Chapter 2
Maximum Likelihood Estimation of a Natural Parameter for a One-Sided TEF

For a one-sided truncated exponential family (oTEF) of distributions with a natural parameter θ and a truncation parameter γ as a nuisance parameter, the maximum likelihood estimators (MLEs) $\hat{\theta}_{ML}^{\gamma}$ and $\hat{\theta}_{ML}$ of θ for known γ and unknown γ and the maximum conditional likelihood estimator (MCLE) $\hat{\theta}_{MCL}$ of θ are asymptotically compared up to the second order.

2.1 Introduction

In the presence of nuisance parameters, the asymptotic loss of the maximum likelihood estimator of an interest parameter was discussed by Akahira and Takeuchi (1982) and Akahira (1986) under suitable regularity conditions from the viewpoint of higher order asymptotics. On the other hand, in statistical estimation in multiparameter cases, the conditional likelihood method is well known as a way of eliminating nuisance parameters (see, e.g., Basu 1997). The consistency, asymptotic normality, and asymptotic efficiency of the MCLE were discussed by Andersen (1970), Huque and Katti (1976), Bar-Lev and Reiser (1983), Bar-Lev (1984), Liang (1984), and others. Further, in higher order asymptotics, asymptotic properties of the MCLE of an interest parameter in the presence of nuisance parameters were also discussed by Cox and Reid (1987) and Ferguson (1992) in the regular case. However, in the non-regular case when the regularity conditions do not necessarily hold, the asymptotic comparison of asymptotically efficient estimators has not been discussed enough in the presence of nuisance parameters in higher order asymptotics yet.

For a truncated exponential family of distributions which is regarded as a typical non-regular case, we consider a problem of estimating a natural parameter θ in the presence of a truncation parameter γ as a nuisance parameter. Let $\hat{\theta}_{ML}^{\gamma}$ and $\hat{\theta}_{ML}$ be

M. Akahira, *Statistical Estimation for Truncated Exponential Families*,
JSS Research Series in Statistics, DOI 10.1007/978-981-10-5296-5_2

the MLEs of θ based on a sample of size n when γ is known and γ is unknown, respectively. Let $\hat{\theta}_{MCL}$ be the MCLE of θ. Then, it was shown by Bar-Lev (1984) that the MLEs $\hat{\theta}^{\gamma}_{ML}$,$\hat{\theta}_{ML}$ and the MCLE $\hat{\theta}_{MCL}$ have the same asymptotic normal distribution, hence they are shown to be asymptotically equivalent in the sense of having the same asymptotic variance. A similar result can be derived from the stochastic expansions of the MLEs $\hat{\theta}^{\gamma}_{ML}$ and $\hat{\theta}_{ML}$ in Akahira and Ohyauchi (2012). But, $\hat{\theta}^{\gamma}_{ML}$ for known γ may be asymptotically better than $\hat{\theta}_{ML}$ for unknown γ in the higher order, because $\hat{\theta}^{\gamma}_{ML}$ has the full information on γ. Otherwise, the existence of a truncation parameter γ as a nuisance parameter is meaningless. So, it is a quite interesting problem to compare asymptotically them up to the higher order.

In this chapter, following mostly the paper by Akahira (2016), we compare them up to the second order, i.e., the order of n^{-1}, in the asymptotic variance. We show that a bias-adjusted MLE $\hat{\theta}_{ML^*}$ and $\hat{\theta}_{MCL}$ are second order asymptotically equivalent, but they are asymptotically worse than $\hat{\theta}^{\gamma}_{ML}$ in the second order. We thus calculate the second-order asymptotic losses on the asymptotic variance among them. Several examples are also given.

2.2 Preliminaries

Suppose that $X_1, X_2, \dots, X_n, \dots$ is a sequence of independent and identically distributed (i.i.d.) random variables according to $P_{\theta,\gamma}$ in a oTEF $\mathscr{P}_o$ with the density (1.7). In Bar-Lev (1984), the asymptotic behavior of the MLE $\hat{\theta}_{ML}$ and MCLE $\hat{\theta}_{MCL}$ of a parameter θ in the presence of γ as a nuisance parameter was compared and also done with that of the MLE $\hat{\theta}^{\gamma}_{ML}$ of θ when γ was known. As the result, it was shown there that, for a sample of size $n(\geq 2)$, the $\hat{\theta}_{ML}$ and $\hat{\theta}_{MCL}$ of θ existed with probability 1 and were given as the unique roots of the appropriate maximum likelihood equations. These two estimators were also shown to be strongly consistent for θ with the limiting distribution which coincides with that of the MLE $\hat{\theta}^{\gamma}_{ML}$ of θ when γ was known. Denote a random vector $(X_1, \dots, X_n)$ by $\boldsymbol{X}$ and let $X_{(1)} \leq \cdots \leq X_{(n)}$ be the corresponding order statistics of a random vector $\boldsymbol{X}$. Then, the density (1.7) is considered to belong to a regular exponential family of distributions with a natural parameter θ for any fixed γ, hence $\log b(\theta, \gamma)$ is strictly convex and infinitely differentiable in $\theta \in \Theta$ and

$$\lambda_j(\theta, \gamma) := \frac{\partial^j}{\partial \theta^j} \log b(\theta, \gamma) \tag{2.1}$$

is the j th cumulant corresponding to (1.7) for $j = 1, 2, \dots$.

In the subsequent sections, we obtain the stochastic expansions of $\hat{\theta}_{ML}^{\gamma}$, $\hat{\theta}_{ML}$, and $\hat{\theta}_{MCL}$ up to the second order, i.e., $o_p(n^{-1})$. We get their second-order asymptotic variances and derive the second-order asymptotic losses on the asymptotic variance among them. The proofs of theorems are located in Appendixes A1 and A2.

2.3 MLE $\hat{\theta}_{ML}^{\gamma}$ of a Natural Parameter θ When a Truncation Parameter γ is Known

For given $\boldsymbol{x} = (x_1, \dots, x_n)$ satisfying $\gamma \le x_{(1)} := \min_{1 \le i \le n} x_i$ and $x_{(n)} := \max_{1 \le i \le n} x_i < d$, the likelihood function of θ is given by

$$L^{\gamma}(\theta; \boldsymbol{x}) := \frac{1}{b^n(\theta, \gamma)} \left\{ \prod_{i=1}^{n} a(x_i) \right\} \exp\left\{ \theta \sum_{i=1}^{n} u(x_i) \right\}.$$

Then, the likelihood equation is

$$\frac{1}{n} \sum_{i=1}^{n} u(X_i) - \lambda_1(\theta, \gamma) = 0. \tag{2.2}$$

Since there exists a unique solution of Eq. (2.2) with respect to θ, we denote it by $\hat{\theta}_{ML}^{\gamma}$ which is the MLE of θ (see, e.g., Barndorff-Nielsen (1978) and Bar-Lev (1984)). Let $\lambda_i = \lambda_i(\theta, \gamma)$ $(i = 2, 3, 4)$ and put

$$Z_1 := \frac{1}{\sqrt{\lambda_2 n}} \sum_{i=1}^{n} \{u(X_i) - \lambda_1\}, \quad U_{\gamma} := \sqrt{\lambda_2 n} \left(\hat{\theta}_{ML}^{\gamma} - \theta \right).$$

Then, we have the following.

Theorem 2.3.1 *For the oTEF $\mathscr{P}_o$ of distributions with densities of the form (1.7) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}_{ML}^{\gamma}$ be the MLE of θ when γ is known. Then, the stochastic expansion of U_{γ} is given by*

$$U_{\gamma} = Z_1 - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} Z_1^2 + \frac{1}{2n} \left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2} \right) Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}} \right),$$

and the second-order asymptotic mean and variance are given by

$$E_\theta\left(U_\gamma\right) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right),$$
$$V_\theta\left(U_\gamma\right) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + O\left(\frac{1}{n\sqrt{n}}\right),$$

respectively.

Since $U_\gamma = Z_1 + o_p(1)$, it is seen that U_γ is asymptotically normal with mean 0 and variance 1, which coincides with the result of Bar-Lev (1984).

2.4 Bias-Adjusted MLE $\hat{\theta}_{ML^*}$ of θ When γ is Unknown

For given $\boldsymbol{x} = (x_1, \ldots, x_n)$ satisfying $\gamma \le x_{(1)}$ and $x_{(n)} < d$, the likelihood function of θ and γ is given by

$$L(\theta, \gamma; \boldsymbol{x}) = \frac{1}{b^n(\theta, \gamma)}\left\{\prod_{i=1}^n a(x_i)\right\}\exp\left\{\theta\sum_{i=1}^n u(x_i)\right\}. \tag{2.3}$$

Let $\hat{\theta}_{ML}$ and $\hat{\gamma}_{ML}$ be the MLEs of θ and γ, respectively. From (2.3), it is seen that $\hat{\gamma}_{ML} = X_{(1)}$ and $L(\hat{\theta}_{ML}, X_{(1)}; \boldsymbol{X}) = \sup_{\theta\in\Theta} L(\theta, X_{(1)}; \boldsymbol{X})$, hence $\hat{\theta}_{ML}$ satisfies the likelihood equation

$$0 = \frac{1}{n}\sum_{i=1}^n u(X_i) - \lambda_1(\hat{\theta}_{ML}, X_{(1)}), \tag{2.4}$$

where $\boldsymbol{X} = (X_1, \cdots, X_n)$. Let $\lambda_2 = \lambda_2(\theta, \gamma)$ and put $\hat{U} := \sqrt{\lambda_2 n}(\hat{\theta}_{ML} - \theta)$ and $T_{(1)} := n(X_{(1)} - \gamma)$. Then, we have the following.

Theorem 2.4.1 *For the oTEF $\mathscr{P}_o$ of distributions with densities of the form (1.7) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}_{ML}$ be the MLE of θ when γ is unknown, and $\hat{\theta}_{ML^*}$ be a bias-adjusted MLE such that $\hat{\theta}_{ML}$ has the same asymptotic bias as that of $\hat{\theta}_{ML}^{\gamma}$, i.e.,*

$$\hat{\theta}_{ML^*} = \hat{\theta}_{ML} + \frac{1}{k(\hat{\theta}_{ML}, X_{(1)})\lambda_2(\hat{\theta}_{ML}, X_{(1)})n}\left\{\frac{\partial\lambda_1}{\partial\gamma}\left(\hat{\theta}_{ML}, X_{(1)}\right)\right\}, \tag{2.5}$$

where $k(\theta, \gamma) := a(\gamma)e^{\theta u(\gamma)}/b(\theta, \gamma)$. Then, the stochastic expansion of $\hat{U}^* := \sqrt{\lambda_2 n}(\hat{\theta}_{ML^*} - \theta)$ is given by

$$\hat{U}^* = \hat{U} + \frac{1}{k\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) - \frac{1}{k\lambda_2 n}\left\{\delta + \frac{1}{k}\left(\frac{\partial k}{\partial\theta}\frac{\partial\lambda_1}{\partial\gamma}\right)\right\}Z_1 + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

where $k = k(\theta, \gamma)$*,*

$$\delta = \frac{\lambda_3}{\lambda_2}\left(\frac{\partial \lambda_1}{\partial \gamma}\right) - \frac{\partial \lambda_2}{\partial \gamma},$$

$$\hat{U} = Z_1 - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} Z_1^2 - \frac{1}{\sqrt{\lambda_2 n}}\left(\frac{\partial \lambda_1}{\partial \gamma}\right) T_{(1)} + \frac{\delta}{\lambda_2 n} Z_1 T_{(1)} + \frac{1}{2n}\left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2}\right) Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

and the second-order asymptotic mean and variance are given by

$$E_{\theta,\gamma}(\hat{U}^*) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right),$$

$$V_{\theta,\gamma}(\hat{U}^*) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + \frac{1}{\lambda_2 n}\{u(\gamma) - \lambda_1\}^2 + O\left(\frac{1}{n\sqrt{n}}\right),$$

respectively.

Since $\hat{U} = \hat{U}^* = Z_1 + o_p(1)$, it is seen that $\hat{U}$ and $\hat{U}^*$ are asymptotically normal with mean 0 and variance 1 in the first order, which coincides with the result of Bar-Lev (1984). But, it is noted from Theorems 2.3.1, and 2.4.1 that there is a difference between $V_\theta(U_\gamma)$ and $V_{\theta,\gamma}(\hat{U}^*)$ in the second order, i.e., the order n^{-1}, which is discussed in Sect. 2.6. It is also remarked that the asymptotic distribution of $T_{(1)}$ is exponential in the first order and given up to the second order (see Lemma 2.9.1 in later Appendix A1).

2.5 MCLE $\hat{\theta}_{MCL}$ of θ When γ is Unknown

First, it is seen from (1.7) that there exists a random permutation, say $Y_2, \cdots, Y_n$ of the $(n-1)!$ permutations of $(X_{(2)}, \ldots, X_{(n)})$ such that conditionally on $X_{(1)} = x_{(1)}$, $Y_2, \ldots, Y_n$ are i.i.d. random variables according to a distribution with density

$$g(y; \theta, x_{(1)}) = \frac{a(y)e^{\theta u(y)}}{b(\theta, x_{(1)})} \quad \text{for } x_{(1)} \le y < d$$

with respect to the Lebesgue measure (see Quesenberry (1975) and Bar-Lev (1984)). For given $X_{(1)} = x_{(1)}$, the conditional likelihood function of θ for $\boldsymbol{y} = (y_2, \ldots, y_n)$ satisfying $x_{(1)} \le y_i < d$ $(i = 2, \ldots, n)$ is

$$L(\theta; \boldsymbol{y}|x_{(1)}) = \frac{1}{b^{n-1}(\theta, x_{(1)})}\left\{\prod_{i=2}^n a(y_i)\right\} \exp\left\{\theta \sum_{i=2}^n u(y_i)\right\}.$$

Then, the likelihood equation is

$$\frac{1}{n-1}\sum_{i=2}^{n} u(y_i) - \lambda_1(\theta, x_{(1)}) = 0. \tag{2.6}$$

Since there exists a unique solution on θ of (2.6), we denote it by the MCLE $\hat{\theta}_{MCL}$, i.e., the value of θ for which $L(\theta; \boldsymbol{y}|x_{(1)})$ attains supremum. Let $\tilde{\lambda}_i := \lambda_i(\theta, x_{(1)})$ $(i = 1, 2, 3, 4)$ and put

$$\tilde{Z}_1 := \frac{1}{\sqrt{\tilde{\lambda}_2(n-1)}}\sum_{i=2}^{n}\left\{u(Y_i) - \tilde{\lambda}_1\right\}, \quad \tilde{U}_0 := \sqrt{\lambda_2 n}\left(\hat{\theta}_{MCL} - \theta\right).$$

Then, we have the following.

Theorem 2.5.1 *For a oTEF $\mathscr{P}_o$ of distributions with densities of the form (1.7) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}_{MCL}$ be the MCLE of θ when γ is unknown. Then, the stochastic expansion of $\tilde{U}_0$ is given by*

$$\begin{aligned}\tilde{U}_0 = \tilde{Z}_1 - &\frac{\tilde{\lambda}_3}{2\tilde{\lambda}_2^{3/2}\sqrt{n}}\tilde{Z}_1^2 + \frac{1}{2n}\left\{1 - \frac{1}{\lambda_2}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T_{(1)}\right\}\tilde{Z}_1 \\ &+ \frac{1}{2n}\left(\frac{\tilde{\lambda}_3^2}{\tilde{\lambda}_2^3} - \frac{\tilde{\lambda}_4}{3\tilde{\lambda}_2^2}\right)\tilde{Z}_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right),\end{aligned}$$

and the second-order asymptotic mean and variance are given by

$$\begin{aligned} E_{\theta,\gamma}\left(\tilde{U}_0\right) &= -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right), \\ V_{\theta,\gamma}\left(\tilde{U}_0\right) &= 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + \frac{1}{\lambda_2 n}\{u(\gamma) - \lambda_1\}^2 + O\left(\frac{1}{n\sqrt{n}}\right). \end{aligned}$$

Remark 2.5.1 From Theorems 2.4.1, and 2.5.1, it is seen that the second-order asymptotic mean and variance of $\tilde{U}_0$ are the same as those of $\hat{U}^* = \sqrt{\lambda_2 n}(\hat{\theta}_{ML^*} - \theta)$. It is noted that $\hat{\theta}_{MCL}$ has an advantage over $\hat{\theta}_{ML}$ in the sense of no need of the bias-adjustment.

Remark 2.5.2 As is seen from Theorems 2.3.1, 2.4.1, and 2.5.1, the first terms of order $1/n$ in $V_\theta(U_\gamma)$, $V_{\theta,\gamma}(\hat{U}^*)$, and $V_{\theta,\gamma}(\tilde{U}_0)$ result from the regular part of the density (1.7), which coincides with the fact that the distribution with (1.7) is considered to belong to a regular exponential family of distributions when γ is known. The second terms of order $1/n$ in $V_{\theta,\gamma}(\hat{U}^*)$ and $V_{\theta,\gamma}(\tilde{U}_0)$ follow from the non-regular (i.e., truncation) part of (1.7) when γ is unknown, which means a ratio of the variance $\lambda_2 = V_{\theta,\gamma}(u(X)) = E_{\theta,\gamma}[\{u(X) - \lambda_1\}^2]$ to the distance $\{u(\gamma) - \lambda_1\}^2$ from the mean λ_1 of $u(X)$ to $u(x)$ at $x = \gamma$.

2.6 Second-Order Asymptotic Comparison Among $\hat{\theta}^{\gamma}_{ML}$, $\hat{\theta}_{ML^*}$, and $\hat{\theta}_{MCL}$

From the results in the previous sections, we can asymptotically compare the estimators $\hat{\theta}^{\gamma}_{ML}$, $\hat{\theta}_{ML^*}$, and $\hat{\theta}_{MCL}$ using their second-order asymptotic variances as follows.

Theorem 2.6.1 *For a oTEF $\mathscr{P}_o$ of distributions with densities of the form (1.7) with a natural parameter θ and a truncation parameter γ, let $\hat{\theta}^{\gamma}_{ML}$, $\hat{\theta}_{ML^*}$, and $\hat{\theta}_{MCL}$ be the MLE of θ when γ is known, the bias-adjusted MLE of θ when γ is unknown and the MCLE of θ when γ is unknown, respectively. Then, the bias-adjusted MLE $\hat{\theta}^*_{ML}$ and the MCLE $\hat{\theta}_{MCL}$ are second order asymptotically equivalent in the sense that*

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{MCL}) := n\left\{V_{\theta,\gamma}(\hat{U}^*) - V_{\theta,\gamma}(\tilde{U}_0)\right\} = o(1) \tag{2.7}$$

as $n \to \infty$ and they are second order asymptotically worse than $\hat{\theta}^{\gamma}_{ML}$ with the following second-order asymptotic losses of $\hat{\theta}_{ML^*}$and $\hat{\theta}_{MCL}$ relative to $\hat{\theta}^{\gamma}_{ML}$:

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}^{\gamma}_{ML}) := n\left\{V_{\theta,\gamma}(\hat{U}^*) - V_{\theta}(U_{\gamma})\right\} = \frac{\{u(\gamma) - \lambda_1\}^2}{\lambda_2} + o(1), \tag{2.8}$$

$$d_n(\hat{\theta}_{MCL}, \hat{\theta}^{\gamma}_{ML}) := n\left\{V_{\theta,\gamma}(\tilde{U}_0) - V_{\theta}(U_{\gamma})\right\} = \frac{\{u(\gamma) - \lambda_1\}^2}{\lambda_2} + o(1) \tag{2.9}$$

as $n \to \infty$, respectively.

The proof is straightforward from Theorems 2.3.1, 2.4.1, and 2.5.1.

Remark 2.6.1 It is seen from (1.6) and (2.8) that the ratio of the asymptotic variance of $\hat{U}^*$ to that of U_{γ} is given by

$$R_n(\hat{\theta}_{ML^*}, \hat{\theta}^{\gamma}_{ML}) = 1 + \frac{\{u(\gamma) - \lambda_1\}^2}{\lambda_2 n} + o\left(\frac{1}{n}\right),$$

and similarly from (1.6) and (2.9)

$$R_n(\hat{\theta}_{MCL}, \hat{\theta}^{\gamma}_{ML}) = 1 + \frac{\{u(\gamma) - \lambda_1\}^2}{\lambda_2 n} + o\left(\frac{1}{n}\right).$$

From the consideration of models in Sect. 1.1, using (1.5), (1.6), and (2.8) we see that the difference between the asymptotic models $M(\hat{\theta}_{ML^*}, \gamma)$ and $M(\hat{\theta}^{\gamma}_{ML}, \gamma)$ is given by $d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ or $R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ up to the second order, through the MLE of θ. In a similar way to the above, the difference between $M(\hat{\theta}_{MCL}, \gamma)$ and $M(\hat{\theta}_{MCL}^{\gamma}, \gamma)$ is given by $d_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma})$ or $R_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma})$ up to the second order.

2.7 Examples

Examples on the second-order asymptotic losses of the estimators are given for a lower-truncated exponential, a lower-truncated normal, and Pareto, a lower-truncated beta and a lower-truncated Erlang type distributions.

Example 2.7.1 (**Lower-truncated exponential distribution**) Let $c = -\infty, d = \infty$, $a(x) = 1$, and $u(x) = -x$ for $-\infty < \gamma \le x < \infty$ in the density (1.7). Since $b(\theta, \gamma) = e^{-\theta\gamma}/\theta$ for $\theta \in \Theta = (0, \infty)$, it follows from (2.1) that

$$\lambda_1 = \frac{\partial}{\partial\theta} \log b(\theta, \gamma) = -\gamma - \frac{1}{\theta},$$
$$\lambda_2 = \frac{\partial^2}{\partial\theta^2} \log b(\theta, \gamma) = \frac{1}{\theta^2}, \quad k(\theta, \gamma) = \theta.$$

From (2.2) and (2.4)–(2.6), we have

$$\hat{\theta}_{ML}^{\gamma} = 1/(\bar{X} - \gamma), \quad \hat{\theta}_{ML} = 1/(\bar{X} - X_{(1)}),$$
$$\hat{\theta}_{ML^*} = \hat{\theta}_{ML} - \frac{1}{n}\hat{\theta}_{ML}, \quad \hat{\theta}_{MCL} = 1 \bigg/ \left(\frac{1}{n-1} \sum_{i=2}^{n} X_{(i)} - X_{(1)} \right).$$

Note that $\hat{\theta}_{ML^*} = \hat{\theta}_{MCL}$. In this case, the first part in Theorem 2.6.1 is trivial, since $d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{MCL}) = 0$. From Theorem 2.6.1, we obtain the second-order asymptotic loss

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma}) = d_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma}) = 1 + o(1)$$

as $n \to \infty$. Note that the loss is independent of γ up to the order $o(1)$. From Remark 2.6.1, we have the ratio

$$R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma}) = R_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma}) = 1 + \frac{1}{n} + o\left(\frac{1}{n}\right).$$

In this case, we have the uniformly minimum variance unbiased (UMVU) estimator

$$\hat{\theta}_{UMVU}^{\gamma} = (n-1) \bigg/ \left(\sum_{i=1}^{n} X_i - n\gamma \right),$$

where γ is known (see Voinov and Nikulin 1993). Then,

$$\hat{\theta}_{ML}^{\gamma} = \left(1 + \frac{1}{n-1}\right) \hat{\theta}_{UMVU}^{\gamma},$$

hence $\hat{\theta}_{ML}^{\gamma}$ is not unbiased for any fixed n. When γ is unknown, we obtain the UMVU estimator

$$\hat{\theta}_{UMVU} = (n-2) \Bigg/ \left\{ \sum_{i=2}^{n} X_{(i)} - (n-1)X_{(1)} \right\}$$

which is derived from the formula in the lower-truncated exponential distribution of a general type discussed by Lwin (1975) and Voinov and Nikulin (1993). Then,

$$\hat{\theta}_{ML^*} = \hat{\theta}_{MCL} = \left(1 + \frac{1}{n-2}\right) \hat{\theta}_{UMVU},$$

hence $\hat{\theta}_{ML^*}$ and $\hat{\theta}_{MCL}$ are not unbiased for any fixed n. Note that $\hat{\theta}_{ML^*}$ and $\hat{\theta}_{MCL}$ are asymptotically compared with $\hat{\theta}_{ML}^{\gamma}$ after such a bias adjustment that $\hat{\theta}_{ML}$ has the same asymptotic bias given in Theorem 2.3.1 as $\hat{\theta}_{ML}^{\gamma}$. Since $\lambda_2 = 1/\theta^2$,

$$V_{\theta}(\hat{\theta}_{UMVU}^{\gamma}) = \frac{\theta^2}{n-2}, \quad V_{\theta,\gamma}(\hat{\theta}_{UMVU}) = \frac{\theta^2}{n-3},$$

we have the second-order asymptotic loss

$$\begin{aligned} d_n(\hat{\theta}_{UMVU}, \hat{\theta}_{UMVU}^{\gamma}) &= n \left\{ V_{\theta,\gamma}\left(\frac{\sqrt{n}}{\theta}(\hat{\theta}_{UMVU} - \theta)\right) - V_{\theta}\left(\frac{\sqrt{n}}{\theta}(\hat{\theta}_{UMVU}^{\gamma} - \theta)\right)\right\} \\ &= \frac{n^2}{(n-2)(n-3)} = 1 + o(1) \end{aligned}$$

as $n \to \infty$.

Example 2.7.2 (**Lower-truncated normal distribution**) Let $c = -\infty$, $d = \infty$, $a(x) = e^{-x^2/2}$ and $u(x) = x$ for $-\infty < \gamma \le x < \infty$ in the density (1.7). Since

$$b(\theta, \gamma) = \sqrt{2\pi} e^{\theta^2/2} \Phi(\theta - \gamma)$$

for $\theta \in \Theta = (-\infty, \infty)$, it follows from (2.1) and Theorem 2.4.1 that

$$\begin{aligned} &\lambda_1(\theta, \gamma) = \theta + \rho(\theta - \gamma), \quad \frac{\partial \lambda_1}{\partial \gamma}(\theta, \gamma) = (\theta - \gamma)\rho(\theta - \gamma) + \rho^2(\theta - \gamma), \\ &\lambda_2(\theta, \gamma) = 1 - (\theta - \gamma)\rho(\theta - \gamma) - \rho^2(\theta - \gamma), \quad k(\theta, \gamma) = \rho(\theta - \gamma), \end{aligned}$$

Table 2.1 Values of $d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ and $R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ for $\tau = \theta - \gamma = 2, 4, 6$

τ	$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$	$R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$
2	$4.7640 + o(1)$	$1 + \dfrac{4.7640}{n} + o\left(\dfrac{1}{n}\right)$
4	$9.1486 + o(1)$	$1 + \dfrac{9.1486}{n} + o\left(\dfrac{1}{n}\right)$
6	$16.0096 + o(1)$	$1 + \dfrac{16.0096}{n} + o\left(\dfrac{1}{n}\right)$

where $\rho(t) := \phi(t)/\Phi(t)$ with $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$ and $\phi(t) = (1/\sqrt{2\pi})e^{-t^2/2}$ for $-\infty < t < \infty$. Then, it follows from (2.2), (2.4), and (2.6) that the solutions of θ of the following equations

$$\theta + \rho(\theta - \gamma) = \bar{X}, \quad \theta + \rho(\theta - X_{(1)}) = \bar{X},$$
$$\theta + \rho(\theta - X_{(1)}) = \frac{1}{n-1}\sum_{i=2}^{n} X_{(i)}$$

become $\hat{\theta}_{ML}^{\gamma}$, $\hat{\theta}_{ML}$, and $\hat{\theta}_{MCL}$, respectively, where $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$. From (2.5), the bias-adjusted MLE is given by

$$\hat{\theta}_{ML^*} = \hat{\theta}_{ML} + \frac{\hat{\theta}_{ML} - X_{(1)} + \rho(\hat{\theta}_{ML} - X_{(1)})}{1 - (\hat{\theta}_{ML} - X_{(1)})\rho(\hat{\theta}_{ML} - X_{(1)}) - \rho^2(\hat{\theta}_{ML} - X_{(1)})}.$$

From Theorem 2.6.1, we obtain the second-order asymptotic losses

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{MCL}) = o(1),$$
$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma}) = d_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma}) = \frac{\{\theta - \gamma + \rho(\theta - \gamma)\}^2}{1 - (\theta - \gamma)\rho(\theta - \gamma) - \rho^2(\theta - \gamma)} + o(1)$$

as $n \to \infty$.

When $\tau := \theta - \gamma = 2, 4, 6$, the value of second-order asymptotic loss $d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ and the ratio $R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ up to the order $1/n$ are obtained from (2.8) and Remark 2.6.1 (see Table 2.1 and Fig. 2.1).

Example 2.7.3 (**Pareto distribution**) Let $c = 0$, $d = \infty$, $a(x) = 1/x$, and $u(x) = -\log x$ for $0 < \gamma \le x < \infty$ in the density (1.7). Then, $b(\theta, \gamma) = 1/(\theta\gamma^{\theta})$ for $\theta \in \Theta = (0, \infty)$. Letting $t = \log x$ and $\gamma_0 = \log \gamma$, we see that (1.7) becomes

$$f(t; \theta, \gamma_0) = \begin{cases} \theta e^{\theta\gamma_0} e^{-\theta t} & \text{for } t \ge \gamma_0, \\ 0 & \text{for } t < \gamma_0. \end{cases}$$

Hence, the Pareto case is reduced to the truncated exponential one in Example 2.7.1. Replacing $\bar{X}$ and $X_{(i)}$ $(i = 1, \ldots, n)$ by $\overline{\log X} := (1/n)\sum_{i=1}^{n} \log X_i$ and $\log X_{(i)}$

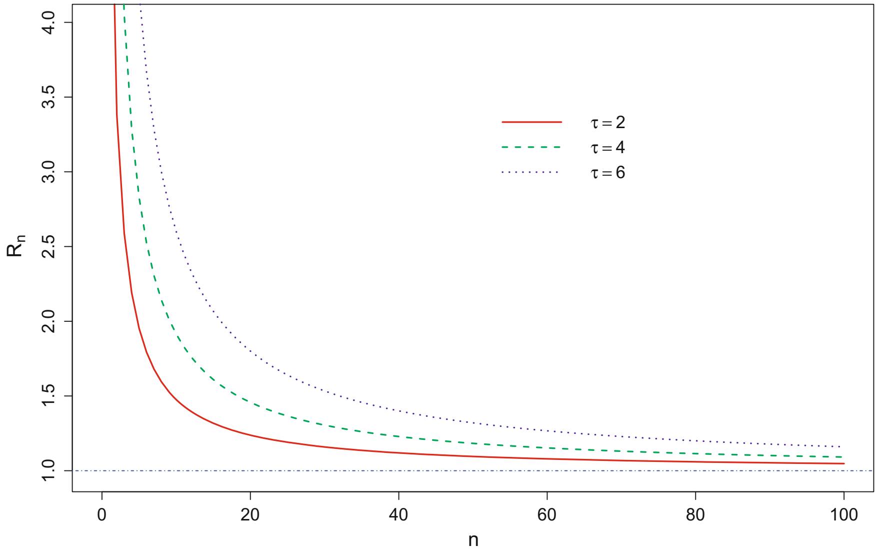

Fig. 2.1 Graph of the ratio $R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ up to the order $1/n$ for $\tau = \theta - \gamma = 2, 4, 6$

$(i = 1, \dots, n)$, respectively, in Example 2.7.1, we have the same results including the UMVU estimation as those in Example 2.7.1. For Pareto distributions, see also Arnold (2015).

Example 2.7.4 (**Lower-truncated beta distribution**) Let $c = 0$, $d = 1$, $a(x) = x^{-1}$ and $u(x) = \log x$ for $0 < \gamma \le x < 1$ in the density (1.7). Since $b(\theta, \gamma) = \theta^{-1}(1-\gamma^{\theta})$ for $\theta \in \Theta = (0, \infty)$, it follows from (2.1) and Theorem 2.4.1 that

$$\lambda_1(\theta, \gamma) = -\frac{1}{\theta} - \frac{(\log\gamma)\gamma^{\theta}}{1-\gamma^{\theta}}, \quad \frac{\partial\lambda_1}{\partial\gamma}(\theta, \gamma) = -\frac{\gamma^{\theta-1}}{(1-\gamma^{\theta})^2}(1 - \gamma^{\theta} + \theta\log\gamma),$$
$$\lambda_2(\theta, \gamma) = \frac{1}{\theta^2} - \frac{(\log\gamma)^2\gamma^{\theta}}{(1-\gamma^{\theta})^2}, \quad k(\theta, \gamma) = \frac{\theta\gamma^{\theta-1}}{1-\gamma^{\theta}}.$$

Then, it follows from (2.2), (2.4), and (2.6) that the solution of θ of the following equations

$$\frac{1}{n}\sum_{i=1}^{n}\log X_i + \frac{1}{\theta} + \frac{(\log\gamma)\gamma^{\theta}}{1-\gamma^{\theta}} = 0,$$
$$\frac{1}{n}\sum_{i=1}^{n}\log X_i + \frac{1}{\theta} + \frac{(\log X_{(1)})X_{(1)}^{\theta}}{1-X_{(1)}^{\theta}} = 0,$$
$$\frac{1}{n-1}\sum_{i=2}^{n}\log X_{(i)} + \frac{1}{\theta} + \frac{(\log X_{(1)})X_{(1)}^{\theta}}{1-X_{(1)}^{\theta}} = 0$$

Table 2.2 Values of $d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ and $R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ for $\theta = 2$ and $\gamma = 1/2, 1/3, 1/5$

γ	$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$	$R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$
$1/2$	$4.9346 + o(1)$	$1 + \dfrac{4.9346}{n} + o\left(\dfrac{1}{n}\right)$
$1/3$	$6.7471 + o(1)$	$1 + \dfrac{6.7471}{n} + o\left(\dfrac{1}{n}\right)$
$1/5$	$10.0611 + o(1)$	$1 + \dfrac{10.0611}{n} + o\left(\dfrac{1}{n}\right)$

becomes $\hat{\theta}_{ML}^{\gamma}$, $\hat{\theta}_{ML}$, and $\hat{\theta}_{MCL}$, respectively. From (2.5), the bias-adjusted MLE of θ is given by

$$\hat{\theta}_{ML^*} = \hat{\theta}_{ML} - \frac{\hat{\theta}_{ML}(1 - X_{(1)}^{\hat{\theta}_{ML}})(1 - X_{(1)}^{\hat{\theta}_{ML}} + \hat{\theta}_{ML} \log X_{(1)})}{n\{(1 - X_{(1)}^{\hat{\theta}_{ML}})^2 - \hat{\theta}_{ML}^2 X_{(1)}^{\hat{\theta}_{ML}} (\log X_{(1)})^2\}}.$$

From Theorem 2.6.1, we obtain the second-order asymptotic losses

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{MCL}) = o(1),$$
$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma}) = d_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma}) = \frac{(1 - \gamma^{\theta} + \theta \log \gamma)^2}{(1 - \gamma^{\theta})^2 - \theta^2 \gamma^{\theta} (\log \gamma)^2}.$$

When $\theta = 2$ and $\gamma = 1/2, 1/3, 1/5$, the values of second-order asymptotic loss $d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ and the ratio $R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ up to the order $1/n$ are obtained from (2.8) and Remark 2.6.1 (see Table 2.2 and Fig. 2.2).

Example 2.7.5 (**Lower-truncated Erlang type distribution**) Let $c = 0, d = \infty$, $a(x) = |x|^{j-1}$ and $u(x) = -|x|$ for $-\infty < \gamma \le x < \infty$ in the density (1.7), where $j = 1, 2, \ldots$. Note that the distribution is a lower-truncated Erlang distribution when $\gamma > 0$ and a one-sided truncated bilateral exponential distribution when $j = 1$. Since for each $j = 1, 2, \ldots,$

$$b_j(\theta, \gamma) = \int_{\gamma}^{\infty} |x|^{j-1} e^{-\theta|x|} dx,$$

it follows that $\Theta = (0, \infty)$. Let j be arbitrarily fixed in $\{1, 2, \ldots\}$ and $\lambda_{ji}(\theta, \gamma) = (\partial^i / \partial\theta^i) \log b_j(\theta, \gamma)$ $(i = 1, 2, \ldots)$. Since $\partial b_j / \partial\theta = -b_{j+1}$, it follows from (2.1) and Theorem 2.4.1 that

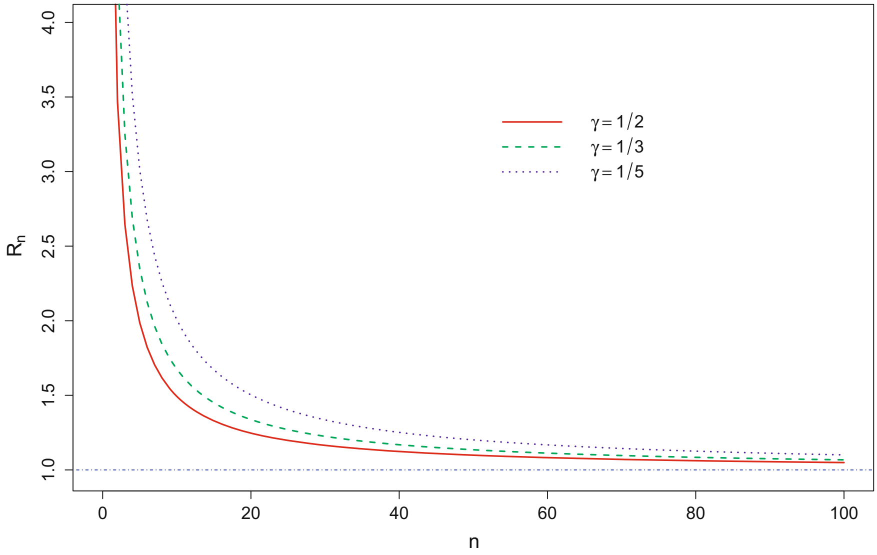

Fig. 2.2 Graph of the ratio $R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma})$ up to the order $1/n$ for $\theta = 2$ and $\gamma = 1/2, 1/3, 1/5$

$$\lambda_{j1}(\theta, \gamma) = -\frac{b_{j+1}(\theta, \gamma)}{b_j(\theta, \gamma)},$$

$$\frac{\partial \lambda_{j1}}{\partial \gamma}(\theta, \gamma) = \begin{cases} \frac{\gamma^{j-1} e^{-\theta\gamma}}{b_j(\theta,\gamma)} \left\{ \frac{b_{j+1}(\theta,\gamma)}{b_j(\theta,\gamma)} + \gamma \right\} & \text{for } \gamma > 0, \\ (-1)^j \frac{\gamma^{j-1} e^{\theta\gamma}}{b_j(\theta,\gamma)} \left\{ \frac{b_{j+1}(\theta,\gamma)}{b_j(\theta,\gamma)} + \gamma \right\} & \text{for } \gamma \leq 0, \end{cases}$$

$$\lambda_{j2}(\theta, \gamma) = \frac{b_{j+2}(\theta, \gamma)}{b_j(\theta, \gamma)} - \left\{ \frac{b_{j+1}(\theta, \gamma)}{b_j(\theta, \gamma)} \right\}^2,$$

$$\lambda_{j3}(\theta, \gamma) = -\frac{b_{j+3}(\theta, \gamma)}{b_j(\theta, \gamma)} + \frac{3b_{j+1}(\theta, \gamma)b_{j+2}(\theta, \gamma)}{b_j^2(\theta, \gamma)} - 2\left\{ \frac{b_{j+1}(\theta, \gamma)}{b_j(\theta, \gamma)} \right\}^3,$$

$$k_j(\theta, \gamma) = \frac{|\gamma|^{j-1} e^{-\theta|\gamma|}}{b_j(\theta, \gamma)}.$$

Then, it follows from (2.2), (2.4), and (2.6) that the solutions of θ of the equations

$$\bar{X} - \frac{b_{j+1}(\theta, \gamma)}{b_j(\theta, \gamma)} = 0, \quad \bar{X} - \frac{b_{j+1}(\hat{\theta}_{ML}, X_{(1)})}{b_j(\hat{\theta}_{ML}, X_{(1)})} = 0,$$

$$\frac{1}{n-1} \sum_{i=2}^{n} X_{(i)} - \frac{b_{j+1}(\theta, X_{(1)})}{b_j(\theta, X_{(1)})} = 0$$

becomes $\hat{\theta}_{ML}^{\gamma}$, $\hat{\theta}_{ML}$, and $\hat{\theta}_{MCL}$, respectively, where $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$. From (2.5), we have the bias-adjusted MLE $\hat{\theta}_{ML^*}$ of θ. From Theorem 2.6.1, we obtain the second-order asymptotic losses

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{MCL}) = o(1),$$
$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma}) = d_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma}) = \left(|\gamma| - \frac{b_{j+1}}{b_j}\right)^2 \Big/ \left\{\frac{b_{j+2}}{b_j} - \left(\frac{b_{j+1}}{b_j}\right)^2\right\},$$

where $b_j = b_j(\theta, \gamma)$. In particular, we consider the case when $\gamma \geq 0$ and $j = 2$. Since

$$b_2 = \frac{1}{\theta}\left(\gamma + \frac{1}{\theta}\right)e^{-\theta\gamma}, \quad b_3 = \frac{1}{\theta}\left(\gamma^2 + \frac{2\gamma}{\theta} + \frac{2}{\theta^2}\right)e^{-\theta\gamma},$$
$$b_4 = \frac{1}{\theta}\left(\gamma^3 + \frac{3\gamma^2}{\theta} + \frac{6\gamma}{\theta^2} + \frac{6}{\theta^3}\right)e^{-\theta\gamma}, \quad b_5 = \frac{1}{\theta}\left(\gamma^4 + \frac{4\gamma^3}{\theta} + \frac{12\gamma^2}{\theta^2} + \frac{24\gamma}{\theta^3} + \frac{24}{\theta^4}\right)e^{-\theta\gamma},$$

we obtain λ_{21}, λ_{22}, and λ_{23}. From (2.2), (2.4), and (2.6), we have

$$\hat{\theta}_{ML}^{\gamma} = 4\left\{\bar{X} - 2\gamma + \sqrt{4\gamma(\bar{X} - \gamma) + \bar{X}^2}\right\}^{-1},$$
$$\hat{\theta}_{ML} = 4\left\{\bar{X} - 2X_{(1)} + \sqrt{4X_{(1)}(\bar{X} - X_{(1)}) + \bar{X}^2}\right\}^{-1},$$
$$\hat{\theta}_{MCL} = 4\left\{\tilde{X} - 2X_{(1)} + \sqrt{4X_{(1)}(\tilde{X} - X_{(1)}) + \tilde{X}^2}\right\}^{-1},$$

where $\tilde{X} = (1/(n-1))\sum_{i=2}^{n} X_{(i)}$. From (2.5), we also obtain the bias-adjusted MLE $\hat{\theta}_{ML^*}$ of θ. Further, we have the second-order asymptotic loss

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma}) = d_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma}) = \frac{(\theta\gamma + 2)^2}{(\theta\gamma + 2)^2 - 2} + o(1)$$

and the ratio

$$R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{\gamma}) = R_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{\gamma}) = 1 + \frac{(\theta\gamma + 2)^2}{n\{(\theta\gamma + 2)^2 - 2\}} + O\left(\frac{1}{n}\right).$$

If $\gamma = 0$, then $\hat{\theta}_{ML}^{0} = 2/\bar{X}$ and

$$d_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{0}) = d_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{0}) = 2 + o(1),$$
$$R_n(\hat{\theta}_{ML^*}, \hat{\theta}_{ML}^{0}) = R_n(\hat{\theta}_{MCL}, \hat{\theta}_{ML}^{0}) = 1 + \frac{2}{n} + O\left(\frac{1}{n}\right).$$

In Vancak et al. (2015), the ratio of the mean squared error (MSE) of $\hat{\theta}_{ML}$ to that of $\hat{\theta}_{ML}^0$ is calculated by simulation and its graph is given as a function of n when $\theta = -1$. Here, we can theoretically obtain the function. Indeed, letting $\gamma = 0$ and

$$U_0 = \sqrt{\lambda_{22}(\theta, 0)}(\hat{\theta}_{ML}^0 - \theta),$$

we have from (2.13) in Appendix A1 later

$$E_\theta(U_0^2) = 1 + \frac{1}{n}\left\{\frac{11\lambda_{23}^2(\theta, 0)}{4\lambda_{22}^3(\theta, 0)} - \frac{\lambda_{24}(\theta, 0)}{\lambda_{22}^2(\theta, 0)}\right\} + O\left(\frac{1}{n\sqrt{n}}\right),$$

hence the MSE of $\hat{\theta}_{ML}^0$ is given by

$$\begin{aligned} MSE_\theta(\hat{\theta}_{ML}^0) = E_\theta[(\hat{\theta}_{ML}^0 - \theta)^2] &= \frac{1}{\lambda_{22}(\theta, 0)n}E_\theta(U_0^2) \\ &= \frac{1}{\lambda_{22}(\theta, 0)n}\left[1 + \frac{1}{n}\left\{\frac{11\lambda_{23}^2(\theta, 0)}{4\lambda_{22}^3(\theta, 0)} - \frac{\lambda_{24}(\theta, 0)}{\lambda_{22}^2(\theta, 0)}\right\} + O\left(\frac{1}{n\sqrt{n}}\right)\right]. \end{aligned}$$

When γ is unknown, letting

$$\hat{U} = \sqrt{\lambda_{22}(\theta, \gamma)}(\hat{\theta}_{ML} - \theta),$$

we have from (2.24) in Appendix A1 given later

$$\begin{aligned} E_\theta(\hat{U}^2) =& 1 - \frac{2}{k_2\lambda_{22}n}\left(\frac{\partial\lambda_{21}}{\partial\gamma}\right)\left\{u(\gamma) - \lambda_{21} + \frac{1}{k_2}\left(\frac{\partial\lambda_{21}}{\partial\gamma}\right)\right\} + \frac{11\lambda_{23}^2}{4\lambda_{22}^3 n} \\ &+ \frac{3\lambda_{23}}{k_2\lambda_{22}^2 n}\left(\frac{\partial\lambda_{21}}{\partial\gamma}\right) - \frac{2}{k_2\lambda_{22}n}\left(\frac{\partial\lambda_{22}}{\partial\gamma}\right) - \frac{\lambda_{24}}{\lambda_{22}^2 n} + O\left(\frac{1}{n\sqrt{n}}\right), \end{aligned}$$

where $k_2 = k_2(\theta, \gamma) = a(\gamma)e^{\theta u(\gamma)}/b_2(\theta, \gamma)$ and $\lambda_{2j} = \lambda_{2j}(\theta, \gamma)$ $(j = 1, 2, 3, 4)$. From (2.28), and (2.29) in Appendix A1 given later, we have

$$\frac{\partial\lambda_{21}}{\partial\gamma}(\theta, \gamma) = k_2(\theta, \gamma)\{\lambda_{21}(\theta, \gamma) - u(\gamma)\}, \qquad \frac{\partial k_2}{\partial\theta}(\theta, \gamma) = k_2(\theta, \gamma)\{u(\gamma) - \lambda_{21}(\theta, \gamma)\}.$$

Since

$$\frac{\partial b_2}{\partial\gamma}(\theta, \gamma) = -k_2(\theta, \gamma)b_2(\theta, \gamma),$$

it follows that

$$\frac{\partial\lambda_{22}}{\partial\gamma}(\theta, \gamma) = -\frac{\partial^2 k_2}{\partial\theta^2}(\theta, \gamma) = -k_2(\theta, \gamma)\{u(\gamma) - \lambda_{21}\}^2 + k_2(\theta, \gamma)\lambda_{22}(\theta, \gamma),$$

hence

$$E_{\theta,\gamma}(\hat{U}^2) = 1 + \frac{11\lambda_{23}^2}{4\lambda_{22}^3 n} - \frac{\lambda_{24}}{\lambda_{22}^2 n} - \frac{3\lambda_{23}}{\lambda_{22}^2 n}(u(\gamma) - \lambda_{21}) + \frac{2}{\lambda_{22} n}(u(\gamma) - \lambda_{21})^2 - \frac{2}{n} + O\left(\frac{1}{n\sqrt{n}}\right).$$

Denote the ratio of the MSE of $\hat{\theta}_{ML}$ at $\gamma = 0$ to that of $\hat{\theta}_{ML}^0$ by

$$R_{MSE}(\hat{\theta}_{ML}, \hat{\theta}_{ML}^0) := [MSE_{\theta,\gamma}(\hat{\theta}_{ML})]_{\gamma=0} \Big/ MSE_{\theta}(\hat{\theta}_{ML}^0).$$

Since

$$MSE_{\theta,\gamma}(\hat{\theta}_{ML}) = E_{\theta,\gamma}[(\hat{\theta}_{ML} - \theta)^2] = \frac{1}{\lambda_{22} n} E_{\theta,\gamma}(\hat{U}^2),$$

we have

$$[MSE_{\theta,\gamma}(\hat{\theta}_{ML})]_{\gamma=0} = \frac{1}{\lambda_{22}(\theta,0)n}\left\{1 + \frac{11\lambda_{23}^2(\theta,0)}{4\lambda_{22}^3(\theta,0)n} - \frac{\lambda_{24}(\theta,0)}{\lambda_{22}^2(\theta,0)n} - \frac{3\lambda_{21}(\theta,0)\lambda_{23}(\theta,0)}{\lambda_{22}^2(\theta,0)n} + \frac{2\lambda_{21}^2(\theta,0)}{\lambda_{22}(\theta,0)n} - \frac{2}{n} + O\left(\frac{1}{n\sqrt{n}}\right)\right\}.$$

Hence, we obtain

$$\begin{aligned} R_{MSE}(\hat{\theta}_{ML}, \hat{\theta}_{ML}^0) =& \left\{1 + \frac{11\lambda_{23}^2(\theta,0)}{4\lambda_{22}^3(\theta,0)n} - \frac{\lambda_{24}(\theta,0)}{\lambda_{22}^2(\theta,0)n} + \frac{3\lambda_{21}(\theta,0)\lambda_{23}(\theta,0)}{\lambda_{22}^2(\theta,0)n} + \frac{2\lambda_{21}^2(\theta,0)}{\lambda_{22}(\theta,0)n} \right. \\ & \left. - \frac{2}{n} + O\left(\frac{1}{n\sqrt{n}}\right)\right\} \cdot \left[1 - \frac{1}{n}\left\{\frac{11\lambda_{23}^2(\theta,0)}{4\lambda_{22}^3(\theta,0)} - \frac{\lambda_{24}(\theta,0)}{\lambda_{22}^2(\theta,0)}\right\} + O\left(\frac{1}{n\sqrt{n}}\right)\right] \\ =& 1 + \frac{1}{n}\left\{\frac{3\lambda_{21}(\theta,0)\lambda_{23}(\theta,0)}{\lambda_{22}^2(\theta,0)} + \frac{2\lambda_{21}^2(\theta,0)}{\lambda_{22}(\theta,0)n} - 2\right\} + O\left(\frac{1}{n\sqrt{n}}\right). \end{aligned}$$

Since

$$\begin{aligned} \lambda_{21}(\theta,0) &= -\frac{b_3(\theta,0)}{b_2(\theta,0)} = -\frac{2}{\theta}, \\ \lambda_{22}(\theta,0) &= \frac{b_4(\theta,0)}{b_2(\theta,0)} - \left\{\frac{b_3(\theta,0)}{b_2(\theta,0)}\right\}^2 = \frac{2}{\theta^2}, \\ \lambda_{23}(\theta,0) &= -\frac{b_5(\theta,0)}{b_2(\theta,0)} + \frac{3b_3(\theta,0)b_4(\theta,0)}{b_2^2(\theta,0)} - 2\left\{\frac{b_3(\theta,0)}{b_2(\theta,0)}\right\}^2 = -\frac{4}{\theta^3}, \end{aligned}$$

it follows that

$$R_{MSE}(\hat{\theta}_{ML}, \hat{\theta}^0_{ML}) = 1 + \frac{8}{n} + O\left(\frac{1}{n\sqrt{n}}\right),$$

which is the required function of n. The ratio $R_{MSE}(\hat{\theta}_{ML}, \hat{\theta}^0_{ML})$ seems to be fit for the simulation result, i.e., Fig. 3 by Vancak et al. (2015).

Example 2.7.6 (**Lower-truncated lognormal distribution**) Let $c = 0, d = \infty$, $a(x) = x^{-1}\exp\{-(1/2)(\log x)^2\}$ and $u(x) = \log x$ for $0 < \gamma \le x < \infty$ in the density (1.7). Then, $b(\theta, \gamma) = \Phi(\theta - \log\gamma)/\phi(\theta)$ for $\theta \in \Theta = (-\infty, \infty)$, where $\Phi(x) = \int_{-\infty}^{x} \phi(t)dt$ with $\phi(t) = (1/\sqrt{2\pi})e^{-t^2/2}$ for $-\infty < t < \infty$. Letting $t = \log x$ and $\gamma_0 = \log\gamma$, we see that (1.7) becomes

$$f(t; \theta, \gamma_0) = \begin{cases} \frac{1}{\sqrt{2\pi}\Phi(\theta-\gamma_0)} e^{-(t-\theta)^2/2} & \text{for} -\infty < \gamma_0 \le \text{t} < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the lower-truncated lognormal case is reduced to the truncated normal one in Example 2.7.2.

For a truncated beta distribution and a truncated Erlang distribution, related results to the above can be found in Vancak et al. (2015).

2.8 Concluding Remarks

In a oTEF of distributions with a two-dimensional parameter (θ, γ), we considered the estimation problem of a natural parameter θ in the presence of a truncation parameter γ as a nuisance parameter. In the paper of Bar-Lev (1984), it was shown that the MLE $\hat{\theta}^{\gamma}_{ML}$ of θ for known γ, the MLE $\hat{\theta}_{ML}$ and the MCLE $\hat{\theta}_{MCL}$ of θ for unknown γ were asymptotically equivalent in the sense that they had the same asymptotic normal distribution. In this chapter, we derived the stochastic expansions of $\hat{\theta}^{\gamma}_{ML}$, $\hat{\theta}_{ML}$, and $\hat{\theta}_{MCL}$. We also obtained the second-order asymptotic loss of the bias-adjusted MLE $\hat{\theta}_{ML^*}$ relative to $\hat{\theta}^{\gamma}_{ML}$ from their second-order asymptotic variances and showed that $\hat{\theta}_{ML^*}$ and $\hat{\theta}_{MCL}$ were second order asymptotically equivalent in the sense that their asymptotic variances were same up to the second order, i.e., $o(1/n)$ as in (2.7). It seems to be natural that $\hat{\theta}^{\gamma}_{ML}$ is second order asymptotically better than $\hat{\theta}_{ML^*}$ after adjusting the bias of $\hat{\theta}_{ML}$ such that $\hat{\theta}_{ML}$ has the same as that of $\hat{\theta}^{\gamma}_{ML}$. The values of the second-order asymptotic losses of $\hat{\theta}_{ML^*}$ and $\hat{\theta}_{MCL}$ given by (2.8) and (2.9) are quite simple, which results from the truncated exponential family $\mathscr{P}_o$ of distributions.

The corresponding results to Theorems 2.3.1, 2.4.1, 2.5.1, and 2.6.1 can be obtained in the case of a two-sided truncated exponential family of distributions with a natural parameter θ and two truncation parameters γ and ν as nuisance parameters, including an upper-truncated Pareto distribution which is important in applications (see Chap. 3). Further, they may be similarly extended to the case of a more general truncated family of distributions from the truncated exponential family $\mathscr{P}_o$. In relation to Theorem 2.4.1, if two different bias-adjustments are introduced, i.e., $\hat{\theta}_{ML} + (1/n)c_i(\hat{\theta}_{ML})$ $(i = 1, 2)$, then the problem whether or not the admissibility result holds may be interesting.

2.9 Appendix A1

The proof of Theorem 2.3.1 Let $\lambda_i = \lambda_i(\theta, \gamma)$ $(i = 1, 2, 3, 4)$. Since

$$Z_1 = \frac{1}{\sqrt{\lambda_2 n}} \sum_{i=1}^{n} \{u(X_i) - \lambda_1\}, \quad U_\gamma := \sqrt{\lambda_2 n}(\hat{\theta}_{ML}^{\gamma} - \theta),$$

by the Taylor expansion, we obtain from (2.2)

$$0 = \sqrt{\frac{\lambda_2}{n}} Z_1 - \sqrt{\frac{\lambda_2}{n}} U_\gamma - \frac{\lambda_3}{2\lambda_2 n} U_\gamma^2 - \frac{\lambda_4}{6\lambda_2^{3/2} n\sqrt{n}} U_\gamma^3 + O_p\left(\frac{1}{n^2}\right),$$

which implies that the stochastic expansion of U_γ is given by

$$U_\gamma = Z_1 - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} Z_1^2 + \frac{1}{2n}\left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2}\right) Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right). \tag{2.10}$$

Since

$$E_\theta(Z_1) = 0, \quad V_\theta(Z_1) = E_\theta(Z_1^2) = 1,$$
$$E_\theta(Z_1^3) = \frac{\lambda_3}{\lambda_2^{3/2}\sqrt{n}}, \quad E_\theta(Z_1^4) = 3 + \frac{\lambda_4}{\lambda_2^2 n}, \tag{2.11}$$

it follows that

$$E_\theta(U_\gamma) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right), \tag{2.12}$$

$$E_\theta(U_\gamma^2) = 1 + \frac{1}{n}\left(\frac{11\lambda_3^2}{4\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + O\left(\frac{1}{n\sqrt{n}}\right), \tag{2.13}$$

hence, by (2.12) and (2.13)

$$V_\theta(U_\gamma) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.14}$$

From (2.10), (2.12), and (2.14), we have the conclusion of Theorem 2.3.1.

Before proving Theorem 2.4.1, we prepare three lemmas (the proofs are given in Appendix A2).

Lemma 2.9.1 *The second-order asymptotic density of $T_{(1)}$ is given by*

$$\begin{aligned} f_{T_{(1)}}(t) =& k(\theta,\gamma)e^{-k(\theta,\gamma)t} \\ &- \frac{1}{2n}\left\{\frac{\partial}{\partial\gamma}\log k(\theta,\gamma)\right\}\left\{k(\theta,\gamma)t^2 - 2t\right\}k(\theta,\gamma)e^{-k(\theta,\gamma)t} + O\left(\frac{1}{n^2}\right) \end{aligned} \tag{2.15}$$

for $t > 0$, where $k(\theta,\gamma) := a(\gamma)e^{\theta u(\gamma)}/b(\theta,\gamma)$ and

$$E_{\theta,\gamma}(T_{(1)}) = \frac{1}{k(\theta,\gamma)} + \frac{A(\theta,\gamma)}{n} + O\left(\frac{1}{n^2}\right), \quad E_{\theta,\gamma}(T_{(1)}^2) = \frac{2}{k^2(\theta,\gamma)} + O\left(\frac{1}{n}\right), \tag{2.16}$$

where

$$A(\theta,\gamma) := -\frac{1}{k^2(\theta,\gamma)}\left\{\frac{\partial}{\partial\gamma}\log k(\theta,\gamma)\right\}.$$

Lemma 2.9.2 *It holds that*

$$E_{\theta,\gamma}(Z_1T_{(1)}) = \frac{1}{k\sqrt{\lambda_2 n}}\left\{u(\gamma) - \lambda_1 + \frac{2}{k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + O\left(\frac{1}{n\sqrt{n}}\right), \tag{2.17}$$

where $k = k(\theta,\gamma)$ and $\lambda_i = \lambda_i(\theta,\gamma)$ $(i = 1, 2)$.

Lemma 2.9.3 *It holds that*

$$E_{\theta,\gamma}(Z_1^2T_{(1)}) = \frac{1}{k} + O\left(\frac{1}{n}\right), \tag{2.18}$$

where $k = k(\theta,\gamma)$.

The proof of Theorem 2.4.1 Since, for $(\theta, \gamma) \in \Theta \times (c, X_{(1)})$

$$
\begin{aligned}
&\lambda_1(\hat{\theta}_{ML}, X_{(1)}) \\
&= \lambda_1(\theta,\gamma) + \left\{ \frac{\partial}{\partial\theta}\lambda_1(\theta,\gamma) \right\} (\hat{\theta}_{ML} - \theta) + \left\{ \frac{\partial}{\partial\gamma}\lambda_1(\theta,\gamma) \right\} (X_{(1)} - \gamma) \\
&+ \frac{1}{2}\left\{ \frac{\partial^2}{\partial\theta^2}\lambda_1(\theta,\gamma) \right\} (\hat{\theta}_{ML} - \theta)^2 + \left\{ \frac{\partial^2}{\partial\theta\partial\gamma}\lambda_1(\theta,\gamma) \right\} (\hat{\theta}_{ML} - \theta)(X_{(1)} - \gamma) \\
&+ \frac{1}{2}\left\{ \frac{\partial^2}{\partial\gamma^2}\lambda_1(\theta,\gamma) \right\} (X_{(1)} - \gamma)^2 + \frac{1}{6}\left\{ \frac{\partial^3}{\partial\theta^3}\lambda_1(\theta,\gamma) \right\} (\hat{\theta}_{ML} - \theta)^3 \\
&+ \frac{1}{2}\left\{ \frac{\partial^2}{\partial\theta^2}\lambda_1(\theta,\gamma) \right\} \left\{ \frac{\partial}{\partial\gamma}\lambda_1(\theta,\gamma) \right\} (\hat{\theta}_{ML} - \theta)^2 (X_{(1)} - \gamma) + \cdots, \qquad (2.19)
\end{aligned}
$$

noting $\hat{U} = \sqrt{\lambda_2 n}(\hat{\theta}_{ML} - \theta)$ and $T_{(1)} = n(X_{(1)} - \gamma)$, we have from (2.4) and (2.19)

$$
\begin{aligned}
0 = &\sqrt{\frac{\lambda_2}{n}} Z_1 - \sqrt{\frac{\lambda_2}{n}} \hat{U} - \frac{1}{n}\left(\frac{\partial\lambda_1}{\partial\gamma} \right) T_{(1)} - \frac{\lambda_3}{2\lambda_2 n}\hat{U}^2 - \frac{1}{\sqrt{\lambda_2 n}n}\left(\frac{\partial\lambda_2}{\partial\gamma} \right) \hat{U} T_{(1)} \\
&- \frac{\lambda_4}{6\lambda_2^{3/2} n\sqrt{n}}\hat{U}^3 + O_p\left(\frac{1}{n^2} \right),
\end{aligned}
$$

where $\lambda_j = \lambda_j(\theta, \gamma)$ $(j = 1, 2, 3, 4)$ are defined by (2.1), hence the stochastic expansion of $\hat{U}$ is given by

$$
\begin{aligned}
\hat{U} = Z_1 &- \frac{1}{\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma} \right) T_{(1)} - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} Z_1^2 + \frac{\delta}{\lambda_2 n} Z_1 T_{(1)} \\
&+ \frac{1}{2n}\left(\frac{\lambda_3^2}{\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2} \right) Z_1^3 + O_p\left(\frac{1}{n\sqrt{n}} \right). \qquad (2.20)
\end{aligned}
$$

It follows from (2.11) and (2.20) that

$$
E_{\theta,\gamma}(\hat{U}) = -\frac{1}{\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma} \right) E_{\theta,\gamma}(T_{(1)}) - \frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + \frac{\delta}{\lambda_2 n} E_{\theta,\gamma}(Z_1 T_{(1)}) + O\left(\frac{1}{n\sqrt{n}} \right). \qquad (2.21)
$$

Substituting (2.16) and (2.17) into (2.21), we obtain

$$
E_{\theta,\gamma}(\hat{U}) = -\frac{1}{\sqrt{\lambda_2 n}}\left\{ \frac{1}{k}\left(\frac{\partial\lambda_1}{\partial\gamma} \right) + \frac{\lambda_3}{2\lambda_2} \right\} + O\left(\frac{1}{n\sqrt{n}} \right), \qquad (2.22)
$$

where $k = k(\theta, \gamma)$ is defined in Lemma 2.9.1. We have from (2.20)

$$E_{\theta,\gamma}(\hat{U}^2) = E_{\theta,\gamma}(Z_1^2) - \frac{1}{\sqrt{\lambda_2 n}}\left\{2\left(\frac{\partial\lambda_1}{\partial\gamma}\right)E_{\theta,\gamma}(Z_1 T_{(1)}) + \frac{\lambda_3}{\lambda_2}E_{\theta,\gamma}(Z_1^3)\right\}$$
$$+ \frac{1}{\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)^2 E_{\theta,\gamma}\left(T_{(1)}^2\right) + \frac{1}{\lambda_2 n}\left\{\frac{\lambda_3}{\lambda_2}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) + 2\delta\right\}E_{\theta,\gamma}(Z_1^2 T_{(1)})$$
$$+ \frac{1}{n}\left(\frac{5\lambda_3^2}{4\lambda_2^3} - \frac{\lambda_4}{3\lambda_2^2}\right)E_{\theta,\gamma}(Z_1^4) + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.23}$$

Substituting (2.11) and (2.16)–(2.18) into (2.23), we have

$$E_{\theta,\gamma}(\hat{U}^2) = 1 - \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left\{u(\gamma) - \lambda_1 + \frac{1}{k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + \frac{11\lambda_3^2}{4\lambda_2^3 n}$$
$$+ \frac{3\lambda_3}{k\lambda_2^2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) - \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right) - \frac{\lambda_4}{\lambda_2^2 n} + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.24}$$

Since

$$\frac{\sqrt{\lambda_2}\{(\partial/\partial\gamma)\lambda_1(\hat{\theta}_{ML}, X_{(1)})\}}{k(\hat{\theta}_{ML}, X_{(1)})\lambda_2(\hat{\theta}_{ML}, X_{(1)})\sqrt{n}}$$
$$= \frac{(\partial/\partial\gamma)\lambda_1(\theta,\gamma)}{k\sqrt{\lambda_2 n}} + \frac{1}{k\lambda_2 n}\left\{\frac{\partial\lambda_2}{\partial\gamma}(\theta,\gamma) - \left(\frac{\lambda_3}{\lambda_2} + \frac{1}{k}\frac{\partial k}{\partial\theta}\right)\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\}\hat{U} + O_p\left(\frac{1}{n\sqrt{n}}\right),$$

it follows from (2.5) that the stochastic expansion of $\hat{U}^*$ is given by

$$\hat{U}^* := \sqrt{\lambda_2 n}(\hat{\theta}_{ML^*} - \theta) = \sqrt{\lambda_2 n}(\hat{\theta}_{ML} - \theta) + \frac{\sqrt{\lambda_2}\{(\partial/\partial\gamma)\lambda_1(\hat{\theta}_{ML}, X_{(1)})\}}{k(\hat{\theta}_{ML}, X_{(1)})\hat{\lambda}_2\sqrt{n}}$$
$$= \hat{U} + \frac{1}{k\sqrt{\lambda_2 n}}\left(\frac{\partial\lambda_1}{\partial\gamma}\right) - \frac{1}{k\lambda_2 n}\left\{\delta + \frac{1}{k}\left(\frac{\partial k}{\partial\theta}\right)\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\}Z_1 + O_p\left(\frac{1}{n\sqrt{n}}\right), \tag{2.25}$$

where $\hat{U}$ is given by (2.20), $\lambda_i = \lambda_i(\theta, \gamma)$ $(i = 1, 2, 3)$ and $k = k(\theta, \gamma)$. From (2.11) and (2.22), we have

$$E_{\theta,\gamma}(\hat{U}^*) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.26}$$

It follows from (2.22), (2.24), and (2.25) that

$$E_{\theta,\gamma}(\hat{U}^{*^2}) = 1 - \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left\{u(\gamma) - \lambda_1 + \frac{3}{2k}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + \frac{11\lambda_3^2}{4\lambda_2^3 n} - \frac{\lambda_4}{\lambda_2^2 n} - \frac{2}{k^2\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left(\frac{\partial k}{\partial\theta}\right) + O\left(\frac{1}{n\sqrt{n}}\right),$$

hence, by (2.26)

$$V_{\theta,\gamma}(\hat{U}^*) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) - \frac{2}{k\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\left\{u(\gamma) - \lambda_1 + \frac{1}{k}\left(\frac{\partial k}{\partial\theta}\right)\right\} - \frac{3}{k^2\lambda_2 n}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)^2 + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.27}$$

Since, by (2.1)

$$\lambda_1(\theta,\gamma) = \frac{\partial}{\partial\theta}\log b(\theta,\gamma) = \frac{1}{b(\theta,\gamma)}\int_\gamma^d a(x)u(x)e^{\theta u(x)}dx,$$

it follows that

$$\frac{\partial\lambda_1(\theta,\gamma)}{\partial\gamma} = \frac{a(\gamma)e^{\theta u(\gamma)}}{b(\theta,\gamma)}\{\lambda_1(\theta,\gamma) - u(\gamma)\} = k(\theta,\gamma)\{\lambda_1(\theta,\gamma) - u(\gamma)\}. \tag{2.28}$$

Since

$$\frac{\partial k}{\partial\theta}(\theta,\gamma) = k(\theta,\gamma)\{u(\gamma) - \lambda_1(\theta,\gamma)\}, \tag{2.29}$$

it is seen from (2.27)–(2.29) that

$$V_{\theta,\gamma}(\hat{U}^*) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + \frac{1}{\lambda_2 n}\{\lambda_1 - u(\gamma)\}^2 + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.30}$$

From (2.25), (2.26) and (2.30), we have the conclusion of Theorem 2.4.1.

The proof of Theorem 2.5.1 Since, from (2.6)

$$0 = \frac{1}{n-1}\sum_{i=2}^n\{u(Y_i) - \lambda_1(\theta, x_{(1)})\} - \frac{1}{\sqrt{n}}\lambda_2(\theta, x_{(1)})\sqrt{n}(\hat{\theta}_{MCL} - \theta) - \frac{1}{2n}\lambda_3(\theta, x_{(1)})n(\hat{\theta}_{MCL} - \theta)^2 - \frac{1}{6n\sqrt{n}}\lambda_4(\theta, x_{(1)})n\sqrt{n}(\hat{\theta}_{MCL} - \theta)^3 + O_p\left(\frac{1}{n^2}\right),$$

letting

$$\tilde{Z}_1 = \frac{1}{\sqrt{\tilde{\lambda}_2(n-1)}} \sum_{i=2}^{n} \{u(Y_i) - \lambda_1(\theta, x_{(1)})\}, \quad \tilde{U} = \sqrt{\tilde{\lambda}_2 n}(\hat{\theta}_{MCL} - \theta),$$

where $\tilde{\lambda}_i := \lambda_i(\theta, x_{(1)})$ $(i = 1, 2, 3, 4)$, we have

$$0 = \sqrt{\frac{\tilde{\lambda}_2}{n-1}}\tilde{Z}_1 - \sqrt{\frac{\tilde{\lambda}_2}{n}}\tilde{U} - \frac{\tilde{\lambda}_3}{2\tilde{\lambda}_2 n}\tilde{U}^2 - \frac{\tilde{\lambda}_4}{6\tilde{\lambda}_2^{3/2} n\sqrt{n}}\tilde{U}^3 + O_p\left(\frac{1}{n^2}\right),$$

hence, the stochastic expansion of $\tilde{U}$ is given by

$$\tilde{U} = \tilde{Z}_1 - \frac{\tilde{\lambda}_3}{2\tilde{\lambda}_2^{3/2}\sqrt{n}}\tilde{Z}_1^2 + \frac{1}{2n}\tilde{Z}_1 + \frac{1}{2n}\left(\frac{\tilde{\lambda}_3^2}{\tilde{\lambda}_2^3} - \frac{\tilde{\lambda}_4}{3\tilde{\lambda}_2^2}\right)\tilde{Z}_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right). \quad (2.31)$$

Since

$$\tilde{\lambda}_2 = \lambda_2(\theta, X_{(1)}) = \lambda_2(\theta, \gamma) + \frac{1}{n}\left(\frac{\partial \lambda_2}{\partial \gamma}\right) T_{(1)} + O_p\left(\frac{1}{n^2}\right),$$

we obtain

$$\tilde{U} = \sqrt{\lambda_2 n}(\hat{\theta}_{MCL} - \theta)\left\{1 + \frac{1}{2n\lambda_2}\left(\frac{\partial \lambda_2}{\partial \gamma}\right) T_{(1)} + O_p\left(\frac{1}{n^2}\right)\right\}, \quad (2.32)$$

where $T_{(1)} = n(X_{(1)} - \gamma)$ and $\lambda_2 = \lambda_2(\theta, \gamma)$. Then, it follows from (2.31) and (2.32) that

$$\begin{aligned} \tilde{U}_0 &= \sqrt{\lambda_2 n}(\hat{\theta}_{MCL} - \theta) \\ &= \tilde{Z}_1 - \frac{\tilde{\lambda}_3}{2\tilde{\lambda}_2^{3/2}\sqrt{n}}\tilde{Z}_1^2 + \frac{1}{2n}\left\{1 - \frac{1}{\lambda_2}\left(\frac{\partial \lambda_2}{\partial \gamma}\right) T\right\}\tilde{Z}_1 \\ &\quad + \frac{1}{2n}\left(\frac{\tilde{\lambda}_3^2}{\tilde{\lambda}_2^3} - \frac{\tilde{\lambda}_4}{3\tilde{\lambda}_2^2}\right)\tilde{Z}_1^3 + O_p\left(\frac{1}{n\sqrt{n}}\right). \end{aligned} \quad (2.33)$$

For given $X_{(1)} = x_{(1)}$, i.e., $T_{(1)} = t := n(x_{(1)} - \gamma)$, the conditional expectation of $\tilde{Z}_1$ and $\tilde{Z}_1^2$ are

$$\begin{aligned}
E_{\theta,\gamma}(\tilde{Z}_1|t) &= \frac{1}{\sqrt{\tilde{\lambda}_2(n-1)}}\sum_{i=2}^{n}\left\{E_{\theta,\gamma}[u(Y_i)|t] - \lambda_1(\theta, x_{(1)})\right\} = 0,\\
E_{\theta,\gamma}(\tilde{Z}_1^2|t) &= \frac{1}{\tilde{\lambda}_2(n-1)}\Bigg[\sum_{i=2}^{n} E_{\theta,\gamma}[\{u(Y_i) - \lambda_1(\theta, x_{(1)})\}^2|t]\\
&\quad + \sum_{\substack{i\neq j\\ 2\le i,j\le n}}\sum E_{\theta,\gamma}\left[\{u(Y_i) - \lambda_1(\theta, x_{(1)})\}\{u(Y_j) - \lambda_1(\theta, x_{(1)})\}\,|\,t\right]\Bigg]\\
&= 1,
\end{aligned} \tag{2.34}$$

hence, the conditional variance of $\tilde{Z}_1$ is equal to 1, i.e., $V_{\theta,\gamma}(\tilde{Z}_1|t) = 1$. In a similar way to the above, we have

$$E_{\theta,\gamma}(\tilde{Z}_1^3|t) = \frac{\tilde{\lambda}_3}{\tilde{\lambda}_2^{3/2}\sqrt{n-1}}, \quad E_{\theta,\gamma}(\tilde{Z}_1^4|t) = 3 + \frac{\tilde{\lambda}_4}{\tilde{\lambda}_2^2(n-1)}. \tag{2.35}$$

Then, it follows from (2.33)–(2.35) that

$$E_{\theta,\gamma}(\tilde{U}_0|T_{(1)}) = -\frac{\tilde{\lambda}_3}{2\tilde{\lambda}_2^{3/2}\sqrt{n}} + O_p\left(\frac{1}{n\sqrt{n}}\right), \tag{2.36}$$

$$\begin{aligned}
E_{\theta,\gamma}(\tilde{U}_0^2|T_{(1)}) &= 1 + \frac{1}{n} + \frac{1}{n}\left(\frac{11\tilde{\lambda}_3^2}{4\tilde{\lambda}_2^3} - \frac{\tilde{\lambda}_4}{\tilde{\lambda}_2^2}\right) - \frac{1}{\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T_{(1)}\\
&\quad + O_p\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned} \tag{2.37}$$

where $\tilde{\lambda}_i = \lambda_i(\theta, X_{(1)})$ $(i = 2, 3, 4)$. Since, for $i = 2, 3, 4$

$$\tilde{\lambda}_i = \lambda_i(\theta, X_{(1)}) = \lambda_i(\theta, \gamma) + O_p\left(\frac{1}{n}\right) = \lambda_i + O_p\left(\frac{1}{n}\right), \tag{2.38}$$

it follows from (2.36) that

$$E_{\theta,\gamma}(\tilde{U}_0) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.39}$$

It is noted from (2.12), (2.26), and (2.39) that

$$E_{\theta,\gamma}(U_\gamma) = E_{\theta,\gamma}(\hat{U}^*) = E_{\theta,\gamma}(\tilde{U}_0) = -\frac{\lambda_3}{2\lambda_2^{3/2}\sqrt{n}} + O\left(\frac{1}{n\sqrt{n}}\right).$$

In a similar way to the above, we obtain from (2.16), (2.37), and (2.38)

$$E_{\theta,\gamma}(\tilde{U}_0^2) = 1 + \frac{1}{n} + \frac{11\lambda_3^2}{4\lambda_2^3 n} - \frac{\lambda_4}{\lambda_2^2 n} - \frac{1}{k\lambda_2 n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right) + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.40}$$

Since, by (2.28) and (2.29)

$$\frac{1}{k}\left(\frac{\partial\lambda_2}{\partial\gamma}\right) = \frac{1}{k}\left\{\frac{\partial k}{\partial\theta}(\lambda_1 - u(\gamma)) + k\left(\frac{\partial\lambda_1}{\partial\theta}\right)\right\} = -(\lambda_1 - u(\gamma))^2 + \lambda_2,$$

it follows from (2.40) that

$$E_{\theta,\gamma}(\tilde{U}_0^2) = 1 + \frac{11\lambda_3^2}{4\lambda_2^3 n} - \frac{\lambda_4}{\lambda_2^2 n} + \frac{1}{\lambda_2 n}\{\lambda_1 - u(\gamma)\}^2 + O\left(\frac{1}{n\sqrt{n}}\right),$$

hence, by (2.39)

$$V_{\theta,\gamma}(\tilde{U}_0) = 1 + \frac{1}{n}\left(\frac{5\lambda_3^2}{2\lambda_2^3} - \frac{\lambda_4}{\lambda_2^2}\right) + \frac{1}{\lambda_2 n}\{\lambda_1 - u(\gamma)\}^2 + O\left(\frac{1}{n\sqrt{n}}\right). \tag{2.41}$$

From (2.33), (2.39) and (2.41), we have the conclusion of Theorem 2.5.1.

2.10 Appendix A2

The proof of Lemma 2.9.1 Since the second-order asymptotic cumulative distribution function of $T_{(1)}$ is given by

$$\begin{aligned} F_{T_{(1)}}(t) &= P_{\theta,\gamma}\left\{T_{(1)} \le t\right\} = P_{\theta,\gamma}\left\{n(X_{(1)} - \gamma) \le t\right\} = 1 - \left\{\frac{b(\theta, \gamma + (t/n))}{b(\theta,\gamma)}\right\}^n \\ &= 1 - e^{-k(\theta,\gamma)t}\left[1 - \frac{t^2}{2n}\left\{\frac{\partial k(\theta,\gamma)}{\partial\gamma}\right\} + O\left(\frac{1}{n^2}\right)\right] \end{aligned}$$

for $t > 0$, we obtain (2.15). From (2.15), we also get (2.16) by a straightforward calculation.

The proof of Lemma 2.9.2 As is seen from the beginning of Sect. 2.5, $Y_2, \dots, Y_n$ are i.i.d. random variables according to a distribution with density

$$g(y;\theta,x_{(1)}) = \frac{a(y)e^{\theta u(y)}}{b(\theta,x_{(1)})} \quad \text{for } x_{(1)} \le y < d \tag{2.42}$$

with respect to the Lebesgue measure. Then, the conditional expectation of Z_1 given $T_{(1)}$ is obtained by

$$E_{\theta,\gamma}(Z_1|T_{(1)}) = \frac{1}{\sqrt{\lambda_2 n}}\left\{u(X_{(1)}) + \sum_{i=2}^{n} E_{\theta,\gamma}[u(Y_i)|T_{(1)}] - n\lambda_1\right\}, \tag{2.43}$$

where $\lambda_i = \lambda_i(\theta,\gamma)$ $(i = 1, 2)$. Since, for each $i = 2, \dots, n$, by (2.42)

$$E_{\theta,\gamma}[u(Y_i)|T_{(1)}] = \frac{\partial}{\partial\theta}\log b(\theta, X_{(1)}) = \lambda_1(\theta, X_{(1)}) =: \hat{\lambda}_1 \quad \text{(say)},$$

it follows from (2.43) that

$$E_{\theta,\gamma}(Z_1|T_{(1)}) = \frac{1}{\sqrt{\lambda_2 n}}\left\{u(X_{(1)}) + (n-1)\hat{\lambda}_1\right\} - \frac{\lambda_1\sqrt{n}}{\sqrt{\lambda_2}},$$

hence, from (2.16) and (2.43)

$$\begin{aligned} E_{\theta,\gamma}(Z_1 T_{(1)}) = & \frac{1}{\sqrt{\lambda_2 n}}\left\{E_{\theta,\gamma}[u(X_{(1)})T_{(1)}] + (n-1)E_{\theta,\gamma}(\hat{\lambda}_1 T_{(1)})\right\} \\ & - \sqrt{\frac{n}{\lambda_2}}\lambda_1\left\{\frac{1}{k} + \frac{A(\theta,\gamma)}{n} + O\left(\frac{1}{n^2}\right)\right\}, \end{aligned} \tag{2.44}$$

where $k = k(\theta, \gamma)$. Since, by the Taylor expansion

$$\begin{aligned} u(X_{(1)}) &= u(\gamma) + \frac{u'(\gamma)}{n}T_{(1)} + \frac{u''(\gamma)}{2n^2}T_{(1)}^2 + O_p\left(\frac{1}{n^3}\right), \\ \hat{\lambda}_1 &= \lambda_1(\theta, X_{(1)}) = \lambda_1(\theta,\gamma) + \frac{1}{n}\left\{\frac{\partial}{\partial\gamma}\lambda_1(\theta,\gamma)\right\}T_{(1)} \\ &\quad + \frac{1}{2n^2}\left\{\frac{\partial^2}{\partial\gamma^2}\lambda_1(\theta,\gamma)\right\}T_{(1)}^2 + O_p\left(\frac{1}{n^3}\right), \end{aligned}$$

it follows from (2.16) that

$$E_{\theta,\gamma}[u(X_{(1)})T_{(1)}] = \frac{u(\gamma)}{k} + \frac{1}{n}\left\{Au(\gamma) + \frac{2u'(\gamma)}{k^2}\right\} + O\left(\frac{1}{n^2}\right), \tag{2.45}$$

$$E_{\theta,\gamma}(\hat{\lambda}_1 T_{(1)}) = \frac{\lambda_1}{k} + \frac{1}{n}\left\{\lambda_1 A + \frac{2}{k^2}\left(\frac{\partial\lambda_1}{\partial\gamma}\right)\right\} + O\left(\frac{1}{n^2}\right), \tag{2.46}$$

where $k = k(\theta,\gamma)$, $A = A(\theta,\gamma)$, and $\lambda_1 = \lambda_1(\theta,\gamma)$. From (2.44)–(2.46), we obtain (2.17).

The proof of Lemma 2.9.3 First, we have

$$
\begin{aligned}
E_{\theta,\gamma}(Z_1^2|T_{(1)}) &= \frac{1}{\lambda_2 n}\left\{u(X_{(1)})-\lambda_1\right\}^2 \\
&+ \frac{2}{\lambda_2 n}\left\{u(X_{(1)})-\lambda_1\right\}\sum_{i=2}^{n} E_{\theta,\gamma}\left[u(Y_i)-\lambda_1|T_{(1)}\right] \\
&+ \frac{1}{\lambda_2 n}\sum_{i=2}^{n} E_{\theta,\gamma}\left[\{u(Y_i)-\lambda_1\}^2|T_{(1)}\right] \\
&+ \frac{1}{\lambda_2 n}\sum\sum_{\substack{i\neq j\\ 2\le i,j\le n}} E_{\theta,\gamma}\left[\{u(Y_i)-\lambda_1\}\left\{u(Y_j)-\lambda_1\right\}|T_{(1)}\right].
\end{aligned} \tag{2.47}
$$

For $2 \le i \le n$, we have

$$
E_{\theta,\gamma}[u(Y_i)-\lambda_1|T_{(1)}] = \left(\frac{\partial\lambda_1}{\partial\gamma}\right)\frac{T_{(1)}}{n} + O_p\left(\frac{1}{n^2}\right) = O_p\left(\frac{1}{n}\right), \tag{2.48}
$$

and for $i \neq j$ and $2 \le i, j \le n$

$$
\begin{aligned}
E_{\theta,\gamma}\left[\{u(Y_i)-\lambda_1\}\left\{u(Y_j)-\lambda_1\right\}|T_{(1)}\right] &= E_{\theta,\gamma}\left[u(Y_i)-\lambda_1 \mid T_{(1)}\right]E_{\theta,\gamma}\left[u(Y_j)-\lambda_1|T_{(1)}\right] \\
&= \left(\frac{\partial\lambda_1}{\partial\gamma}\right)^2\frac{T_{(1)}^2}{n^2} + O_p\left(\frac{1}{n^3}\right) = O_p\left(\frac{1}{n^2}\right).
\end{aligned} \tag{2.49}
$$

Since, for $i = 2, \dots, n$

$$
E_{\theta,\gamma}[u^2(Y_i)|T_{(1)}] = \hat{\lambda}_1^2 + \hat{\lambda}_2,
$$

where $\hat{\lambda}_i = \lambda_i(\theta, X_{(1)})$ $(i = 1, 2)$, we have for $i = 2, \dots, n$

$$
E_{\theta,\gamma}[\{u(Y_i)-\lambda_1\}^2|T_{(1)}] = \lambda_2 + \frac{1}{n}\left(\frac{\partial\lambda_2}{\partial\gamma}\right)T_{(1)} + O_p\left(\frac{1}{n^2}\right) = \lambda_2 + O_p\left(\frac{1}{n}\right). \tag{2.50}
$$

From (2.47)–(2.50), we obtain

$$
E_{\theta,\gamma}(Z_1^2|T_{(1)}) = 1 + O_p\left(\frac{1}{n}\right),
$$

hence, by (2.16)

$$E_{\theta,\gamma}(Z_1^2 T_{(1)}) = E_{\theta,\gamma}[T_{(1)} E_{\theta,\gamma}(Z_1^2 | T_{(1)})] = E_{\theta,\gamma}(T_{(1)}) + O\left(\frac{1}{n}\right) = \frac{1}{k} + O\left(\frac{1}{n}\right).$$

Thus, we get (2.18).

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Statistical Estimation for Truncated Exponential Families
Akahira, M.
2017, XI, 122 p. 10 illus., Softcover
ISBN: 978-981-10-5295-8