Chapter 2
Core of a Sequence and the Matrix Class (ℓ, ℓ)

In this chapter, we present a study of the core of a sequence and properties of the matrix class (ℓ, ℓ). This chapter is divided into 4 sections. In the first section, we introduce the core of a sequence and study some of its properties. In the second section, we prove an improvement of Sherbakoff's result. This result leads to a short and elegant proof of Knopp's core theorem. The third section is devoted to a study of the matrix class (ℓ, ℓ) in the context of a convolution product *. In the final section, we prove a Mercerian theorem for the Banach algebra (ℓ, ℓ) under the convolution product *.

2.1 Core of a Sequence

The core of a complex sequence is defined as follows.

Definition 2.1 If $x = \{x_k\}$ is a complex sequence, we denote by $K_n(x)$, $n = 0, 1, 2, \ldots$, the smallest closed convex set containing $x_n, x_{n+1}, \ldots$.

$$\mathscr{K}(x) = \bigcap_{n=0}^{\infty} K_n(x)$$

is defined as the core of x.

It is known [1] that if $x = \{x_k\}$ is bounded,

$$\mathscr{K}(x) = \bigcap_{z \in \mathbb{C}} C_{\overline{\lim}_{n\to\infty} |z - x_n|}(z),$$

where $C_r(z)$ is the closed ball centered at z and radius r. Sherbakhoff [1] generalized the notion of the core of a bounded complex sequence by introducing the idea of the generalized α-core $\mathscr{K}^{(\alpha)}(x)$ of a bounded complex sequence as

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$$\mathscr{K}^{(\alpha)}(x) = \bigcap_{z \in \mathbb{C}} C_{\alpha \overline{\lim}_{n\to\infty} |z - x_n|}(z), \quad \alpha \geq 1. \tag{2.1}$$

When $\alpha = 1$, $\mathscr{K}^{(\alpha)}(x)$ reduces to the usual core $\mathscr{K}(x)$. Sherbakhoff [1] showed that under the condition

$$\overline{\lim_{n\to\infty}} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) = \alpha, \quad \alpha \geq 1, \tag{2.2}$$

$$\mathscr{K}(A(x)) \subseteq \mathscr{K}^{(\alpha)}(x).$$

Natarajan [2] improved Sherbakhoff's result by showing that his result works with the less stringent precise condition

$$\overline{\lim_{n\to\infty}} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) \leq \alpha, \quad \alpha \geq 1, \tag{2.3}$$

(2.3) being also necessary besides the regularity of A for

$$\mathscr{K}(A(x)) \subseteq \mathscr{K}^{(\alpha)}(x),$$

for any bounded complex sequence x. This result for the case $\alpha = 1$ yields a simple and very elegant proof of Knopp's core theorem (see, for instance, [3]).

2.2 Natarajan's Theorem and Knopp's Core Theorem

Natarajan's theorem is

Theorem 2.1 ([2, Theorem 2.1]) $A = (a_{nk})$ *is such that*

$$\mathscr{K}(A(x)) \subseteq \mathscr{K}^{(\alpha)}(x), \quad \alpha \geq 1,$$

for any bounded sequence x if and only if A is regular and satisfies (2.3),

$$i.e., \ \overline{\lim_{n\to\infty}} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) \leq \alpha, \quad \alpha \geq 1.$$

Proof Let $x = \{x_n\}$ be a bounded sequence. If $y \in \mathscr{K}(A(x))$, for any z,

$$|y - z| \leq \overline{\lim_{n\to\infty}} |z - (Ax)_n|.$$

If A is a regular matrix satisfying (2.3), then

$$
\begin{aligned}
|y - z| &\le \overline{\lim_{n\to\infty}} |z - (Ax)_n| \\
&= \overline{\lim_{n\to\infty}} \left| \sum_{k=0}^{\infty} a_{nk}(z - x_k) \right| \\
&\le \alpha \overline{\lim_{k\to\infty}} |z - x_k|,
\end{aligned}
$$

$$
i.e., \;\; y \in C_{\alpha \overline{\lim_{k\to\infty}} |z - x_k|}(z) \;\; \text{for any } z,
$$

which implies that

$$
\mathscr{K}(A(x)) \subseteq \mathscr{K}^{(\alpha)}(x).
$$

Conversely, let

$$
\mathscr{K}(A(x)) \subseteq \mathscr{K}^{(\alpha)}(x).
$$

Then, it is clear that A is regular by considering convergent sequences for which

$$
\mathscr{K}^{(\alpha)}(x) = \left\{ \lim_{n\to\infty} x_n \right\}.
$$

It remains to prove (2.3). Let, if possible,

$$
\overline{\lim_{n\to\infty}} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) > \alpha.
$$

Then,

$$
\overline{\lim_{n\to\infty}} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) = \alpha + h, \;\; \text{for some } h > 0.
$$

Using the hypothesis and the fact that A is regular, we can choose two strictly increasing sequences $\{n(i)\}$ and $\{k(n(i))\}$ of positive integers such that

$$
\sum_{k=0}^{k(n(i-1))} |a_{n(i),k}| < \frac{h}{8},
$$

$$
\sum_{k=k(n(i-1))+1}^{k(n(i))} |a_{n(i),k}| > \alpha + \frac{h}{4}
$$

and

$$
\sum_{k=k(n(i))+1}^{\infty} |a_{n(i),k}| < \frac{h}{8}.
$$

Define the sequence $x = \{x_k\}$ by

$$x_k = sgn(a_{n(i),k}),\; k(n(i-1)) \le k < k(n(i)), i = 1, 2, \ldots.$$

Now,

$$\begin{aligned}
|(Ax)_{n(i)}| &\ge \sum_{k=k(n(i-1))+1}^{k(n(i))} |a_{n(i),k}| - \sum_{k=0}^{k(n(i-1))} |a_{n(i),k}| \\
&\quad - \sum_{k=k(n(i))+1}^{\infty} |a_{n(i),k}| \\
&> \alpha + \frac{h}{4} - \frac{h}{8} - \frac{h}{8} \\
&= \alpha, \quad i = 1, 2, \ldots.
\end{aligned} \tag{2.4}$$

By the regularity of A, $\{(Ax)_{n(i)}\}_{i=1}^{\infty}$ is a bounded sequence. It has a convergent subsequence whose limit cannot be in $C_\alpha(0)$, in view of (2.4). Using (2.1), we have, $\mathscr{K}^{(\alpha)}(x) \subseteq C_\alpha(0)$ for the sequence x chosen above. This leads to a contradiction of the fact that $\mathscr{K}(A(x)) \subseteq \mathcal{K}^{(\alpha)}(x)$, completing the proof of the theorem. □

Remark 2.1 Condition (2.3) cannot be relaxed if $\mathcal{K}(A(x))$ were to be contained in $\mathcal{K}^{(\alpha)}(x)$. The infinite matrix

$$\begin{pmatrix} 1 & \lambda & -\lambda & 0 & 0 & 0 & \cdots \\ 0 & 1 & \lambda & -\lambda & 0 & 0 & \cdots \\ 0 & 0 & 1 & \lambda & -\lambda & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where $|\lambda| > \alpha$, transforms the sequence $\{1, 0, 1, 0, \ldots\}$ into the sequence $\{\lambda, 1 - \lambda, \lambda, 1 - \lambda, \ldots\}$. $\mathcal{K}^{(\alpha)}(x) \subset C_\alpha(0)$ while $\lambda \in \mathscr{K}(A(x))$ and $\lambda \notin C_\alpha(0)$.

Remark 2.2 For a regular matrix $A = (a_{nk})$, note that

$$\overline{\lim_{n\to\infty}} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) \le 1$$

is equivalent to

$$\overline{\lim_{n\to\infty}} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) = 1.$$

Remark 2.3 The proof of Theorem 2.1, for the case $\alpha = 1$, yields a very simple and elegant proof of Knopp's core theorem. This proof is much simpler than the proofs of Knopp's core theorem known earlier (for instance, see [3, p.149]).

2.3 Some Results for the Matrix Class (ℓ, ℓ)

We recall that

$$\ell = \left\{ x = \{x_k\} : \sum_{k=0}^{\infty} |x_k| < \infty \right\}.$$

Note that ℓ is a linear space with respect to coordinatewise addition and scalar multiplication and it is a Banach space with respect to the norm defined by

$$\|x\| = \sum_{k=0}^{\infty} |x_k|, \quad x = \{x_k\} \in \ell.$$

$(\ell, \ell; P)$ denotes the set of all infinite matrices $A = (a_{nk}) \in (\ell, \ell)$ such that

$$\sum_{n=0}^{\infty} (Ax)_n = \sum_{k=0}^{\infty} x_k, \quad x = \{x_k\} \in \ell.$$

We recall the following results (see [4–6])

Theorem 2.2 $A = (a_{nk}) \in (\ell, \ell)$ *if and only if*

$$\sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |a_{nk}| \right) < \infty. \tag{2.5}$$

Further, $A \in (\ell, \ell; P)$ *if and only if* $A \in (\ell, \ell)$ *and (1.29) holds.*

Theorem 2.3 *The matrix class* (ℓ, ℓ) *is a Banach algebra under the norm*

$$\|A\| = \sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |a_{nk}| \right), \quad A = (a_{nk}) \in (\ell, \ell), \tag{2.6}$$

with the usual matrix addition, scalar multiplication, and multiplication.

We shall now prove a few results for the matrix class (ℓ, ℓ) (see [7]).

Theorem 2.4 *The class* $(\ell, \ell; P)$*, as a subset of* (ℓ, ℓ)*, is a closed convex semigroup with identity, the multiplication being the usual matrix multiplication.*

Proof Let $A = (a_{nk})$, $B = (b_{nk}) \in (\ell, \ell; P)$ and $\lambda + \mu = 1$, λ, μ being nonnegative real numbers. Then, there exists $M > 0$ such that

$$\sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |a_{nk}| \right), \sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |b_{nk}| \right) \leq M.$$

Now,

$$\begin{aligned}\sup_{k\geq 0}\sum_{n=0}^{\infty}|\lambda a_{nk}+\mu b_{nk}| &\leq \lambda \sup_{k\geq 0}\left(\sum_{n=0}^{\infty}|a_{nk}|\right)+\mu \sup_{k\geq 0}\left(\sum_{n=0}^{\infty}|b_{nk}|\right)\\ &\leq (\lambda+\mu)M\\ &= M,\ \text{since } \lambda+\mu=1.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{n=0}^{\infty}(\lambda a_{nk}+\mu b_{nk}) &= \lambda\left(\sum_{n=0}^{\infty}a_{nk}\right)+\mu\left(\sum_{n=0}^{\infty}b_{nk}\right)\\ &= \lambda(1)+\mu(1)\\ &= \lambda+\mu\\ &= 1,\ k=0,1,2,\dots,\end{aligned}$$

since $\sum_{n=0}^{\infty} a_{nk} = \sum_{n=0}^{\infty} b_{nk} = 1, k = 0, 1, 2, \dots,$ using (1.29). In view of Theorem 2.2, $\lambda A + \mu B \in (\ell, \ell; P)$ so that $(\ell, \ell; P)$ is a convex subset of (ℓ, ℓ).

Let, now, $A = (a_{nk}) \in \overline{(\ell, \ell; P)}$. Then, there exist $A^{(m)} = (a_{nk}^{(m)}), m = 0, 1, 2, \dots$ such that

$$\|A^{(m)} - A\| \to 0, m \to \infty.$$

Thus, given $\epsilon > 0$, there exists a positive integer N such that

$$\|A^{(m)} - A\| < \infty, m \geq N,$$

$$i.e., \sup_{k\geq 0}\left(\sum_{n=0}^{\infty}|a_{nk}^{(m)} - a_{nk}|\right) < \epsilon, m \geq N. \tag{2.7}$$

Now,

$$\begin{aligned}\sup_{k\geq 0}\left(\sum_{n=0}^{\infty}|a_{nk}|\right) &\leq \sup_{k\geq 0}\left(\sum_{n=0}^{\infty}|a_{nk}-a_{nk}^{(N)}|\right)+\sup_{k\geq 0}\left(\sum_{n=0}^{\infty}|a_{nk}^{(N)}|\right)\\ &< \epsilon+\sup_{k\geq 0}\left(\sum_{n=0}^{\infty}|a_{nk}^{(N)}|\right),\ \text{using (2.7)}\\ &< \infty,\end{aligned}$$

since $A^{(N)} \in (\ell, \ell; P)$, so that $A \in (\ell, \ell)$. Again,

$$
\begin{aligned}
\left|\sum_{n=0}^{\infty} a_{nk} - 1\right| &= \left|\sum_{n=0}^{\infty} a_{nk} - \sum_{n=0}^{\infty} a_{nk}^{(N)}\right|, \text{ since } A^{(N)} \in (\ell, \ell; P) \\
&= \left|\sum_{n=0}^{\infty} (a_{nk} - a_{nk}^{(N)})\right| \\
&\leq \sum_{n=0}^{\infty} |a_{nk} - a_{nk}^{(N)}| \\
&\leq \sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |a_{nk} - a_{nk}^{(N)}|\right) \\
&< \epsilon, \ k = 0, 1, 2, \ldots, \text{ in view of (2.7)}
\end{aligned}
$$

so that

$$
\left|\sum_{n=0}^{\infty} a_{nk} - 1\right| < \epsilon \text{ for all } \epsilon > 0.
$$

Consequently,

$$
\sum_{n=0}^{\infty} a_{nk} = 1, \ k = 0, 1, 2, \ldots.
$$

Thus, $A \in (\ell, \ell; P)$ and so $(\ell, \ell; P)$ is a closed subset of (ℓ, ℓ).

It is clear that the unit matrix is in $(\ell, \ell; P)$, and it is the identity element of $(\ell, \ell; P)$.

To complete the proof, it suffices to check closure under matrix multiplication. If $A = (a_{nk})$, $B = (b_{nk}) \in (\ell, \ell; P)$, using Theorem 2.3, $AB \in (\ell, \ell)$. In fact, $AB \in (\ell, \ell; P)$, since

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_{nk} &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} a_{ni} b_{ik}\right) \\
&= \sum_{i=0}^{\infty} b_{ik} \left(\sum_{n=0}^{\infty} a_{ni}\right) \\
&= \sum_{i=0}^{\infty} b_{ik}, \text{ since } \sum_{n=0}^{\infty} a_{ni} = 1, i = 0, 1, 2, \ldots \\
&= 1, \text{ since } \sum_{i=0}^{\infty} b_{ik} = 1, k = 0, 1, 2, \ldots.
\end{aligned}
$$

This completes the proof of the theorem. □

Remark 2.4 $(\ell, \ell; P)$ is not an algebra since the sum of two elements of $(\ell, \ell; P)$ is not in $(\ell, \ell; P)$.

We now introduce a convolution product (see [5]).

Definition 2.2 For $A = (a_{nk})$, $B = (b_{nk})$, define

$$(A * B)_{nk} = \sum_{i=0}^{n} a_{ik} b_{n-i,k}, \quad n, k = 0, 1, 2, \ldots . \tag{2.8}$$

$A * B = ((A * B)_{nk})$ is called the convolution product of A and B.

We keep the usual norm structure in (ℓ, ℓ) as defined by (2.6) and replace matrix product by the convolution product as defined by (2.8) and prove the following result.

Theorem 2.5 *(ℓ, ℓ) is a commutative Banach algebra, with identity, under the convolution product * as defined by (2.8). Furthermore, $(\ell, \ell; P)$, as a subset of (ℓ, ℓ), is a closed convex semigroup with identity.*

Proof Recall that it was proved in Theorem 2.4 that $(\ell, \ell; P)$ is a convex subset of (ℓ, ℓ). We will first prove closure under the convolution product *. Let $A = (a_{nk})$, $B = (b_{nk}) \in (\ell, \ell)$, and $A * B = (c_{nk})$. Then,

$$\begin{aligned}
\sum_{n=0}^{\infty} |c_{nk}| &= \sum_{n=0}^{\infty} \left| \sum_{i=0}^{n} a_{ik} b_{n-i,k} \right| \\
&\leq \sum_{n=0}^{\infty} \sum_{i=0}^{n} |a_{ik}||b_{n-i,k}| \\
&= \left(\sum_{n=0}^{\infty} |a_{nk}| \right) \left(\sum_{n=0}^{\infty} |b_{nk}| \right) \\
&\leq \sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |a_{nk}| \right) \sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |b_{nk}| \right) \\
&= \|A\| \, \|B\|, \ k = 0, 1, 2, \ldots,
\end{aligned}$$

so that

$$\sup_{k \geq 0} \left(\sum_{n=0}^{\infty} |c_{nk}| \right) < \infty$$

and so $A * B \in (\ell, \ell)$. Also,

$$\|A * B\| \leq \|A\| \, \|B\|.$$

It is clear that $A * B = B * A$. The identity element is the matrix $E = (e_{nk})$, whose first row consists of 1's and which has 0's elsewhere,

$$\begin{aligned} i.e., \; e_{0k} &= 1, \; k = 0, 1, 2, \ldots; \\ e_{nk} &= 0, \; n = 1, 2, \ldots; k = 0, 1, 2, \ldots. \end{aligned}$$

We note that $E \in (\ell, \ell; P)$ and $\|E\| = 1$. It now suffices to prove that $(\ell, \ell; P)$ is closed under the convolution product *. Now,

$$\begin{aligned} \sum_{n=0}^{\infty} c_{nk} &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_{ik} b_{n-i,k} \right) \\ &= \left(\sum_{n=0}^{\infty} a_{nk} \right) \left(\sum_{n=0}^{\infty} b_{nk} \right) \\ &= 1, \; k = 0, 1, 2, \ldots, \end{aligned}$$

where $A, B \in (\ell, \ell; P)$. This completes the proof of the theorem. □

2.4 A Mercerian Theorem

We close the present chapter by proving a Mercerian theorem for the Banach algebra (ℓ, ℓ) under the convolution product *.

Theorem 2.6 *If*

$$y_n = x_n + \lambda(c^n x_0 + c^{n-1} x_1 + \cdots + c x_{n-1} + x_n),$$

$|c| < 1$ *and if* $\{y_n\} \in \ell$*, then* $\{x_n\} \in \ell$*, provided*

$$|\lambda| < 1 - c.$$

Proof Since (ℓ, ℓ) is a Banach algebra under the convolution product *, if $|\lambda| < \frac{1}{\|A\|}$, $A \in (\ell, \ell)$, then $E - \lambda A$, where E is the identity element of (ℓ, ℓ) under *, has an inverse in (ℓ, ℓ). We recall that

$$E = (e_{nk}) = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

We note that the equations

$$y_n = x_n + \lambda(c^n x_0 + c^{n-1} x_1 + \cdots + c x_{n-1} + x_n), \; |c| < 1, n = 0, 1, 2, \ldots$$

can be written in the form

$$(E + \lambda A) * x' = y',$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ c & 0 & 0 & \cdots \\ c^2 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$x' = \begin{pmatrix} x_0 & 0 & 0 & \cdots \\ x_1 & 0 & 0 & \cdots \\ x_2 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$y' = \begin{pmatrix} y_0 & 0 & 0 & \cdots \\ y_1 & 0 & 0 & \cdots \\ y_2 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

It is clear that $A \in (\ell, \ell)$ with $\|A\| = \frac{1}{1-c}$. So, if $|\lambda| < 1 - c$, $(E + \lambda A)$ has an inverse in (ℓ, ℓ). Consequently, it follows that

$$x' = (E + \lambda A)^{-1} * y'.$$

Since $y' \in (\ell, \ell)$ and $(E + \lambda A)^{-1} \in (\ell, \ell)$, we have, $x' \in (\ell, \ell)$. In view of Theorem 2.2, it follows that $\{x_n\} \in \ell$, completing the proof of the theorem. $\square$

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