Chapter 2
Basic Introductory Results

Abstract The goal of this chapter is to present basic introductory techniques of model theory. The main results presented are Łoś fundamental lemma on ultraproduct of structures, compactness theorem and quantifier elimination. These are cornerstones of model theory. A large number of applications given in this chapter bear testimony to the importance of these results. We also introduce the notion of independence and dimension in minimal sets. Finally we give several applications of the results proved in this section in algebra and geometry. A large number of examples and exercises are given as we go along.

2.1 Ultraproduct of Structures

In this section, we introduce ultraproduct of models. It is a notion of the product of structures and a basic technique of constructing new models from old ones. It made its first appearance in Skolem [57]. The fundamental lemma was proved by Łoś in [36]. Since then ultraproduct has become a basic tool in model theory.

Let L be a first-order language and $\mathcal{F}$ a filter on a non-empty set I. Suppose for each $i \in I$ we are given an L-structure M_i of L. Set

$$M = \times_{i \in I} M_i.$$

For $\alpha, \beta \in M$, define

$$\alpha \sim \beta \Leftrightarrow \{i \in I : \alpha(i) = \beta(i)\} \in \mathcal{F}.$$

Since $I \in \mathcal{F}$, $\sim$ is reflexive. Clearly, it is symmetric. Since $\mathcal{F}$ is closed under finite intersections and supersets, $\sim$ is transitive. Thus, $\sim$ is an equivalence relation on $\times_i M_i$. For $\alpha \in M$, $[\alpha]$ will denote the $\sim$-equivalence class containing α. We set

$$M(\mathcal{F}) = M/\sim \; = \{[\alpha] : \alpha \in M\}.$$

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We interpret the nonlogical symbols of L as follows:

1. If c is a constant symbol, $c^{M(\mathcal{F})} = [\alpha]$, where $\alpha(i) = c^{M_i}$, $i \in I$.
2. If p is an n-ary relation symbol,

$$p^{M(\mathcal{F})}([\alpha_1], \ldots, [\alpha_n]) \Leftrightarrow \{i \in I : p^{M_i}(\alpha_1(i), \ldots, \alpha_n(i))\} \in \mathcal{F}.$$

3. If f is an n-ary function symbol, we define

$$[\beta] = f^{M(\mathcal{F})}([\alpha_1], \ldots, [\alpha_n]),$$

where

$$\beta(i) = f^{M_i}(\alpha_1(i), \ldots, \alpha_n(i)),\ i \in I.$$

We need to show that $p^{M(\mathcal{F})}$ and $f^{M(\mathcal{F})}$ are well defined. Suppose $\alpha_j \sim \beta_j$, $1 \leq j \leq n$. Since $\mathcal{F}$ is closed under finite intersections, there is an $X \in \mathcal{F}$ such that $\alpha_j(i) = \beta_j(i)$ for all $1 \leq j \leq n$ and all $i \in X$. This implies the well-definedness of $p^{M(\mathcal{F})}$ and $f^{M(\mathcal{F})}$.

Proposition 2.1.1 *For every term $t[\overline{x}]$ and every $\alpha_0, \ldots, \alpha_{n-1}, \beta \in M$,*

$$t^{M(\mathcal{F})}[[\alpha_0], \ldots, [\alpha_{n-1}]] = [\beta] \Leftrightarrow \{i \in I : t^{M_i}[\alpha_0(i), \ldots, \alpha_{n-1}(i)] = \beta(i)\} \in \mathcal{F}.$$

Proof The result is proved easily by induction on the length of t. The details are left for the reader as an easy exercise. □

Proposition 2.1.2 *For every atomic formula $\varphi[\overline{x}]$ and every $\overline{\alpha} \in M$,*

$$M(\mathcal{F}) \models \varphi[[\alpha_0], \ldots, [\alpha_{n-1}]] \Leftrightarrow \{i \in I : M_i \models \varphi[\alpha_0(i), \ldots, \alpha_{n-1}(i)]\} \in \mathcal{F}. \quad (*)$$

Proof Let $t[\overline{x}], s[\overline{x}]$ be terms and $\alpha_0, \ldots, \alpha_{n-1} \in M$. Define

$$\beta(i) = t^{M_i}[\alpha_0(i), \ldots, \alpha_{n-1}(i)],\ i \in I$$

and

$$\gamma(i) = s^{M_i}[\alpha_0(i), \ldots, \alpha_{n-1}(i)],\ i \in I.$$

By the last Proposition 2.1.1,

$$t^{M(\mathcal{F})}[[\alpha_0], \ldots, [\alpha_{n-1}]] = [\beta]$$

and

$$s^{M(\mathcal{F})}[[\alpha_0], \ldots, [\alpha_{n-1}]] = [\gamma].$$

Thus, $(*)$ holds for $t[\overline{x}] = s[\overline{x}]$ and $\alpha_0, \ldots, \alpha_{n-1}$.

Let $\varphi[\overline{x}]$ be an atomic formula $p[t_1[\overline{x}], \ldots, t_m[\overline{x}]]$ and every $\alpha_0, \ldots, \alpha_{n-1} \in M$. Set

$$\beta_j(i) = t_j^{M_i}[\alpha_0(i), \ldots, \alpha_{n-1}(i)],\ i \in I, 1 \leq j \leq m.$$

Then

$$\begin{aligned} & M(\mathcal{F}) \models p[t_1^{M(\mathcal{F})}([\alpha_0], \ldots, [\alpha_{n-1}]), \ldots, t_m^{M(\mathcal{F})}([\alpha_0], \cdots, [\alpha_{n-1}])] \\ \Leftrightarrow\ & M(\mathcal{F}) \models p[[\beta_1], \ldots, [\beta_m]]] \\ \Leftrightarrow\ & \{i \in I : M_i \models p[t_1^{M_i}[\alpha_0(i), \ldots, \alpha_{n-1}(i)], \ldots, t_m^{M_i}[\alpha_0(i), \ldots, \alpha_{n-1}(i)]]\} \in \mathcal{F} \end{aligned}$$

The first equivalence holds by the last Proposition 2.1.1 and the last equivalence holds by definition. □

In a fundamental contribution to model theory Łoś showed that $(*)$ holds for every formula if $\mathcal{F}$ is an ultrafilter on I.

Theorem 2.1.3 (Łoś Fundamental Lemma) *Let $\mathcal{U}$ be an ultrafilter on I, $\varphi[\overline{x}]$ an L-formula and $[\alpha_0], \ldots, [\alpha_{n-1}] \in M(\mathcal{U})$. Then*

$$M(\mathcal{U}) \models \varphi[[\alpha_0], \ldots, [\alpha_{n-1}]] \Leftrightarrow \{i \in I : M_i \models \varphi[\alpha_0(i), \ldots, \alpha_{n-1}(i)]\} \in \mathcal{U}. \tag{$**$}$$

Proof For atomic φ, $(**)$ follows from the last Proposition 2.1.2. Suppose φ satisfies $(**)$ and ψ is the formula $\neg\varphi$. Take $[\alpha_0], \ldots, [\alpha_{n-1}] \in M(\mathcal{U})$. Then

$$\begin{aligned} M(\mathcal{U}) \models \psi[[\alpha_0], \ldots, [\alpha_{n-1}]] & \Leftrightarrow M(\mathcal{U}) \not\models \varphi[[\alpha_0], \ldots, [\alpha_{n-1}]] \\ & \Leftrightarrow \{i \in I : M_i \models \varphi[\overline{\alpha}(i)]\} \notin \mathcal{U} \\ & \Leftrightarrow \{i \in I : M_i \models \psi[\overline{\alpha}(i)]\} \in \mathcal{U}, \end{aligned}$$

where $\overline{\alpha}(i) = (\alpha_0(i), \ldots, \alpha_{n-1}(i))$. The second equivalence holds because φ satisfies $(**)$ whereas the third equivalence holds because $\mathcal{U}$ is an ultrafilter. Similarly we show that if φ and ψ satisfy $(**)$, so does $\varphi \vee \psi$.

Now assume that $(**)$ holds for $\psi[x_0, x_1, \ldots, x_n]$, $n \geq 0$ and all $(\alpha_0, \ldots, \alpha_n) \in M^{n+1}$. Consider $\varphi = \exists x_0 \psi$. Take any $\alpha_1, \ldots, \alpha_n \in M$ such that

$$M(\mathcal{U}) \models \varphi[[\alpha_1], \ldots, [\alpha_n]].$$

Then there exists $[\alpha_0] \in M(\mathcal{U})$ such that

$$M(\mathcal{U}) \models \psi[[\alpha_0], \ldots, [\alpha_n]].$$

By our hypothesis,

$$\{i \in I : M_i \models \psi[\alpha_0(i), \ldots, \alpha_n(i)]\} \in \mathcal{U}.$$

This clearly implies that

$$\{i \in I : M_i \models \varphi[\alpha_1(i), \ldots, \alpha_n(i)]\} \in \mathcal{U}.$$

To prove the converse, assume that the set

$$U = \{i \in I : M_i \models \varphi[\alpha_1(i), \ldots, \alpha_n(i)]\} \in \mathcal{U}.$$

So, for each $i \in U$ there exists an $\alpha_0(i) \in M_i$ such that

$$M_i \models \psi[\alpha_0(i), \ldots, \alpha_n(i)].$$

Take any extension α_0 of $i \to \alpha_0(i)$, $i \in U$, to I. Then by our assumption

$$M(\mathcal{U}) \models \psi[[\alpha_0], \ldots, [\alpha_n]].$$

Thus,

$$M(\mathcal{U}) \models \varphi[[\alpha_1], \ldots, [\alpha_n]].$$

The result is thus seen by induction on the rank of φ. □

If $\mathcal{U}$ is an ultrafilter on I, the structure $M(\mathcal{U})$ is called the *ultraproduct* of M_i's. If each $M_i = M$, it is denoted by $M^{\mathcal{U}}$ and is called an *ultrapower* of M.

Let $\{M_i : i \in I\}$ and $\{N_i : i \in I\}$ be families of sets and $\mathcal{U}$ an ultrafilter on I. Let $g_i, h_i : M_i \to N_i$, $i \in I$, be arbitrary maps. Define

$$\{g_i : i \in I\} \sim_{\mathcal{U}} \{h_i : i \in I\} \Leftrightarrow \{i \in I : g_i = h_i\} \in \mathcal{U}.$$

It is easy to see that $\sim_{\mathcal{U}}$ is an equivalence relation.

Fix $\{g_i : i \in I\} \sim_{\mathcal{U}} \{h_i : i \in I\}$ and $\overline{a} = (a_i) \sim (a_i') = \overline{a}'$. Then $(g_i(a_i)) \sim (h_i(a_i'))$. Hence, we have a well-defined map

$$(\Pi_i g_i)^{\mathcal{U}}([(a_i)]) = [(g_i(a_i))].$$

We make a series of simple observations whose proofs are left to the reader as a simple exercise.

1. If $\{i \in I : g_i \text{ is onto}\} \in \mathcal{U}$, then $(\Pi_i g_i)^{\mathcal{U}}$ is onto.
2. If $\{i \in I : g_i \text{ is one-to-one}\} \in \mathcal{U}$, then $(\Pi_i g_i)^{\mathcal{U}}$ is one-to-one.

Next assume that each M_i and each N_i, $i \in I$, are L-structures.

3. If $\{i \in I : g_i \text{ is a homomorphism}\} \in \mathcal{U}$, then $(\Pi_i g_i)^{\mathcal{U}}$ is a homomorphism. It follows that if $\{i \in I : g_i \text{ is an embedding (isomorphism)}\} \in \mathcal{U}$, then $(\Pi_i g_i)^{\mathcal{U}}$ is an embedding (isomorphism).
4. Using Łoś theorem (Theorem 2.1.3), it is easy to see that if $\{i \in I : g_i \text{ is elementary}\} \in \mathcal{U}$, then $(\Pi_i g_i)^{\mathcal{U}}$ is elementary.

Corollary 2.1.4 *Let T be an L-theory and $\{M_i : i \in I\}$ a family of models of T. Then for every ultrafilter $\mathcal{U}$ on I, the ultraproduct $M(\mathcal{U})$ is a model of T.*

Remark 2.1.5 Ultraproduct gives a new notion of product in the category of models of T; in particular, in any category of algebraic structures such as groups, rings, fields, etc. Further, by choosing the ultrafilter $\mathcal{U}$ suitably, one gets a model $M(\mathcal{U})$ with some desired properties.

Since $M(\mathcal{U})$ is in a sense a limit of $\{M_i : i \in I\}$, in general, no reasonable converse of the corollary exists. However, if T has only finitely many axioms, a converse of the corollary is true.

Proposition 2.1.6 *Let $\{M_i : i \in I\}$ be a family of L-structures and $\mathcal{U}$ an ultrafilter on I. Suppose an L-theory T has finitely many axioms only and $M(\mathcal{U}) \models T$. Then $\{i \in I : M_i \models T\} \in \mathcal{U}$.*

Proof Let $\varphi_1, \dots, \varphi_n$ be all the axioms of T. Since $M(\mathcal{U}) \models T$, for each $1 \leq k \leq n$, the set $A_k = \{i \in I : M_i \models \varphi_k\} \in \mathcal{U}$. Then $A = \cap_{1 \leq k \leq n} A_k \in \mathcal{U}$ and for every $i \in A$, $M_i \models T$. $\square$

Corollary 2.1.7 *Let $\{\mathbb{K}_i : i \in I\}$ be a family of rings, $\mathcal{U}$ an ultrafilter on I and $p > 0$ a prime. Then the ultraproduct $\mathbb{K}(\mathcal{U})$ is a field of characteristic p if and only if $\{i \in I : \mathbb{K}_i$ is a field of characteristic $p\}$ is in $\mathcal{U}$.*

Example 2.1.8 For each prime $p > 0$, let $\mathbb{K}_p$ be a field of characteristic p and $\mathcal{U}$ a free ultrafilter on the set of all primes. Then the ultraproduct $\mathbb{K}(\mathcal{U})$ is a field of characteristic 0. To see this, let P denote the set of all primes. Fix a prime p. Since $\mathcal{U}$ is free, $\{q \in P : q > p\} \in \mathcal{U}$. Since $\mathbb{K}_q \models \underline{p} \neq 0$ for every $q > p$, $char(\mathbb{K}(\mathcal{U})) \neq p$ by Łoś Theorem 2.1.3. Our claim follows.

Proposition 2.1.9 *A class $\mathcal{C}$ of L-structures is elementary if and only if $\mathcal{C}$ is closed under elementary equivalences and ultraproducts.*

Proof The only if part is clear from Łoś theorem (Theorem 2.1.3). So, assume that $\mathcal{C}$ is closed under elementary equivalences and ultraproducts and $T = Th(\mathcal{C})$. We now show that $\mathcal{C}$ is precisely the class of all models of T. Clearly, if $M \in \mathcal{C}$, $M \models T$.

Now assume that $M \models T$. Let I denote the set of all non-empty finite subsets of $Th(M)$. Note that for each $i \in I$ there is a $M_i \in \mathcal{C}$ such that $M_i \models \wedge i$. If not, then $\neg(\wedge i) \in T$. But then both $\wedge i$ and $\neg(\wedge i)$ are true in M which is a contradiction. For each sentence $\varphi \in Th(M)$, set

$$A_\varphi = \{i \in I : \varphi \in i\}.$$

Given $\varphi_1, \dots, \varphi_k$,

$$\{\varphi_1, \dots, \varphi_k\} \in \wedge_{j=1}^{k} A_{\varphi_j}.$$

This implies that there is an ultrafilter $\mathcal{U}$ containing each A_φ, $\varphi \in Th(M)$. Set

$$N = \times_{i \in I} M_i / \mathcal{U}.$$

By our hypothesis $N \in \mathcal{C}$. Our proof will be complete if we show that M is elementarily equivalent to N. But for any $\varphi \in Th(M)$,

$$A_\varphi \subset \{i \in I : M_i \models \varphi\}.$$

Thus, $Th(M) \subset Th(N)$. This implies that these two sets are equal, i.e. M and N are elementarily equivalent. □

This result can be easily used to show that various classes of structures are not elementary. To illustrate this let L have no nonlogical symbol. So any non-empty set is an L-structure. Let $\mathcal{C}$ be the class of all finite sets. For $k > 0$, let $X_k = \{0, \ldots, k-1\}$. Take any free ultrafilter $\mathcal{U}$ on the set of all positive integers. Now consider $X = \times_k X_k/\mathcal{U}$. For any positive integer m, let $\alpha_m \in \times_k X_k$ be a sequence which is eventually m. Then for $m \neq n$, $[\alpha_m] \neq [\alpha_n]$. Thus, X is infinite. Hence, $\mathcal{C}$ is not closed under ultraproducts. So, $\mathcal{C}$ is not elementary.

We saw earlier that the class of all fields of positive characteristic is not closed under ultraproducts. Hence, the class of all fields of positive characteristic is not elementary.

Exercise 2.1.10 Let $\mathcal{U}$ be an ultrafilter on I with $\cap\mathcal{U} = \{j\}$. Suppose $\{M_i : i \in I\}$ is a family of L-structures. Show that $M(\mathcal{U})$ is isomorphic to M_j.

Exercise 2.1.11 Let M be an L-structure and $\mathcal{U}$ an ultrafilter on I. Define the inclusion map $j : M \to M^{\mathcal{U}}$ by

$$j(x) = [c_x], x \in M,$$

where $c_x : I \to M$ is the constant map $c_x(i) = x, i \in I$. Show that j is an elementary embedding.

2.2 Compactness Theorem

In this section, we prove the compactness theorem for first-order theories. Because of its great importance, we also give several variants of this theorem. This was first proved for countable theories by Gödel in [15]. For general theories, it was independently proved by Mal'tsev in [39, 40] and by Henkin in [18].

Theorem 2.2.1 (Compactness theorem) *An L-theory T has a model if and only if each finite $T' \subset T$ has a model.*

Proof If part: For each finite $i \subset T$, let M_i be a model of i. Set $I = \{i : i \subset T \text{ finite}\}$. For each sentence φ, set

$$B_\varphi = \{i \in I : \varphi \in i\}.$$

Let $\varphi_1, \ldots, \varphi_n \in T$. $\{\varphi_1, \ldots, \varphi_n\} \in \cap_{i=1}^n B_{\varphi_i}$. Thus, the family $\{B_\varphi : \varphi \in T\}$ has finite intersection property. Hence, it is contained in an ultrafilter $\mathcal{U}$.

We claim that $M(\mathcal{U}) \models T$. Let $\varphi \in T$. Then for every $i \in B_\varphi$, $M_i \models \varphi$. Hence, by Łoś theorem, $M(\mathcal{U}) \models \varphi$.

The only if part is entirely trivial. This completes the proof of the compactness theorem. □

We give an alternative proof of compactness theorem. This is essentially the semantic version of the syntactical proof given originally. This proof gives yet another technique of building models which will be used later also.

Let T be a finitely satisfiable set of L-sentences. The following observation is trivially seen.

Fact. For any L-sentence φ, at least one of $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ is finitely satisfiable.

A finitely satisfiable set T of L-sentences will be called complete if for every sentence φ, φ or $\neg\varphi$ is in T. Using Zorn's lemma, it is immediately seen that.

Theorem 2.2.2 (Lindenbaum Theorem) *Every finitely satisfiable set of L-sentences is contained in a complete set of finitely satisfiable L-sentences.*

We leave the detail for the reader as a simple exercise.

A set of finitely satisfiable L-sentences T will be called *Henkin* if whenever a closed sentence of the form $\exists x\varphi \in T$, there is a constant symbol c such that $\varphi_x[c] \in T$. Note that if T is complete and finitely satisfiable, then the sentence $\exists x(x = x) \in T$. Otherwise, $\neg\exists x(x = x) \in T$, contradicting that T is finitely satisfiable. This, in particular, implies that L has constant symbols.

The main idea of the proof is the following.

Theorem 2.2.3 *Every complete, Henkin set of finitely satisfiable L-sentences T has a model.*

Proof Let M' denote the set of all variable-free L-terms. By the above remark, $M' \neq \emptyset$. If t and s are variable-free terms, define

$$t \sim s \text{ if } t = s \in T.$$

Using finite satisfiability and completeness of T, it can be easily proved that $\sim$ is an equivalence relation on M'. For instance, if t_1, t_2, t_3 are variable-free L-terms and $t_1 \sim t_2$ and $t_2 \sim t_3$ hold, then $t_1 \sim t_3$ must hold. For otherwise, by completeness of T, $t_1 = t_2, t_2 = t_3, t_1 \neq t_3 \in T$. This contradicts the finite satisfiability of T.

Let $M = M'/\sim$, the set of all $\sim$-equivalence classes. For any variable-free term t, let $[t]$ denote the equivalence class containing t. For any constant symbol c, take $c^M = [c]$. Let f be a n-ary function symbol, R a n-ary relation symbol and $[t_1], \ldots, [t_n] \in M$. Define

$$f^M([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$$

and

$$R^M([t_1], \ldots, [t_n]) \Leftrightarrow R[t_1, \ldots, t_n] \in T.$$

Using completeness and finite satisfiability of T it is easy to see that these are well defined. Thus, we have defined an L-structure M.

By induction on the rank of φ, we now show that for every L_M-sentence φ,

$$M \models \varphi \Leftrightarrow \varphi \in T. \tag{$*$}$$

$(*)$ is true for all atomic φ by the definition of M. Suppose $(*)$ holds for φ and $\psi = \neg\varphi$. Then

$$M \models \psi \Leftrightarrow M \not\models \varphi \Leftrightarrow \varphi \notin T \Leftrightarrow \psi \in T.$$

The second equivalence holds by the induction hypothesis and the third equivalence holds because T is complete.

Next, we assume that $(*)$ holds for φ and ψ and $\xi = \varphi \vee \psi$. Suppose $M \models \xi$. Then $M \models \varphi$ or $M \models \psi$. Without any loss of generality, assume that $M \models \varphi$. Then by induction hypothesis, $\varphi \in T$. Hence, by the completeness of T, $\xi \in T$. Conversely, let $\xi \in T$. Then by the completeness of T, $\varphi \in T$ or $\psi \in T$. Hence, by induction hypothesis, $M \models \varphi$ or $M \models \psi$. In either case, $M \models \xi$.

Finally, let $(*)$ holds for all L_M-sentences of length less than the length of $\exists x\varphi[x]$ which is assumed to be closed. Suppose $M \models \exists x\varphi$. Then there exists $[t] \in M$ such that $M \models \varphi[[t]]$. So, by induction hypothesis, $\varphi[[t]] \in T$. Hence, by completeness of T, $\exists x\varphi[x] \in T$. Now assume that $\exists x\varphi[x] \in T$. Since T is Henkin, there is a constant c such that $\varphi_x[c] \in T$. So, by induction hypothesis, $M \models \varphi_x[c]$. Thus, $M \models \exists x\varphi[x]$. □

The model of T obtained in the last proposition is called the canonical model of T. To complete the proof of compactness theorem, we need one more result.

Proposition 2.2.4 *Let T be a finitely satisfiable set of L-sentences. Then there is an extension L_∞ of L obtained by adding new constant symbols only and a finitely satisfiable, Henkin set of L_∞-sentences T_∞ that contains T.*

Proof Set $L_0 = L$ and $T_0 = T$. Suppose L_n and a finitely satisfiable set of L_n-sentences T_n have been defined. For each L_n-sentence of the form $\exists x\varphi[x]$ which is not an L_m-sentence for any $m < n$, we add a new constant symbol $c_{\exists x\varphi}$ to L_n and the sentence $\exists x\varphi[x] \to \varphi_x[c_{\exists x\varphi}]$ to T_n. Call the resulting language L_{n+1} and resulting set of L_{n+1}-sentences T_{n+1}. It is straightforward to check that T_{n+1} is finitely satisfiable.

We put $L_\infty = \cup_n L_n$ and $T_\infty = \cup_n T_n$. These satisfy the conclusions of the proposition. □

Proof of the compactness theorem. Let T be a finitely satisfiable set of L-sentences. Then we obtain L_∞ and T_∞ as in the last proposition. By Lindenbaum Theorem 2.2.2, there is a complete finitely satisfiable set of L_∞-sentences T' containing T_∞. Then T' is Henkin. The canonical model of T' is a model of T. □

Exercise 2.2.5 Let L be a first-order language and $\mathcal{T}$ the set of all complete L-theories. This set of exercises defines a topology on $\mathcal{T}$ making it into a compact, Hausdorff, zero-dimensional space. For each L-sentence φ, set

$$B_\varphi = \{T \in \mathcal{T} : \varphi \in T\}$$

and $\mathcal{B} = \{B_\varphi : \varphi \text{ an } L - \text{sentence}\}$. Show the following.

1. $\mathcal{B}$ is closed under finite intersection and complementation. Thus, it is a base of a zero-dimensional, topology τ on $\mathcal{T}$.
2. Show that $(\mathcal{T}, \tau)$ is a compact, Hausdorff topological space.
3. Show that $(\mathcal{T}, \tau)$ is metrizable if the language L is countable.

There are some variants of compactness theorem which are quite useful.

Theorem 2.2.6 *For any sentence φ, $T \models \varphi$ if and only if $T' \models \varphi$ for some finite $T' \subset T$.*

Proof The if part is clear. For only if part, suppose for no finite T', $T' \models \varphi$. This implies that every finite part of $T'' = T \cup \{\neg\varphi\}$ has a model. Hence, by compactness theorem, T'' has a model, say M. But then $M \models T$ and $M \not\models \varphi$. So, $T \not\models \varphi$. □

Let L be a first-order language and Φ a set of formulas of L. Let $v_0, v_1, \ldots$ be all the variables (finitely or countably many), v_i's distinct, that has a free occurrence in a $\varphi \in \Phi$. We say that Φ is *satisfiable* if there is a structure M for L and $a_0, a_1, \ldots \in M$ such that for all $\varphi[v_0, \ldots, v_{n-1}] \in \Phi$, $M \models \varphi[\overline{a}]$. We say that Φ is finitely satisfiable if every finite $\Phi' \subset \Phi$ is satisfiable.

Proposition 2.2.7 *Every finitely satisfiable Φ is satisfiable.*

Proof Introduce in L a new constant c_i corresponding to each v_i that has a free occurrence in Φ and call the resulting language L'. Now consider

$$\Phi' = \{\varphi[\overline{c}] : \varphi[\overline{v}] \in \Phi\}.$$

Note that Φ is satisfiable if and only if Φ' has a model. By the compactness Theorem 2.2.1, it is sufficient to prove that each finite part of Φ' has a model. This follows because Φ is finitely satisfiable. □

Proposition 2.2.8 *Let M be an L-structure and Φ a set of L-formulas such that every finite $\Phi' \subset \Phi$ is satisfiable in an elementary extension of M. Then there is an elementary extension N of M in which Φ is satisfiable.*

Proof Consider $\Psi = \Phi \cup Diag_{el}(M)$. By our hypothesis, Ψ is finitely satisfiable. Hence, there is an L-structure N in which Ψ is satisfiable. Since $N \models Diag_{el}(M)$, N is an elementary extension of M. The proof is complete. □

Remark 2.2.9 Let L be a first-order language with uncountably many variables and Φ a set of L-formulas. In this case also the notion of finite satisfiability and satisfiability for Φ makes sense. Further, the last two propositions are seen to be true.

2.3 Some Consequences of Compactness Theorem

Proposition 2.3.1 *Let L be the language with constants* 0, 1*, binary function symbols* $+$ *and* $\cdot$ *and a binary relation symbol* $<$*. Let* $\mathbb{N}$ *denote the standard model of natural numbers. There is a structure M for L elementarily equivalent to the standard model* $\mathbb{N}$ *and having an element b such that for every natural number n,* $n < b$*.*

Proof Introduce a new constant symbol c to $L_{\mathbb{N}}$. For each natural number m, let A_m be the formula $\underline{m} < c$. Now consider the theory

$$N' = Diag_{el}(\mathbb{N}) \cup \{A_m : m \in \mathbb{N}\}.$$

Since every finite set of natural numbers has an upper bound in $\mathbb{N}$, $\mathbb{N}$ is a model of each finite part of N'. Hence, by the compactness theorem, N' has a model M. This model has the required properties with $b = c_M$. □

Proposition 2.3.2 *There is a non-Archimedean ordered field* $^*\mathbb{R}$ *elementarily equivalent to the ordered field* $\mathbb{R}$*.*

Proof Let L denote the language of the theory of ordered fields. Add a new constant symbol c to $L_{\mathbb{R}}$. For natural numbers n, let A_n be the formula $\underline{n} < c$ and consider

$$T = Diag_{el}(\mathbb{R}) \cup \{A_n : n \in \mathbb{N}\}.$$

Since the real line $\mathbb{R}$ is a model of each finite $T' \subset T$, by the compactness theorem, T has a model. Any model $^*\mathbb{R}$ of T does the job. □

Proposition 2.3.3 *The class of all well-ordered sets is not elementary.*

Proof If possible, suppose there is a first-order theory T whose models are precisely well-ordered sets. Add to T a sequence $\{c_n\}$ of distinct and new constants and set $T' = T \cup \{c_{n+1} < c_n : n \in \omega\}$. Then, T' is finitely satisfiable. Hence, by compactness theorem, T' has a model, say M. But then $\{c_n^M\}$ is a non-empty subset of M with no least element. This is a contradiction. □

Proposition 2.3.4 *The class of all fields of characteristic* 0 *is not finitely axiomatizable.*

Proof Let T be the theory of fields and φ_n denote the sentence $\underline{n} \neq 0$, $n > 1$. If possible, suppose ψ is a sentence in the language of rings such that $M \models \psi$ if and only if M is a field of characteristic 0. So, $T[\{\varphi_n : n > 1\}] \models \psi$. By compactness theorem, there is a positive integer N such that

$$T[\wedge_{i=2}^{N} \varphi_i] \models \psi.$$

Let $p > N$ be prime. It follows that $\mathbb{F}_p \models \psi$, a contradiction. □

Exercise 2.3.5 Show that the class of all algebraically closed fields is not finitely axiomatizable.

(Hint: Use Proposition B.1.4)

Exercise 2.3.6 Show that the class of all archimedean ordered fields is not elementary.

Exercise 2.3.7 A graph (V, E) is called connected if for every $x \neq y \in V$, there exist $x_0, \ldots, x_n \in V$ such that $x_0 = x$, $x_n = y$ and for all $i < n$, $E[x_i, x_{i+1}]$. Show that the class of all connected graphs is not elementary.

Exercise 2.3.8 Show that the class of all torsion-free groups is not finitely axiomatizable.

Exercise 2.3.9 Show that a class $\mathcal{C}$ of L-structures is finitely axiomatizable if and only if both $\mathcal{C}$ and its complement are elementary.

Exercise 2.3.10 Let $\mathcal{F}$ denote the class of all finite fields. Call a field $\mathbb{F}$ *pseudofinite* if it is infinite and a model of $Th(\mathcal{F})$. Show that the class of all pseudofinite fields is elementary and non-empty.

Using compactness theorem we now show that every field is a subfield of an algebraically closed field. By easy algebra arguments, this will imply the existence of the algebraic closure of each field.

We shall use a standard fact from algebra. Let $\mathbb{F}$ be a field and $f(X) \in \mathbb{F}[X]$ an irreducible polynomial. Let (f) denote the ideal in $\mathbb{F}[X]$ generated by f. Then $\mathbb{F}[X]/(f)$ is a field extension of $\mathbb{F}$ in which f has a root. It then follows that given finitely many polynomials $f_1, \ldots, f_n \in \mathbb{F}[X]$ there is a field extension $\mathbb{K}$ of $\mathbb{F}$ in which each of $f_1, \ldots, f_n$ has a root.

Proposition 2.3.11 *Let $\mathbb{F}$ be a field. Then there is a field extension $\mathbb{K}$ of $\mathbb{F}$ such that every polynomial $f(X) \in \mathbb{F}[X]$ has a root in $\mathbb{K}$.*

Proof Let T denote the theory of fields in the language L of rings with identity. For each polynomial $f(X) \in \mathbb{F}[X]$ introduce a new constant symbol c_f to $L_{\mathbb{F}}$. Let φ_f be the sentence $f(c_f) = 0$ of $L_{\mathbb{F}} \cup \{c_f : f \in \mathbb{F}[X]\}$. By the above observation, each finite subset of the theory

$$T' = T \cup Diag(\mathbb{F}) \cup \{\varphi_f : f \in \mathbb{F}[X]\}$$

has a model. Hence, by compactness theorem, T' has a model, say $\mathbb{K}$. Such a $\mathbb{K}$ does our job. $\square$

Proposition 2.3.12 *Every field is a subfield of an algebraically closed field.*

Proof Let $\mathbb{F}_0$ be a field. By repeatedly applying the last Proposition 2.3.11, we get a chain of fields $\mathbb{F}_0 \sqsubseteq \mathbb{F}_1 \sqsubseteq \mathbb{F}_2 \sqsubseteq \ldots$ such that for each n, every polynomial $f(X) \in \mathbb{F}_n[X]$ has a root in $\mathbb{F}_{n+1}$. Now take $\mathbb{K} = \cup_n \mathbb{F}_n$. $\square$

2.4 Preservation Results

In this section, we use compactness theorem and prove several so-called preservation results.

Proposition 2.4.1 *Let N be a substructure of an L-structure M. Then N is existentially closed in M if and only if there is an extension M' of M in which N is elementarily embedded.*

Proof 'If part' is easy and was left as an exercise in Chap. 1. Assume then N is existentially closed in M. Take $T = Diag_{el}(N) \cup Diag(M)$. Sufficient to prove that T has a model. If not, then by the compactness theorem, there is an open L-formula $\varphi[\overline{x}]$ and $\overline{a} \in M$ such that $M \models \varphi[\overline{a}]$ and $Diag_{el}(N) \not\models \varphi[\overline{a}]$. Hence, there exists an elementary extension N' of N such that $\varphi[\overline{a}]$ is not satisfiable in N'. This implies that $N' \not\models \exists\overline{x}\varphi[\overline{x}]$. Hence, $N \not\models \exists\overline{x}\varphi[\overline{x}]$. This contradicts that N is existentially closed in M. □

Proposition 2.4.2 *1. Let T be a first-order theory. Then $M \models T_\forall$ if and only if it is a substructure of a model N of T.*

2. A theory T is universal if and only if every substructure of a model of T is a model of T.

Proof (1): 'If part' was given as an exercise in Chap. 1. Conversely, let $M \models T_\forall$. Set $T' = T \cup Diag(M)$. It is sufficient to show that T' is consistent. If not, then by compactness theorem, there is a finite set $\Gamma \subset Diag(M)$ such that $T[\Gamma]$ has no model. Let $\varphi_1[\overline{x}], \ldots, \varphi_n[\overline{x}]$ be open formulas and $\overline{c} \in M$ such that $\Gamma = \{\varphi_1[\overline{c}], \ldots, \varphi_n[\overline{c}]\}$. It now follows that $T[\exists\overline{x} \wedge_{i=1}^{n} \varphi_i[\overline{x}]]$ has no model. So, $T \models \forall\overline{x}\neg \wedge_{i=1}^{n} \varphi_i[\overline{x}]$. In other words, $\forall\overline{x}\neg \wedge_{i=1}^{n} \varphi_i[\overline{x}] \in T_\forall$. So, $M \models \forall\overline{x}\neg \wedge_{i=1}^{n} \varphi_i[\overline{x}]$. This contradicts that $\wedge_{i=1}^{n}\varphi_i[\overline{c}] \in Diag(M)$.

(2) follows from (1) because T is universal if and only if T and $T_\forall$ have the same class of models. □

Proposition 2.4.3 *Let T be a theory and $\varphi[\overline{x}]$ a formula. The following are equivalent:*

1. *There is a universal formula $\psi[\overline{x}]$ such that $T \models \forall\overline{x}(\varphi[\overline{x}] \leftrightarrow \psi[\overline{x}])$.*
2. *Whenever $M, N \models T$, $N \sqsubseteq M$ and $\overline{a} \in N$, $M \models \varphi[\overline{a}] \Rightarrow N \models \varphi[\overline{a}]$.*

Proof (1) implies (2) is easy and was given as an exercise in Chap. 1. So, assume (2). Add new constants $\overline{c}$ to the language of T and consider the theories, $T_1 = T[\varphi[\overline{c}]]$ and $T_2 = T[\neg\varphi[\overline{c}]]$. Then (2) says that no substructure of a model of T_1 can be a model of T_2. But substructures of models of T_1 are precisely models of $(T_1)_\forall$. Thus, by (2), $(T_1)_\forall \cup T_2$ is inconsistent. Since a finite conjunction of universal sentences is tautologically equivalent to a universal sentence, by compactness theorem, we get a $\psi[\overline{c}] \in (T_1)_\forall$ such that $T_2[\psi[\overline{c}]]$ has no model. It follows that

$$T[\varphi[\overline{c}]] \models \psi[\overline{c}] \ \& \ T[\neg\varphi[\overline{c}]] \models \neg\psi[\overline{c}].$$

Hence,

$$T \models \forall \overline{x}(\varphi[\overline{x}] \leftrightarrow \psi[\overline{x}]).$$

$\square$

Proposition 2.4.4 *$M \models T_{\forall\exists}$ if and only if there is a $N \models T$ such that M is an existentially closed substructure of N.*

Proof ‘If part’ is easy and was given as an exercise in Chap. 1. For the converse, let $M \models T_{\forall\exists}$ and T' be the set of all universal L_M-sentences true in M.

Sufficient to show that $T \cup T'$ has a model, say N: Then $N \models T$. Since T' contains the atomic diagram of M, M has an embedding in N. Let φ be an existential L_M-sentence true in N. If possible suppose φ is not true in M. Then $\neg\varphi$, a universal L_M-sentence, is true in M. But then $N \models \neg\varphi$ which is a contradiction.

If possible, suppose $T \cup T'$ is inconsistent. By compactness theorem, there exist universal L_M-sentences $\varphi_1, \ldots, \varphi_k$ true in M such that $T \models \neg \wedge_{i=1}^{k} \varphi_i$. Since $\neg \wedge_{i=1}^{k} \varphi_i$ is equivalent to a closed existential formula, it belongs to $T_{\forall\exists}$. So, $M \models \neg \wedge_{i=1}^{k} \varphi_i$. Hence, $M \models \neg\varphi_i$ for some $1 \leq i \leq k$. This contradicts that $M \models \varphi_i$. $\square$

A model M of a theory T is called an *existentially closed model* of T if M is existentially closed in every extension $N \sqsupseteq M$ which is a model of T.

Corollary 2.4.5 *Let T be a $\forall\exists$ theory and $T' = T_\forall$. Then every existentially closed model of T' is a model of T.*

Proof Let M be an existentially closed model of $T' = T_\forall$. By Proposition 2.4.2, there is an extension N of M that models T. Let $\forall\overline{x}\exists\overline{y}\varphi[\overline{y}, \overline{x}]$, φ open, be in $T_{\forall\exists}$. Take any $\overline{a} \in M$. Then $N \models \exists\overline{y}\varphi[\overline{y}, \overline{a}]$. Note that $M, N \models T'$. Since M is an existentially closed model of T', $M \models \exists\overline{y}\varphi[\overline{y}, \overline{a}]$. $\square$

Corollary 2.4.6 *A theory T is $\forall\exists$ if and only if T is inductive.*

Proof ‘Only if’ part is easy and was proved in Proposition 1.5.12. So, assume that the class of models of T is closed under unions of chains. Let $M_0 \models T_{\forall\exists}$. We shall find an elementary extension M_∞ of M_0 which is a model of T. This will prove that $M_0 \models T$ and the proof will be complete.

Applying Propositions 2.4.4 and 2.4.1 alternatively, we have

$$M_0 \sqsubseteq N_0 \sqsubseteq M_1 \sqsubseteq N_1 \sqsubseteq M_2 \sqsubseteq \cdots$$

such that for each k, M_k is existentially closed in N_k, $N_k \models T$ and M_{k+1} is an elementary extension of M_k. Set $N_\infty = \cup_k N_k$ and $M_\infty = \cup_k M_k$. By our hypothesis, $N_\infty \models T$. But $M_\infty = N_\infty$. So, $M_\infty \models T$. Further, $M_0 \preceq M_\infty$. $\square$

2.5 Extensions of Partial Elementary Maps

In this section using compactness theorem, we prove results on the extensions of partial elementary maps.

Proposition 2.5.1 *Let M, N be L-structures, $A \subset M$, $f : A \to N$ a partial elementary map and $a \in M$. Then there is an elementary extension N' of N and a partial elementary map $g : A \cup \{a\} \to N'$ that extends f. Moreover, if L, A and N are countable, we can choose N' to be countable.*

Proof Suppose $\overline{a} \in A$ and $\varphi[x, \overline{x}]$, an L-formula, is such that $M \models \varphi[a, \overline{a}]$. Then $M \models \exists x \varphi[x, \overline{a}]$. Since f is partial elementary, $N \models \exists x \varphi[x, f(\overline{a})]$. From this it is entirely routine to see that every finite subset of

$$T = Diag_{el}(N) \cup \{\varphi[x, f(\overline{a})] : \overline{a} \in A \wedge M \models \varphi[a, \overline{a}]\}$$

is finitely satisfiable in N. Hence, by compactness theorem, it is satisfiable. Therefore, there is an elementary extension N' of N and a $b \in N'$ such that $N' \models \varphi[b, f(\overline{a})]$ whenever $M \models \varphi[a, \overline{a}]$. Now take $g = f \cup \{(a, b)\}$.

In case L, A and N are countable, T is countable. Therefore, a countable model N' of T exists. $\square$

Applying this result repeatedly, by transfinite induction, we also have the following result.

Proposition 2.5.2 *Let M, N_0 be L-structures, $A \subset M$ and $f_0 : A \to N_0$ partial elementary. Then there exists an elementary extension N_∞ of N_0 such that f_0 can be extended to an elementary embedding $f_\infty : M \to N_\infty$. Moreover, if L, M and N_0 are countable, we can choose N_∞ to be countable.*

Proof Fix an enumeration $\{a_\alpha : \alpha < |M|\}$ of M. By transfinite induction, for each $\alpha < |M|$, we shall get an L-structure N_α and a partial elementary map $f_\alpha : A \cup \{a_\beta : \beta < \alpha\} \to N_\alpha$ satisfying the following conditions:

1. $N_{\alpha+1}$ is an elementary extension of N_α, $N_\alpha = \cup_{\beta<\alpha} N_\beta$ if α is a limit ordinal.
2. $f_{\alpha+1}$ extends f_α and $f_\alpha = \cup_{\beta<\alpha} f_\beta$ if α limit.

Suppose f_α, N_α satisfying the desired properties have been defined. If $a_\alpha \in domain(f_\alpha)$, we set $N_{\alpha+1} = N_\alpha$ and $f_{\alpha+1} = f_\alpha$. Otherwise, by the last Proposition 2.5.1, there is an elementary extension $N_{\alpha+1}$ of N_α and a partial elementary map

$$f_{\alpha+1} : A \cup \{a_\beta : \beta \le \alpha\} \to N_{\alpha+1}$$

extending f_α. Finally take $N_\infty = \cup_{\alpha<|M|} N_\alpha$ and $f_\infty = \cup_\alpha f_\alpha$.

In case L, M and N_0 are countable, enumerate $M = \{a_n\}$ and proceed as above but choose at each stage N_n countable. $\square$

Proposition 2.5.3 *Let M, N be L-structures and $A \subset M$, $B \subset N$. Suppose $f : A \to B$ is a partial elementary map. Then f has a partial elementary extension $f' : acl(A) \to acl(B)$. Moreover, if f is surjective, we can choose f' to be surjective also.*

Proof By Zorn's lemma, there is a maximal elementary extension $f' : A' \to B'$ of f with $A' \subset acl(A)$ and $B' \subset acl(B)$. Note that $acl(A') = acl(A)$. If possible, suppose there exists $a \in acl(A') \setminus A'$. Get an L-formula $\varphi[x, \overline{y}]$, $\overline{a} \in A'$ and $n \geq 1$ such that

$$M \models \varphi[a, \overline{a}] \wedge \exists_{=n} x \varphi[x, \overline{a}].$$

Choose φ and $\overline{a}$ such that n is minimal possible. Since $f' : A' \to B'$ is partial elementary,

$$N \models \exists_{=n} x \varphi[x, f'(\overline{a})].$$

Clearly, there exists $b \in acl(B)$ such that $N \models \varphi[b, f'(\overline{a})]$.

We claim that $f' \cup \{(a, b)\}$ is partial elementary. (This will complete the proof of the first part of the result.) Let $\psi[x, \overline{y}]$ be an L-formula, $\overline{b} \in A'$ such that $M \models \psi[a, \overline{b}]$. By the minimality of n,

$$M \models \forall x (\varphi[x, \overline{a}] \to \psi[x, \overline{b}]).$$

Hence,

$$N \models \forall x (\varphi[x, f'(\overline{a})] \to \psi[x, f'(\overline{b})]).$$

Thus, $N \models \psi[b, f'(\overline{b})]$.

Now assume that f is surjective. Then $acl(B) = acl(B')$. Let $b \in acl(B)$. Since f is surjective, there exist an L-formula $\varphi[x, \overline{y}]$, an $\overline{a} \in A$ and a $n \geq 1$ such that

$$N \models \varphi[b, f(\overline{a})] \wedge \exists_{=n} x \varphi[x, f(\overline{a})].$$

Then $M \models \exists_{=n} x \varphi[x, \overline{a}]$. Let $a_1, \ldots, a_n$ be all $a \in acl(A)$ such that $M \models \varphi[a, \overline{a}]$. Since f' is defined on $acl(A)$, $b = f'(a_i)$ for some i. □

2.6 Upward Löwenheim–Skolem Theorem

In Theorem 1.7.6 we proved Downward Löwenheim–Skolem Theorem which can be viewed as a method for building models of smaller cardinalities. In this section, we present a technique for building large models. First such result was proved by Tarski in 1928 who showed that every first-order theory with an infinite model has an uncountable model. The so-called Upward Löwenheim–Skolem theorem (Theorem 2.6.3) appeared in a paper by Tarski and Vaught in [63].

Proposition 2.6.1 *If a theory T has arbitrarily large finite models, it has an infinite model.*

Proof Let $\{c_n : n \in \mathbb{N}\}$ be a sequence of distinct symbols not appearing in L. Let T' be the extension of T obtained by adding each c_n as a new constant symbol and for each $m < n$, let the formula $c_n \neq c_m$ be an axiom.

Since T has arbitrarily large finite models, each finite $T'' \subset T'$ has a model. Hence, by the compactness theorem, T' has a model. Clearly, any model of T' is infinite and a model of T. □

Theorem 2.6.2 *Let κ be an infinite cardinal and T a consistent κ-theory. Assume that T has an infinite model M. Then T has a model of cardinality κ.*

Proof Fix a set $\{c_\alpha : \alpha < \kappa\}$ of cardinality κ of distinct symbols not appearing in L. Let L' be the extension of L obtained by adding each c_α as a constant symbol. Set $\Gamma = \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}$ and consider the theory $T' = T[\Gamma]$ with language L'.

We claim that T' is finitely satisfiable. To see this, fix a finite subset Γ' of Γ. Let $c_{\alpha_1}, \ldots, c_{\alpha_k}$ be all the new constants that appear in a formula in Γ'. Since M is infinite, there exist distinct elements $b_1, \ldots, b_k$ of M. Interpret c_{α_i} by b_i, $1 \leq i \leq k$. Thus we get a model of $T[\Gamma']$. Hence, by the compactness theorem, T' has a model. Now note that any model of T' is of cardinality at least κ and a model of T.

Fix a model M of T'. By downward Löwenheim–Skolem Theorem 1.7.6, M has an elementary substructure N of cardinality at most κ. Evidently $|M| = \kappa$. □

Theorem 2.6.3 (Upward Löwenheim–Skolem theorem) *Let κ be an infinite cardinal and L a κ-language. Then every infinite structure N of L of cardinality at most κ has an elementary extension M of cardinality κ.*

Proof Note that elementary diagram $Diag_{el}(N)$ of N is a consistent κ-theory. Further, N is an infinite model of $Diag_{el}(N)$. Hence, $Diag_{el}(N)$ has a model M of cardinality κ by the last theorem. Since $M \models Diag_{el}(N)$, M is an elementary extension of N. □

Exercise 2.6.4 Show that there are structures of arbitrarily large infinite cardinality elementarily equivalent to $\mathbb{N} \models PA$.

2.7 Some Complete Theories

The following theorem was independently proved by Łoś in [37] and Vaught in [65].

Theorem 2.7.1 (Vaught's Categoricity Theorem) *Let κ be an infinite cardinal and T a consistent κ-theory all of whose models are infinite. If T is κ-categorical, T is complete.*

Proof Suppose a sentence φ is not decidable in T. The theories $T_1 = T[\varphi]$ and $T_2 = T[\neg\varphi]$ are consistent. Since T has no finite models, both T_1 and T_2 have infinite models. So, T_1 and T_2 have models M_1 and M_2 respectively of cardinality κ by Theorem 2.6.2. Hence, by the hypothesis of the theorem, they are isomorphic. But φ is true in M_1 and false in M_2 contradicting that T is κ-categorical. Hence, T is complete. $\square$

Example 2.7.2 The theory T of infinite sets is κ-categorical for every infinite cardinal κ, Hence, it is complete.

We saw in Chap. 1 that DLO is $\aleph_0$-categorical and DAG and $ACF(p)$, $p = 0$ or prime, are κ-categorical for all uncountable κ. Further, these three are countable theories with all models infinite. Hence,

Example 2.7.3 DLO, DAG and $ACF(p)$, $p = 0$ or prime, are complete theories. In particular, any two models of these theories are elementarily equivalent.

Exercise 2.7.4 Let G be an infinite group and T the theory of free G-spaces. Show that T is complete.

Exercise 2.7.5 Show that the theory of random graphs is complete.

2.8 Amalgamation

We continue with applications of compactness theorem and give quite handy conditions under which two structures have a common elementary extension.

Proposition 2.8.1 *Let A and B be elementarily equivalent L-structures. Then there is an elementary extension C of A such that there is an elementary embedding $g : B \to C$.*

Proof Let B' be an L-structure, $f : B \to B'$ an isomorphism and $A \cap B' = \emptyset$. Take

$$T = Diag_{el}(A) \cup Diag_{el}(B').$$

Let $\psi_1[\overline{b}], \ldots, \psi_n[\overline{b}] \in Diag_{el}(B')$. Then $B' \models \exists\overline{y} \wedge_{j=1}^{n} \psi_j[\overline{y}]$. Since A and B' are elementarily equivalent, $A \models \exists\overline{y} \wedge_{j=1}^{n} \psi_j[\overline{y}]$. Thus, A is a model of each finite part of T. Hence, by compactness theorem, T has a model, say C. Take $g = i \circ f$. Then $g : B \to C$ is elementary and C an elementary extension of A. $\square$

The following theorem is due to Abraham Robinson ([51], Theorem 4.2.2).

Theorem 2.8.2 (Elementary Amalgamation Theorem) *Let A and B be L-structures and $\overline{a} \in A$, $\overline{b} \in B$ be such that $(A, \overline{a})$ is elementarily equivalent to $(B, \overline{b})$. Let $\langle\overline{a}\rangle_A$ be the substructure of A generated by $\overline{a}$ and $f : \langle\overline{a}\rangle_A \to B$ the embedding such that $f(\overline{a}) = \overline{b}$. Then there is an elementary extension C of A and an elementary embedding $g : B \to C$ such that $g(f(\overline{a})) = g(\overline{b}) = \overline{a}$.*

Proof Replacing B by an isomorphic copy if necessary, without loss of generality, we assume that $A \cap B = \emptyset$. Set

$$T = Diag_{el}(A) \cup \{\varphi[\overline{a}, \overline{c}] : \varphi[\overline{b}, \overline{c}] \in Diag_{el}(B) \ \wedge \ \overline{b} \cap \overline{c} = \emptyset\}.$$

Note that since $(A, \overline{a})$ and $(B, \overline{b})$ are elementarily equivalent, $A \models \exists \overline{y} \varphi[\overline{a}, \overline{y}]$, whenever $\varphi[\overline{b}, \overline{c}] \in Diag_{el}(B)$ and $\overline{b} \cap \overline{c} = \emptyset$. Now, it is fairly routine to see that A models every finite part of T. Hence, by compactness theorem, T has a model.

Let $C \models T$. Then C is an elementary extension of A. Define $g : B \to C$ be $g(\overline{b}) = \overline{a}$ and $g(b) = b^C$, if $c \notin \overline{b}$. Then g is an elementary embedding of B into C. □

Let A, B, C and D be L-structures such that A is a common elementary substructure of B and C and B and C are elementary substructures of D. We call D a *heir-coheir amalgamation* of B and C over A or a *coheir-heir amalgamation* of C and B over A if for all L-formulas $\varphi[\overline{x}, \overline{y}]$, whenever $\overline{b} \in B, \overline{c} \in C$ and $D \models \varphi[\overline{c}, \overline{b}]$, there is an $\overline{a} \in A$ such that $B \models \varphi[\overline{a}, \overline{b}]$.

The following theorem is due to Lascar and Poizat [35].

Theorem 2.8.3 *Let A, B, C be L-structures with A a common elementary substructure of B and C. Then, there is a common elementary extension D of B and C which is a heir-coheir amalgamation of B and C over A.*

Proof Replacing B by an isomorphic copy if necessary, without loss of generality, we assume that $B \cap C = A$. Let T' be the theory

$$\{\neg\varphi[\overline{c}, \overline{b}] : \overline{b} \in B \ \wedge \ \overline{c} \in C \ \wedge \ \forall \overline{a} \in A(B \models \neg\varphi[\overline{a}, \overline{b}])\},$$

and

$$T = Diag_{el}(B) \cup Diag_{el}(C) \cup T'.$$

Clearly, it is sufficient to show that T has a model. This will follow if we show that B models every finite part of $Diag_{el}(C) \cup T'$.

Let $\overline{a} \in A$, $\overline{b} \in B$, $\overline{c} \in C \backslash A$, $\neg\varphi_1[\overline{a}, \overline{c}, \overline{b}], \ldots, \neg\varphi_k[\overline{a}, \overline{c}, \overline{b}] \in T'$ and $\psi[\overline{a}, \overline{c}] \in Diag_{el}(C)$. So, for all $\overline{a'}, \overline{a''} \in A$, $B \models \neg\varphi_i[\overline{a'}, \overline{a''}, \overline{b}]$, $1 \leq i \leq k$.

Now, $C \models \psi[\overline{a}, \overline{c}]$ implies that $C \models \exists \overline{y} \psi[\overline{a}, \overline{y}]$. Hence, $A \models \exists \overline{y} \psi[\overline{a}, \overline{y}]$. So, there exists $\overline{a''} \in A$ such that $A \models \psi[\overline{a}, \overline{a''}]$. Thus, $B \models \psi[\overline{a}, \overline{a''}]$. Clearly, $B \models \neg\varphi_i[\overline{a}, \overline{a''}, \overline{b}]$, $1 \leq i \leq k$. □

Remark 2.8.4 By interchanging the role of B and C in the above proof, we get a coheir-heir amalgamation of B and C over A. We shall see later that in a stable theory, every heir-coheir amalgam is a coheir-heir amalgam.

Remark 2.8.5 Let D be a heir-coheir amalgamation of B and C over A. Suppose $b \in B, c \in C$ and $D \models b = c$. Then there exists $a \in A$ such that $B \models b = a$. So, the overlap of B and C in D remains A. Such amalgamations are called *strong*.

2.9 Quantifier Elimination

In this section, we introduce yet another important technique in model theory, namely quantifier elimination. This was introduced and systematically studied by Tarski [62]. Results and examples that follow are due to him.

Let T be an L-theory. We say that T has *quantifier elimination* if for every L-formula $\varphi[\overline{x}]$ there is an open L-formula $\psi[\overline{x}]$ such that

$$T \models \forall\overline{x}(\varphi[\overline{x}] \leftrightarrow \psi[\overline{x}]).$$

Example 2.9.1 Let φ be a sentence decidable in T and the language of T have a constant symbol, say c. Then $T \models \varphi \leftrightarrow c = c$ if $T \models \varphi$, else $T \models \varphi \leftrightarrow c \neq c$.

In the rest of this section, we present some necessary and sufficient conditions for T to have quantifier elimination. Some examples of theories having quantifier elimination are given in the next section.

Proposition 2.9.2 *A theory T has quantifier elimination if and only if for every open formula $\varphi[x, \overline{y}]$, there is an open formula $\psi[\overline{y}]$ such that*

$$T \models \forall\overline{y}((\exists x\varphi[x, \overline{y}]) \leftrightarrow \psi[\overline{y}]).$$

Proof Since only if part of the result is clear, we need to prove if part only. By induction on the rank of formulas, we prove that for every formula $\varphi[\overline{x}]$ of L there is an open formula $\psi[\overline{x}]$ such that

$$T \models \forall\overline{x}(\varphi[\overline{x}] \leftrightarrow \psi[\overline{x}]). \qquad (*)$$

$(*)$ is clearly true for open φ. It is easy to prove that if $(*)$ is true for φ, it is true for $\neg\varphi$. If $(*)$ holds for $\varphi = \varphi_1$ and $\varphi = \varphi_2$, it holds for $\varphi_1 \vee \varphi_2$.

To complete the proof, assume that $(*)$ holds for $\varphi[x, \overline{y}]$. Get an open formula $\eta[x, \overline{y}]$ such that

$$T \models \forall x\forall\overline{y}(\varphi[x, \overline{y}] \leftrightarrow \eta[x, \overline{y}]).$$

This implies that

$$T \models \forall\overline{y}((\exists x\varphi[x, \overline{y}]) \leftrightarrow \exists x\eta[x, \overline{y}]).$$

By our hypothesis, there is an open formula $\psi[\overline{y}]$ such that

$$T \models \forall\overline{y}((\exists x\eta[x, \overline{y}]) \leftrightarrow \psi[\overline{y}]).$$

Now it is clear that

$$T \models \forall\overline{y}((\exists x\varphi[x, \overline{y}]) \leftrightarrow \psi[\overline{y}]).$$

Our proof is complete. $\square$

Theorem 2.9.3 *Let T be a theory with a constant symbol c and $\varphi[\overline{x}]$ a formula of T. The following are equivalent:*

(1) *There is an open formula $\psi[\overline{x}]$ such that*

$$T \models \forall\overline{x}(\varphi[\overline{x}] \leftrightarrow \psi[\overline{x}]). \qquad (*)$$

(2) *For any two models $M, N \models T$, for any common substructure A of M, N and for any $\overline{a} \in A$,*

$$M \models \varphi[\overline{a}] \Leftrightarrow N \models \varphi[\overline{a}].$$

(3) *For any two models $M, N \models T$, for any common finitely generated substructure A of M, N and for any $\overline{a} \in A$,*

$$M \models \varphi[\overline{a}] \Leftrightarrow N \models \varphi[\overline{a}].$$

Proof (1) implies (2): Take M, N, A and $\overline{a}$ as in (2). By (1), there is an open formula $\psi[\overline{x}]$ such that $T \models \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$. So,

$$M \models \varphi(\overline{a}) \Leftrightarrow M \models \psi[\overline{a}]$$

and

$$N \models \varphi(\overline{a}) \Leftrightarrow N \models \psi[\overline{a}].$$

But A being a common substructure of M and N, since $\overline{a} \in A$ and ψ is open,

$$M \models \psi(\overline{a}) \Leftrightarrow A \models \psi(\overline{a}) \Leftrightarrow N \models \psi(\overline{a}).$$

Hence,

$$M \models \varphi(\overline{a}) \Leftrightarrow N \models \varphi(\overline{a}).$$

(3) is a special case of (2).

(3) implies (1): Assume that $\varphi[\overline{x}]$ satisfies (3). When a closed formula φ satisfies (3), φ is either true in all models or in none. Now note that $T \models \varphi \leftrightarrow c = c$ if $T \models \varphi$. Otherwise $T \models \neg\varphi$ when $T \models \varphi \leftrightarrow c \neq c$. The same argument works when $\varphi[\overline{x}]$ is not closed but decidable in T, i.e. $\forall\overline{x}\varphi[\overline{x}]$ is decidable in T.

It remains to prove the result in case both $T[\varphi[\overline{x}]]$ and $T[\neg\varphi[\overline{x}]]$ are satisfiable. Introduce new constants $\overline{c}$ to the language to get a new language, say L'. Let T' be the new theory whose language is L' but no new nonlogical axiom. Consider

$$\Gamma = \{\psi[\overline{c}] : T' \models \varphi[\overline{x}] \to \psi[\overline{x}], \psi \text{ open}\}.$$

We first see that it is sufficient to prove that

$$T'[\Gamma] \models \varphi[\overline{c}]. \qquad (*)$$

Then by compactness theorem, there exist $\psi_0[\overline{c}], \ldots, \psi_{n-1}[\overline{c}] \in \Gamma$ such that

$$T' \models \wedge_{i<n}\psi_i[\overline{c}] \to \varphi[\overline{c}].$$

Since $\overline{c}$ are new constants, it follows that

$$T \models \forall\overline{x}(\varphi[\overline{x}] \leftrightarrow \wedge_{i<n}\psi_i[\overline{x}])$$

and $\wedge_{i<n}\psi_i[\overline{x}]$ is open.

We prove $(*)$ by contradiction. So, assume that

$$T'[\Gamma] \not\models \varphi[\overline{c}].$$

Let

$$M \models T'[\Gamma] \cup \{\neg\varphi[\overline{c}]\}.$$

Let A be the substructure of M generated by $\overline{c}^M$. So A is finitely generated. Now consider

$$\Delta = T \cup Diag(A) \cup \{\varphi[\overline{c}]\}.$$

We claim that Δ has a model. If not, then by compactness theorem, there exist $\psi_1[\overline{c}], \ldots, \psi_n[\overline{c}] \in Diag(A)$ such that

$$T' \models \wedge_{i=1}^{n}\psi_i[\overline{c}] \to \neg\varphi[\overline{c}].$$

Since $\overline{c}$ are new constants,

$$T \models \wedge_{i=1}^{n}\psi_i[\overline{x}] \to \neg\varphi[\overline{x}].$$

Set $\psi[\overline{x}] = \neg \wedge_{i=1}^{n} \psi_i[\overline{x}]$. Note that ψ is open. We have,

$$T \models \varphi[\overline{x}] \to \psi[\overline{x}].$$

Thus, $\psi[\overline{c}] \in \Gamma$. Hence, $M \models \psi[\overline{c}]$. Since ψ is open and $\overline{c}^M \in A$, $A \models \psi[\overline{c}]$, contradicting that $\psi_1[\overline{c}], \ldots, \psi_n[\overline{c}] \in Diag(A)$.

Now take a model $N \models \Delta$. By the Atomic diagram Theorem 1.5.13, A is a substructure of N. But $M \models \neg\varphi[\overline{c}]$ and $N \models \varphi[\overline{c}]$. This contradicts (3) and proves $(*)$.

□

Since every open formula is equivalent to an open formula in disjunctive normal form (DNF), we now easily see that

Proposition 2.9.4 *Let T be a theory with a constant. The following are equivalent:*

(1) T has quantifier elimination.

(2) *For every conjunction of literals* $\varphi[x, \overline{y}]$, *for any two models* $M, N \models T$, *for every common substructure* A *of* M, N *and for every* $\overline{a} \in A$, *if there is a* $b \in M$ *such that* $M \models \varphi[b, \overline{a}]$, *there is a* $c \in N$ *such that* $N \models \varphi[c, \overline{a}]$.

The simple proof of this result is left to the reader as a simple exercise.

Let T be an L theory, $M, N \models T$, $A \subset M$ and $B \subset N$. A map $f : A \to B$ is called a *partial isomorphism* if f is onto and for every atomic L-formula $\varphi[\overline{x}]$ and every $\overline{a} \in A$,

$$M \models \varphi[\overline{a}] \Leftrightarrow N \models \varphi[f(\overline{a})]. \tag{$*$}$$

It is easy to see that every partial isomorphism $f : A \to B$ is a bijection and for every open L-formula $\varphi[\overline{x}]$ and every $\overline{a} \in A$,

$$M \models \varphi[\overline{a}] \Leftrightarrow N \models \varphi[f(\overline{a})].$$

If, moreover, $(*)$ is satisfied for every formula $\varphi[\overline{x}]$ and every $\overline{a} \in A$, we call f a *partial elementary*. In the next chapter, we shall study partial elementary maps in detail.

Theorem 2.9.5 *A theory* T *has quantifier elimination if and only if for every pair of models* M, N *of* T *every finite partial isomorphism* $M \ni \overline{a} \to \overline{b} \in N$ *is partial elementary.*

Proof The only if part of the result being clear, we prove the if part only. Take an L-formula $\varphi[\overline{x}]$, $\overline{x} = (x_0, \ldots, x_{n-1})$.

$$\Gamma[\overline{x}] = \{\psi[\overline{x}] : \psi[\overline{x}] \text{ an open } L - \text{formula } \& \ T \models \forall\overline{x}(\varphi[\overline{x}] \to \psi[\overline{x}])\}.$$

Add new constants $c_0, \ldots, c_{n-1}$ and consider

$$\Gamma[\overline{c}] = \{\psi[\overline{c}] : \psi[\overline{x}] \in \Gamma[\overline{x}]\}.$$

Claim. $T[\Gamma[\overline{c}]] \models \varphi[\overline{c}]$.

Assuming the claim, we complete the proof first. Since $\Gamma[\overline{c}]$ is closed under conjunctions, by compactness theorem, there is a $\psi[\overline{c}] \in \Gamma[\overline{c}]$ such that $T[\psi[\overline{c}]] \models \varphi[\overline{c}]$. It follows that

$$T \models \forall\overline{x}(\psi[\overline{x}] \to \varphi[\overline{x}]).$$

But we already have

$$T \models \forall\overline{x}(\varphi[\overline{x}] \to \psi[\overline{x}]).$$

Hence,

$$T \models \forall\overline{x}(\varphi[\overline{x}] \leftrightarrow \psi[\overline{x}]).$$

Proof of the claim. Suppose the claim does not hold. Then there exists a $M \models T[\Gamma[\overline{c}]] \cup \{\neg\varphi[\overline{c}]\}$. Let $a_i = c_i^M$, $i < n$. Set

$$p[\overline{x}] = \{\xi[\overline{x}] : \xi[\overline{x}] \text{ an open } L - \text{formula } \& \ M \models \xi[\overline{a}]\}.$$

Then $T \cup p[\overline{x}] \cup \{\varphi[\overline{x}]\}$ is satisfiable: If not, then it is not finitely satisfiable. Hence, there is a formula $\xi[\overline{x}] \in p[\overline{x}]$ such that

$$T \models \forall\overline{x}(\varphi[\overline{x}] \to \neg\xi[\overline{x}]).$$

This forces $\neg\xi[\overline{x}] \in \Gamma[\overline{x}]$ which is a contradiction.

Thus, there exist $N \models T$ and $\overline{b} \in N$ such that $N \models \varphi[\overline{b}]$ and for every open formula $\xi[\overline{x}]$,

$$M \models \xi[\overline{a}] \Leftrightarrow N \models \xi[\overline{b}].$$

Since $M \models \neg\varphi[\overline{a}]$, we have arrived at a contradiction. □

We close this section by giving an application of partial elementary maps to quantifier elimination. Let M be a model of a theory T and $A \sqsubseteq M$. We say that M is *prime over* A or that M is a *prime model extension* of A if for every model N of T and every partial elementary map $h : A \to N$, there is an elementary embedding $g : M \to N$ such that $h = g|A$. We say that T has *algebraically prime models* if every model A of $T_\forall$ has an extension $M \models T$ such that M is prime over A. Recall that $A \models T_\forall$ if and only if it has an extension to a model of T (Proposition 2.4.2).

Example 2.9.6 Consider the theory ACF of algebraically closed fields. Let D be an integral domain and $\mathbb{F}$ the algebraic closure of the fraction field of D. We know that given any $\mathbb{K} \models ACF$ and a partial elementary map $h : D \to \mathbb{K}$ (an embedding, in particular), there is an embedding $g : \mathbb{F} \to \mathbb{K}$ such that $h = g|D$. Since ACF has quantifier elimination, g is elementary.

Example 2.9.7 Consider the theory $RCOF$ of real closed fields. Let D be an ordered integral domain and $\mathbb{F}$ the real closure of the ordered fraction field of D. We know that given any $\mathbb{K} \models RCF$ and an elementary map $h : D \to \mathbb{K}$, there is an embedding $g : \mathbb{F} \to \mathbb{K}$ such that $h = g|D$. Since $RCOF$ has quantifier elimination, g is elementary.

Example 2.9.8 Consider the theory DLO of dense linearly ordered sets with no end points. Let $(A, <)$ be a linearly ordered sets. We define a dense linearly ordered set A^* as follows: If A has a least element, say x, add a copy of $\mathbb{Q}$ with the usual order to the left of x, if A has a greatest element, say y, add a copy of $\mathbb{Q}$ with the usual order to the right of y and if $x < y$ are two elements of A with no element in between, add a copy of $\mathbb{Q}$ with the usual order between x and y. There is a canonical inclusion map $f : A \hookrightarrow A^*$. Now given any $B \models DLO$ and a partial elementary map $h : A \to B$, it is easy to define an embedding $g : A^* \to B$ such that $h = g \circ f$. Since DLO has quantifier elimination, g is elementary.

We leave the proof of following theorem for readers as an exercise:

Theorem 2.9.9 *Let T be a theory such that*

1. *T has algebraically prime models, and*
2. *for any two $M, N \models T$ with $M \sqsubseteq N$, for any conjunction of literals $\varphi[x, \overline{y}]$ and for every $\overline{a} \in M$,*
$$N \models \exists x \varphi[x, \overline{a}] \Rightarrow M \models \exists x \varphi[x, \overline{a}].$$

Then T has quantifier elimination.

Exercise 2.9.10 Show that the theory T of vector spaces over a fixed field has quantifier elimination.

2.10 Examples of Quantifier Elimination

In the following examples, we use Proposition 2.9.4 without mentioning it.

Example 2.10.1 The theory DLO of dense linear orders without end points has quantifier elimination.

Proof Let $\varphi[x, \overline{y}]$ be a conjunction of literals. For instance, suppose

$$\varphi[x, \overline{y}] = y_1 < \cdots < y_{i-1} < x < y_i < \cdots < y_n.$$

Suppose $M, N \models DLO$, A is a common substructure of M, N, $\overline{a} \in A$ and there is a $b \in M$ satisfying

$$a_1 < \cdots < a_{i-1} < b < a_i < \cdots < a_n.$$

This, in particular, implies that

$$a_1 < \cdots < a_{i-1} < a_i < \cdots < a_n.$$

Since $N \models DLO$, there is a $c \in N$ such that

$$a_1 < \cdots < a_{i-1} < c < a_i < \cdots < a_n.$$

Cases when $\varphi[x, \overline{y}]$ is "$x < y_1 < \cdots < y_n$" or "$y_1 < \cdots < y_n < x$" are dealt with similarly because N has no end points. $\square$

Example 2.10.2 The theory DAG of torsion-free divisible abelian groups has quantifier elimination.

Proof We take $G_1, G_2 \models DAG$, a common subgroup $H \subset G_1, G_2$. Let $\varphi[x, \overline{y}]$ be a conjunction of literals. Suppose $\overline{a} \in H$. Replacing H by its divisible hull considered as a common subgroup of both G_1 and G_2, we further assume that H too is divisible. Now $\varphi[x, \overline{y}]$, being a conjunction of literals, it can be assumed to be of the form

$$\wedge_{i=0}^{k-1} \sum_{j=1}^{m_i} (n_{ij} y_j + n_i x = 0) \wedge \wedge_{p=0}^{l-1} \sum_{j=1}^{r_p} (n'_{pj} y_j + n'_p x \neq 0)). \qquad (*)$$

Assume that there is a $b \in G_1$ such that

$$G_1 \models \varphi[b, \overline{a}].$$

We need to show that there is a $c \in G_2$ such that

$$G_2 \models \varphi[c, \overline{a}].$$

Since H is a substructure of G_2, it is sufficient to show that there is such a c in H.

If any $n_i \neq 0$, as H is divisible,

$$b = -\frac{\sum_{j=1}^{m_i} n_{ij} a_j}{n_i} \in H$$

and we are done. So, assume that all $n_i = 0$. Then b disappears from the equalities appearing in $(*)$. Since H is infinite, we can certainly find a $c \in H$ satisfying all inequalities in $(*)$. $\square$

Example 2.10.3 The theory $ODAG$ of ordered divisible abelian groups has quantifier elimination.

Proof As in the above case, we take ordered divisible abelian groups G_1 and G_2, a common subgroup H, a conjunction of literals $\varphi[x, \overline{y}]$ and an $\overline{a} \in H$. Assume that there is a $b \in G_1$, such that $G_1 \models \varphi[b, \overline{a}]$. Again, as in the last example, it is sufficient to show that if H' is the ordered divisible hull of H, there is a $c \in H'$ such that $H' \models \varphi[c, \overline{a}]$. Towards showing this, note that we can assume that $\varphi[x, \overline{y}]$ is of the form

$$\wedge_{i=0}^{k-1} \sum_{j=1}^{m_i} (n_{ij} y_j + n_i x = 0) \wedge \wedge_{p=0}^{l-1} (\sum_{j=1}^{r_p} n'_{pj} y_j < n'_p x).$$

Since H' is order-dense, arguing as in the last example, we get a required $c \in H'$. $\square$

Example 2.10.4 Let $\mathbb{K}$ be a field. Then the theory T of infinite vector spaces over $\mathbb{K}$ has quantifier elimination.

Proof Let $V_1, V_2 \models T$ and V be a common subspace of V_1 and V_2. Let $\varphi[x]$ be an open L_V-formula and there exists an $a \in V_1$ such that $V_1 \models \varphi[a]$. We need to show that $V_2 \models \exists x \varphi[x]$.

If $a \in V$, since V is a substructure of V_1 and V_2 and φ open, $V_2 \models \varphi[a]$. Next assume that $a \notin V$. If $V = V_2$, since V_2 is infinite, it has a proper elementary extension, say V_2'. If $V_2' \models \exists x\varphi[x]$, $V_2 \models \exists x\varphi[x]$. Hence, without any loss of generality, we assume that $V \neq V_2$. Let $b \in V_2 \setminus V$. Set $L_1 = span(V \cup \{a\})$ and $L_2 = span(V \cup \{b\})$. There is a linear isomorphism $f : L_1 \to L_2$ fixing V pointwise and $f(a) = b$. This implies that $L_2 \models \varphi[b]$. Since φ is open, $V_2 \models \varphi[b]$. □

Example 2.10.5 The theory ACF of algebraically closed fields has quantifier elimination.

Proof Note that a substructure of a field is an integral domain. Also, recall that if D is an integral domain, its quotient field embeds into every field in which D is embedded. Therefore, as in the last two cases, we only need to show that whenever $\mathbb{F} \subset \mathbb{K}$ are algebraically closed fields, $\varphi[x, \overline{y}]$ a conjunction of literals and $\overline{a} \in \mathbb{F}$, if there is a $b \in \mathbb{K}$ such that $\mathbb{K} \models \varphi[b, \overline{a}]$, there is a $c \in \mathbb{F}$ such that $\mathbb{F} \models \varphi[c, \overline{a}]$. Now note that we can take $\varphi[x, \overline{a}]$ in the form

$$\wedge_{i=0}^{k-1}(P_i(x) = 0) \wedge \wedge_{j=0}^{l-1}(Q_j(x) \neq 0),$$

$P_i[X]$'s and $Q_j[X]$'s are polynomials over the smallest subfield of $\mathbb{F}$ generated by $\overline{a}$. If $k \geq 1$, $b \in \mathbb{F}$ because it is algebraically closed. Otherwise, since $\mathbb{F}$ is infinite, it certainly has a c which is not a root of any $Q_j[X]$ which works for us. □

It is interesting to ask if the converse of Proposition 1.9.17 is true? We shall come back to this question later.

Corollary 2.10.6 *Let $\mathbb{K}$ be an algebraically closed field and $A \subset \mathbb{K}$. Then $a \in acl(A)$ if and only if a is algebraic in usual algebra sense over the subfield k generated by A.*

Proof Let $a \in acl(A)$. By quantifier elimination and the fact that every open formula is equivalent to a formula in disjunctive normal form, there exist polynomial terms $p_i(x, \overline{y})$, $i < n$, $q_j(x, \overline{y})$, $j < m$, and $\overline{a} \in A$ such that

$$\wedge_i p_i(a, \overline{a}) = 0 \wedge \wedge_j q_j(a, \overline{a}) \neq 0,$$

and that this equation has only finitely many solutions. But then $n > 0$. Hence a is algebraic over k. If part is straight forward. □

Exercise 2.10.7 Let $G \models DAG$ and $A \subset G$. Show that $acl(A) = dcl(A)$ and it equals the smallest divisible subgroup of G generated by A.

Exercise 2.10.8 Let V be an infinite vector space over a field $\mathbb{K}$ and $A \subset V$. Show that $acl(A) = dcl(A)$ and it equals the vector subspace of V generated by A.

Example 2.10.9 The theory $RCOF$ of real closed fields has quantifier elimination.

Proof As in the cases of say $ODAG$ and ACF etc. we only need to show that if $\varphi[x, \overline{y}]$ is a conjunction of literals, $\mathbb{F} \subset \mathbb{K} \models RCOF$ and $\overline{a} \in \mathbb{F}$, then

$$\mathbb{K} \models \exists x \varphi[x, \overline{a}] \Rightarrow \mathbb{F} \models \exists x \varphi[x, \overline{a}].$$

We can assume that $\varphi[x, \overline{y}]$ is of the form

$$\wedge_{i=1}^{n}(p_i(x, \overline{y}) = 0) \wedge \wedge_{j=1}^{m}(q_j(x, \overline{y}) > 0),$$

with p_i, q_j being terms.

Choose a $b \in \mathbb{K}$ such that

$$\mathbb{K} \models \varphi[b, \overline{a}].$$

If any of the equality term is present, since $\mathbb{F}$ has no proper real algebraic extension (Theorem B.3.10), $b \in \mathbb{F}$.

So, assume no p_i is present. Since $\mathbb{F}$ has no proper real algebraic extension, roots of q_j's, if any, belong to $\mathbb{F}$. If a q_j has no root in the field and since $q_j(b, \overline{a}) > 0$, by Weierstrass Nullstellensatz (Theorem B.3.9), $q_j(c, \overline{a}) > 0$ for all $c \in \mathbb{F}$. By considering finitely many roots of all q_j's (all of which belong to $\mathbb{F}$), we find a non-empty open interval I in $\mathbb{K}$ with end points in $\mathbb{F}$ such that $b \in I$ and $q_j(x, \overline{a}) > 0$ for all $x \in I$ and for all $1 \leq j \leq m$. Using the order-denseness of $\mathbb{F}$, we have a $b \in \mathbb{F}$ that lies in I. This b witnesses $\mathbb{F} \models \varphi[b, \overline{a}]$. □

Exercise 2.10.10 Show that the theory of random graphs has quantifier elimination and it is complete.

Exercise 2.10.11 Let $\mathbb{K}$ be a field. Show that the theory of infinite vector spaces over $\mathbb{K}$ is complete.

2.11 Strongly Minimal and O-Minimal Theories

As a consequence of the fact that ACF has quantifier elimination, we get

Proposition 2.11.1 *Let $\mathbb{F}$ be an algebraically closed field. Then $\mathbb{F}$ is infinite and $D \subset \mathbb{F}$ is definable if and only if D is either finite or cofinite in $\mathbb{F}$.*

Proof Note that a subset D of $\mathbb{F}$ is defined by an atomic formula if and only if it is the set of all roots of a polynomial in $\mathbb{F}$. Hence, such a set $D \subset \mathbb{F}$ is finite. Boolean algebra of subsets of $\mathbb{F}$ generated by all finite sets consists of all finite and cofinite sets. These are precisely sets defined by open formulas. Our claim is followed by Example 2.10.5. □

The same argument shows the following.

Proposition 2.11.2 *Let G be a torsion-free divisible abelian group. Then G is infinite and $D \subset G$ is definable if and only if D is either finite or cofinite in G.*

Corollary 2.11.3 *$\mathbb{R}$ is not a definable subset of the field $\mathbb{C}$ of complex numbers.*

Corollary 2.11.4 *$\mathbb{Z}$ and $\mathbb{N}$ are not definable subsets of the group $\mathbb{Q}$ of rational numbers.*

Remark 2.11.5 In a remarkable discovery, J. Robinson produced a formula $\varphi[x]$ in the language of rings such that for a rational number r,

$$\mathbb{Q} \models \varphi[r] \Leftrightarrow r \in \mathbb{N}.$$

(See [13, 52]).

Let M be an L-structure and $A \subset M^n$. We call A *minimal* if A is infinite and if for every L_M-formula $\varphi[\overline{x}]$ either $A \cap \varphi(M)$ or $A \backslash \varphi(M) = A \cap \neg\varphi(M)$ is finite. Thus, M is a minimal structure if and only if M is infinite and every definable subset of M is either finite or cofinite in M.

An L_M-formula $\varphi[\overline{x}]$ is called *minimal in M* if $\varphi(M)$ is minimal; it is called *strongly minimal in M* if φ is minimal in every elementary extension of M.

A theory T is called *strongly minimal* if every $M \models T$ is minimal. It follows that if T is strongly minimal, every model of T is strongly minimal. Whatever may be the language L, clearly all finite subsets and their complements in an L-structure M is definable. Thus definable subsets of models of a strongly minimal theory have simplest possible structure. This notion was introduced by Marsh [42]. Its importance was shown by Baldwin and Lachlan to give a simpler proof of Morley categoricity theorem [5].

Example 2.11.6 ACF and DAG are strongly minimal.

Remark 2.11.7 Consider the theory RCF of real closed fields (without order relation). The field of real numbers $\mathbb{R}$ is a model of it. We also have

$$x \geq 0 \Leftrightarrow \exists y(x = y \cdot y).$$

This shows that the real closed field $\mathbb{R}$ is not minimal. Hence, RCF does not admit quantifier elimination.

Exercise 2.11.8 Show that the theory T of vector spaces over a fixed field is strongly minimal.

Proposition 2.11.9 *Let M be an L-structure and $\varphi[\overline{x}]$ an L-formula. The following conditions are equivalent:*

1. *φ is strongly minimal in M.*
2. *φ is minimal in every structure N which is elementarily equivalent to M.*

Proof Since every elementary extension of M is elementarily equivalent to M, clearly (2) implies (1).

Now assume (1) and let N be elementarily equivalent to M. By Proposition 2.8.1, there exists an elementary extension A of M and an elementary embedding $g : N \to A$. Let $\psi[\overline{x}, \overline{y}]$ be an L-formula and $\overline{a} \in N$. By (a), either $\varphi(A) \cap \psi(A, g(\overline{a}))$ or $\varphi(A) \cap \neg\psi(A, g(\overline{a}))$ is finite. Since g is elementary, either $\varphi(N) \cap \psi(N, \overline{a})$ or $\varphi(N) \cap \neg\psi(N, \overline{a})$ is finite. Thus, (1) implies (2). $\square$

Let $(X, <)$ be a linearly ordered set. An interval in X is a subset I of X such that whenever $x \leq y$ are in I and $x \leq z \leq y$, $z \in I$.

Here is a very important class of theories. Let T be a theory whose language has a binary relation symbol $<$ such that for every $M \models T$, $<^M$ is a linear order on M. We call T *O-minimal* if for every $M \models T$, $D \subset M$ is definable if and only if D is a finite union of intervals. Here 'O' stands for order. This concept was defined by Pillay and Steinhorn in [48, 49]. Today O-minimality is a major tool in geometry.

Since the theories DLO, $ODAG$ and $RCOF$ have quantifier elimination, we have the following example.

Example 2.11.10 Theories DLO, $ODAG$ and $RCOF$ are O-minimal.

Example 2.11.11 The theory $ODAG$ of ordered abelian groups has definable Skolem functions. To see this, let $\varphi[\overline{x}, y]$ be a formula. By quantifier elimination, we know that "$\{y : \varphi[\overline{x}, y]\}$ is a finite union of intervals and singletons." We define $\psi[\overline{x}, y]$ as the disjunction of following formulas:

$$(\forall z \neg\varphi[\overline{x}, z] \vee \forall z \varphi[\overline{x}, z]) \wedge y = 0,$$

$$\exists z(\forall u < z \varphi[\overline{x}, u] \wedge \forall w > z \exists v < w \neg\varphi[\overline{x}, v] \wedge y = z - 1),$$

$$\exists z(\forall u > z \varphi[\overline{x}, u] \wedge \forall w < z \exists v > w \neg\varphi[\overline{x}, v] \wedge y = z + 1),$$

$$\exists z_1, z_2(\forall u < z_1 \neg\varphi[\overline{x}, u] \wedge ((\forall z_1 < u < z_2 \varphi[\overline{x}, u]) \vee (z_1 = z_2 \wedge \varphi[\overline{x}, z_1]))$$

$$\wedge \forall v > z_2 \exists z_2 \leq u < v \neg\varphi[\overline{x}, u] \wedge y = \frac{z_1 + z_2}{2}).$$

Then

$$ODAG \models \forall\overline{x}(\exists_{=1} y \psi[\overline{x}, y] \wedge (\exists y \varphi[\overline{x}, y] \to \forall y(\psi[\overline{x}, y] \to \varphi[\overline{x}, y]))).$$

Further, $\psi[\overline{x}, y]$ defines a function F whose graph is the set defined by $\psi[\overline{x}, y]$. We also have

$$ODAG \models \forall\overline{x}\forall\overline{x'}(\forall y(\varphi[\overline{x}, y] \leftrightarrow \varphi[\overline{x'}, y]) \to F(\overline{x}) = F(\overline{x'})).$$

In this sense, we call F invariant.

Example 2.11.12 We extend the idea contained in the last Example further. Let $\varphi[\overline{x}, \overline{y}]$ be a formula of $ODAG$. By induction on the arity n of $\overline{y}$, we show that there exists an invariant definable function $\overline{y} = F(\overline{x})$ such that

$$ODAG \models \forall \overline{x}(\exists \overline{y}\varphi[\overline{x}, \overline{y}] \rightarrow \varphi[\overline{x}, F(\overline{x})]).$$

For $n = 1$, this is done above.

For inductive step, take a formula $\varphi[\overline{x}, y_1, \ldots, y_{n+1}]$. By induction hypothesis, there exists an invariant, definable Skolem function $f(\overline{x}, y_1)$ such that

$$ODAG \models \forall y_1 \forall \overline{x}(\exists y_2 \ldots \exists y_{n+1}\varphi[\overline{x}, \overline{y}] \rightarrow \varphi[\overline{x}, y_1, f(\overline{x}, y_1)]).$$

By case $n = 1$, there exists an invariant definable Skolem function $g(\overline{x})$ such that

$$ODAG \models \forall \overline{x}(\exists y_1\varphi[\overline{x}, y_1, f(\overline{x}, y_1)] \rightarrow \varphi[\overline{x}, g(\overline{x}), f(\overline{x}, g(\overline{x}))]).$$

Now take

$$F(\overline{x}) = (g(\overline{x}), f(\overline{x}, g(\overline{x}))).$$

Then $F(\overline{x})$ is an invariant function such that

$$ODAG \models \forall \overline{x}(\exists \overline{y}\varphi[\overline{x}, \overline{y}] \rightarrow \varphi[\overline{x}, F(\overline{x})]).$$

Further note that if $\varphi[\overline{x}, \overline{y}]$ is an equivalence formula, then $F(\overline{x})$ is a definable section of φ.

Example 2.11.13 Exactly the same arguments as in the last two examples show that $RCOF$ has definable Skolem functions. Further, since we can introduce $<$ in an extension by definition of RCF, we see that RCF too has definable Skolem functions.

Example 2.11.14 By Theorem 1.12.3 it follows that $ODAG$, $RCOF$ and RCF admit uniform elimination of imaginaries.

The theory of algebraically closed fields ACF also admits uniform elimination of imaginaries. However, it requires considerable work. This will be proved in Sect. 4.3.

2.12 Independence and Dimension in Minimal Sets

In this section, we generalise the notions of independence and basis to models of strongly minimal theories.

Theorem 2.12.1 (Exchange Lemma) *Let M be an L-structure, $A \subset M$ and X an L_A-definable minimal set. Let $a, b \in X$ be such that $b \in acl(A \cup \{a\}) \setminus acl(A)$. Then $a \in acl(A \cup \{b\})$.*

Proof If possible, suppose there exists $b \in acl(A \cup \{a\}) \setminus acl(A)$ such that $a \notin acl(A \cup \{b\})$. We shall arrive at a contradiction.

Since $b \in acl(A \cup \{a\})$, there exists an L_A-formula $\varphi[x, y]$ and $n \geq 1$ such that

$$M \models \varphi[b, a] \wedge \exists_{=n} x\varphi[x, a].$$

Since X is minimal A-definable and $a \in X \backslash acl(A \cup \{b\})$, there exists a finite set $Y \subset M$ such that for all $c \in X \backslash Y$,

$$M \models \varphi[b, c] \wedge \exists_{=n} x\varphi[x, c].$$

Let $\psi[y]$ be an L_A-formula that defines X and $|Y| = m$. We have

$$M \models \exists y_1 \dots \exists y_m (\forall y ((\wedge_i (y \neq y_i) \wedge \psi[y]) \rightarrow (\varphi[b, y] \wedge \exists_{=n} x\varphi[x, y])).$$

Since $b \notin acl(A)$, there exists an infinite set $Z \subset M$ such that for all $b' \in Z$,

$$M \models \exists y_1 \dots \exists y_m (\forall y ((\wedge_i (y \neq y_i) \wedge \psi[y]) \rightarrow (\varphi[b', y] \wedge \exists_{=n} x\varphi[x, y])).$$

Take distinct elements $b_0, \dots, b_n \in Z$. Then there exists a $c \in X$ such that

$$M \models \wedge_{i=0}^{n} \varphi[b_i, c] \wedge \exists_{=n} x\varphi[x, c].$$

This is a contradiction. □

We say that $A \subset M$ is *independent* if for every $a \in A, a \notin acl(A \backslash \{a\})$. If $C \subset M$, we say that A is *independent over C* if for every $a \in A, a \notin acl(C \cup (A \backslash \{a\}))$. This, in particular, implies that $A \cap C = \emptyset$. A subset B of A is called a *basis* of A if B is independent and $acl(B) = acl(A)$. Equivalently, B is a maximal independent subset of A.

Proposition 2.12.2 *Let X be an $\emptyset$-definable minimal subset of an L-structure M and A, B independent subsets of X with $A \subset acl(B)$. Then*

1. *Let $A_0 \subset A$, $B_0 \subset B$ and $A_0 \cup B_0$ a basis for $acl(B)$. Then for every $a \in A \backslash A_0$, there is a $b \in B_0$ such that $A_0 \cup \{a\} \cup (B_0 \backslash \{b\})$ is a basis of $acl(B)$.*
2. *$|A| \leq |B|$.*
3. *For every $Y \subset X$, any two bases of Y have the same cardinality.*

Proof Let $C \subset B_0$ be a set of minimum cardinality such that $a \in acl(A_0 \cup C)$. Since A is independent, $C \neq \emptyset$. Take a $b \in C$. Because C is of minimum possible cardinality,

$$a \in acl(A_0 \cup C) \setminus acl((A_0 \cup C) \setminus \{b\}).$$

Therefore, by exchange lemma (Theorem 2.12.1),

$$b \in acl((A_0 \cup \{a\}) \cup (C\backslash\{b\})).$$

Hence,

$$acl(B) = acl((A_0 \cup \{a\}) \cup (B_0\backslash\{b\})).$$

We claim that $a \notin acl(A_0 \cup (B_0\backslash\{b\}))$. For otherwise, $b \in acl(A_0 \cup (B_0\backslash\{b\}))$ which contradicts that $A_0 \cup B_0$ is a basis of $acl(B)$. Using exchange lemma (Theorem 2.12.1) it is easy to see that $(A_0 \cup \{a\}) \cup (B_0\backslash\{b\})$ is independent. Thus, $(A_0 \cup \{a\}) \cup (B_0\backslash\{b\})$ is a basis of $acl(B)$. This proves (1).

First we prove (2) when B is finite. Let $|B| = n$. Set $A_0 = \emptyset$. Take any $a_1 \in A$. Get $b_1 \in B$ such that $\{a_1\} \cup (B\backslash\{b_1\})$ is a basis of $acl(B)$. Such an a_1 exists by (1). Suppose $1 \leq i < n$ and $a_1, \ldots, a_i \in A$ and $b_1, \ldots, b_i \in B$ be such that $\{a_1, \ldots, a_i\} \cup (B\backslash\{b_1, \ldots, b_i\})$ is a basis of $acl(B)$. If $A \neq \{a_1, \ldots, a_i\}$, take any $a_{i+1} \in A\backslash\{a_1, \cdots, a_i\}$. Get $b_{i+1} \in B\backslash\{b_1, \ldots, b_i\}$ such that $\{a_1, \ldots, a_{i+1}\} \cup (B\backslash\{b_1, \ldots, b_{i+1}\}$ is a basis of $acl(B)$. Such an a_{i+1} exists by (1). This process must stop in a maximum of n steps. Thus, $|A| \leq n$. If B is infinite

$$A = \cup\{A \cap acl(B_0) : B_0 \subset B \ \text{ finite}\}.$$

Hence, $|A| \leq |B|$. Thus, (2) is proved.

(3) is a direct corollary of (2). □

Let M be an L-structure, X a $\emptyset$-definable minimal set in M and $A \subset X$. Then any two bases of A have the same cardinality which we call the *dimension* of A, denoted by $dim(A)$.

Proposition 2.12.3 *Let M and N be L-structures, $X \subset M$, $Y \subset N$ and $g : X \to Y$ partial elementary. Suppose $\psi[x]$ is an L-formula minimal in both M and N, $\{a_\alpha : \alpha < \kappa\}$ a sequence in $\psi(M)$ independent over X and $\{b_\alpha : \alpha < \kappa\}$ a sequence in $\psi(N)$ independent over Y. Then the extension $g : X \cup \{a_\alpha : \alpha < \kappa\} \to Y \cup \{b_\alpha : \alpha < \kappa\}$ of g (which we denote by g itself) defined by $g(a_\alpha) = b_\alpha$, $\alpha < \kappa$, is partial elementary.*

Proof Set $g_\beta = g|(X \cup \{a_\alpha : \alpha < \beta\})$, $\beta < \kappa$. Suffices to show that each g_β is partial elementary. This will follow if we show that whenever g_β is partial elementary, so is $g_{\beta+1}$.

Assume that $\beta < \kappa$ and g_β is partial elementary. Take an L-formula $\varphi[\overline{x}, \overline{y}, z]$, $\overline{a} \in \{a_\alpha : \alpha < \beta\}$ and $\overline{b} \in X$. Suppose

$$M \models \varphi[\overline{a}, \overline{b}, a_\beta].$$

Since $\psi(M)$ is minimal and $a_\beta \notin acl(X \cup \{a_\alpha : \alpha < \beta\})$, there exists a natural number m such that

$$M \models \exists_{=m} z(\psi[z] \wedge \neg\varphi[\overline{a}, \overline{b}, z]).$$

Since g_β is partial elementary, we have

$$N \models \exists_{=m} z(\psi[z] \wedge \neg\varphi[g_\beta(\overline{a}), g_\beta(\overline{b}), z].$$

As $\psi(N)$ is minimal and $b_\beta \notin acl(Y \cup \{b_\alpha : \alpha < \beta\})$, we must have

$$N \models \varphi[g_\beta(\overline{a}), g_\beta(\overline{b}), b_\beta].$$

If $M \not\models \varphi[\overline{a}, \overline{b}, a_\beta]$, we repeat the above argument with $\neg\varphi$ to see that $N \not\models \varphi[g_\beta(\overline{a}), g_\beta(\overline{b}), b_\beta]$. Our proof is complete now. □

Our next few exercises show that these notion of independence and basis generalise corresponding notions in vector spaces and fields.

Exercise 2.12.4 Let $\mathbb{K}$ be a field, V an infinite vector space over $\mathbb{K}$ and $A \subset V$. Show the following:

1. A is an independent set if and only if A is linearly independent.
2. A is a basis of V if and only if A is a basis of V in linear algebra sense.
3. $dim(V)$ equals the vector space dimension of V.

Exercise 2.12.5 Let $\mathbb{F}$ be an algebraically closed field and $A \subset \mathbb{F}$. Show the following:

1. A is an independent set if and only if A is algebraically independent.
2. A is a basis of $\mathbb{F}$ if and only if A is a transcendence basis of $\mathbb{F}$.
3. $dim(\mathbb{F})$ equals the transcendence degree of $\mathbb{F}$ over the prime field.

2.13 More Complete Theories

Quantifier elimination can be used to prove completeness of theories.

Proposition 2.13.1 *Let T have quantifier elimination and M an L-structure such that $T \cup Diag(M)$ is consistent. Then $T \cup Diag(M)$ is complete.*

Proof Let $M_1, M_2 \models T \cup Diag(M)$. Then $M \sqsubseteq M_1, M_2$ and $M_1, M_2 \models T$. Take a sentence φ. By quantifier elimination of T, there is an open sentence ψ such that $T \models \varphi \leftrightarrow \psi$. Now

$$M_1 \models \varphi \Leftrightarrow M_1 \models \psi \Leftrightarrow M \models \psi \Leftrightarrow M_2 \models \psi \Leftrightarrow M_2 \models \varphi.$$

This completes the proof. □

An L-structure M is called a *prime structure* of an L-theory T if M is embeddable in every model of T.

Corollary 2.13.2 *If T has quantifier elimination and a prime structure M, then T is complete.*

Proof This follows from the fact that if $N \models T$, then $N \models T \cup Diag(M)$. □

Now note the following:

1. $\mathbb{Q} \models DLO$ and it embeds into all models of DLO.
2. $\mathbb{Q} \models DAG$ and it embeds into all models of DAG.
3. $\mathbb{Q} \models ODAG$ and it embeds into all models of $ODAG$.
4. The field of all algebraic numbers is a model of $ACF(0)$ that embeds into all models of $ACF(0)$.
5. Let p be a prime and $\overline{\mathbb{F}}_p$ the algebraic closure of the field $\mathbb{F}_p$. Then $\overline{\mathbb{F}}_p$ is a model of $ACF(p)$ that embeds into all models of $ACF(p)$.
6. The field $\mathbb{R}_{alg}$ is a real closed field that embeds into all models of RCF.

Thus,

Theorem 2.13.3 *The theories DLO, DAG, $ODAG$, $ACF(p)$, $p = 0$ or prime, and RCF are all complete. Hence, models of these theories are elementarily equivalent.*

A model M of a theory T is called a *prime model* of T if it is elementarily embeddable into every $N \models T$. If T has quantifier elimination, then every model of T which is a prime structure of T is a prime model of T. So, DLO, DAG, $ODAG$, $ACF(p)$, $p = 0$ or prime, and RCF have prime models.

Remark 2.13.4 A word on decidability of theories and decidable structures: Suppose T is a theory with finitely many nonlogical symbols. Then Gödel coded each formula of T, a finite sequence of logical and nonlogical symbols, by a natural number. The theory T is called *axiomatised* if the set of codes of its axioms is computable. In a landmark result, Gödel showed that a complete, axiomatised theory is decidable. It follows that every model of such a T is decidable. Thus, we get many examples of classical structures such as $\mathbb{R}$ as a real closed field, $\mathbb{C}$, $\overline{\mathbb{F}}_p$, p a prime, etc. which are decidable. All these results are due to Tarski. Since this topic is beyond the scope of this book, we refer the reader to [59] for details.

2.14 Model Completeness

A theory T is called *model complete* if whenever $M, N \models T$ and N is a substructure of M, N is an elementary substructure of M. This notion was introduced and used, for instance, to prove Hilbert Nullstellensatz (Theorem 2.15.8) and give a model theoretic proof of Artin's theorem on Hilbert's seventeenth problem (Theorem 2.15.9) in [51].

Proposition 2.14.1 *If T is model complete and has a model which is a prime structure of T, then T is complete.*

Proposition 2.14.2 *If T has quantifier elimination, it is model complete.*

Proof Let $M, N \models T$ and M be a substructure of N. We need to show that the inclusion map $i : M \hookrightarrow N$ is an elementary embedding. Take a formula $\varphi[\overline{x}]$ and an $\overline{a} \in M$. By elimination of quantifiers, there is an open formula $\psi[\overline{x}]$ such that

$$T \models \forall \overline{x}(\varphi[\overline{x}] \leftrightarrow \psi[\overline{x}]).$$

So,

$$M \models \varphi[\overline{a}] \Leftrightarrow M \models \psi[\overline{a}],$$

$$N \models \varphi[\overline{a}] \Leftrightarrow N \models \psi[\overline{a}]$$

and since M is a substructure of N,

$$M \models \psi[\overline{a}] \Leftrightarrow N \models \psi[\overline{a}].$$

The result follows now. □

Corollary 2.14.3 *The theories DLO, DAG, $ODAG$, ACF, RCF and $RCOF$ are model complete.*

Proposition 2.14.4 *Let T be a model complete theory. Then*

1. *The class of all models of T is closed under unions of chains.*
2. *T is a $\forall\exists$ theory.*

Proof By model completeness, every chain of models of T is an elementary chain. Hence, their unions are models of T. By Corollary 2.4.6, (1) implies (2). □

Proposition 2.14.5 *An L-theory T is model complete if and only if for every model M of T, $T \cup Diag(M)$ is a complete theory.*

Proof Note that $T \cup Diag(M)$ is complete if and only if every model of $T \cup Diag(M)$ is elementarily equivalent to M. Further, every model of $T \cup Diag(M)$ is elementarily equivalent to M if and only if T is model complete. The result follows. □

Proposition 2.14.6 *Let T be a theory. The following statements are equivalent:*

1. *T is model complete.*
2. *For every $M, N \models T$ with $N \sqsubseteq M$, for every formula $\varphi[\overline{x}]$ without parameters, for every $\overline{a} \in N$,*
$$M \models \varphi[\overline{a}] \Rightarrow N \models \varphi[\overline{a}].$$
3. *Every model of T is an existentially closed model of T.*
4. *Every existential formula is equivalent in T to a universal formula.*

5. *Every formula* $\varphi[\overline{x}]$ (*without parameters*) *is equivalent in* T *to a universal formula* $\psi[\overline{x}]$ (*without parameters*).
6. *Every formula* $\varphi[\overline{x}]$ (*without parameters*) *is equivalent in* T *to a existential formula* $\xi[\overline{x}]$ (*without parameters*).

Proof (1) implies (2) because for a model complete theory T, every submodel of a model of T is an elementary submodel. (3) is a special case of (2).

Now assume (3). Take $M, N \models T$ with $N \sqsubseteq M$. Let $\varphi[\overline{x}]$ be an existential formula, and $\overline{a} \in N$. By (3), N is existentially closed in M. Hence, $M \models \varphi[\overline{a}] \Rightarrow N \models \varphi[\overline{a}]$. Therefore, by Proposition 2.4.3, φ is equivalent to an universal formula.

Clearly, (4), (5) and (6) are equivalent. (5) and (6) together imply that T is model complete. □

Let T be an L-theory. An L-theory T' is called a *model companion* of T if it satisfies the following three conditions:

1. T' is model complete.
2. Every model T has an extension which is a model of T'.
3. Every model T' has an extension which is a model of T.

Example 2.14.7 1. The theory of infinite sets is a model companion of the empty theory.
2. DLO is a model companion of the theory of linearly ordered sets.
3. DAG is a model companion of the theory of torsion-free abelian groups.
4. $ODAG$ is a model companion of the theory of ordered groups.
5. ACF is a model companion of the theory of integral domains.

Proposition 2.14.8 *A theory* T *can have at most one model companion.*

Proof Let T_0 and T_1 be model companions of T. Start with a model M_0 of T_0. Get an extension M of M_0 that models T. Then get a model N_0 of T_1 that extends M. There exists a model N of T that extends N_0, Now get a model M_1 of T_0 that extends N. Proceeding similarly, we get a chain of L-structures

$$M_0 \sqsubseteq N_0 \sqsubseteq M_1 \sqsubseteq N_1 \sqsubseteq \cdots$$

such that $\{M_k\}$ is a chain of models of T_0 and $\{N_k\}$ is a chain of models of T_1. But T_0 and T_1 are model complete. Hence these two chains are elementary. Let $M' = \cup_k M_k = \cup_k N_k$. Then M_0 is an elementary substructure of M' and $M' \models T_1$. Thus, every model of T_0 is a model of T_1. Likewise, every model of T_1 is a model of T_0. □

Exercise 2.14.9 A linearly ordered set $(D, <)$ is called *discrete* if every element of D that is not the least element has an immediate predecessor and every element that is not the greatest element has an immediate successor. Show that the theory of discrete linear orders with no least element and no greatest element is not model complete. In Exercise 4.7.8 it is shown that this theory is complete.

2.15 Some Applications to Algebra and Geometry

Let $\mathbb{F}$ be a field. A set $C \subset \mathbb{F}^n$ is called constructible if and only if it belongs to the algebra of subsets of $\mathbb{F}^n$ generated by sets of the form $\{\overline{a} \in \mathbb{F}^n : f(\overline{a}) = 0\}$, $f \in \mathbb{F}[X_1, \ldots, X_n]$. Since ACF has quantifier elimination, we have the following result:

Proposition 2.15.1 *For every algebraically closed field $\mathbb{F}$, $C \subset \mathbb{F}^n$ is constructible if and only if it is definable.*

This is a generalisation of

Theorem 2.15.2 (Chevalley Projection Theorem) *If $\mathbb{F}$ is an algebraically closed field and $C \subset \mathbb{F}^{n+1}$ constructible, then its projection $\pi_{\mathbb{F}^n}(C) \subset \mathbb{F}^n$ is constructible.*

If $\mathbb{F}$ is a real closed ordered field, then $D \subset \mathbb{F}^n$ is definable if and only if it belongs to the algebra $\mathcal{A}_n$ of subsets of $\mathbb{F}^n$ generated by sets of the form $\{\overline{a} \in \mathbb{F}^n : p(\overline{a}) < 0\}$, where $p \in \mathbb{F}[X_1, \ldots, X_n]$. Geometers call sets in $\mathcal{A}_n$, $n \geq 1$, *semi-algebraic*. A function $f : \mathbb{F}^n \to \mathbb{F}^m$ is called *semi-algebraic* if its graph is semi-algebraic. So, semi-algebraic sets and functions in a real closed field are precisely those which are definable. This can be thought of as the counterpart of Chevalley's theorem in real case. We now have the following result of Tarski and Seidenberg.

Theorem 2.15.3 (Tarski–Seidenberg Theorem) *If $\mathbb{F}$ is a real closed field and $f : \mathbb{F}^n \to \mathbb{F}^m$, $C \subset \mathbb{F}^n$ and $D \subset \mathbb{F}^m$ semi-algebraic, then $f(C)$ and $f^{-1}(D)$ are semi-algebraic.*

Since RCF is complete, every model of RCF is elementarily equivalent to the ordered field of reals $\mathbb{R}$ or of real algebraic numbers $\mathbb{R}_{alg}$. Hence, $Th(\mathbb{R}) = Th(\mathbb{R}_{alg})$ is the set of all theorems of RCF. This is very useful in proving many theorems of RCF. We illustrate it by proving Rolle's theorem for real closed fields.

Let $\mathbb{F}$ be any field and $\sum_{i=0}^{n} a_i X^i \in \mathbb{F}[X]$. Then the *formal derivative* of f is the polynomial $f'(X) = \sum_{i=1}^{n} i a_i X^{i-1}$.

Theorem 2.15.4 (Rolle's Theorem for Real Closed Fields) *Let $\mathbb{F}$ be a real closed field, $a < b$ in $\mathbb{F}$ and $f \in \mathbb{F}[X]$ be such that $f(a) = f(b)$. Then there is $a < c < b$ such that $f'(c) = 0$.*

Proof For each $d \geq 1$, consider the sentence φ given by

$$\forall\overline{x}\forall x\forall y((x < y \wedge \sum_{i=0}^{d} x_i x^i = \sum_{i=0}^{d} x_i y^i) \to \exists z(x < z < y \wedge \sum_{i=0}^{d-1} i x_i z^i = 0)).$$

By classical Rolle's theorem for $\mathbb{R}$, $\varphi \in Th(\mathbb{R})$. Since RCF is complete, it follows that $RCF \models \varphi$. $\square$

Theorem 2.15.5 *Let φ be a sentence of the language of the theory of fields. The following statements are equivalent:*

(i) $\mathbb{C} \models \varphi$.
(ii) φ *is true in some algebraically closed field of characteristic* 0.
(iii) $ACF(0) \models \varphi$.
(iv) *There is an* m *such that for all prime* $p > m$, $ACF(p) \models \varphi$.
(v) *There is an* m *such that for all prime* $p > m$, φ *is true in some algebraically closed field of characteristic* p.
(vi) $ACF(p) \models \varphi$ *for infinitely many primes* p.

Proof Clearly (i) implies (ii). Since any two models of $ACF(0)$ are elementarily equivalent, (ii) implies (iii). Clearly (iii) implies (i).

Now assume (iii). Then by the compactness theorem, $T \models \varphi$, where T consists of some finitely many axioms of $ACF(0)$. Hence, there is an m such that for no prime $p > m$, $\underline{p} \neq 0$ belongs to T. Thus, $ACF(p) \models \varphi$ for all $p > m$. Thus, (iii) implies (iv).

Clearly (iv) implies (v). The statement (v) implies (iv) because each $ACF(p)$ is complete. (iv) clearly implies (vi).

We now show that (vi) implies (iii). Let $ACF(0) \not\models \varphi$. Since $ACF(0)$ is complete, it follows that $ACF(0) \models \neg\varphi$. Since (iii) implies (v), there is an m such that for all primes $p > m$, $ACF(p) \models \neg\varphi$. This completes the proof. □

Let $p > 0$ be a prime and $\overline{\mathbb{F}}_p$ the algebraic closure of the field with p elements. It is a standard fact of algebra that every finitely generated subfield of $\overline{\mathbb{F}}_p$ is finite. Using this we easily get the following result.

Proposition 2.15.6 *Let* $f_1, \ldots, f_n \in \overline{\mathbb{F}}_p[X_1, \ldots, X_n]$ *be such that* $\overline{f} = (f_1, \ldots, f_n) : \overline{\mathbb{F}}_p^n \to \overline{\mathbb{F}}_p^n$ *is injective. Then* $\overline{f}$ *is surjective.*

Proof Assume that $\overline{f}$ is not surjective. Take any $\overline{b} \notin range(\overline{f})$. Let $\mathbb{K}$ be the smallest subfield of $\overline{\mathbb{F}}_p$ that contains $\overline{b}$ and coefficients of $f_1, \ldots, f_n$. As observed above $\mathbb{K}$ is finite. But then $\overline{f} : \mathbb{K}^n \to \mathbb{K}^n$ is one-to-one but not onto. This is a contradiction since $\mathbb{K}^n$ is finite. □

Theorem 2.15.7 (Ax [1]) *Let* $\mathbb{F}$ *be an algebraically closed field and* $f_1, \ldots, f_n \in \mathbb{F}[X_1, \ldots, X_n]$ *be such that* $\overline{f} = (f_1, \ldots, f_n) : \mathbb{F}^n \to \mathbb{F}^n$ *is injective. Then* $\overline{f}$ *is surjective.*

Proof Let each f_i be of degree at most d. It is not hard to see that there is a sentence φ of the language of fields saying that if $f_1, \ldots, f_n$ are polynomials of degree at most d and if the map $f = (f_1, \ldots, f_n)$ is injective, it is surjective.

Let $\mathbb{F}$ be of characteristic p for some prime $p > 1$. By the last proposition $\overline{\mathbb{F}}_p \models \varphi$. Since any two models of $ACF(p)$ are elementarily equivalent, $\mathbb{F} \models \varphi$. As $\overline{\mathbb{F}}_p \models \varphi$ for all prime $p > 1$, by the above theorem, $ACF(0) \models \varphi$ also. □

We now give some applications of model completeness.

Recall that for an ideal $I \subset \mathbb{K}[\overline{X}]$,

$$\sqrt{I} = \{f \in \mathbb{K}[\overline{X}] : f^n \in I \text{ for some } n \geq 1\}.$$

Then

$$\mathcal{V}(I) = \mathcal{V}(\sqrt{I}).$$

Theorem 2.15.8 (Hilbert Nullstellensatz) *Let $\mathbb{K}$ be an algebraically closed field and I an ideal in $\mathbb{K}[\overline{X}]$. Then*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

Proof We clearly have $\sqrt{I} \subset \mathcal{I}(\mathcal{V}(I))$. If possible, suppose there is an $f \in \mathcal{I}(\mathcal{V}(I)) \setminus \sqrt{I}$. By prime decomposition theorem (Theorem B.2.4), there is a prime ideal $P \supset \sqrt{I}$ not containing f. Since P is a prime ideal in $\mathbb{K}[\overline{X}]$, $\mathbb{K}[\overline{X}]/P$ is an integral domain.

Let $\mathbb{F}$ be the algebraic closure of the quotient field of $\mathbb{K}[\overline{X}]/P$. By Hilbert's basis theorem (Theorem B.2.3), we fix a basis $g_1, \ldots, g_k \in \sqrt{I}$ generating $\sqrt{I}$. Note that each X_i can be regarded as an element of $\mathbb{K}[\overline{X}]$. Because $f \notin P$ and $g_1, \ldots, g_k \in \sqrt{I}$, we have

$$\mathbb{F} \models \wedge_{i=1}^{k} g_i([X_1], \ldots, [X_n]) = 0 \wedge f([X_1], \ldots, [X_n]) \neq 0.$$

In particular,

$$\mathbb{F} \models \exists \overline{y}(\wedge_{i=1}^{k} g_i(\overline{y}) = 0 \wedge f(\overline{y}) \neq 0).$$

By model completeness of RCF,

$$\mathbb{K} \models \exists \overline{y}(\wedge_{i=1}^{k} g_i(\overline{y}) = 0 \wedge f(\overline{y}) \neq 0).$$

This gives an $\overline{a} \in \mathbb{K}$ such that for all $1 \leq i \leq k$, $g_i(\overline{a}) = 0$ and $f(\overline{a}) \neq 0$. But if $g_i(\overline{a}) = 0$ for all $1 \leq i \leq k$, as $g_1, \ldots, g_k$ generate $\sqrt{I}$, $\overline{a} \in \mathcal{V}(\sqrt{I}) = \mathcal{V}(I)$. Since $f \in \mathcal{I}(\mathcal{V}(I))$, $f(\overline{a}) = 0$. This contradiction proves the result. □

17th problem in Hilbert's famous list of 23 problems was

Hilbert's Seventeenth problem *Let $f \in \mathbb{R}(\overline{X})$ be a rational function such that for no $\overline{x} \in \mathbb{R}^n$, $f(\overline{x}) < 0$. Then is it true that f is a sum of squares of finitely many rational functions?*

This problem was answered in the affirmative by Artin. Abraham Robinson pointed out a strikingly beautiful proof of Artin's theorem using model completeness of *RCOF*. We refer the reader to the appendix in algebra and geometry for relevant definitions and results on real closed fields.

Theorem 2.15.9 *Let $\mathbb{F}$ be a real closed field and $f \in \mathbb{F}(\overline{X}) = \mathbb{F}(X_1, \ldots, X_n)$ a rational function over $\mathbb{F}$ in n variables such that for no $\overline{x} \in \mathbb{F}^n$, $f(\overline{x}) < 0$. Then f is a sum of squares of rational functions over $\mathbb{F}$.*

Proof By Proposition B.3.4, the field of rational functions $\mathbb{F}(\overline{X})$ is real. Suppose f is not a sum of squares. By Theorem B.3.7, there is a linear order $<$ on the field $\mathbb{F}(\overline{X})$ of rational functions over $\mathbb{F}$ making it into an ordered field such that $f < 0$.

Let $\mathbb{K}$ be the real closure of $\mathbb{F}(\overline{X})$ order compatible with $<$. Then

$$\mathbb{K} \models \exists \overline{x}(f(\overline{x}) < 0).$$

(Take $x_i = X_i \in \mathbb{K}$.) By model completeness of RCF,

$$\mathbb{F} \models \exists \overline{x}(f(\overline{x}) < 0).$$

But there is no $\overline{a} \in \mathbb{F}$ such that $f(\overline{a}) < 0$. Hence, f must be a sum of squares of rational functions over $\mathbb{F}$. $\square$

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