Chapter 2
New Generation Parametric Optimality

Applying the new notion of the generalized $(\phi, \eta, \rho, \theta, \tilde{m})$-invexity, a set of higher order parametric necessary optimality conditions and several sets of higher order sufficient optimality conditions in semi-infinite framework for a discrete minmax fractional programming problem applying various classes of $(\phi, \eta, \rho, \theta, \tilde{m})$-invexity assumptions are presented. In this chapter, the presented results are new, especially on the semi-infinite aspects for the discrete minmax fractional programming problems. The obtained results offer greater opportunities for the interdisciplinary collaborative research and beyond.

1 The Significance of Semi-infinite Fractional Programming

In this section, we aim at describing the significance of semi-infinite fractional programming for a set of second-order necessary optimality conditions in conjunction with numerous sets of second-order sufficient optimality conditions using the generalized $(\phi, \eta, \rho, \theta, \tilde{m})$-invexities of higher orders to the context of the following semi-infinite discrete minmax fractional programming problem:

$$(P) \qquad \text{Minimize} \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)}$$

subject to

$$G_j(x, t) \le 0 \text{ for all } t \in T_j, \quad j \in \underline{q},$$

$$H_k(x, s) = 0 \text{ for all } s \in S_k, \quad k \in \underline{r},$$

$$x \in X,$$

R.U. Verma, *Semi-Infinite Fractional Programming*, Infosys Science Foundation Series in Mathematical Sciences, DOI 10.1007/978-981-10-6256-8_2

where $p,\ q$, and r are positive integers, X is a nonempty open convex subset of $\mathbb{R}^n$ (n-dimensional Euclidean space), for each $j \in \underline{q} \equiv \{1, 2, \ldots, q\}$ and $k \in \underline{r},\ T_j$ and S_k are compact subsets of complete metric spaces, for each $i \in \underline{p},\ f_i$ and g_i are twice continuously differentiable real-valued functions defined on X, for each $j \in \underline{q},\ z \to G_j(z, t)$ is a twice continuously differentiable real-valued function defined on X for all $t \in T_j$, for each $k \in \underline{r},\ z \to H_k(z, s)$ is a twice continuously differentiable real-valued function defined on X for all $s \in S_k$, for each $j \in \underline{q}$ and $k \in \underline{r},\ t \to G_j(x, t)$ and $s \to H_k(x, s)$ are continuous real-valued functions defined, respectively, on T_j and S_k for all $x \in X$, and for each $i \in \underline{p},\ g_i(x) > 0$ for all x satisfying the constraints of (P).

There is an enormous amount of research work available in the literature on the general fractional programming problems, while there are plenty of opportunities for the advanced research to the context of semi-infinite aspect of the fractional programming problems. We plan first to examine and explore the direct impact of the new version of the notion $(\phi, \eta, \rho, \theta, \tilde{m})$-sonvexities (which we present in the next section) on the semi-infinite fractional programming in general, and then we apply it to the context of the second-order necessary and sufficient optimality conditions for minmax fractional programming problem (P), especially, we intend to discuss the second-order optimality aspects of our principal problem (P) to the context of the semi-infinite discrete fractional programming. Our second-order sufficient optimality results will be established using the properties of $(\phi, \eta, \rho, \theta, \tilde{m})$-sonvexities. Sometimes, second-order $(\phi, \eta, \rho, \theta, \tilde{m})$-invexities are referred to as "sonvexities" in the literature.

The optimality results thus obtained in the present chapter can further be applied for constructing several second-order parametric and nonparametric duality models for (P) and proving numerous duality theorems. Our observation at this point is that the field of semi-infinite discrete fractional programming is still fast-expanding in the literature, the results established in this chapter would impact constructively to developing several second-order parametric and nonparametric duality models for (P) and achieving numerous duality theorems. We remark that the most of optimality results obtained for (P) are also applicable under appropriate specialized settings to other classes of problems with semi-infinite discrete maxm, fractional, and conventional objective functions.

2 Basic Concepts and Auxiliary Results

In this section, we introduce some new definitions of certain classes of generalized sonvexities of functions of higher orders. For more details on invex functions, we refer the reader [1]. Recall that a function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is said to be superlinear if $\mathcal{F}(x+y) \geq \mathcal{F}(x) + \mathcal{F}(y)$ for all $x, y \in \mathbb{R}^n$, and $\mathcal{F}(ax) = a\mathcal{F}(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}_+ \equiv [0, \infty)$. Let $x^* \in X$ and assume that the function $f : X \to \mathbb{R}$ is twice differentiable at x^*.

Definition 2.1 The function f is said to be (*strictly*) $(\phi, \eta, \rho, \theta, \tilde{m})$-*sonvex* at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer $\tilde{m}$ such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*)\big)(>) \geq \langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$

$$+\frac{1}{4}\langle z^*, \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}.$$

The function f is said to be (strictly) $(\phi, \eta, \rho, \theta, m)$-sonvex on X if it is (strictly) $(\phi, \eta, \rho, \theta, \tilde{m})$-sonvex at each $x^* \in X$.

Definition 2.2 The function f is said to be (*strictly*) $(\phi, \eta, \rho, \theta, \tilde{m})$-*pseudosonvex* at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer $\tilde{m}$ such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$

$$+ \ \frac{1}{4}\langle z^*, \nabla^2 f(x^*)z\rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}} \quad \Rightarrow \quad \phi\big(f(x) - f(x^*)\big)(>) \geq 0,$$

equivalently,

$$\phi\big(f(x) - f(x^*)\big)(\leq) < 0 \Rightarrow \langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$

$$+\frac{1}{4}\langle z^*, \nabla^2 f(x^*)z\rangle < -\rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}.$$

The function f is said to be (strictly) $(\phi, \eta, \rho, \theta, m)$-pseudosonvex on X if it is (strictly) $(\phi, \eta, \rho, \theta, m)$-pseudosonvex at each $x^* \in X$.

Definition 2.3 The function f is said to be (*prestrictly*) $(\phi, \eta, \rho, \theta, \tilde{m})$-*quasisonvex* at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer $\tilde{m}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*)\big)(<) \leq 0 \quad \Rightarrow \quad \langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$

$$+\frac{1}{4}\langle z^*, \nabla^2 f(x^*)z\rangle \leq -\rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}},$$

equivalently

$$\langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$

$$+\frac{1}{4}\langle z^*, \nabla^2 f(x^*)z\rangle > -\rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}} \Rightarrow \phi\big(f(x) - f(x^*)\big)(\geq) > 0.$$

We conclude this section by recalling a set of parametric necessary optimality conditions for (P) based on the following result.

Theorem 2.4 [2] *Let $x^* \in \mathbb{F}$ and $\lambda^* = \max_{1\leq i\leq p} f_i(x^*)/g_i(x^*)$, for each $i \in \underline{p}$, let f_i and g_i be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$. If x^* is an optimal solution of (P), if the second-order generalized Abadie constraint qualification holds at x^*, and if for any critical direction y, the set cone*

$$\{\Big(\nabla G_j(x^*, t), \langle y, \nabla^2 G_j(x^*, t)y\rangle\Big) : t \in \hat{T}_j(x^*), j \in \underline{q}\}$$
$$+ \, span\{\Big(\nabla H_k(x^*, s), \langle y, \nabla^2 H_k(x^*, s)y\rangle\Big) : s \in S_k, k \in \underline{r}\},$$
$$where \ \hat{T}_j(x^*) \equiv \{t \in T_j \ : \ G_j(x^*, t) = 0\},$$

is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u \geq 0, \ \sum_{i=1}^p u_i = 1\}$ and integers ν_0^* and ν^*, with $0 \leq \nu_0^* \leq \nu^* \leq n+1$, such that there exist ν_0^* indices j_m, with $1 \leq j_m \leq q$, together with ν_0^* points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0^*}$, $\nu^* - \nu_0^*$ indices k_m, with $1 \leq k_m \leq r$, together with $\nu^* - \nu_0^*$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu^*}\backslash\underline{\nu_0^*}$, and ν^* real numbers v_m^*, with $v_m^* > 0$ for $m \in \underline{\nu_0^*}$, with the property that

$$\sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^*(\nabla g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^*[\nabla G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla H_k(x^*, s^m) = 0, \tag{1}$$

$$\langle y, \Big[\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0^*} v_m^* \nabla^2 G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0^*+1}^{\nu^*} v_m^* \nabla^2 H_k(x^*, s^m)\Big]y\rangle \geq 0, \tag{2}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \quad i \in \underline{p}(x^*), \tag{3}$$

$$v_j^* G_j(x^*) = 0, \ j \in \underline{q}(x^*), \tag{4}$$

where $\underline{\nu} \setminus \underline{\nu_0}$ is the complement of the set $\underline{\nu_0}$ relative to the set $\underline{\nu}$.

3 Sufficient Optimality Theorems

In this section, we shall present several second-order sufficiency results in which various generalized $(\phi, \eta, \rho, \theta, \tilde{m})$-sonvexity assumptions are imposed on the individual as well as certain combinations of the problem functions. Now we need to introduce the following suitable notations for our work on hand.

$$\mathcal{C}(x, v) = \sum_{m=1}^{\nu_0} v_m G_{j_m}(x, t^m),\ m \in \underline{\nu_0}$$
$$\mathcal{D}_{k_m}(x, s^m) = w_m H_{k_m}(x, s^m),\ m \in \underline{\nu} \setminus \underline{\nu_0}$$
$$\mathcal{D}(x, s^m) = \sum_{m=\nu_0+1}^{\nu} w_m H_{k_m}(x, s^m),\ m \in \underline{\nu} \setminus \underline{\nu_0},$$
$$\mathcal{E}_i(x, \lambda) = f_i(x) - \lambda g_i(x),$$
$$\mathcal{E}(x, u, \lambda) = \sum_{i=1}^{p} u_i [f_i(x) - \lambda g_i(x)],$$
$$\mathcal{G}(x, v, w) = \sum_{m=1}^{\nu_0} v_m G_{j_m}(x, t^m) + \sum_{m=\nu_0+1}^{\nu} w_m H_{k_m}(x, s^m),$$

$$I_+(u) = \{i \in \underline{p} : u_i > 0\}.$$

In the proofs of our sufficiency theorems, we shall make frequent use of the following auxiliary result which provides an alternative expression for the objective function of (P).

Lemma 2.5 [2] *For each $x \in X$,*

$$\varphi(x) = \max_{1 \le i \le p} \frac{f_i(x)}{g_i(x)} = \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)}.$$

Theorem 2.6 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \ge 0$, and assume that the functions f_i, g_i, $i \in \underline{p}$, be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$, and let us assume that for each critical direction z^*, there exist $u^* \in U$ and, integers ν_0 and ν with $0 \le \nu_0 \le \nu \le n + 1$ such that there are ν_0 indices j_m with $1 \le j_m \le q$ together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$ for $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m with $1 \le k_m \le r$ together with $\nu - \nu_0$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu} \setminus \underline{\nu_0}$, and ν real numbers $v^* \in \mathbb{R}_+^q$ for $m \in \underline{\nu_0}$ such that*

$$\sum_{i=1}^{p} u_i[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(x^*, t^m) \tag{5}$$

$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla H_{k_m}(x^*, s^m) = 0,$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(x^*, t^m)$$

$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla^2 H_{k_m}(x^*, s^m) \Big\} z^* \Big\rangle \geq 0, \tag{6}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \quad i \in \underline{p}, \tag{7}$$

$$v_m^* G_{j_m}(x^*, t^m) = 0, \quad m \in \underline{\nu_0},$$

$$w_m^* \nabla H_{k_m}(x^*, s^m) \geq 0 \ \ \textit{for all } m \in \underline{\nu} \setminus \underline{\nu_0}. \tag{8}$$

Assume, furthermore, that any one of the following six sets of conditions holds:

(a) (i) for each $i \in I_+ \equiv I_+(u^*)$, f_i is $(\phi, \eta, \bar{\rho}_i, \theta, \tilde{m})$-sonvex and $-g_i$ is $(\phi, \eta, \tilde{\rho}_i, \theta, \tilde{m})$-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;
  (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}_m, \eta, \hat{\rho}_j, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;
  (iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_k, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
  (iv) $\rho^*(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, where $\rho^*(x, x^*) = \sum_{i \in I_+} u_i^*[\bar{\rho}_i(x, x^*) + \lambda^* \tilde{\rho}_i(x, x^*)]$;

(b) (i) for each $i \in I_+$, f_i is $(\phi, \eta, \bar{\rho}_i, \theta, m)$-sonvex and $-g_i$ is $(\phi, \eta, \tilde{\rho}_i, \theta, m)$-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;
  (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;
  (iii) the function $z \to w_m^* H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
  (iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+$, f_i is $(\phi, \eta, \bar{\rho}_i, \theta, \tilde{m})$-sonvex and $-g_i$ is $(\phi, \eta, \tilde{\rho}_i, \theta, \tilde{m})$-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;
  (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}_m, \eta, \hat{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;
  (iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;

(iv) $\rho^*(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+$, f_i is $(\phi, \eta, \bar{\rho}_i, \theta, m\tilde{m}m)$-sonvex and $-g_i$ is $(\phi, \eta, \tilde{\rho}_i, \theta, \tilde{m})$-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;

(iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;

(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) for each $i \in I_+$, f_i is $(\phi, \eta, \bar{\rho}_i, \theta, \tilde{m})$-sonvex and $-g_i$ is $(\phi, \eta, \tilde{\rho}_i, \theta, \tilde{m})$-sonvex at x^*, ϕ is superlinear, and $\phi(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, v^*)$ is $(\hat{\phi}, \eta, \zeta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(f) (i) the Lagrangian-type function

$$\xi \to L(\xi, u^*, v^*, w^*, \lambda^*) = \sum_{i=1}^{p} u_i^*[f_i(\xi) - \lambda^* g_i(\xi)]$$
$$+ \sum_{m=1}^{\nu} v_m^* G_{j_m}(\xi, t^m) + \sum_{m=\nu_0+1}^{\nu} v_m^* H_{k_m}(\xi, s^m)$$

is $(\phi, \eta, \rho, \theta, \tilde{m})$-pseudosonvex at x^*, $\rho(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, and $\phi(a) \geq 0 \Rightarrow a \geq 0$.

Then x^* is an optimal solution of (P).

Proof Let x be an arbitrary feasible solution of (P).

(a): Using the hypotheses specified in (i), we have for each $i \in I_+$,

$$\phi\big(f_i(x) - f_i(x^*)\big) \geq \langle \nabla f_i(x^*) + \frac{1}{4}\nabla^2 f_i(x^*)z^*, \eta(x, x^*)\rangle + \frac{1}{4}\langle z^*, \nabla^2 f_i(x^*)z^*\rangle$$
$$+\bar{\rho}_i(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}$$

and

$$\phi\big(-g_i(x) + g_i(x^*)\big) \geq \langle -\nabla g_i(x^*) - \frac{1}{4}\nabla^2 g_i(x^*)z^*, \eta(x, x^*)\rangle - \frac{1}{4}\langle z^*, \nabla^2 g_i(x^*)z^*\rangle$$
$$+\tilde{\rho}_i(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}.$$

In as much as $\lambda^* \geq 0$, $u^* \geq 0$, $\sum_{i=1}^p u_i^* = 1$, and ϕ is superlinear, we deduce from the above inequalities that

$$\begin{aligned}
&\phi\Big(\sum_{i=1}^p u_i^*[f_i(x) - \lambda^* g_i(x)] - \sum_{i=1}^p u_i^*[f_i(x^*) - \lambda^* g_i(x^*)]\Big) \qquad (9)\\
&\geq \Big\langle \sum_{i=1}^p u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \frac{1}{4}\sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)]z^*, \eta(x, x^*)\Big\rangle\\
&\quad + \frac{1}{4}\Big\langle z^*, \sum_{i=1}^p u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)]z^*\Big\rangle +\\
&\quad \sum_{i \in I_+} u_i^*[\bar{\rho}_i(x, x^*) + \lambda^* \tilde{\rho}_i(x, x^*)]\|\theta(x, x^*)\|^{\tilde{m}}.
\end{aligned}$$

Since $x \in \mathbb{F}$ and (7) holds, it follows from the properties of the functions $\hat{\phi}_j$ that for each $m \in \bar{\nu_0}$, $\hat{\phi}_m\big(G_{j_m}(x, t^m) - G_{j_m}(x^*, t^m)\big) \leq 0$ which in view of (ii) implies that

$$\begin{aligned}
&\langle \nabla G_{j_m}(x^*, t^m) + \frac{1}{4}\nabla^2 G_{j_m}(x^*, t^m)z^*, \eta(x, x^*)\rangle + \frac{1}{4}\langle z^*, \nabla^2 G_{j_m}(x^*, t^m)z^*\rangle\\
&\leq -\hat{\rho}_m(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}.
\end{aligned}$$

As $v_j^* \geq 0$ for each $m \in \underline{\nu_0}$, the above inequalities yield

$$\begin{aligned}
&\Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m)\\
&+ \frac{1}{4}\sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m)z^*, \eta(x, x^*)\Big\rangle\\
&+ \frac{1}{4}\Big\langle z^*, \sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m)z^*\Big\rangle\\
&\leq -\sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}. \qquad (10)
\end{aligned}$$

Similarly, we can show that (iii) leads to the following inequality:

$$\begin{aligned}
&\Big\langle \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla H_{k_m}(x^*, s^m)\\
&+ \frac{1}{4}\sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m)z^*, \eta(x, x^*)\Big\rangle
\end{aligned}$$

$$+\frac{1}{4}\Big\langle z^*, \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m) z^* \Big\rangle$$
$$\leq - \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \|\theta(x, x^*)\|^{\tilde{m}}. \tag{11}$$

Now, using (4), (5), and (8)–(10), we see that

$$\phi\Big(\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] - \sum_{i=1}^{p} u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] \Big)$$
$$\geq -\Big[\Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \frac{1}{4} \sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m) z^*, \eta(x, x^*) \Big\rangle$$
$$+ \frac{1}{4}\Big\langle z^*, \sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m) z^* \Big\rangle$$
$$+ \Big\langle \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla H_{k_m}(x^*, s^m) + \frac{1}{4} \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m) z^*, \eta(x, x^*) \Big\rangle$$
$$+ \frac{1}{4}\Big\langle z^*, \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m) z^* \Big\rangle \Big]$$
$$+ \sum_{i \in I_+} u_i^*[\bar{\rho}_i(x, x^*) + \lambda^* \tilde{\rho}_i(x, x^*)] \|\theta(x, x^*)\|^{\tilde{m}} \quad \text{(by (5), (6), and (9))}$$
$$\geq \Big\{ \sum_{i \in I_+} u_i^*[\bar{\rho}_i(x, x^*) + \lambda^* \tilde{\rho}_i(x, x^*)]$$
$$+ \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \Big\} \|\theta(x, x^*)\|^{\tilde{m}}$$

(by (10) and (11))

$$\geq 0 \quad \text{(by (iv))}.$$

But $\phi(a) \geq 0 \Rightarrow a \geq 0$, and hence we have

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] \geq \sum_{i=1}^{p} u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \tag{12}$$

where the equality follows from (6). Now using (11) and Lemma 2.5, we find that

$$\varphi(x^*) = \lambda^* \leq \frac{\sum_{i=1}^{p} u_i^* f_i(x)}{\sum_{i=1}^{p} u_i^* g_i(x)} \leq \max_{u \in U} \frac{\sum_{i=1}^{p} u_i f_i(x)}{\sum_{i=1}^{p} u_i g_i(x)} = \varphi(x).$$

Since $x \in \mathbb{F}$ is arbitrary, we conclude from this inequality that x^* is an optimal solution of (P).

(b): Based on part (a), for each $m \in \nu_0$, we have $G_{j_m}(x, t^m) - G_{j_m}(x^*, t^m) \leq 0$, and hence using the properties of the function $\hat{\phi}$, we get

$$\hat{\phi}\Big(\sum_{m=1}^{\nu_0} v_m^* G_{j_m}(x, t^m) - \sum_{m=1}^{\nu_0} v_m^* G_{j_m}(x^*, t^m)\Big) \leq 0,$$

which in view of (ii) implies that

$$\begin{aligned}
&\Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \frac{1}{4} \sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m) z^*, \eta(x, x^*) \Big\rangle \\
&+ \frac{1}{4} \Big\langle z^*, \sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m) z^* \Big\rangle \\
&\leq -\hat{\rho}(x, x^*) \|\theta(x, x^*)\|^{\tilde{m}}.
\end{aligned}$$

From now on, proceeding as in the proof of part (a) and using this inequality instead of (9), we arrive at (11), that leads to the desired conclusion that x^* is an optimal solution of (P).

(c)–(e): The proofs using (c)–(e) are similar to those of parts (a) and (b).

(f): Since $\rho(x, x^*) \geq 0$, (4) and (5) yield

$$\begin{aligned}
&\langle \nabla L(x^*, u^*, v^*, w^*, \lambda^*) + \frac{1}{4} \nabla^2 L(x^*, u^*, v^*, w^*, \lambda^*) z^*, \eta(x, x^*) \rangle \\
&+ \frac{1}{4} \langle z^*, \nabla^2 L(x^*, u^*, v^*, w^*, \lambda^*) z^* \rangle \geq \ 0 \ \geq -\rho(x, x^*) \|\theta(x, x^*)\|^{\tilde{m}},
\end{aligned}$$

which in view of our $(\phi, \eta, \rho, \theta, \tilde{m})$-pseudosonvexity assumption implies that

$$\phi\big(L(x, u^*, v^*, w^*, \lambda^*) - L(x^*, u^*, v^*, w^*, \lambda^*)\big) \geq 0.$$

But $\phi(a) \geq 0 \Rightarrow a \geq 0$ and hence we have

$$L(x, u^*, v^*, w^*, \lambda^*) \geq L(x^*, u^*, v^*, w^*, \lambda^*).$$

Because $x, x^* \in \mathbb{F}$, $v^* \geq 0$, and (2) and (3) hold, the right-hand side of the above inequality is equal to zero, and so we get

$$\sum_{i=1}^{p} u_i^*[f_i(x) - \lambda^* g_i(x)] \geq 0,$$

that is, (11). Based on the proof of part (a), we conclude that x^* is an optimal solution of (P). □

In Theorem 2.6, separate $(\phi, \eta, \rho, \theta, \tilde{m})$-sonvexity assumptions were imposed on the functions f_i and $-g_i$, $i \in \underline{p}$. It seems to establish a wide range of additional sufficient optimality results in which various generalized $(\phi, \eta, \rho, \theta, m)$-sonvexity constraints are placed on certain combinations of these functions. Next, we examine a series of sufficiency theorems in which appropriate generalized $(\phi, \eta, \rho, \theta, \tilde{m})$-sonvexity assumptions are imposed on the functions involved.

Theorem 2.7 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions $f_i, g_i, i \in \underline{p}$, be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$ and let us assume that for each critical direction z^*, there exist $u^* \in U$ and, integers ν_0 and ν with $0 \leq \nu_0 \leq \nu \leq n+1$ such that there are ν_0 indices j_m with $1 \leq j_m \leq q$ together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$ for $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m with $1 \leq k_m \leq r$ together with $\nu - \nu_0$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu} \setminus \underline{\nu_0}$, and ν real numbers $v^* \in \mathbb{R}_+^q$ for $m \in \underline{\nu_0}$ and $w_m^* \neq 0$ for $m \in \underline{\nu} \setminus \underline{\nu_0}$ such that*

$$\sum_{i=1}^{p} u_i[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla H_{k_m}(x^*, s^m) = 0, \tag{13}$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla^2 H_{k_m}(x^*, s^m) \Big\} z^* \Big\rangle \geq 0, \tag{14}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \tag{15}$$

$$v_m^* G_{j_m}(x^*, t^m) = 0, \quad m \in \underline{\nu_0},$$

$$w_m^* \nabla H_{k_m}(x^*, s^m) \geq 0 \ \textit{for all } m \in \underline{\nu} \setminus \underline{\nu_0}. \tag{16}$$

Assume, further that any one of the following six sets of conditions holds:

(a) (i) for each $i \in I_+ \equiv I_+(u^*)$, f_i is $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-pseudosonvex and $-g_i$ is $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-pesudosonvex at x^*, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
(ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}_m, \eta, \hat{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;
(iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
(iv) $\rho^*(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$, where $\rho^*(x, x^*) = \sum_{i \in I_+} u_i^*[\bar{\rho}_i(x, x^*) + \lambda^* \bar{\rho}_i(x, x^*)]$;

(b) (i) for each $i \in I_+$, f_i is $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-pseudosonvex and $-g_i$ is $(\bar{\phi}, \eta, \bar{\rho}, \theta, m)$-pseudosonvex at x^*, ϕ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
(ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;
(iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
(iv) $\rho^*(x, x^*) + \hat{\rho}(x, x^*) + \sum_{m=\nu_0+1}^{\nu} \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+$, f_i is $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-pseudosonvex and $-g_i$ is $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-pseudosonvex at x^*, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
(ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}_m, \eta, \hat{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;
(iii) the function $z \to v_m^* H_{k_m}(z, s^m)$ is $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
(iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+$, f_i is $(|bar\phi, \eta, \bar{\rho}, \theta, \tilde{m})$-pseudosonvex and $-g_i$ is $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-pseudosonvex at x^*, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
(ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;
(iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
(iv) $\bar{\rho}^*(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ is $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-pseudosonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
(ii) $\xi \to \mathcal{G}(\xi, v^*, w^*)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;
(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof (a): Based on (ii) and (iii), applying (12), (13) and (iv), we have

$$\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \frac{1}{4}\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)]z, \eta(x, x^*)\Big\rangle$$
$$+\frac{1}{4}\Big\langle z^*, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)]z\Big\rangle$$
$$\geq -\Big[\Big\langle \sum_{m=1}^{\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \frac{1}{4}\sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m) z^*, \eta(x, x^*)\Big\rangle$$
$$+\frac{1}{4}\Big\langle z^*, \sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m) z^*\Big\rangle$$
$$+\Big\langle \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla H_{k_m}(x^*, s^m) + \frac{1}{4}\sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m) z^*, \eta(x, x^*)\Big\rangle$$
$$+\frac{1}{4}\Big\langle z^*, \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m) z^*\Big\rangle\Big]$$
$$\geq [\sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*)]\|\theta(x, x^*)\|^{\tilde{m}}$$
$$\geq -\bar{\rho}(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}.$$

This implies using (i) that

$$f_i(x^*) - \lambda^* g_i(x^*) \geq 0,$$

which means x^* is an optimal solution to (P). On the other hand, the proofs using (b)–(e) are similar to that of (a). □

Theorem 2.8 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions f_i, g_i, $i \in \underline{p}$, be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$ and let us assume that for each critical direction z^*, there exist $u^* \in U$ and, integers ν_0 and ν with $0 \leq \nu_0 \leq \nu \leq n+1$ such that there are ν_0 indices j_m with $1 \leq j_m \leq q$ together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$ for $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m with $1 \leq k_m \leq r$ together with $\nu - \nu_0$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu} \setminus \underline{\nu_0}$, and ν real numbers $v^* \in \mathbb{R}_+^q$ for $m \in \underline{\nu_0}$ such that*

$$\sum_{i=1}^{p} u_i[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla H_{k_m}(x^*, s^m) = 0, \tag{17}$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(x^*, t^m)$$

$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla^2 H_{k_m}(x^*, s^m) \Big\} z^* \Big\rangle \geq 0, \tag{18}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \tag{19}$$

$$v_m^* G_{j_m}(x^*, t^m) = 0, \quad m \in \underline{\nu_0},$$

$$w_m^* \nabla H_{k_m}(x^*, s^m) \geq 0 \ \textit{for all}\ m \in \underline{\nu} \setminus \underline{\nu_0}. \tag{20}$$

Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) for each $i \in I_+ \equiv I_+(u^*)$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
 (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}_m, \eta, \hat{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;
 (iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
 (iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(b) (i) for each $i \in I_+$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
 (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;
 (iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
 (iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
 (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}_m, \eta, \hat{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;
 (iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
 (iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
 (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;

(iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;

(iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$;

(e) (i) $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) $\xi \to \mathcal{G}(\xi, v^*, v^*)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;

(iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) > 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof Let x be an arbitrary feasible solution of (P).

(a): In view of our assumptions specified in (ii) and (iii), (16) and (17) remain valid for the present case. From (16)–(19), and (iv) we deduce that

$$
\begin{aligned}
&\Big\langle \sum_{i=1}^{p} u_i^*[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \frac{1}{4}\sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)]z^*, \eta(x, x^*)\Big\rangle \\
&+ \frac{1}{4}\Big\langle z^*, \sum_{i=1}^{p} u_i^*[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)]z^*\Big\rangle \\
&\geq -\Big[\Big\langle \sum_{m=1\nu_0} v_m^* \nabla G_{j_m}(x^*, t^m) + \frac{1}{4}\sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m)z^*, \eta(x, x^*)\Big\rangle \\
&+ \frac{1}{4}\Big\langle z^*, \sum_{m=1}^{\nu_0} v_m^* \nabla^2 G_{j_m}(x^*, t^m)z^*\Big\rangle \\
&+ \Big\langle \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla H_{k_m}(x^*, s^m) + \frac{1}{4}\sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m)z^*, \eta(x, x^*)\Big\rangle \\
&\frac{1}{4}\Big\langle z^*, \sum_{m=\nu_0+1}^{\nu} w_m^* \nabla^2 H_{k_m}(x^*, s^m)z^*\Big\rangle\Big] \\
&\geq \Big[\sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*)\Big]\|\theta(x, x^*)\|^{\tilde{m}} \quad \text{(by (10) and (11))} \\
&> -\bar{\rho}(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}} \quad \text{(by (iv))},
\end{aligned}
$$

which in view of (i) implies that

$$\bar{\phi}\big(\mathcal{E}(x, u^*, \lambda^*) - \mathcal{E}(x^*, u^*, \lambda^*)\big) \geq 0.$$

It follows using the properties of the function $\bar{\phi}$ that

$$\mathcal{E}(x, u^*, \lambda^*) \geqq \mathcal{E}(x^*, u^*, \lambda^*) = 0,$$

where the equality follows from (19). Now based on the proof of Theorem 2.6, we conclude that x^* is an optimal solution to (P).

(b)–(e) : The proofs are similar to that of part (a). $\square$

Theorem 2.9 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions f_i, g_i, $i \in \underline{p}$, be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at x^* for all $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$ and let us assume that for each critical direction z^*, there exist $u^* \in U$ and, integers ν_0 and ν with $0 \leq \nu_0 \leq \nu \leq n+1$ such that there are ν_0 indices j_m with $1 \leq j_m \leq q$ together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$ for $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m with $1 \leq k_m \leq r$ together with $\nu - \nu_0$ points $s^m \in \overline{S_{k_m}}$ for $m \in \underline{\nu} \setminus \underline{\nu_0}$, and ν real numbers $v^* \in \mathbb{R}^q_+$ for $m \in \underline{\nu_0}$ such that*

$$\sum_{i=1}^{p} u_i[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} w_m \nabla G_{j_m}(x^*, t^m)$$

$$+ \sum_{m=\nu_0+1}^{\nu} v_m \nabla H_{k_m}(x^*, s^m) = 0, \tag{21}$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(x^*, t^m)$$

$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla^2 H_{k_m}(x^*, s^m) \Big\} z^* \Big\rangle \geq 0, \tag{22}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \tag{23}$$

$$v_m^* G_{j_m}(x^*, t^m) = 0, \quad m \in \underline{\nu_0},$$

$$w_m^* \nabla H_{k_m}(x^*, s^m) \geq 0 \ \textit{for all } m \in \underline{\nu} \setminus \underline{\nu_0}. \tag{24}$$

Assume, furthermore, that any one of the following five sets of hypotheses is satisfied:

(a) (i) for each $i \in I_+ \equiv I_+(u^*)$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;

(ii) the function $z \to G_{j_m}(z, t^m)$ is strictly $(\hat{\phi}_m, \eta, \hat{\rho}_m, \theta, \tilde{m})$-pseudosonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;

(iii) the function $z \to v_m^* H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;

(iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(b) (i) for each $i \in I_+$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
   (ii) the function $z \to G_{j_m}(z, t^m)$ is strictly $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-pseudosonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;
   (iii) the function $z \to v_m^* H_{k_m}(z, s^m)$ is $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
   (iv) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(c) (i) for each $i \in I_+$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\bar{\phi}$ is superlinear, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
   (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}_m, \eta, \hat{\rho}_m, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}_m$ is increasing, and $\hat{\phi}_m(0) = 0$ for each $m \in \underline{\nu_0}$;
   (iii) the function $z \to H_{k_m}(z, s^m)$ is strictly $(\breve{\phi}_m, \eta, \breve{\rho}_m, \theta, \tilde{m})$-pseudosonvex at x^*, $\breve{\phi}_m$ is increasing, and $\breve{\phi}_m(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
   (iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}_m(x, x^*) + \sum_{m=\nu_0+1}^{\nu} w_m^* \breve{\rho}_m(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(d) (i) for each $i \in I_+$, f_i is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex and $-g_i$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
   (ii) the function $z \to G_{j_m}(z, t^m)$ is $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$ for each $m \in \underline{\nu_0}$;
   (iii) the function $z \to H_{k_m}(z, s^m)$ is strictly $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-pseudosonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
   (iv) $\bar{\rho}(x, x^*) + \sum_{m=1}^{\nu_0} v_m^* \hat{\rho}(x, x^*) + \sum_{m=\nu_0)+1}^{\nu} w_m^8 \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$;

(e) (i) $\xi \to \mathcal{E}(\xi, u^*, \lambda^*)$ is prestrictly $(\bar{\phi}, \eta, \bar{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, and $\bar{\phi}(a) \geq 0 \Rightarrow a \geq 0$;
   (ii) $\xi \to \mathcal{G}(\xi, v^*, v^*)$ is strictly $(\hat{\phi}, \eta, \hat{\rho}, \theta, \tilde{m})$-pseudosonvex at x^*, $\hat{\phi}$ is increasing, and $\hat{\phi}(0) = 0$;
   (iii) the function $z \to H_{k_m}(z, s^m)$ is $(\breve{\phi}, \eta, \breve{\rho}, \theta, \tilde{m})$-quasisonvex at x^*, $\breve{\phi}$ is increasing, and $\breve{\phi}(0) = 0$ for each $m \in \underline{\nu} \setminus \underline{\nu_0}$;
   (iii) $\bar{\rho}(x, x^*) + \hat{\rho}(x, x^*) + \breve{\rho}(x, x^*) \geq 0$ for all $x \in \mathbb{F}$.

Then x^* is an optimal solution of (P).

Proof The proof is similar to that of Theorem 2.6 with suitable adjustments to involved constraints. □

Finally, we present the following variant of Theorem 2.6, while its proof is almost identical to that of Theorem 2.6 and hence omitted.

Theorem 2.10 *Let $x^* \in \mathbb{F}$, let $\lambda^* = \varphi(x^*) \geq 0$, and assume that the functions f_i, g_i, $i \in \underline{p}$, be twice continuously differentiable at x^*, for each $j \in \underline{q}$, let the function $z \to G_j(z, t)$ be twice continuously differentiable at x^* for all $t \in T_j$, and for each $k \in \underline{r}$, let the function $z \to H_k(z, s)$ be twice continuously differentiable at*

x^* for all $s \in S_k$, $j \in \underline{q}$, $k \in \underline{r}$ and let us assume that for each critical direction z^*, there exist $u^* \in U$ and, integers ν_0 and ν with $0 \leq \nu_0 \leq \nu \leq n+1$ such that there are ν_0 indices j_m with $1 \leq j_m \leq q$ together with ν_0 points $t^m \in \hat{T}_{j_m}(x^*)$, $m \in \underline{\nu_0}$ for $m \in \underline{\nu_0}$, $\nu - \nu_0$ indices k_m with $1 \leq k_m \leq r$ together with $\nu - \nu_0$ points $s^m \in S_{k_m}$ for $m \in \underline{\nu} \setminus \underline{\nu_0}$, and ν real numbers $v^* \in \mathbb{R}^q_+$ for $m \in \underline{\nu_0}$ such that

$$\Big\langle \sum_{i=1}^{p} u_i[\nabla f_i(x^*) - \lambda^* \nabla g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla H_{k_m}(x^*, s^m), \eta(x, x^*)\Big\rangle \geq 0 \, \forall \, x \in \mathbb{F}, \tag{25}$$

$$\Big\langle z^*, \Big\{ \sum_{i=1}^{p} u_i[\nabla^2 f_i(x^*) - \lambda^* \nabla^2 g_i(x^*)] + \sum_{m=1}^{\nu_0} v_m \nabla^2 G_{j_m}(x^*, t^m)$$
$$+ \sum_{m=\nu_0+1}^{\nu} w_m \nabla^2 H_{k_m}(x^*, s^m) \Big\} z^* \Big\rangle \geq 0, \tag{26}$$

$$u_i^*[f_i(x^*) - \lambda^* g_i(x^*)] = 0, \tag{27}$$

$$v_m^* G_{j_m}(x^*, t^m) = 0, \quad m \in \underline{\nu_0},$$

$$w_m^* \nabla H_{k_m}(x^*, s^m) \geq 0 \;\; \textit{for all } m \in \underline{\nu} \setminus \underline{\nu_0}. \tag{28}$$

Furthermore, assume that any one of the six sets of conditions specified in Theorem 2.6 is satisfied. Then x^* is an optimal solution of (P).

Proof The proof is similar to that of Theorem 2.6 with some appropriate manipulations. □

In this section, we made some observations and remarks on the results and applications for the future research and applications in the interdisciplinary sense.

4 General Remarks

There exists an enormous amount of investigations on discrete minmax fractional programming problems ranging from generalized invexities to generalized univexities, notably the recent work of Zalmai [3], while we have established a set of higher order parametric necessary optimality conditions and numerous sets of second-order sufficient criteria to the context of a semi-infinite discrete minmax fractional programming problem using a variety of generalized $(\phi, \eta, \rho, \theta, \tilde{m})$-sonvexity constraints.

Note that the field of semi-infinite discrete minmax fractional programming is still developing, the obtained results have a greater potential for applications to higher order generalized univexity in general semi-infinite discrete minmax fractional programming.

We further remark that the results presented in this chapter can be further applied in generalizing by using the some new upgrades for definitions of certain classes of generalized sonvexities of functions of higher orders as follows. Let $x^* \in X$ and assume that the function $f : X \to \mathbb{R}$ is twice differentiable at x^*.

Definition 2.11 The function f is said to be (*strictly*) $(\phi, \eta, \zeta, \rho, \theta, \tilde{m})$-*sonvex* at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta, \zeta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer $\tilde{m}$ such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\phi\big(f(x) - f(x^*)\big)(>) \geq \langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$

$$+\frac{1}{4}\langle \zeta(x, x^*), \nabla^2 f(x^*)z\rangle + \rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}.$$

The function f is said to be (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-sonvex on X if it is (strictly) $(\phi, \eta, \zeta, \rho, \theta, \tilde{m})$-sonvex at each $x^* \in X$.

Definition 2.12 The function f is said to be (*strictly*) $(\phi, \eta, \zeta, \rho, \theta, \tilde{m})$-*pseudosonvex* at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta, \zeta : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer $\tilde{m}$ such that for each $x \in X$ $(x \neq x^*)$ and $z \in \mathbb{R}^n$,

$$\langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$
$$+\frac{1}{4}\langle \zeta(x, x^*), \nabla^2 f(x^*)z\rangle \geq -\rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}} \quad \Rightarrow \quad \phi\big(f(x) - f(x^*)\big)(>) \geq 0,$$

equivalently,

$$\phi\big(f(x) - f(x^*)\big)(\leq) < 0 \Rightarrow \langle \nabla f(x^*) + \frac{1}{4}\nabla^2 f(x^*)z, \eta(x, x^*)\rangle$$

$$+\frac{1}{4}\langle \zeta(x, x^*), \nabla^2 f(x^*)z\rangle < -\rho(x, x^*)\|\theta(x, x^*)\|^{\tilde{m}}.$$

The function f is said to be (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex on X if it is (strictly) $(\phi, \eta, \zeta, \rho, \theta, m)$-pseudosonvex at each $x^* \in X$.

Definition 2.13 The function f is said to be (*prestrictly*) $(\phi, \eta, \zeta, \rho, \theta, \tilde{m})$-*quasisonvex* at x^* if there exist functions $\phi : \mathbb{R} \to \mathbb{R}$, $\eta, \zeta, : X \times X \to \mathbb{R}^n$, $\rho : X \times X \to \mathbb{R}$, and $\theta : X \times X \to \mathbb{R}^n$, and a positive integer $\tilde{m}$ such that for each $x \in X$ and $z \in \mathbb{R}^n$,

$$\phi\big(f(x)-f(x^*)\big)(<)\leq 0 \quad\Rightarrow\quad \langle \nabla f(x^*)+\frac{1}{4}\nabla^2 f(x^*)z, \eta(x,x^*)\rangle$$

$$+\frac{1}{4}\langle \zeta(x,x^*), \nabla^2 f(x^*)z\rangle \leq -\rho(x,x^*)\|\theta(x,x^*)\|^{\tilde{m}},$$

equivalently

$$\langle \nabla f(x^*)+\frac{1}{4}\nabla^2 f(x^*)z, \eta(x,x^*)\rangle$$

$$+\frac{1}{4}\langle \zeta(x,x^*), \nabla^2 f(x^*)z\rangle > -\rho(x,x^*)\|\theta(x,x^*)\|^{\tilde{m}} \Rightarrow \phi\big(f(x)-f(x^*)\big)(\geq) > 0.$$

References

1. Verma, R.U., Zalmai, G.J.: Generalized parametric duality models in discrete minmax fractional programming based on second-order optimality conditions. Commun. Appl. Nonlinear Anal. **22**(2), 17–36 (2015)
2. Verma, R.U., Zalmai, G.J.: Generalized second-order parameter-free optimality conditions in discrete minmax fractional programming. Commun. Appl. Nonlinear Anal. **22**(2), 57–78 (2015)
3. Zalmai, G.J.: Generalized second-order ($\mathcal{F}$, β, ϕ, ρ, θ)-univex functions and parametric duality models in semiinfinite discrete minmax fractional programming. Adv. Nonlinear Variational Inequalities **15**(2), 63–91 (2012)

http://www.springer.com/978-981-10-6255-1

Semi-Infinite Fractional Programming
Verma, R.U.
2017, XI, 291 p., Hardcover
ISBN: 978-981-10-6255-1