The Non-symmetric L-Nash Bargaining Solution

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Dedicated to Ferenc Szidarovszky for his academic and research achievements in the last 50 years.

Abstract It is demonstrated how the concept of the Limit-Nash bargaining solution as defined in Forgó and Szidarovszky (Eur J Oper Res 147:108–116, 2003) can be carried over to the non-symmetric case. It is studied how externally given weights of the players and the relative magnitude of penalties for not being able to come to an agreement influence the solution.

1 Introduction

The Nash bargaining solution as introduced by Nash (1950) is a fundamental concept in game theory and conflict resolution. In its most simple form it is about two players trying to come to an agreement on choosing an element from a given set of feasible outcomes. The outcomes are evaluated according to the individual utility functions of the players. Normally, this leads to a conflict. To resolve the conflict, mutual concessions have to be made, otherwise a bad outcome (disagreement outcome) will realize where both players are penalized for not having been able to agree. Nash approached the problem from two directions. One is the axiomatic approach, Nash (1950) where reasonable properties (axioms) are required of a solution to hold. Nash showed that his axioms uniquely determine what is now called the Nash bargaining solution. The other, Nash (1953), aims at devising a suitable bargaining process which realizes in subgame perfect equilibrium the same outcome that the axiomatic approach prescribes. This dual approach was later termed the "Nash program" see e.g. Thomson (1994), Serrano (2005).

Since the Nash bargaining solution depends on both the feasible set of outcomes and the disagreement point, it is a valid question to ask how it behaves if either of

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them changes in some way. Forgó and Szidarovszky (2003) showed that if the disagreement point goes to negative infinity in a given direction, then the Nash bargaining solution converges to a unique outcome which they termed L-Nash bargaining solution (L stands for limit). It was also shown that the L-Nash bargaining solution can also be obtained as a solution of a multiple criteria decision making problem with weights that are the reciprocals of the components of the disagreement direction. The L-Nash solution can also be axiomatized within the context of multiple criteria decision making.

One way of generalizing Nash's bargaining model is to allow assigning weights to the players meant to indicate their "importance" or "power" in the conflict. This is outside information (just as the disagreement point), and critically influences the final outcome. Several axiomatizations of the non-symmetric Nash bargaining solution have emerged throughout the years e.g. Harsanyi and Selten (1972), Kalai (1997), Roth (1979), Anbarci and Sun (2013) as well as non-cooperative bargaining models that implement it, e.g. Kalai (1997), Laruelle and Valenciano (2008), Britz et al. (2010), Anbarci and Sun (2013).

In this paper it is demonstrated how the non-symmetric Nash bargaining solution behaves when the disagreement point goes to negative infinity in a fixed direction. It turns out that in certain cases the two pieces of outside information, the power of the players and the disagreement direction can be treated as one, while in other instances they cannot.

2 Preliminaries

We consider two-person bargaining games. Let $B(F, d, p, q)$ be a two-person non-symmetric bargaining problem with convex, compact feasible set $F \subset \mathbb{R}^2$ which is assumed to have at least one positive element, d is a non-positive disagreement point, and the positive integers p, q represent the "power" of the players. The game is played as follows. If both players agree, then they choose a feasible point $f \in F, f = (f_1, f_2)$ in which the players get the components of f accordingly. If they cannot come to an agreement, then a usually "bad" disagreement point $d = (d_1, d_2)$ realizes.

Consider the following constrained maximization problem

$$P: \quad \max\, (x_1 - d_1)^p\, (x_2 - d_2)^q$$

$$x \in F$$

where p, q are positive and $p + q = 2$.

This problem has a unique solution $\varphi \in F$ which is called the non-symmetric (asymmetric) Nash bargaining solution (*NSNBS*). In the special case $p = q = 1$ we have the classical, symmetric bargaining solution (*NBS*) of Nash (1950) which is uniquely determined by a set of axioms (feasibility, rationality, Pareto-optimality, independence of irrelevant alternatives, scale independence and symmetry). Among the several axiomatizations of the *NSNBS* we only quote Roth's (1979). From among the Nash axioms Roth changed Pareto optimality and symmetry to the following (Roth's axiom).

Consider a bargaining problem $B(G, 0, p, q)$, where the feasible set is defined by

$$G = \{g = (g_1, g_2) \in G, g_1 + g_2 \leq 2\}.$$

It is required by Roth's axiom that the solution of $B(G, 0, p, q)$ be (p, q). Then the unique solution of $B(F, d, p, q)$ is the *NSNBS*.

3 The Main Result

Parametrize $B(F, d, p, q)$ by taking $d = -\alpha r$, where $r > 0$ is the so-called disagreement direction and the positive parameter α represents how far we push the disagreement point in the direction $-r$. Forgó and Szidarovszky (2003) introduced the L-Nash bargaining solution as the limit of the Nash bargaining solution as $\alpha \to \infty$. Several interesting issues were addressed in Forgó and Szidarovszky (2003) concerning the behavior of the L-Nash solution. It is a natural question to ask: what happens if we consider non-symmetric bargaining problems and approach negative infinity with the disagreement point in a given direction?

Let $B(F, -\alpha r, p, q)$ be a two-person bargaining problem with positive parameter α. The parametrized two-person non-symmetric Nash bargaining solution $NSNBS(\alpha)$ is the unique solution of the maximization problem

$$P(\alpha): \quad \max\, (x_1 + \alpha r_1)^p\, (x_2 + \alpha r_2)^q$$

$$x \in F.$$

It is not a significant loss of generality if we confine ourselves to rational weights which amounts to allowing p and q to be positive integers.

For any given x and r, the objective function of $P(\alpha)$ is a polynomial of order $p + q$ of the parameter α. The coefficient of the leading term α^{p+q} is $r_1^p r_2^q$, independent of x thus having no role in the maximization of the objective function of $P(\alpha)$. The coefficient $h(x)$ of α^{p+q-1}, however, does depend on x. In particular, as can be shown by the application of the binomial formula

$$h(x) = p r_1^{p-1} r_2^q x_1 + q r_1^p r_2^{q-1} x_2,$$

or equivalently

$$h(x) = r_1^p r_2^q (\frac{p}{r_1} x_1 + \frac{q}{r_2} x_2).$$

If α is large enough, then the linear function $h(x)$ should be as large as possible in order to maximize $P(\alpha)$. If

$$\begin{aligned} &\max h(x) \\ &x \in F \end{aligned} \tag{1}$$

has a unique solution, then terms with degree less then $p + q - 1$ do not count if α is large enough. If the above maximization problem has multiple solutions, then the coefficient $g(x)$ of the term α^{p+q-2} comes into play. In particular, $g(x)$ should be maximized over the optimal set of (1) i.e. the following maximization problem should be solved

$$\begin{aligned} &\max g(x) \\ h(x) = &\max_{x \in F} h(x) \\ x \in &F. \end{aligned} \tag{2}$$

Again, by using the binomial formula, it can easily be seen that

$$g(x) = \frac{p(p-1)}{2} r_1^{p-2} r_2^q x_1^2 + pq r_1^{p-1} r_2^{q-1} + \frac{q(q-1)}{2} r_1^p r_2^{q-2} x_2^2.$$

Define

$$f(x) = r_1^{2p} r_2^{2q} (\frac{p}{r_1^2} x_1^2 + \frac{q}{r_2^2} x_2^2).$$

Then, with simple algebra one can verify that

$$g(x) = \frac{1}{2 r_1^p r_2^q} ((h(x))^2 - f(x)).$$

This means that problem (2) is equivalent to

$$\begin{aligned} &\min f(x) \\ h(x) = &\max_{x \in F} h(x) \\ x \in &F \end{aligned} \tag{3}$$

whose objective function is a strictly convex quadratic function, the feasible set is convex, compact implying that problem (3) has a unique optimal solution x^2.

Define now $x^0 = x^1$ if problem (1) has the unique optimal solution x^1, and $x^0 = x^2$ otherwise.

We can now state the main result.

Theorem 1 *The NSNBS of the two-person nonsymmetric bargaining problem $B(F, -\alpha r, p, q)$ converges to x^0 if $\alpha \to \infty$.*

Proof Along the lines of Theorem 1 in Forgó and Szidarovszky (2003).

x^0 can rightly be called the non-symmetric limit-Nash bargaining solution, *L-NSNBS*. For polyhedral feasible sets there is no need to go to infinity with α to obtain the *L-NSNBS*.

Theorem 2 *If F is a polytope, then there is an α_0 such that for all $\alpha \geq \alpha_0$, the two-person NSNBS coincides with the L-NSNBS.*

Proof Along the lines of Theorem 2 in Forgó and Szidarovszky (2003).

4 Example

Let us consider a very simple example of firm-union bargaining over wage and employment as in McDonald and Solow (1981). The firm has a profit function (revenue less labor cost) $R(L) - wL$, where w denotes wage per worker and L denotes the number of employed workers. The union's utility function is given by $L[U(w) - U(w')]$, where w' denotes benefits if worker is unemployed, and U is each union member's utility function. Bargaining takes place in the region constrained by the bounds $0 \leq w \leq W$, $0 \leq L \leq N$. The utility function of the union (total wage) is increasing in both arguments and in order to have a conflict, the profit function of the firm should be decreasing in w and L. For the model to be meaningful we assume that wage is at least as high as the marginal revenue of labor, $w \geq R'(L)$. Bargaining power of the two parties are p and q and we suppose that $p < q$ (union is less powerful than the firm). On the other hand, the firm is more vulnerable to the failure of negotiations, i.e. $r_1 < r_2$.

We will determine the *L-NSNBS* for specific values of the parameters and specific forms of the functions involved. In particular, let

$$
\begin{aligned}
U(w) - U(w') &= w \\
R(L) &= 320L - 10L^2 \\
W &= 400, N = 200 \\
p &= 2, q = 3 \\
r_1 &= 1, r_2 = 3.
\end{aligned}
$$

Then, to determine the *L-NSNBS*, problem (1) has first to be solved, which takes now the form

$$\max 2wL + \frac{3}{2}(320L - L^2 - wL)$$
$$320 - 20L \leq w \leq 400$$
$$0 \leq L \leq 10.$$

Notice that the objective of the above problem is linear in the utilities of the parties but nonlinear (quadratic) in the original decision variables w, L. The solution is in favor of the union:

$L = 10, w = 400$, full employment and highest possible wages.

5 Discussion

Consider the case when problem (1) has a unique solution. As pointed out and also observed in the context of this paper, in this case the *L-NSNBS* is the solution of a multi-criteria decision problem (*MCDP*) by the method of linear weighting where the weights are represented by the coefficients in the linear objective function of problem (1). In the symmetric case, the additional information is supplied by the relative magnitude of the components in the disagreement direction. The less the first player is hurt relative to the other (r_1 is small) by disagreement getting more costly, the more weight her interest carries through the large coefficient $\frac{1}{r_1}$in the objective function of problem (1). In the non-symmetric case there seem to be two reference points (outside information indicating the weight or importance of the players). One is the direct weights p, q, the other is the relative costs of disagreement r_1, r_2. Our analysis reveals, however, that when combining these together and using only the disagreement directions $\frac{r_1}{p}, \frac{r_2}{q}$ in the symmetric bargaining model, we get the same limiting solution.

This is not the case when problem (1) has multiple optima. Then the coefficients of the quadratic terms in the objective function of problem (3) are $\frac{p}{r_1^2}$ and $\frac{q}{r_2^2}$ while in the corresponding symmetric bargaining model they would be $\frac{p^2}{r_1^2}$ and $\frac{q^2}{r_2^2}$ giving rise to different solutions. It should also be noticed that if F is a polyhedron, then problem (1) is a linear programming problem and multiple optima are unlikely to occur in unstructured problems. Nevertheless, theoretically, *L-NSNBS* is determined by two reference points.

There is, however, a significant difference between the "importance indicators": the direct weights and the disagreement direction. Direct weights do not explicitly require interpersonal comparison of utilities since they are completely external to the model. As we have interpreted the components of the disagreement direction vector as an expression of the relative damage caused by prolonged negotiations, they implicitly mean comparison in utility (damage interpreted as disutility). Comparison of power is less closely related to utilities. It is therefore somewhat of a surprise, that these two things merge together in the *L-NSNBS*.

The whole analysis can be done for an arbitrary number of players and results remain the same if adjustments are made accordingly. We confined ourselves to two players in order to keep technicalities within reasonable bounds and because the two-player case is of interest in its own.

Forgó and Fülöp (2008) showed how the L-Nash solution can be implemented by proper adjustment of Rubinstein's alternating offer bargaining scenario, Rubinstein (1982) either exactly or asymptotically depending on F, r, and exactly by Howard's scheme, Howard (1992) for any F, r. There does not seem any special difficulty to extend their results to the non-symmetric case if the weights p, q are externally given. Howard's implementation makes it possible to internalize not only the penalty parameter α but the weights p, q as well. How this can technically be done in the framework of a bargaining process remains an issue and calls for further research. It is also left for further research how other bargaining processes for *NSNBS* can be adjusted so that they implement *L-NSNBS*.

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