Chapter 2
Lax-Sato Formulation of the KP Hierarchy

Abstract In this chapter, we briefly review the Lax formulation of the KP hierarchy, which consists of an infinite set of linear equations whose compatibility conditions give rise to the flows corresponding to the KP hierarchy. The main purpose of this section is to highlight the basic framework of integrability underlying the KP theory. Then we show that the multi-component Burgers hierarchy discussed in the previous chapter appears as a finite reduction in the Sato theory. In particular, we emphasize the importance of the τ-function and explain the central role of the τ-function in the KP hierarchy. The materials discussed in this chapter can also be found in [27, 35, 37, 91, 100, 111–113, 132].

2.1 Lax Formulation of the KP Equation

Let L be a pseudo-differential operator of order one defined by

$$L = \partial + u_1 + u_2\partial^{-1} + u_3\partial^{-2} + \cdots ,$$

where the coefficients $u_i = u_i(\mathbf{t})$ depend on infinitely many variables $\mathbf{t} = (t_1, t_2, t_3, \ldots)$. The symbol ∂ is a differential operator whereas ∂^{-1} is a formal integration, satisfying $\partial\partial^{-1} = \partial^{-1}\partial = 1$, i.e. ∂^{-1} is a formal inverse of ∂. The operation of ∂^{ν} with $\nu \in \mathbb{Z}$ follows the generalized Leibnitz rule,

$$\partial^{\nu} f = \sum_{j\ge 0} \binom{\nu}{j} \partial_x^j(f)\partial^{\nu-j}.$$

For example, we have $\partial f = f_x + f\partial$ and $\partial^{-1} f = f\partial^{-1} - f_x\partial^{-2} + f_{xx}\partial^{-3} - \cdots$ (the latter expression follows from the usual formula of integration by parts). Here we define the *weights* for the functions u_i and ∂^{ν} as

$$\mathrm{wt}(u_i) = i, \qquad \mathrm{wt}(\partial^{\nu}) = \nu,$$

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such that the L has a homogeneous weight of one. We also remark that the term u_1 in L can be eliminated by a gauge transformation with a function g such that $u_1 = -g^{-1}\partial_x(g) = -g_x/g$,

$$L \longrightarrow g^{-1}Lg = \partial + \tilde{u}_2\partial^{-1} + \tilde{u}_3\partial^{-2} + \cdots .$$

This can be extended so that we can eliminate *all* u_j with an appropriate gauge operator W, which will be discussed in Sect. 2.2. We will henceforth consider L without the u_1 term,

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots . \tag{2.1}$$

The Lax form of the KP hierarchy is defined by the infinite set of nonlinear equations

$$\partial_{t_n}(L) = [B_n, L] \qquad \text{with} \quad B_n = (L^n)_{\geq 0} \quad n = 1, 2, \ldots, \tag{2.2}$$

where $(L^n)_{\geq 0}$ represents the polynomial part of L^n in ∂, i.e. B_n is a differential operator of order n, and $[B_n, L] := B_nL - LB_n$ is the usual commutator of operators. Since $[B_n, L] = [L^n - (L^n)_{<0}, L] = [L, (L^n)_{<0}]$, each side of the Eq. (2.2) is a pseudo-differential operator of order ≤ -1. Here $(L^n)_{<0}$ is the negative part of L^n in ∂, and note $[\partial, (L^n)_{<0}]$ has no polynomial part. Thus for $n > 1$, each equation in (2.2) is a consistent but infinite system of coupled $(1 + 1)$-evolution equations in t_n and x, for the variables $\{u_i : i = 2, 3, \ldots\}$. The case $n = 1$ yields the equations $\partial_{t_1}u_i = \partial_x u_i$, so we identify t_1 with x. The infinite system (2.2) is compatible, as prescribed by the following theorem:

Theorem 2.1 *The differential operators $B_n = (L^n)_{\geq 0}$ satisfy*

$$\partial_{t_m}(B_n) - \partial_{t_n}(B_m) + [B_n, B_m] = 0. \tag{2.3}$$

Consequently, the flows defined by (2.2) commute i.e., for any $n, m \geq 1$,

$$\partial_{t_n}(\partial_{t_m}(L)) = \partial_{t_m}(\partial_{t_n}(L)).$$

Proof It follows from (2.2) that $\partial_{t_m}(L^n) = [B_m, L^n]$. Then using the decomposition $L^n = B_n + (L^n)_{<0}$, we have

$$\begin{aligned}\partial_{t_m}(L^n) - \partial_{t_n}(L^m) &= [B_m, L^n] - [B_n, L^m] \\ &= [B_m, B_n] - [(L^m)_{<0},\ (L^n)_{<0}].\end{aligned}$$

The differential part (≥ 0) of the above equation gives (2.3).

To prove the commutability of the flows, we compute using (2.2) once again

$$\begin{aligned}\partial_{t_m}(\partial_{t_n}(L)) - \partial_{t_n}(\partial_{t_m}(L)) &= [\partial_{t_m}(B_n), L] + [B_n, \partial_{t_m}(L)] - [\partial_{t_n}(B_m), L] - [B_m, \partial_{t_n}(L)] \\ &= [\partial_{t_m}(B_n) - \partial_{t_n}(B_m), L] + [B_n, [B_m, L]] - [B_m, [B_n, L]] .\end{aligned}$$

Applying the Jacobi identity for commutators, the right hand side of the above equations becomes $[\partial_{t_m}(B_n) - \partial_{t_n}(B_m) + [B_n, B_m], L]$, which vanishes due to (2.3). □

Equation (2.3) are called the *Zakharov-Shabat* equations for the KP hierarchy. Note that given pair (n, m) with $n > m$, (2.3) gives a system of $n - 1$ equations for $u_2, u_3, \ldots, u_n$, in the variables t_m, t_n and x. For example, consider the case with $n = 3$ and $m = 2$, i.e. $B_2 = (L^2)_{\geq 0} = \partial^2 + 2u_2$ and $B_3 = (L^3)_{\geq 0} = \partial^3 + 3u_2 + 3(u_{2,x} + u_3)$. Then the Zakharov-Shabat equation (2.3) for B_2 and B_3 gives the system

$$\begin{cases} u_{2,t_2} = u_{2,xx} + 2u_{3,x} \\ 2u_{2,t_3} = 3(u_{2,x} + u_3)_{t_2} - (u_{2,xx} - 3u_{3,x} + 3u_2^2)_x \end{cases}$$

After setting $t_2 = y$, $t_3 = t$ and eliminating u_3 from the system, $u = 2u_2$ satisfies the KP equation (1.2).

We also remark that the KP hierarchy (2.2) is given by the compatibility of the linear system

$$\begin{cases} L\phi = k\phi\,, \\ \partial_{t_n}\phi = B_n\phi\,, \qquad n = 1, 2, \ldots\,, \end{cases} \tag{2.4}$$

where $k \in \mathbb{C}$, the spectral parameter, and the eigenfunction $\phi(\mathbf{t}; k)$ with $\mathbf{t} = (t_1, t_2, \ldots)$ will be referred to as the wave function of the KP hierarchy. The compatibility among the second set of equations gives the Zakharov-Shabat equations (2.3).

2.2 The Dressing Transformation

As we mentioned in the previous section, the Lax operator L can be gauge-transformed into the trivial operator ∂, i.e.

$$L \longrightarrow \partial = W^{-1} L W, \tag{2.5}$$

where the operator of the gauge transformation is defined by

$$W = 1 - w_1\partial^{-1} - w_2\partial^{-2} - w_3\partial^{-3} + \cdots\,. \tag{2.6}$$

The coefficient functions w_i are related to the coefficients u_j of L via the relation $LW = W\partial$ in (2.5), and we have

$$u_2 = w_{1,x}\,, \quad u_3 = w_{2,x} + w_1 w_{1,x}\,, \quad u_4 = w_{3,x} + (w_1 w_2)_x - w_{1,x}^2 + w_1^2 w_{1,x}\,, \quad \cdots$$
$$u_{j+1} = w_{j,x} + F_{j+1}(w_1, w_2, \ldots, w_{j-1})\,, \quad \cdots\,,$$

where F_{j+1} are differential polynomials of weight $j + 1$ (note $\mathrm{wt}(w_j) = j$). Thus w_j can be considered as primary variables whose x-derivatives determine the KP variables. The evolutions of w_j with respect to the time variables t_n can be prescribed in a consistent fashion by requiring that the gauge operator W satisfy the following system of equations:

$$\partial_{t_n}(W) = B_n W - W\partial^n \qquad \text{for} \quad n = 1, 2, \dots, \tag{2.7}$$

where B_n is now given by $B_n = (W\partial^n W^{-1})_{\geq 0}$. Notice that this expression for B_n as a differential operator is a consequence of the equations, $[\partial_{t_n}(W)W^{-1}]_{\geq 0} = 0$. Equation (2.7) is sometimes referred to as the Sato equation.

The following theorem asserts that the KP hierarchy is generated by the dressing of the trivial commutation relation $[\partial_{t_n} - \partial^n, \partial] = 0$ by the operator W. Because of this result, the (inverse) gauge transformation, $\partial \to L$, is called the *dressing* transformation for the KP hierarchy, and W is sometimes called the dressing operator.

Theorem 2.2 *If the operator W satisfies the Sato equation (2.7), then the operator $L = W\partial W^{-1}$ satisfies the Lax equation (2.2) for the KP hierarchy, and the operators $B_n = (W\partial^n W^{-1})_{\geq 0} = (L^n)_{\geq 0}$ satisfy the Zakharov-Shabat equations (2.3).*

Proof First, a direct calculation using $L = W\partial W^{-1}$ and $L^n = W\partial^n W^{-1} = B_n + (L^n)_{<0}$ shows that

$$\begin{aligned} W(\partial_{t_n} - \partial^n)W^{-1} &= \partial_{t_n} - \partial_{t_n}(W)W^{-1} - W\partial^n W^{-1} \\ &= \partial_{t_n} - (\partial_{t_n}(W) + W\partial^n)W^{-1} = \partial_{t_n} - B_n\,, \end{aligned}$$

where the last equality is due to (2.7). Then Eqs. (2.2) and (2.3) follow from the commutator relations

$$\begin{aligned} 0 &= W[\partial_{t_n} - \partial^n, \partial]W^{-1} = \partial_{t_n}(L) - [B_n, L]\,, \\ 0 &= W[\partial_{t_n} - \partial^n, \partial_{t_m} - \partial^m]W^{-1} = [\partial_{t_n} - B_n, \partial_{t_m} - B_m], \end{aligned}$$

which give the desired formulas. □

It follows from Theorem 2.2 that the flows defined by the Sato equation (2.7) are commutative, i.e. they satisfy the compatibility condition $\partial_{t_m}(\partial_{t_n}(W)) = \partial_{t_n}(\partial_{t_m}(W))$. Indeed, the compatibility condition for (2.7) is equivalent to $[\partial_{t_n} + (L^n)_{<0}, \partial_{t_m} + (L^m)_{<0}] = 0$. Using $L^n = B_n + (L^n)_{<0}$, the commutator term on the left hand side can be decomposed as $[\partial_{t_n} - B_n, \partial_{t_m} - B_m] + [\partial_{t_n} - B_n, L^m] - [\partial_{t_m} - B_m, L^n]$, which vanish due to Theorem 2.2. Finally we note that the KP linear system (2.4) is obtained by the dressing action: $\phi = W\phi_0$ where the (vacuum) wave function ϕ_0 satisfies the *bare* linear system

$$\begin{cases} \partial\phi_0 &= k\phi_0\,, \\ \partial_{t_n}\phi_0 = \partial^n\phi_0 = k^n\phi_0\,, & n = 1, 2, \dots. \end{cases} \tag{2.8}$$

This equation together with the Sato equation (2.7) forms the basic ingredients of the dressing transformation. We will use the vacuum wave function in the normalized form,

$$\phi_0(\mathbf{t}; k) = e^{\theta(\mathbf{t};k)} \qquad \text{with} \quad \theta(\mathbf{t}; k) = \sum_{n=1}^{\infty} k^n t_n. \tag{2.9}$$

2.3 Wave Function ϕ and the τ-Function

The τ-function introduced in Sect. 1.2 plays a fundamental role in the Sato theory of the KP hierarchy. In this section we demonstrate explicitly how the τ-function is related to the dressing operator W, which satisfies the Sato equation (2.7). We restrict our discussion to a finite truncation of the infinite order pseudo-differential operator W for simplicity only, while capturing the flavor of the general theory. Let us then consider the dressing operator for a finite N,

$$W = 1 - w_1\partial^{-1} - w_2\partial^{-2} - \cdots - w_N\partial^{-N} ,$$

and define the differential operator,

$$W_N := W\partial^N = \partial^N - w_1\partial^{N-1} - w_2\partial^{N-2} - \cdots - w_N .$$

The equation $W_N f = 0$ gives (1.9) in Sect. 1.2. Since W satisfies the Sato equation, the operator W_N also satisfies

$$\partial_{t_n}(W_N) = B_n W_N - W_N\partial^n \qquad \text{with} \qquad B_n = (W_N\partial^n W_N^{-1})_{\geq 0} \ .$$

The following proposition establishes the compatibility conditions leading to the multi-component Burgers hierarchy introduced in Sect. 1.2.

Proposition 2.1 *The N-th order differential equation $W_N f = 0$ is invariant under any flow of the linear heat hierarchy,* $\{\partial_{t_n} f = \partial_x^n f : n = 1, 2, \ldots\}$.

Proof It suffices to show that $\partial_{t_n}(W_N f) = 0$. A direct computation shows

$$\begin{aligned}\partial_{t_n}(W_N f) &= \partial_{t_n}(W_N) f + W_N\partial_{t_n} f \\ &= (B_n W_N - W_N\partial^n) f + W_N\partial_{t_n} f \\ &= B_n(W_N f) + W_N\left(\partial_{t_n} f - \partial_x^n f\right) = 0.\end{aligned}$$

Then the desired result follows from the uniqueness of the differential equation. $\square$

Proposition 2.1 provides the compatible system considered in Sect. 1.2,

$$\begin{cases} W_N f = 0, \\ \partial_{t_n} f = \partial_x^n f, \qquad n = 1, 2, \ldots . \end{cases} \tag{2.10}$$

Furthermore, a set $\{f_j : j = 1, 2, \ldots, N\}$ of linearly independent solutions of $W_N f = 0$ in (1.9) can be employed to explicitly construct the coefficient functions w_i of the dressing operator W in the form,

$$w_i = \frac{(-1)^{i+1}}{\tau} \begin{vmatrix} f_1 & \cdots & f_1^{(N-i-1)} & f_1^{(N-i+1)} & \cdots & f_1^{(N)} \\ f_2 & \cdots & f_2^{(N-i-1)} & f_2^{(N-i+1)} & \cdots & f_2^{(N)} \\ \vdots & & \vdots & \vdots & & \vdots \\ f_N & \cdots & f_N^{(N-i-1)} & f_N^{(N-i+1)} & \cdots & f_N^{(N)} \end{vmatrix}, \tag{2.11}$$

with $\tau = \mathrm{Wr}(f_1, \ldots, f_N)$ (see Sect. 1.2). Note that since the t_n-dependence of the w_i is given via the evolution equations $\partial_{t_n} f_j = \partial_x^n f_j$ for $j = 1, 2, \ldots, N$, one also has an explicit solution of the Sato equation (2.7).

The formula (2.11) of the coefficients w_i can be exploited to obtain an elegant expression for the wave function ϕ via the τ-function (see also e.g. [9, 37, 91, 132]).

Proposition 2.2 *The wave function ϕ of the linear system (2.4) can be expressed as*

$$\phi(\mathbf{t}; k) = \frac{\tau(\mathbf{t} - [k^{-1}])}{\tau(\mathbf{t})} \phi_0(\mathbf{t}; k), \tag{2.12}$$

where $\phi_0 = e^{\theta(\mathbf{t};k)}$ with $\theta(\mathbf{t}; k) = \sum_{n=1}^{\infty} k^n t_n$, and

$$\left(\mathbf{t} - [k^{-1}]\right) := \left(t_1 - \frac{1}{k},\ t_2 - \frac{1}{2k^2},\ t_3 - \frac{1}{3k^3}, \ldots\right).$$

Proof First note that using (2.11), the wave function ϕ can be written as

$$\begin{aligned} \phi = W_N \phi_0 &= \left(1 - \frac{w_1}{k} - \frac{w_2}{k^2} - \cdots - \frac{w_N}{k^N}\right) \phi_0 \\ &= \frac{1}{\tau} \begin{vmatrix} f_1 & f_1^{(1)} & \cdots & f_1^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ f_N & f_N^{(1)} & \cdots & f_N^{(N)} \\ k^{-N} & k^{-N+1} & \cdots & 1 \end{vmatrix}. \end{aligned}$$

Using the elementary column operations, the determinant in the numerator of the above expression can be re-written as

$$\frac{(-1)^N}{k^N}\left|\left(f_i^{(j)}-kf_i^{(j-1)}\right)_{1\le i,j\le N}\right| .$$

From the integral representation of the functions f_i,

$$f_i(\mathbf{t})=\int_C e^{\theta(\mathbf{t};\lambda)}\rho_i(\lambda)\,d\lambda \qquad \text{for} \quad i=1,2,\ldots,N\,,$$

each matrix element in this determinant is given by

$$\begin{aligned} f_i^{(j)}(\mathbf{t})-kf_i^{(j-1)}(\mathbf{t}) &= -k\int_C \lambda^{j-1}\left(1-\frac{\lambda}{k}\right)e^{\theta(\mathbf{t};\lambda)}\,\rho_i(\lambda)\,d\lambda \\ &= -k\int_C \lambda^{j-1}e^{-\sum\limits_{n=1}^{\infty}\frac{\lambda^n}{nk^n}}\,e^{\theta(\mathbf{t};\lambda)}\rho_i(\lambda)\,d\lambda \\ &= -ke^{-\sum\limits_{n=1}^{\infty}\frac{1}{nk^n}\partial_{t_n}}\,f_i^{(j-1)}(\mathbf{t})=-kf_i^{(j-1)}\left(\mathbf{t}-[k^{-1}]\right)\,, \end{aligned}$$

where we have used $\ln(1-\frac{\lambda}{k})=-\sum\limits_{n=1}^{\infty}\frac{\lambda^n}{nk^n}$. This completes the proof. □

Since the expression of ϕ in Proposition 2.2 does not explicitly depend on N, this formula holds in the general case of the full untruncated version of the operator W (see also [112, 113]). Expanding this formula with respect to k, we have an explicit formula for w_i in (2.11),

$$w_i=-\frac{1}{\tau}\,p_i(-\tilde{\partial})\tau, \tag{2.13}$$

where $\tilde{\partial}:=(\partial_{t_1},\frac{1}{2}\partial_{t_2},\frac{1}{3}\partial_{t_3},\ldots)$ and $p_n(\mathbf{z})$'s are the elementary Schur polynomials in (1.16) (see Problem 2.2). Note that for an N-truncated operator W_N (i.e. $w_n=0$ if $n>N$) the τ-function satisfies the constraints

$$p_n(-\tilde{\partial})\tau=0 \qquad \text{for} \quad n>N. \tag{2.14}$$

Proposition 2.2 will now be used to derive a set of first integrals of the KP hierarchy that will prove to be useful in our classification of the line-soliton solutions in Chap. 6. The integrability of the KP hierarchy may be demonstrated by the existence of an infinite number of conservation laws in the form,

$$\partial_{t_n}h_j=\partial_x g_{j,n}\,,$$

for some conserved densities h_j and the corresponding conserved fluxes $g_{j,n}$ for any $j, n \geq 1$. These functions are differential polynomials of u_i's in the Lax operator L, and they can be found as follows: Differentiating the quantity $\phi^{-1}\partial_x\phi$ with respect to t_n and using the evolution equation $\partial_{t_n}\phi = B_n\phi$, we first derive the conservation law,

$$\partial_{t_n}\left(\phi^{-1}\partial_x\phi\right) = \partial_x\left(\phi^{-1}B_n\phi\right). \tag{2.15}$$

Next we invert (2.1) using the generalized Leibnitz rule to express the differential operator ∂ in $\partial_x\phi$ in terms of L as

$$\partial = L - v_2L^{-1} - v_3L^{-2} - \cdots,$$

where v_j's are the *differential polynomials* of u_i's. Then the conserved density $\phi^{-1}\partial_x\phi$ can be written as an infinite series after using $L^n\phi = k^n\phi,\ n \in \mathbb{Z}$,

$$\phi^{-1}\partial_x\phi = k - \frac{v_2}{k} - \frac{v_3}{k^2} - \cdots.$$

Each function v_j is a conserved density of the KP hierarchy. The first few are given by

$$v_2 = u_2, \qquad v_3 = u_3, \qquad v_4 = u_4 + u_2^2, \qquad v_5 = u_5 - 3u_2u_3 + u_2u_{2,x}, \quad \cdots.$$

Note that these can be expressed simply as sums of the derivatives of $w_1 = \partial_x \ln\tau$. Namely, using (2.12), we have

$$\begin{aligned}\phi^{-1}\partial_x\phi &= \partial_x \ln\phi = \partial_x\left[\theta(\mathbf{t};k) + \ln\tau(\mathbf{t} - [k^{-1}]) - \ln\tau(\mathbf{t})\right]\\ &= k + \partial_x\left[\exp\left(-\sum_{n=1}^{\infty}\frac{1}{nk^n}\partial_{t_n}\right) - 1\right]\ln\tau(\mathbf{t}) = k + \sum_{n=1}^{\infty}\frac{1}{k^n}\,\partial_x p_n(-\tilde{\partial})\,\ln\tau,\end{aligned}$$

which leads to

$$v_{n+1} = -\partial_x p_n(-\tilde{\partial})\,\ln\tau = -p_n(-\tilde{\partial})\,w_1. \tag{2.16}$$

For example, we have

$$\begin{aligned}&v_2 = \partial_x w_1, \quad v_3 = \frac{1}{2}\left(\partial_{t_2} - \partial_x^2\right)w_1, \quad v_4 = \frac{1}{3}\left(\partial_{t_3} - \frac{3}{2}\partial_x\partial_{t_2} + \frac{1}{2}\partial_x^3\right)w_1, \cdots\\ &v_{n+1} = \frac{1}{n}\left(\partial_{t_n} + (\text{h.o.d.})\right)\,w_1 \qquad \cdots,\end{aligned}$$

where (h.o.d) indicates the terms including higher powers of the derivatives. Moreover, if the solutions u_i's of the KP hierarchy decrease rapidly to zero as $|x| \to \infty$, one can define the integrals C_n by

$$C_n := \int_{-\infty}^{\infty} v_{n+1}(x, \ldots)\, dx \qquad \text{for} \quad n = 1, 2, \ldots. \tag{2.17}$$

In particular, if the τ-function gives one line-soliton of $[i, j]$-type, i.e. $\tau(\mathbf{t}) = E_i(\mathbf{t}) + aE_j(\mathbf{t})$ with $E_i(\mathbf{t}) = \exp(\theta(\mathbf{t}; \kappa_i))$ for $\kappa_i < \kappa_j$, then from (2.16) the integrals are

$$C_n = -p_n(-\tilde{\partial}) \ln \tau \Big|_{x=-\infty}^{x=\infty} = \frac{1}{n} \left(\kappa_j^n - \kappa_i^n\right). \tag{2.18}$$

Notice here that since $\kappa_i < \kappa_j$, $\tau \approx E_i = e^{\theta(\mathbf{t};\kappa_i)}$ for $x \ll 0$ and $\tau \approx aE_j = ae^{\theta(\mathbf{t};\kappa_j)}$ for $x \gg 0$. In general, if there are N line-solitons of $[i_l, j_l]$-type for $l = 1, \ldots, N$, then we have

$$C_n = \sum_{l=1}^{N} \frac{1}{n} \left(k_{j_l}^n - k_{i_l}^n\right),$$

(see Chap. 6 for the proof). This expression is similar to the N-soliton solutions of the KdV equation (see e.g. [136]). Note in particular that for the KP equation, the C_n's are also independent of y. This fact will be used to prove a part of Theorem 6.1 below.

An alternative set of the conserved densities can be found by observing the following tautological equations (see p. 99 in [37]) in the form of the conservation laws,

$$\partial_{t_m}(\partial_x \partial_{t_n} \ln \tau) = \partial_x(\partial_{t_m} \partial_{t_n} \ln \tau). \tag{2.19}$$

That is, the conserved densities obtained from those equations are given by

$$\tilde{v}_{n+1} := \frac{1}{n} \partial_x \partial_{t_n} \ln \tau,$$

and one can show that the integrals $\tilde{C}_n = \int_{-\infty}^{\infty} \tilde{v}_{n+1}\, dx = \int_{-\infty}^{\infty} v_{n+1} = C_n$ for all n.

In Chap. 6, we will discuss the classification problem of the solutions obtained from the τ-function with finite dimensional solutions of the f-equation and will show that the classification is completely characterized by the asymptotic behavior of the τ-function.

2.4 Bilinear Identity of the τ-Function

There is a unified formulation for the KP hierarchy in terms of the τ-function developed in [35] called *bilinear identity* of the τ-function (later generalized in [128], see also Chap. 3). In this section, we briefly summarize this formula (see also [6, 7]).

Let us first recall the Lax-Sato formulation of the KP hierarchy,

$$L = W\partial W^{-1}, \qquad \partial_{t_n}(W) = B_n W - W\partial^n, \tag{2.20}$$

where $B_n = (W\partial^n W^{-1})_{\geq 0}$. The wave function $\phi = W\phi_0$ with $\phi_0 = e^\theta$ in (2.9) satisfies

$$\begin{cases} L\phi &= k\phi\,, \\ \partial_{t_n}\phi &= B_n\phi. \end{cases}$$

The wave function ϕ can be expressed in terms of the τ-function, i.e. (2.12) in Proposition 2.2.

We now define an adjoint system of the Lax pair, denoted by (L^*, B_n^*),

$$\begin{cases} L^*\phi^* = k\phi^*\,, \\ \partial_{t_n}\phi^* = -B_n^*\phi^*. \end{cases}$$

Here the (formal) adjoint operator is defined as follows:

- $(\partial^\nu)^* = (-1)^\nu\partial^\nu$ for $\nu \in \mathbb{Z}$,
- For the product of two pseud-differential operators A and B, $(AB)^* = B^*A^*$.

Then the adjoint wave function ϕ^* can be written in the form

$$\phi^*(\mathbf{t}; k) = (W^*)^{-1}e^{-\theta(\mathbf{t};k)}, \tag{2.21}$$

which is derived by taking the adjoint of (2.20). Then one can show the following main theorem [35, 111] (see also [37, 132]).

Theorem 2.3 *The pair $\{\phi(\mathbf{t}; k), \phi^*(\mathbf{t}'; k)\}$ for arbitrary $\mathbf{t}$ and $\mathbf{t}'$ satisfies the bilinear relation, referred to as the bilinear identity*

$$\oint_{C_\infty} \frac{dk}{2\pi i}\,\phi(\mathbf{t}; k)\phi^*(\mathbf{t}'; k) = 0,$$

where C_∞ is taken to be a large circle in $\mathbb{C}$.

Then one can express ϕ^* in terms of the τ-function (cf. (2.12)),

$$\phi^*(\mathbf{t}; k) = \frac{\tau(\mathbf{t} + [k^{-1}])}{\tau(\mathbf{t})}e^{-\theta(\mathbf{t};k)},$$

and the bilinear identity in Theorem 2.3 has the following form in terms of the τ-function,

$$\oint_{C_\infty} \frac{dk}{2\pi i}\,\tau(\mathbf{t} - [k^{-1}])\,\tau(\mathbf{t}' + [k^{-1}])\,e^{\theta(\mathbf{t}-\mathbf{t}',k)} = 0. \tag{2.22}$$

Calculating the residue of (2.22), we can derive the KP hierarchy in terms of the τ-function as follows: First set $\mathbf{t} \to \mathbf{t} - \mathbf{y}$ and $\mathbf{t}' \to \mathbf{t} + \mathbf{y}$. Then we have

$$
\begin{aligned}
0 &= \oint \frac{dk}{2\pi i}\,\tau(\mathbf{t}-\mathbf{y}-[k^{-1}])\tau(\mathbf{t}+\mathbf{y}+[k^{-1}])e^{-2\theta(\mathbf{y};k)} \\
&= \oint \frac{dk}{2\pi i}\left(\exp\left[\sum_{n=1}^{\infty}(y_n+\frac{1}{nk^n})D_n\right]\tau(\mathbf{t})\cdot\tau(\mathbf{t})\right)\sum_{l=0}^{\infty}p_l(-2\mathbf{y})k^l \\
&= \oint \frac{dk}{2\pi i}\left(\left[e^{\sum y_nD_n}\sum_{m=0}^{\infty}p_m(\tilde{D})k^{-m}\right]\tau(\mathbf{t})\cdot\tau(\mathbf{t})\right)\sum_{l=0}^{\infty}p_l(-2\mathbf{y})k^l \\
&= \oint \frac{dk}{2\pi i}\left(\sum_{m=0}\sum_{l=0}\frac{1}{k^{m-l}}p_l(-2\mathbf{y})p_m(\tilde{D})e^{\sum y_nD_n}\right)\tau(\mathbf{t})\cdot\tau(\mathbf{t}) \\
&= \sum_{l=0}^{\infty}p_l(-2\mathbf{y})p_{l+1}(\tilde{D})e^{\sum y_nD_n}\tau(\mathbf{t})\cdot\tau(\mathbf{t}).
\end{aligned}
$$

Here the operator D_n is the Hirota derivative defined by

$$
D_n^m f\cdot g = (\partial_{t_n}-\partial_{s_n})^m f(t_n)g(s_n)\Big|_{t_n=s_n},
$$

so that we have

$$
\exp(\alpha D_n)f(t_n)\cdot g(t_n) = f(t_n+\alpha)g(t_n-\alpha).
$$

We have also defined

$$
\tilde{D} := \left(D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \dots\right).
$$

Then expanding the bilinear equation in terms of $\mathbf{y}$, we can obtain a (Hirota) bilinear equation for the τ-function as the coefficient of each monomial $y_1^{i_1}\cdots y_k^{i_k}$. In particular, we obtain the following equation at the coefficient of y_k,

$$
\left[-2p_{k+1}(\tilde{D})+D_1D_k\right]\tau\cdot\tau = 0 \qquad \text{for} \qquad k=1,2,\dots. \tag{2.23}
$$

Here, we note that the equations for $k = 1, 2$ are trivial, and at $k = 3$, we have

$$
\begin{aligned}
0 &= \left[-2p_4(\tilde{D})+D_1D_3\right]\tau\cdot\tau \\
&= \left[-2\left(\frac{1}{4}D_4+\frac{1}{3}D_1D_3+\frac{1}{4}D_1^2D_2+\frac{1}{8}D_2^2+\frac{1}{24}D_1^4\right)+D_1D_3\right]\tau\cdot\tau \\
&= -\frac{1}{12}\left(-4D_1D_3+3D_2^2+D_1^4\right)\tau\cdot\tau\,.
\end{aligned}
$$

which is the KP equation in the τ-function form (1.12).

Remark 2.1 In [55], Hirota found a bilinear difference equation,

$$\left[Z_1 \exp(D_1) + Z_2 \exp(D_2) + Z_3 \exp(D_3)\right] \tau \cdot \tau = 0, \tag{2.24}$$

where Z_i are constants satisfying $Z_1 + Z_2 + Z_3 = 0$. By taking suitable limits, this equation gives several integrable systems including the KP equation [90]. Related to this bilinear equation, one can also construct other formulation of the τ-function (see e.g. [14, 51], also [122] for a further development).

2.5 Hirota Perturbation Method for N-Soliton Solutions

Here we briefly explain how one gets an N-soliton solution from the bilinear form using a *perturbation method* introduced by Hirota (see [56] for a comprehensive review on the method). The Hirota bilinear equation of the KP equation (1.12) is given by

$$P(D_x, D_y, D_t)\tau \cdot \tau := (-4D_xD_t + D_x^4 + 3D_y^2)\tau \cdot \tau = 0. \tag{2.25}$$

Note that the function $P(D_x, D_y, D_t)$ gives the soliton dispersion relation (1.6) (see also [97].),

$$P(K_{[i,j]}^x, K_{[i,j]}^y, \Omega_{[i,j]}) = -4K_{[i,j]}^x \Omega_{[i,j]} + (K_{[i,j]}^x)^4 + 3(K_{[i,j]}^y)^2 = 0.$$

The perturbation method assumes τ in the following N-term expansion with a parameter ε,

$$\tau = 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \cdots + \varepsilon^N g_N.$$

The first few terms in the coefficients of the powers of ε are given by

$$\begin{aligned}
&P(D_x, D_y, D_t)1 \cdot 1 = 0,\\
&P(D_x, D_y, D_t)1 \cdot g_1 = 0,\\
&P(D_x, D_y, D_t)g_1 \cdot g_1 + 2P(D_x, D_y, D_t)1 \cdot g_2 = 0,\\
&P(D_x, D_y, D_t)g_1 \cdot g_2 + P(D_x, D_y, D_t)1 \cdot g_3 = 0.
\end{aligned}$$

The first equation is trivially satisfied. The second equation implies

$$P(\partial_x, \partial_y, \partial_t)g_1 = -4\partial_x\partial_t g_1 + \partial_x^4 g_1 + 3\partial_y^2 g_1 = 0,$$

which is nothing but the linearized KP equation. The solution g_1 can be given by

$$g_1 = \iint \rho_1(p, q) e^{px+qy+\omega t}\, dp dq,$$

where (p, q, ω) satisfies the dispersion relation, $P(p, q, \omega) = -4p\omega + p^4 + 3q^2 = 0$.

For one-soliton solution ($N = 1$), we choose

$$g_1 = e^{px+qy+\omega t+c}.$$

Then one can easily show that

$$P(D_x, D_y, D_t)g_1 \cdot g_1 = 0.$$

This implies that $\tau = 1 + \varepsilon g_1$ (i.e. $g_n = 0$ for $n \geq 2$) is an exact solution, and the solution u of the KP equation is given by

$$u = 2\partial_x^2 \ln \tau = \frac{1}{2}p^2 \mathrm{sech}^2 \frac{1}{2}(px + qy + \omega t + c).$$

Note here that the dispersion relation (1.6) has a parametrization,

$$(p, q, \omega) = (\kappa_i - \kappa_j, \kappa_i^2 - \kappa_j^2, \kappa_i^3 - \kappa_j^3),$$

where κ_i and κ_j are arbitrary constants (cf. (1.4)).

For 2-soliton solution ($N = 2$), we take

$$g_1 = E_1 + E_2 \qquad \text{with} \qquad E_i := e^{p_i x + q_i y + \omega_i t + c_i},$$

where (p_i, q_i, ω_i) satisfies the dispersion relation for each $i = 1, 2$. Then the third equation at the order ε^2 requires

$$P(D_x, D_y, D_t)E_1 \cdot E_2 + P(D_x, D_y, D_t)1 \cdot g_2 = 0.$$

The solution g_2 can be found in the form

$$g_2 = A_{12}E_1E_2 \qquad \text{with} \qquad A_{12} = -\frac{P(p_1 - p_2, q_1 - q_2, \omega_1 - \omega_2)}{P(p_1 + p_2, q_1 + q_2, \omega_1 + \omega_2)}.$$

One can also show that

$$P(D_x, D_y, D_t)g_1 \cdot g_2 = A_{12}P(D_x, D_y, D_t)(E_1 + E_2) \cdot E_1E_2 = 0,$$
$$P(D_x, D_y, D_t)g_2 \cdot g_2 = A_{12}^2 P(D_x, D_y, D_t)E_1E_2 \cdot E_1E_2 = 0.$$

The solution τ is then given by

$$\tau = 1 + e^{\xi_1} + e^{\xi_2} + A_{12}e^{\xi_1+\xi_2} \qquad \text{with} \qquad \xi_i = p_i x + q_i y + \omega_i t + c_i',$$

where $c_i' = c_i + \ln \varepsilon$ are arbitrary constants.

Using the dispersion relation (1.6), one can set

$$
\begin{aligned}
p_1 &= \kappa_1 - \kappa_2, \qquad & q_1 &= \kappa_1^2 - \kappa_2^2, \qquad & \omega_1 &= \kappa_1^3 - \kappa_2^3, \\
p_2 &= \kappa_3 - \kappa_4, \qquad & q_2 &= \kappa_3^2 - \kappa_4^2, \qquad & \omega_2 &= \kappa_3^3 - \kappa_4^3,
\end{aligned}
$$

with arbitrary parameters $(\kappa_1, \ldots, \kappa_4)$. Then the τ-function can be written in the form

$$
\tau = 1 + e^{\xi_1} + e^{\xi_2} + A_{12} e^{\xi_1 + \xi_2} = \frac{1}{abE_{2,4}} \left(E_{1,3} + aE_{1,4} + bE_{2,3} + abE_{2,4} \right),
$$

where $E_{i,j} = (\kappa_j - \kappa_i) e^{\theta_i + \theta_j}$ with $\theta_i = \kappa_i x + \kappa_i^2 y + \kappa_i^3 t$ and $A_{12} = \frac{(\kappa_1 - \kappa_3)(\kappa_2 - \kappa_4)}{(\kappa_1 - \kappa_4)(\kappa_2 - \kappa_3)}$. The constants c_i's in ξ_i's are given by

$$
c_1' = -\ln \left(b \frac{\kappa_4 - \kappa_2}{\kappa_4 - \kappa_1} \right), \qquad c_2' = -\ln \left(a \frac{\kappa_4 - \kappa_2}{\kappa_3 - \kappa_2} \right),
$$

with arbitrary constants a and b. Since $u = 2\partial_x^2 \ln \tau$, the following function can be used for the same solution u,

$$
\tilde{\tau} = E_{1,3} + aE_{1,4} + bE_{2,3} + abE_{2,4},
$$

which is given by the Wronskian form for $N = 2$ and $M = 4$ with the 2×4 matrix

$$
A = \begin{pmatrix} 1 & b & 0 & 0 \\ 0 & 0 & 1 & a \end{pmatrix}.
$$

That is, the $\tilde{\tau}$-function is given by the Wronskian form,

$$
\tilde{\tau} = \begin{vmatrix} f_1 & \partial_x f_1 \\ f_2 & \partial_x f_2 \end{vmatrix} \qquad \text{with} \qquad (f_1, f_2) = (E_1, E_2, E_3, E_4)\, A^T,
$$

where $E_i = e^{\theta_i}$ and A^T is the transpose of the matrix A. Notice that this matrix A is *not* a generic form of the general 2×4 matrix. Thus the solutions obtained by Hirota's perturbation method gives only a *special* class of the KP solitons. Note also that the τ-function generated by the generic 2×4 matrix A contains *six* exponential terms. We will discuss this issue in Chap. 6 where we construct more general KP solitons and classify these solutions based on the Wronskian structure of the τ-function.

Problems

2.1 Let P_2 be the differential operator of order two given by

$$P_2 = \partial^2 + u,$$

where u is a function of multi-variables $\mathbf{t} = (t_1, t_2, \dots)$.

(a) Find the pseudo-differential operator $L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots$ in (2.1), so that $L^2 = P_2$. That is, find each u_k in terms of a differential polynomial of u.
(b) Define B_n as in (2.2), i.e. $B_n = (L^n)_{\geq 0}$. Then show that $u_{t_n} = 0$ if n is even, and the Lax equations $\partial_{t_{2k+1}}(L) = [B_{2k+1}, L]$ for $k = 1, 2, \dots$ give the KdV hierarchy. Note here that the Lax form of the KdV hierarchy can be written as a pair of differential operators (P, B_n), i.e. $\partial(P) = [B_n, P]$, which is the original formulation given in [78].
(c) Discuss the generalization of this procedure. That is, for a given differential operator of order N, $P_N = \partial^N + v_2\partial^{N-2} + v_3\partial^{N-2} + \cdots + v_N$, find L so that $L^N = P_N$, and define $B_n = (L^n)_{\geq 0}$. Then one can construct a hierarchy in the Lax form, $\partial_{t_n}(L) = [B_n, L]$, which is called the *Gel'fand-Dikey* hierarchy. Note that $[B_n, L] = 0$ if $n = kN$ for $k = 1, 2, \dots$ (see [37]).

2.2 Derive the formula (2.13) of the function w_i in the dressing operator W, and the formula (2.14) for the truncated operator W_N.

2.3 Let ϕ be a particular solution of (2.4), and let W be a dressing operator satisfying the Sato equation (2.7). Then prove the following.

Proposition 2.3 *Let $G = \partial - \phi^{-1}\phi_x$. Then the operator defined by*

$$\widetilde{W} = GW\partial^{-1} = 1 - \tilde{w}_1\partial^{-1} - \tilde{w}_2\partial^{-2} - \cdots$$

also satisfies the Sato equation (2.7), and the new coefficient functions $\tilde{w}_n$ are given by

$$\tilde{w}_1 = w_1 + \phi^{-1}\phi_x, \qquad \tilde{w}_n = w_n + \phi(\phi^{-1}w_{n-1})_x \quad \textit{for} \quad n > 1.$$

This defines the *Darboux* transformation for the KP hierarchy, and using this transformation, one can also find the KP solution in the Wronskian form (see [28, 84]).

2.4 Derive the formula ϕ^* in (2.21) and prove Theorem 2.3.

2.5 Identify the Plücker relation for the next member of the KP hierarchy,

$$\left[-2p_4(\tilde{D}) + D_1D_4\right]\tau\cdot\tau = 0.$$

2.6 First prove the following identity of the bilinear form,

$$p_l(\tilde{D})\tau \cdot \tau = \sum_{i+j=l} \left(p_i(\tilde{\partial})\tau\right) \left(p_j(-\tilde{\partial})\tau\right),$$

where p_l is the elementary Schur polynomial. Then show that the KP hierarchy (2.23) is just a Plücker relation for each k when we express the derivative of τ as $S_Y(\tilde{\partial})\tau = \tau_Y$ with the corresponding Young diagram Y.

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