented knots and the band B is coherent, a connected sum $K_1 \# K_2$ is uniquely determined up to equivalence; that is, it is independent of the choice of B.*

Exercise 2.4.2 (1) Prove Proposition 2.4.1.
(2) Show that $K_1 \# K_2 \cong K_2 \# K_1$ and $(K_1 \# K_2) \# K_3 \cong K_1 \# (K_2 \# K_3)$.

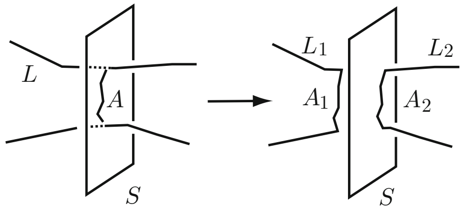

Fig. 2.24 A decomposition of a link

Let L be a link and S a 2-sphere embedded in $\mathbb{R}^3$ such that L intersects with S transversely in two points p_1 and p_2. Take a simple arc in S connecting p_1 and p_2, say A. Let L' be a link obtained from L by removing a neighborhood of $\{p_1, p_2\}$ in L and connecting the endpoints with two arcs A_1 and A_2 that are parallel to A. Then $L' = L_1 \circ L_2$. When neither L_1 nor L_2 is a trivial knot, we say that L is **decomposed** to L_1 and L_2, and S is a **decomposing sphere** (Fig. 2.24). This operation is the right inverse of the operation taking a connected sum, i.e., when L is decomposed into L_1 and L_2 then $L = L_1 \# L_2$.

A knot is called a **composite knot** if it is equivalent to a connected sum of two non-trivial knots. In other words, a composite knot is a knot that can be decomposed into two non-trivial knots. A knot is called a **prime knot** if it is not a composite knot.

Theorem 2.4.3 (The prime decomposition theorem) *Any non-trivial knot is equivalent to a connected sum of a finite number of prime knots. Such a prime decomposition is unique, i.e., if* $K_1 \# K_2 \# \cdots \# K_m \cong K'_1 \# K'_2 \# \cdots \# K'_n$ *for prime knots* K_i $(i = 1, \ldots, m)$ *and* K'_j $(n = 1, \ldots, n)$*, then* $m = n$ *and there is a permutation* σ *on* $\{1, 2, \ldots, m\}$ *such that* $K_1 \cong K'_{\sigma(1)}, \ldots, K_m \cong K'_{\sigma(m)}$.[19]

2.5 Knot Groups

Let K be a knot and $N(K)$ a regular neighborhood of K. The space $E(K) := \mathbb{R}^3 \setminus \operatorname{int} N(K)$ is called the **exterior** of K, and the space $\mathbb{R}^3 \setminus K$ is called the **complement** of K.

The fundamental group $\pi_1(\mathbb{R}^3 \setminus K)$ of the complement is called the **knot group** of K and is denoted by $G(K)$. For a link, it is also called the **link group**.

If two knots are weakly equivalent, they have isomorphic knot groups. If two knots have homeomorphic complements, then they have isomorphic knot groups.

Theorem 2.5.1 *(1) Two knots having homeomorphic complements are weakly equivalent.*
(2) Two prime knots having isomorphic knot groups are weakly equivalent.

This theorem is due to C. McA. Gordon and J. Luecke [49].

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[19] This theorem is due to H. Schubert [163]. For a link case, refer to Y. Hashizume [51].

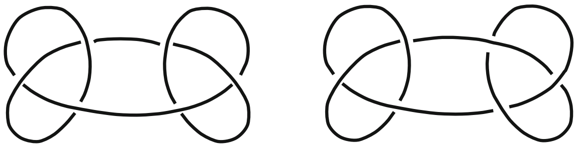

Fig. 2.25 Non-equivalent knots having isomorphic knot groups

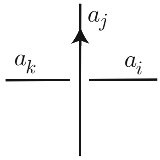

Fig. 2.26 The relation rel(v) at a crossing v: $x_j^{-1}x_ix_j = x_k$

The two composite knots illustrated in Fig. 2.25 have isomorphic knot groups. However they are not weakly equivalent.

Theorem 2.5.2 *A knot is trivial if and only if the knot group is an infinite cyclic group.*[20]

The knot group can be calculated from a knot diagram. Let K be an oriented knot. Consider a diagram D of K, that is a union of some mutually disjoint oriented arcs. (The diagram D may have some loops without crossings. For simplicity, they are also called arcs here.) Let $\mathrm{Arc}(D) = \{a_1, \ldots, a_m\}$ be the set of oriented arcs of D.

We assign the letter x_i to the arc a_i (for $i = 1, \ldots, m$), and consider the free group $\langle x_1, \ldots, x_m\rangle$. Let v be a crossing of D and let a_i, a_j, a_k be the arcs appearing around v. Suppose that a_j is the over-arc at v. When we face v in the orientation of a_j, suppose that the arc on the right side is a_i and the arc on the left side is a_k. See Fig. 2.26. Then we define the relation rel(v) by $x_j^{-1}x_ix_j = x_k$. (Note that in the definition of rel(v), we do not use the orientations of a_i and a_k.) Consider the relations rel$(v_1), \ldots,$ rel(v_n) for all crossings $v_1, \ldots, v_n$ of D.

Now we have a group presentation

$$\langle x_1, \ldots, x_m \mid \mathrm{rel}(v_1), \ldots, \mathrm{rel}(v_n)\rangle.$$

The group determined by this presentation is called the group determined from the diagram D and is denoted by $G(D)$.

Theorem 2.5.3 *Let K be an oriented knot and D a diagram of it. The knot group $G(K)$ is isomorphic to the group $G(D)$ determined from D.*

Proof Move K by an ambient isotopy of $\mathbb{R}^3$ to obtain a knot K' as follows. The intersection of K' and the xy-plane is D, and the remaining part of K' is in the open lower-half space $\{(x, y, z) \in \mathbb{R}^3 \mid z < 0\}$ as in Fig. 2.27 (Right). We regard the arcs a_i $(i = 1, \ldots, m)$ as subsets of K'. The knot groups $G(K')$ and $G(K)$ are isomorphic.

[20]This is due to C.D. Papakriakopoulos [134].

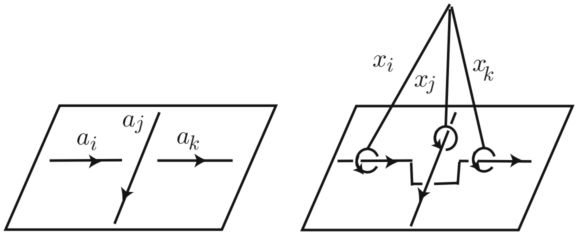

Fig. 2.27 The relation at a crossing: $x_j^{-1}x_ix_j = x_k$

Take a base point p of $G(K')$ in the open upper-half space $\{(x, y, z) \in \mathbb{R}^3 \mid z > 0\}$. For each arc a_i $(i = 1, \ldots, m)$, let B_i be a meridian disk at an interior point of a_i. Take a point q_i on ∂B_i in the open upper-half space. Let α_i be a straight path from q_i to p. Let m_i be a positive meridian loop that starts at q_i and goes along ∂B_i. Then $\alpha_i^{-1}m_i\alpha_i$ is a loop in $\mathbb{R}^3 \setminus K'$ with base point p. Put $x_i := [\alpha_i^{-1}m_i\alpha_i] \in G(K') = \pi_1(\mathbb{R}^3 \setminus K', p)$. The fundamental group $G(K')$ is generated by $x_1, \ldots, x_m$, and for each crossing point, we have $\mathrm{rel}(v) : x_j^{-1}x_ix_j = x_k$. By van-Kampen's theorem, we see that $G(K')$ is isomorphic to $G(D)$. For the details, refer to R.H. Crowell and R.H. Fox [30]. $\square$

Let K be an oriented knot (or link), $G(K)$ the knot group with base point p. Let B be a meridian disk of K, and q a point on ∂B. Let α be a path from q to p in $\mathbb{R}^3 \setminus K$. We call the loop $\alpha^{-1}m\alpha$ a **meridian loop** or simply a **meridian** associated with α. Here m is a loop with base point q that goes along ∂B in the positive direction. An element of $G(K)$ represented by a meridian loop is called a **meridian**, a **meridian element** or a **meridional element**.

The generators $x_1, \ldots, x_m$ of $G(D)$ in the proof of Theorem 2.5.3 are meridians of K'.

The knot group $G(K)$ of a knot K is isomorphic to the group $G(D)$ determined from a diagram D, and the latter group has a presentation whose generators are $x_1, \ldots, x_m$, and each of the defining relations is in a form of $x_k^{-1}x_j^{-1}x_ix_j$. Such a group presentation is called a **Wirtinger presentation**.

2.6 Seifert Matrices

Let K be an oriented knot or link, and S a connected Seifert surface of K. The positive normal orientation of S is determined from the orientation of S as in Fig. 2.19.

For a union ℓ of loops on S, we denote by ℓ^+ a parallel copy of ℓ obtained by pushing out ℓ in the positive normal orientation of S, and by ℓ^- a parallel copy of ℓ obtained by pushing out ℓ in the negative normal orientation of S.

Let $\phi : H_1(S) \times H_1(S) \to \mathbb{Z}$ be the map defined by

$$\phi(x, y) = \mathrm{Lk}(\ell_x^+, \ell_y) \quad (= \mathrm{Lk}(\ell_x, \ell_y^-))$$

where ℓ_x is a union of simple loops on S representing the homology class x, and ℓ_y is one representing the homology class y. We call $\phi : H_1(S) \times H_1(S) \to \mathbb{Z}$ the **Seifert form** of K associated with S.

A presentation matrix V of the Seifert form ϕ with respect to a basis of $H_1(S)$ is called a **Seifert matrix** of K.

A Seifert matrix V depends on a connected Seifert surface S and a basis of $H_1(S)$. However, when we consider an equivalence relation on matrices called **S-equivalence**, the S-equivalence class of V is determined from K.[21]

Let K be an oriented knot or link, and V a Seifert matrix of K. Let ${}^TV$ denote the transpose of V. The signature $\mathrm{sign}(V + {}^TV)$ of the symmetric matrix $V + {}^TV$ is uniquely determined from K. It is called the **signature** of K, and is denoted by $\sigma(K)$.[22]

For a symmetric matrix M, after transforming M into a diagonal matrix, the number of 0s appearing in the diagonal is called the nullity of M. The nullity of $V + {}^TV$ is called the **nullity** of K, and is denoted by $n(K)$.

The absolute value $|\det(V + {}^TV)|$ of the determinant of the matrix $V + {}^TV$ is called the **determinant** of K, and is denoted by $\mathrm{Det}(K)$.

The signature $\sigma(K)$, the nullity $n(K)$ and the determinant $\mathrm{Det}(K)$ are invariants of K.[23]

We denote by $\mathbb{Z}[t, t^{-1}]$ the ring of integral Laurent polynomials in variable t.[24]

An **Alexander polynomial** of K is defined by $\det(V - t\,{}^TV) \in \mathbb{Z}[t, t^{-1}]$, which is denoted by $\Delta_K(t)$. It is determined up to multiplication by units $\pm t^m$ $(m \in \mathbb{Z})$ of the ring $\mathbb{Z}[t, t^{-1}]$.[25]

Exercise 2.6.1 For a trefoil knot and a figure-eight knot, compute their signatures, determinants and Alexander polynomials.

2.7 Skein Relations and Polynomial Invariants

In this section we introduce some invariants of knots and links that can be computed by using skein relations on diagrams. We assume that links are oriented.

A triple (D_+, D_-, D_0) of link diagrams D_+, D_- and D_0 is called a **skein triple** if there exists a 2-disk, say M, in $\mathbb{R}^2$ such that (i) D_+, D_- and D_0 are identical outside M, (ii) the restrictions of D_+, D_- and D_0 to M are as in Fig. 2.28, respectively.

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[21] This is due to J. Levine [109]. Refer to A. Kawauchi [96] for details.

[22] Refer to H.F. Trotter [175], K. Murasugi [127], J. Levine [109], C. McA. Gordon and R.A. Litherland [48] for signatures of knots and links.

[23] When K is a knot, $n(K) = 0$.

[24] A Laurent polynomial is a polynomial that may have negative powers of the variable. When the coefficients are integers, we call it an integral Laurent polynomial. For example, $3t^2 - 6 + 2t^{-1} - t^{-3}$.

[25] Refer to J.W. Alexander [3].

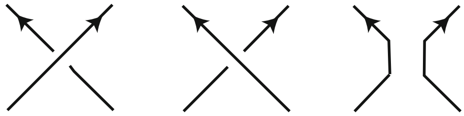


Fig. 2.28 The local pictures of D_+, D_- and D_0 in a skein triple

A triple (L_+, L_-, L_0) of links L_+, L_- and L_0 is called a **skein triple** if they have diagrams in a skein triple.

In what follows, O means a trivial knot and (L_+, L_-, L_0) means any skein triple of links.

The **Conway polynomial** $\nabla_L(z)$ of L is a link invariant that satisfies

$$\nabla_O(z) = 1, \quad \nabla_{L_+}(z) - \nabla_{L_-}(z) = z\nabla_{L_0}(z).$$

The latter equality is understood to hold for every skein triple (L_+, L_-, L_0). This condition is called the skein relation for the Conway polynomial.

The **Alexander–Conway polynomial** $\Delta_L(t)$ of L is a link invariant that satisfies

$$\Delta_O(t) = 1, \quad \Delta_{L_+}(t) - \Delta_{L_-}(t) = \left(t^{1/2} - t^{-1/2}\right) \Delta_{L_0}(t).$$

This is an Alexander polynomial of L defined in the previous section using a Seifert matrix. When we define $\Delta_L(t)$ by the skein relation, we do not need to consider it up to multiplication by units of the integral Laurent polynomial ring. It is also obtained from the Conway polynomial by the relation $\Delta_L(t) = \nabla_L(t^{1/2} - t^{-1/2})$.[26]

The **Jones polynomial** $V_L(t)$ of L is a link invariant valued in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ that satisfies[27]

$$V_O(t) = 1, \quad t^{-1}V_{L_+}(t) - tV_{L_-}(t) = \left(t^{1/2} - t^{-1/2}\right) V_{L_0}(t).$$

We denote by $\mathbb{Z}[\ell, \ell^{-1}, m, m^{-1}]$ the ring of integral Laurent polynomials in two variables ℓ and m.

The **HOMFLY-PT polynomial**[28] $P_L(\ell, m)$ is a link invariant that takes values in $\mathbb{Z}[\ell, \ell^{-1}, m, m^{-1}]$ and satisfies

$$P_O(\ell, m) = 1, \quad \ell P_{L_+}(\ell, m) + \ell^{-1} P_{L_-}(\ell, m) + m P_{L_0}(\ell, m) = 0.$$

[26]The invariant $\nabla_L(z)$ was defined by J.H. Conway [29], where it was called the potential function and the relathion between the Alexander polynomial and the potential funtion was given there.

[27]Refer to V.F.R. Jones [60, 61]. L.H. Kauffman [87, 88] introduced a state model for the Jones polynomial.

[28]HOMFLY is the initials of the authors, P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millett, and A. Ocneanu of [36] (cf. [110]) and PT is that of the authors, J.H. Przytycki and P. Traczyk of [142].

The HOMFLY-PT polynomial is a generalization of the Jones polynomial, and it is sometimes called the 2-variable Jones polynomial. The Conway polynomial is also obtained from the HOMFLY-PT polynomial.

Exercise 2.7.1 For a trivial link with two components, a trefoil knot and a figure eight knot, compute the Conway polynomials, the Jones polynomials and the HOMFLY-PT polynomials.

2.8 2-Bridge Knots, Torus Knots, Satellite Knots

Let $p : \mathbb{R}^3 \to \mathbb{R}$, $(x, y, z) \mapsto z$ be the projection.

Any knot K in $\mathbb{R}^3$ is equivalent to a knot K' such that all maximal points of the restriction map $p|_{K'} : K' \to \mathbb{R}$ are in $\mathbb{R}^2 \times \{a\}$ and all minimal points are in $\mathbb{R}^2 \times \{b\}$ for some $a > b$. Then we say that K' is a **bridge presentation** of K. The number of maximal points of K' is called the **index** of the bridge presentation.

The **bridge index** of a knot K is defined to be the minimum among all indices of bridge presentations of K.

An **m-bridge knot** is a knot whose bridge index is m.

For links, one can also define the notions of a bridge presentation, the bridge index and an m-bridge link. Note that if a link with μ components has a bridge presentation of index m, then $\mu \leq m$. Thus the braid index of a link is greater than or equal to the number of components.

A 1-bridge knot is a trivial knot. There are a lot of 2-bridge knots and 2-bridge links.

For a sequence $(a_1, a_2, \ldots, a_n)$ of non-zero integers, let $C(a_1, a_2, \ldots, a_n)$ be the knot or link illustrated in Fig. 2.29. Here a box indicated by a stands for $|a|$ times twists as in Fig. 2.30. For example, $C(3, 4, -2, 3, 3)$ is depicted in Fig. 2.31.

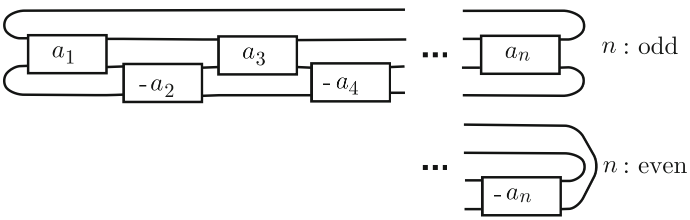


Fig. 2.29 $C(a_1, a_2, a_3, \ldots, a_n)$

Fig. 2.30 A notation presenting twists

Fig. 2.31 $C(3, 4, -2, 3, 3)$

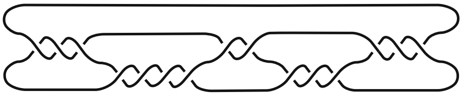
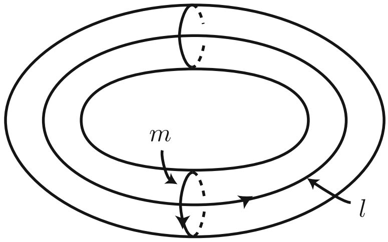


Fig. 2.32 A meridian m and a longitude l on a standard torus



It is known that any 2-bridge knots and links are equivalent to $C(a_1, a_2, \ldots, a_n)$ for some n and a sequence $(a_1, a_2, \ldots, a_n)$. It is said to be in **Conway's normal form**.[29]

There is another method of describing 2-bridge knots and links called **Schubert's normal form**.[30] By H. Schubert [164], all 2-bridge knots and links are classified.

Let T be a standardly embedded torus in $\mathbb{R}^3$, and let m and l be simple loops illustrated in Fig. 2.32, which we call a **meridian** and a **longitude**. The first homology group $H_1(T)$ of T is an free abelian group with basis $\{[m], [l]\}$.

Proposition 2.8.1 *(1) Let C be a simple loop on T such that $[C] \neq 0 \in H_1(T)$. When $[C] = p[m] + q[l] \in H_1(T)$, p and q are co-prime.*

(2) For any co-prime integers p and q, there exists a simple loop C on T such that $[C] = p[m] + q[l] \in H_1(T)$.

(3) In (2), the ambient isotopy class of C in T is uniquely determined from p and q.

For a proof of this proposition, refer to D. Rolfsen [150].

A simple loop C on T is said to be of **type-(p,q)** if $[C] = p[m] + q[l] \neq 0 \in H_1(T)$.

A knot is called a **torus knot** of type-(p, q) if it is equivalent to a knot on T that is of type-(p, q) (Fig. 2.33).

A link L with μ components is called a **torus link** of type-(a, b) if a/μ and b/μ are co-prime integers and L is equivalent to a link on T each of whose components is of type-$(a/\mu, b/\mu)$.

A torus knot or a torus link of type-(a, b) is denoted by $T(a, b)$ in this book.

[29] The length n and the sequence $(a_1, a_2, \ldots, a_n)$ are not determined uniquely. One may take such a sequence $(a_1, a_2, \ldots, a_n)$ in even numbers. Especially, when we consider oriented 2-bridge knots and links, a sequence $(a_1, a_2, \ldots, a_n)$ in even numbers is preferred.

[30] Refer to A. Kawauchi [94].

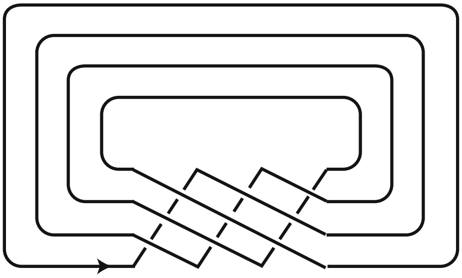


Fig. 2.33 A torus knot of type-(3, 4)

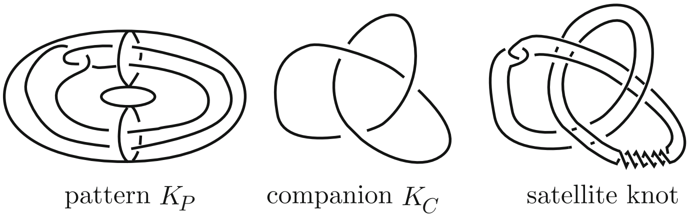


Fig. 2.34 A pattern K_P, a companion K_C and a satellite knot

Exercise 2.8.2 Show the following.

(1) $T(-m, -n) \cong -T(m, n) \cong T(m, n)$.
(2) $T(m, -n) \cong T(m, n)!$.

Theorem 2.8.3 *Let p and q are co-prime integers. The following holds:*[31]

(1) $T(p, q)$ is a trivial knot if and only if $p = \pm 1$ or $q = \pm 1$.
(2) Let $T(p, q)$ and $T(p', q')$ be non-trivial knots. $T(p, q)$ and $T(p', q')$ are equivalent if and only if (p', q') equals one of (p, q), (q, p), $(-p, -q)$ and $(-q, -p)$.
(3) A non-trivial torus knot $T(p, q)$ is chiral.

Let T be a standard torus in $\mathbb{R}^3$ and m and l be a meridian and a longitude as in Fig. 2.32. Let V be the solid torus in $\mathbb{R}^3$ bounded by T.

Let K_C be an oriented knot and $N(K_C)$ a regular neighborhood of K_C, and let m_C and l_C be a meridian and a preferred longitude on $\partial N(K_C)$. Let $f : V \to N(K_C)$ be a homeomorphism sending m to m_C and l to m_C.

For a knot K_P in the interior int V of V, the image $f(K_P)$ is a knot in $\mathbb{R}^3$ that is contained in int $N(K_C)$.

Suppose that K_C is a non-trivial knot, K_P is not ambient isotopic in V to the core of V, and m is not homotopic in $V \setminus K_P$ to a point (i.e., $[m] \neq 1 \in \pi_1(V \setminus K_P)$). Then we call the knot $f(K_P)$ a **satellite knot**, and call K_P and K_C the **pattern** and the **companion**, respectively (Fig. 2.34).

[31] Refer to A. Kawauchi [96].

Let K be a knot in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. When the complement $S^3 \setminus K$ admits a complete hyperbolic structure of finite volume, K is called a **hyperbolic knot**. The figure-eight knot is a hyperbolic knot. There exist many hyperbolic knots. In fact, the following is known.

Theorem 2.8.4 *A knot K is a hyperbolic knot if and only if K is neither a torus knot nor a satellite knot.*[32]

[32]It follows from Thurston's hyperbolization theorem. G. Perelman proved Thurston's geometrization conjecture, that implies Thurston's hyperbolization theorem and the Poincaré conjecture. Refer to J. Morgan and G. Tian [125, 126].

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