Chapter 2
Random Ordinary Differential Equations

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, where $\mathscr{F}$ is a σ-algebra on Ω and $\mathbb{P}$ is a probability measure, and let $\eta : [0, T] \times \Omega \to \mathbb{R}^m$ be an $\mathbb{R}^m$-valued stochastic process with continuous sample paths. In addition, let $g : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ be a continuous function.

A random ordinary differential equation (RODE) in $\mathbb{R}^d$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x, \eta_t(\omega)), \qquad x \in \mathbb{R}^d, \tag{2.1}$$

is a nonautonomous ordinary differential equation (ODE)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G_\omega(t, x) := g(x, \eta_t(\omega)) \tag{2.2}$$

for almost every realization $\omega \in \Omega$.

A simple example of a scalar RODE is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + \sin W_t(\omega),$$

where W_t is a scalar Wiener process. Here $g(x, z) = -x + \sin z$ and $d = m = 1$. RODEs with other kinds of noise such as fractional Brownian motion have also been used.

For convenience, it will be assumed that the RODE (2.2) holds for all $\omega \in \Omega$, by restricting Ω to a subset of full probability if necessary, and that g is infinitely often continuously differentiable in its variables, although k-times continuously differentiable with k sufficiently large would suffice. In particular, g is then locally Lipschitz in x, so the initial value problem

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$$\frac{dx}{dt} = g(x(t,\omega), \eta_t(\omega)), \qquad x(0,\omega) = x_0(\omega), \tag{2.3}$$

where the initial value x_0 is an $\mathbb{R}^d$-valued random variable, has a unique pathwise solution $x(t,\omega)$ for every $\omega \in \Omega$, which will be assumed to exist on the finite time interval $[0,T]$ under consideration. Sufficient conditions that guarantee the existence and uniqueness of such solutions are similar to those for ODEs and will be considered in Sect. 2.1 of this chapter. The situation is more complicated when the sample paths of the driving noise process η_t are only measurable in t, because then function $G_\omega(t,x)$ is only measurable in t and existence and uniqueness of solutions must now be understood in the sense of Carathéodory.

The solution of the RODE (2.3) is a stochastic process X_t on the interval $[0,T]$. Its sample paths $t \mapsto X_t(\omega)$ are continuously differentiable, but need not be further differentiable, since the vector field $G_\omega(t,x)$ of the nonautonomous ODE (2.2) is usually only at most continuous, but not differentiable in t, no matter how smooth the function g is in its variables.

2.1 Existence and Uniqueness Theorems

Once a sample path of the noise has been fixed a RODE (2.1) is an ODE, in fact a nonautonomous ODE (2.2) since the noise changes the vector field with time.

If the vector field function g in the RODE (2.1) is continuous in both of its variables and the sample paths of the noise process η_t are continuous too, then the vector field function $G_\omega(t,x) := g(x, \eta_t(\omega))$ of the corresponding nonautonomous ODE (2.2) is continuous in both of its variables for each fixed ω. Classical existence and uniqueness theorems for ODEs apply to RODEs in this case. On the other hand, if the sample paths of the noise process η_t are only measurable in t, then function $G_\omega(t,x)$ is only measurable in t and the existence and uniqueness of solutions are to be understood in the sense of Carathéodory.

2.1.1 Classical Assumptions

Suppose that the vector field g in the RODE (2.1) is at least continuous in both of its variables and the sample paths of the noise η_t are continuous. Fix a sample path, i.e., ω, write $G(t,x) := g(x, \eta_t(\omega))$, and consider the initial value problem (IVP)

$$\frac{dx}{dt} = G(t,x), \quad x(t_0) = x_0, \qquad x \in \mathbb{R}^d. \tag{2.4}$$

where G is at least continuous.

A solution of the IVP (2.4) is a continuously differentiable function $x : [t_0, T] \to \mathbb{R}^d$ with $x(t_0) = x_0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = G(t, x(t)) \quad \text{for all} \quad t \in (t_0, T).$$

Integrating the RODE (2.4) gives the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} G(s, x(s))\mathrm{d}s, \qquad t \in [t_0, T]. \tag{2.5}$$

A solution of the IVP (2.4) is thus a solution of the integral equation (2.5). The converse also holds.

Lemma 2.1 *A continuous function $x : [t_0, T] \to \mathbb{R}^d$ satisfying the integral equation (2.5) is a solution of the IVP (2.4). In particular, it is continuously differentiable in (t_0, T).*

Proof The mapping $t \mapsto G(t, x(t))$ is continuous, because the mappings $t \mapsto x(t)$ and $(t, x) \mapsto G(t, x)$ are continuous. Hence, the fundamental theorem of integral and differential calculus applies here and gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} G(s, x(s))\,\mathrm{d}s = G(t, x(t)) \quad \text{for each } t \in (t_0, T).$$

This means that the right-hand side of the integral equation (2.5) is continuously differentiable. Hence, when $x(t)$ satisfies the integral equation (2.5), it too is continuously differentiable and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = G(t, x(t)) \quad \text{for each } t \in (t_0, T).$$

Finally, for $t = t_0$, the integral equation reduces to $x(t_0) = x_0$, so x is a solution of the Eq. (2.4). □

The IVP (2.4) and the integral equation (2.5) are equivalent, but the integral equation is theoretically more convenient because it requires only continuity and not continuous differentiability. Another advantage is that the solution is a fixed point of an integral operator in the integral equation representation of the IVP.

The classical existence and uniqueness theorem due to Picard and Lindelöf is proved using a convergent sequence of successive approximations. The result holds under a local Lipschitz assumption, which is satisfied if G is continuously differentiable in x, but existence may only hold on a smaller time interval, the length of which may depend on the initial value.

Theorem 2.1 (The Picard–Lindelöf Theorem) *Let $G : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be continuous on a parallelepiped $R := \{(t, x) : t_0 \le t \le t_0 + a,\ |x - x_0| \le b\}$ and* *uniformly Lipschitz continuous in x and continuous in t. In addition, let M be a bound for $|G(t, x)|$ on R and denote by $\mu := \min\{a, b/M\}$. Then the initial value problem (2.4) has a unique solution $x^* = x^*(t)$ on $[t_0, t_0 + \mu]$.*

Proof Let $x_0(t) = x_0$. Suppose that $x_k(t)$ has been defined on $[t_0, t_0 + \mu]$, is continuous, and satisfies $|x_k(t) - x_0| \le b$ for $k = 0, \ldots, n$. Put

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} G(s, x_n(s))\mathrm{d}s. \tag{2.6}$$

Then $x_{n+1}(t)$ is defined and continuous on $[t_0, t_0 + \mu]$, since $G(t, x_n(t))$ is so. Also it is clear that

$$|x_{n+1}(t) - x_0| \le \int_{t_0}^{t} |G(s, x_n(s))\mathrm{d}s| \le M\mu \le b.$$

Hence $x_0(t), x_1(t), \ldots$ are defined and continuous on $[t_0, t_0+\mu]$, and satisfy $|x_n(t) - x_0| \le b$. It will be shown next by induction that

$$|x_{n+1}(t) - x_n(t)| \le \frac{M\kappa^n(t - t_0)^{n+1}}{(n + 1)!} \quad \text{for } t_0 \le t \le t_0 + \mu,\ \ n = 0, 1, \ldots, \tag{2.7}$$

where κ is a Lipschitz constant for G in its x component.

First it is straightforward to see that (2.7) holds for $n = 0$. Assume that (2.7) holds for $1, \ldots, n - 1$. By (2.6), for $n \ge 1$:

$$x_{n+1}(t) - x_n(t) = \int_{t_0}^{t} [G(s, x_n(s)) - G(s, x_{n-1}(s))]\mathrm{d}s.$$

Hence the Lipschitz condition of G implies that

$$\begin{aligned} |x_{n+1}(t) - x_n(t)| &\le \kappa \int_{t_0}^{t} |x_n(s) - x_{n-1}(s)|\mathrm{d}s \\ &\le \frac{M\kappa^n}{n!} \int_{t_0}^{t} (s - t_0)^n \mathrm{d}s = \frac{M\kappa^n(t - t_0)^{n+1}}{(n + 1)!}, \end{aligned}$$

which proves (2.7).

Now consider

$$x(t) = x_0 + \sum_{n=0}^{\infty}[x_{n+1}(t) - x_n(t)].$$

It follows from (2.7) that $x(t)$ is uniformly convergent on $[t_0, t_0 + \mu]$, i.e.,

$$\lim_{n\to\infty} x_n(t) = x^*(t) \quad \text{exists uniformly.}$$

Since $G(t, x)$ is uniformly continuous on R due to its continuity and boundedness it follows that $G(t, x_n(t))$ converges to $G(t, x^*(t))$ uniformly on $[t_0, t_0+\mu]$ as $n \to \infty$. Thus (2.6) can be integrated term by term to give

$$x^*(t) = x_0 + \int_{t_0}^{t} G(s, x^*(s))\mathrm{d}s,$$

i.e., $x^*(t) = \lim_{n\to\infty} x_n(t)$ is a solution to (2.4).

To prove uniqueness, let $y(t)$ be any other solution to (2.4) on $[t_0, t_0 + \mu]$, then

$$y(t) = x_0 + \int_{t_0}^{t} G(s, y(s))\mathrm{d}s,$$

and it follows from induction that

$$|x_n(t) - y(t)| \leq \frac{M\kappa^n (t - t_0)^{n+1}}{(n+1)!} \quad \text{for } t_0 \leq t \leq t_0 + \mu, \; n = 0, 1, \ldots. \tag{2.8}$$

Letting $n \to \infty$ in (2.8) gives immediately that $|x^*(t) - y(t)| = 0$, so $y(t) \equiv x^*(t)$. □

Remark 2.1 If the vector field satisfies a global Lipschitz condition, then existence on the entire time interval is obtained. The above proof can be used in this case but requires the solutions on sufficiently small subintervals to be patched together. An alternative proof uses the Banach contraction mapping theorem on the Banach space $\mathscr{C}([t_0, T], \mathbb{R}^d)$ of continuous functions $x : [t_0, T] \to \mathbb{R}^d$ with the supremum norm

$$\|x\|_\infty = \max_{t_0 \leq t \leq T} |x(t)|.$$

The existence can be obtained on the entire interval in one step using the exponential norm

$$\|x\|_{\exp} = \max_{t_0 \leq t \leq T} \left\{ |x(t)|\, e^{-2\kappa t} \right\},$$

which is equivalent to the supremum norm on the space $\mathscr{C}([t_0, T], \mathbb{R}^d)$. (Here κ is the Lipschitz constant).

The next theorem drops the Lipschitz assumption and sacrifices the uniqueness of solutions.

Theorem 2.2 (Peano's Existence Theorem) *Let $G : [t_0, T] \times \in \mathbb{R}^d \to \mathbb{R}^d$ be continuous on a parallelepiped $R := \{(t, x) : t_0 \leq t \leq t_0 + a,\; |x - x_0| \leq b\}$. In addition, let M be an upper bound for $|G(t, x)|$ on R and denote by $\mu := \min\{a, b/M\}$. Then the IVP* (2.4) *has at least one solution $x = x(t)$ on $[t_0, t_0 + \mu]$.*

Proof Let $\delta > 0$ and $x_0(t)$ be a continuously differentiable d-dimensional vector-valued function on $[t_0 - \delta, t_0]$ satisfying $x_0(t_0) = x_0$, $|x_0(t) - x_0| \leq b$ and $|x_0(t) - x_0(s)| \leq M|t - s|$ for all $t, s \in [t_0 - \delta, t_0]$.

For any $0 < \varepsilon \leq \delta$, define a function $x_\varepsilon(t)$ on $[t_0 - \delta, t_0 + \mu]$ by

$$x_\varepsilon(t) = \begin{cases} x_0(t), & t \in [t_0 - \delta, t_0], \\ x_0 + \int_{t_0}^{t} G(s, x_\varepsilon(s - \varepsilon))\mathrm{d}s, & t \in [t_0, t_0 + \mu]. \end{cases} \tag{2.9}$$

Note that (2.9) defines an extension $x_\varepsilon(t)$ of $x_0(t)$ from $[t_0 - \delta, t_0]$ to $[t_0 - \delta, t_0 + \min\{\mu, \varepsilon\}]$ and satisfies on this interval

$$|x_\varepsilon(t) - x_0| \leq b, \quad |x_\varepsilon(t) - x_\varepsilon(s)| \leq M|t - s|. \tag{2.10}$$

By (2.9), $x_\varepsilon(t)$ can be extended as a $\mathscr{C}^0$ function over $[t_0 - \delta, t_0 + \min\{\mu, 2\varepsilon\}]$ so it satisfies (2.10). Continuing in this manner, (2.9) serves to define $x_\varepsilon(t)$ on $[t_0 - \delta, t_0 + \mu]$ such that $x_\varepsilon(t)$ is a $\mathscr{C}^0$ function on $[t_0 - \delta, t_0 + \mu]$ and satisfies (2.10).

It follows that the family of functions, $\{x_\varepsilon(t)\}_{0<\varepsilon\leq\delta}$ is equicontinuous. Hence by the Arzelà Selection Theorem, there exists a sequence $\varepsilon_1 > \varepsilon_2 > \cdots$, such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\lim_{n\to\infty} x_{\varepsilon_n}(t) = x(t) \quad \text{exists uniformly}$$

on $[t_0 - \delta, t_0 + \mu]$. By the uniform continuity of G, it follows that $G(t, x_{\varepsilon_n}(t - \varepsilon_n))$ converges to $G(t, x(t))$ uniformly as $n \to \infty$. Hence, integrating (2.9) with $\varepsilon = \varepsilon_n$ term by term gives

$$x(t) = x_0 + \int_{t_0}^{t} G(s, x(s))\mathrm{d}s,$$

i.e., $x(t)$ is a solution to the IVP (2.4). □

2.1.2 Measurability of Solutions

Let $(\Omega, \mathscr{F})$ be a measurable space and let $\mathscr{B}(\mathbb{R}^d)$ denote the σ−algebra of the Borel subsets of $\mathbb{R}^d$. A mapping $\phi : \Omega \to \mathbb{R}^d$ is said to be measurable if for any $B \in \mathscr{B}(\mathbb{R}^d)$,

$$\phi^{-1}(B) = \{\omega \in \Omega | \phi(\omega) \in B\} \in \mathscr{F}.$$

The vector field G in the IVP (2.4), in fact, depends on ω. Then the IVP can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = G(\omega, t, x), \quad x(t_0) = x_0, \qquad x \in \mathbb{R}^d, \tag{2.11}$$

where G is measurable in ω and continuous in (t, x), since $G(\omega, t, x) := g(x, \eta_t(\omega))$ and g is at least continuous in both variables, and $\eta_t(\omega)$ is measurable in ω and has continuous sample paths.

Lemma 2.2 *Suppose that the IVP (2.11) has a unique solution, denoted by $x(t,\omega)$. Then the mapping $\Omega \to \mathbb{R}^d$ defined by $\omega \mapsto x(t,\omega)$ is measurable for each t.*

Proof Choose and fix a function $x_0(t) \in \mathscr{C}([t_0, T], \mathbb{R}^d)$ such that $x_0(t_0) = x_0$. Define a sequence of functions $x_n : [t_0, T] \times \Omega \to \mathbb{R}^d$ by

$$x_{n+1}(t,\omega) = x_0 + \int_{t_0}^{t} G(\omega, s, x_n(s,\omega))\mathrm{d}s.$$

Similar to the proof of Theorem 2.2, it follows that

$$x(t,\omega) = \lim_{n\to\infty} x_n(t,\omega), \quad \text{for all } \omega \in \Omega.$$

Hence it is sufficient to prove that the mappings $x_n(t,\cdot) : \Omega \to \mathbb{R}^d$ are measurable for all $t \in [t_0, T]$ and $n \in \mathbb{N}$, which can be done by induction.

First, the statement holds for $n = 0$. Next suppose that for some $n \in \mathbb{N}$ and all $t \in [t_0, T]$, the function $x_n(t,\cdot) : \Omega \to \mathbb{R}^d$ is measurable. Define $x_n^{(k)} : [t_0, t) \times \Omega \to \mathbb{R}^d$ by

$$x_n^{(k)}(s,\omega) = \sum_{i=0}^{k-1} \chi_{\left[\frac{it}{k}, \frac{(i+1)t}{k}\right)}(s) \cdot x_n\left(\frac{it}{k}, \omega\right) \quad \text{for all } (s,\omega) \in [t_0, t) \times \Omega,$$

where χ_I is an indicator function with value 1 for $x \in I$ and 0 otherwise.

Using the fact that $G : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function,

$$\lim_{k\to\infty} \int_{t_0}^{t} G(\omega, s, x_n^{(k)}(s,\omega))\mathrm{d}s = \int_{t_0}^{t} G(\omega, s, x_n(s,\omega))\mathrm{d}s.$$

Therefore the mapping $\omega \mapsto \int_{t_0}^{t} G(s, x_n(s,\omega))\mathrm{d}s$ is $\mathscr{B}(\mathbb{R}^d)$-measurable for each $t \in [t_0, T]$, which implies that the function $x_{n+1}(t,\cdot)$ is $\mathscr{B}(\mathbb{R}^d)$-measurable for each $t \in [t_0, T]$. By induction, the mapping $x_n(t,\cdot)$ is $\mathscr{B}(\mathbb{R}^d)$-measurable for all $t \in [t_0, T]$ and $n \in \mathbb{N}$. □

The above result holds also if the initial value is measurable, i.e., a random variable with values $x_0(\omega)$.

2.1.3 Carathéodory Assumptions

The equivalence of the IVP (2.4) and the integral equation (2.5) for a RODE whose vector field is continuous in both variables does not hold when its vector field function is only measurable, but not continuous in time. In fact, the concept of a solution also needs to be modified in this case to one defined in the sense of Carathéodory. It is based on absolutely continuous functions.

Definition 2.1 A function $x : [t_0, T] \to \mathbb{R}^d$ is said to be *absolutely continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_{i=1}^{n} |x(t_i) - x(s_i)| < \varepsilon$ whenever $\{[s_i, t_i) : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals in $[t_0, T]$ that satisfy $\sum_{i=1}^{n}(t_i - s_i) < \delta$.

An absolutely continuous function is uniformly continuous. In fact, it is also of bounded variation, hence the difference of two monotone functions, from which it follows that it is weakly differentiable almost everywhere in $[t_0, T]$ and its derivative is Lebesgue integrable. Recall that the weak derivative here is defined by

$$\int_{t_0}^{T} x'(s)\phi(s)\,ds = -\int_{t_0}^{T} x(s)\phi'(s)\,\mathrm{d}s$$

for all $\phi \in \mathscr{C}_0^\infty((t_0, T), \mathbb{R})$, i.e., $\mathscr{C}^\infty$ functions with compact support [146].

Lemma 2.3 (Lemma 4.11 and Theorem 4.12 [52]) *Let $f : [t_0, T] \to \mathbb{R}$ be bounded and Lebesgue measurable, resp., Lebesgue integrable. If $x(t) = \int_{t_0}^{t} f(s)\,ds$ for each $t \in [t_0, T]$, then x is absolutely continuous in $[t_0, T]$ and its weak derivative $x'(t) = f(t)$ for almost all $t \in [t_0, T]$.*

Lemma 2.3 can be used to show that the solution of Eq. (2.4) satisfies the integral equation (2.5) and vice versa, i.e., a counterpart of Lemma 2.1 holds here. Note that if $x : [t_0, T] \to \mathbb{R}^d$ is continuous then the mapping $t \mapsto G(t, x(t))$ is Lebesgue integrable on $[t_0, T]$. The right-hand side of Eq. (2.4) can be integrated in the sense of Lebesgue with the given initial condition to give the right-hand side of the integral equation (2.5). By Lemma 2.3 the function

$$t \mapsto x_0 + \int_{t_0}^{t} G(s, x(s))\,\mathrm{d}s$$

is absolutely continuous on $[t_0, T]$ and its weak derivative is equal to $G(t, x(t))$ for Lebesgue almost all $t \in [t_0, T]$.

Definition 2.2 A function $G : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ of the IVP (2.4) is said to satisfy the *Carathéodory conditions* if

(C1) (continuity) $G(t, x)$ is continuous in x for almost every $t \in [t_0, T]$;
(C2) (measurability) $G(t, x)$ is Lebesgue measurable in t for each $x \in \mathbb{R}^d$;
(C3) (boundedness) $|G(t, x)| \le m(t)$ for each $x \in \mathbb{R}^d$ and almost every $t \in [t_0, T]$ for some absolutely continuous function $m(t)$.

Let $G : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy the Carathéodory conditions. Then a solution (in the extended sense) of the IVP (2.4) with vector field G is an absolutely continuous function $x : [t_0, T] \to \mathbb{R}^d$ with $x(t_0) = x_0$ for which the weak derivative satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = G(t, x(t)) \quad \text{for Lebesgue almost all } \ t \in [t_0, T]. \tag{2.12}$$

The counterpart of the Picard–Lindelöf Theorem 2.1 holds for initial value problems under the Carathéodory conditions with solutions defined in this way. In 1918 Carathéodory [29] established a local existence result under assumptions (C1)–(C3), stated as follows.

Theorem 2.3 (Theorem 1.1, Chap. 2 [32]) *Let G be defined on $R := \{(t, x) : t_0 \leq t \leq T, |x - x_0| < b\}$ and satisfy the Carathéodory conditions. Then the IVP (2.4) has a solution $x^* : [t_0, t_0 + \delta] \to \mathbb{R}^d$ in the extended sense of Eq.* (2.12).

Proof For any $t \in [t_0, T]$, define $M(t)$ by

$$M(t) := \int_{t_0}^{t} m(s)\mathrm{d}s. \tag{2.13}$$

Then $M(t)$ is a continuous nondecreasing function satisfying $M(t_0) = 0$. Therefore $(t, x_0 \pm M(t)) \in R$ on some interval $t_0 \leq t \leq t_0 + \delta \leq T$, where δ is some positive constant. For this $\delta > 0$, define function $x_n(t)$, $n = 1, 2, \ldots$, by

$$x_n(t) := x_0, \qquad t_0 \leq t \leq t_0 + \frac{\delta}{n}, \tag{2.14}$$

$$x_n(t) := x_0 + \int_{t_0}^{t-\delta/n} G(s, x_n(s))\mathrm{d}s, \qquad t_0 + \frac{\delta}{n} < t \leq t_0 + \delta. \tag{2.15}$$

It is clear that $x_1(t) = x_0$ is defined on $[t_0, t_0 + \delta]$. For any $n \geq 1$, formula (2.14) defines x_n on $[t_0, t_0 + \delta/n]$ and since $(t, x_0) \in R$ for $t \in [t_0, t_0 + \delta/n]$, formula (2.15) defines x_n as a continuous function on $(t_0 + \delta/n, t_0 + 2\delta/n]$. Furthermore, due to (C3) and (2.14) for any $t \in (t_0 + \delta/n, t_0 + 2\delta/n]$,

$$|x_n(t) - x_0| \leq M\left(t - \delta/n\right). \tag{2.16}$$

Assume that x_n is defined on $[t_0, t_0 + k \cdot \delta/n]$ for $1 < k < n$. Since the measurability of the integrand in (2.15) is only required on $[t_0, t_0 + k \cdot \delta/n]$, formula (2.15) defines x_n for $t \in (t_0 + k \cdot \delta/n, t_0 + (k + 1) \cdot \delta/n]$. In addition, due to (C3) and (2.13), $x_n(t)$ satisfies (2.16) on $(t_0 + k \cdot \delta/n, t_0 + (k + 1) \cdot \delta/n]$. Therefore by induction, (2.14) and (2.15) define all $x_n(t)$, $n = 1, 2, \ldots$, as continuous functions on $t \in [t_0, t_0 + \delta]$ satisfying

$$x_n(t) = x_0, \qquad t_0 \leq t \leq t_0 + \frac{\delta}{n}, \tag{2.17}$$

$$|x_n(t) - x_0| \leq M\left(t - \frac{\delta}{n}\right), \qquad t_0 + \frac{\delta}{n} < t \leq t_0 + \delta. \tag{2.18}$$

For any $t_1, t_2 \in [t_0, t_0 + \delta]$, according to (C3), (2.13)–(2.15),

$$|x_n(t_1) - x_n(t_2)| \leq \left| M\left(t_1 - \frac{\delta}{n}\right) - M\left(t_2 - \frac{\delta}{n}\right) \right|. \tag{2.19}$$

Since M is continuous on $[t_0, t_0 + \delta]$, it is uniformly continuous on $[t_0, t_0 + \delta]$. Thus by (2.17)–(2.19), the sequence $\{x_n\}$ is equicontinuous and uniformly bounded on $[t_0, t_0 + \delta]$. It then follows from the Ascoli Lemma that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges uniformly to a continuous limit function, namely, $x(t)$, on $[t_0, t_0 + \delta]$ as $j \to \infty$.

By (C3), $|G(t, x_{n_j}(t))| \le m(t)$ for any $t \in [t_0, t_0 + \delta]$. Also, by (C1), G is continuous in x for almost every fixed t, so

$$\lim_{j\to\infty} G(t, x_{n_j}(t)) = G(t, x(t)) \quad \text{for almost every fixed} \quad t \in [t_0, t_0 + \delta].$$

Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{j\to\infty} \int_{t_0}^{t} G(s, x_{n_j}(s))\mathrm{d}s = \int_{t_0}^{t} G(s, x(s))\mathrm{d}s, \qquad \text{for all } t \in [t_0, t_0 + \delta]. \tag{2.20}$$

Noticing that

$$x_{n_j}(t) = x_0 + \int_{t_0}^{t} G(s, x_{n_j}(s))\mathrm{d}s - \int_{t-\delta/n_j}^{t} G(s, x_{n_j}(s))\mathrm{d}s,$$

and that

$$\lim_{j\to\infty} \int_{t-\delta/n_j}^{t} G(s, x_{n_j}(s))\mathrm{d}s = 0,$$

it follows from (2.20) that

$$x(t) = x_0 + \int_{t_0}^{t} G(s, x(s))\mathrm{d}s, \quad \text{for all } t \in [t_0, t_0 + \delta]. \qquad \square$$

2.1.4 Positivity of Solutions

A solution $x(t) = (x_1(t), x_2(t), \ldots, x_d(t))^\top$ of (2.12) is called *positive* (*strongly positive*, resp.) if

$$x_i(t) \ge 0 \ (> 0, \text{ resp.}) \quad \text{for all } \ t \text{ and } i = 1, 2, \ldots, d.$$

The positivity of solutions is important in biological models like those that will be considered in Part IV (see, in particular, Chap. 18) as well as in physics, chemistry and engineering. The following conditions guarantee the positivity of solutions to (2.4).

Definition 2.3 A function $u = (u_1, u_2, \ldots, u_d)^\top : \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^d$ is called *quasi-positive*, or *off-diagonal positive*, if, for each $i = 1, 2, \ldots, d$,

$$u_i\,(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) \geq 0,$$

whenever $x_j \geq 0$ for $j \neq i$.

A function $u = (u_1, u_2, \ldots, u_d)^\top$ is called *strongly quasipositive*, or *strongly off-diagonal positive*, if, for each $i = 1, 2, \ldots, d$,

$$u_i\,(t, x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_d) > 0,$$

whenever $x_j \geq 0$ for all j and $\sum_j x_j > 0$.

Theorem 2.4 *If the vector field G of (2.4) is quasipositive, then the solution $x(t)$ of (2.4) satisfying the initial condition $x(0) = x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,d})$ is positive for $t \geq 0$ whenever $x_{0,i} \geq 0$ for $i = 1, 2, \ldots, d$.*

If the vector field G of (2.4) is strongly quasipositive, then the solution $x(t)$ of (2.4) satisfying the initial condition $x(0) = x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,d})$ is strongly positive for $t \geq 0$ whenever $x_{0,i} > 0$ for $i = 1, 2, \ldots, d$.

The proof of Theorem 2.4 for the case that $G(t, x)$ is continuous in t and x can be found, e.g., in Krasnosel'skii [95, Lemma 4.1]. It was noted in Szarski [131] that similar statements are also valid when $G(t, x)$ satisfies the Carathéodory conditions.

2.2 RODEs with Canonical Noise

RODEs typically involve given stochastic processes in their vector fields which can differ from example to example. The theory of random dynamical systems, in contrast, is formulated abstractly in terms of a canonical noise process. This allows greater generality and is, in particular, independent of the dimension of the driving noise process. The canonical noise process is represented by a measurable theoretical autonomous dynamical system θ on the sample space Ω of some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Specifically, it is a group under composition of measure preserving transformations $\theta_t : \Omega \to \Omega$, $t \in \mathbb{R}$, i.e., satisfying

(i) $\theta_0 = Id$ (identity) and $\theta_t \circ \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$,
(ii) the map $(t, \omega) \mapsto \theta_t(\omega)$ is measurable and invariant with respect to $\mathbb{P}$ in the sense that $\theta_t(\mathbb{P}) = \mathbb{P}$ for all $t \in \mathbb{R}$.

The notation $\theta_t(\mathbb{P}) = \mathbb{P}$ for the measure preserving property of θ_t with respect to $\mathbb{P}$ is just a compact way of writing $\mathbb{P}(\theta_t(A)) = \mathbb{P}(A)$ for all $A \in \mathscr{F}$ and $t \in \mathbb{R}$.

In this context RODEs have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(x, \theta_t(\omega)), \tag{2.21}$$

where the vector field function $g : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is assumed to be suitably smooth in its first variable and measurable in the second.

Consider the simple scalar RODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + W_t(\omega)\,,$$

where the driving noise process is a two-sided[1] Wiener process, i.e., defined for $t \in \mathbb{R}$. The canonical noise system θ is not given directly in terms of the Wiener process W_t, but is defined in terms of shift operators θ_t on the canonical sample space $\Omega := \mathscr{C}_0(\mathbb{R}, \mathbb{R})$ of continuous functions $\omega : \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$, i.e., with

$$\theta_t(\omega(\cdot)) := \omega(t + \cdot) - \omega(\cdot) \quad \text{for all } t \in \mathbb{R}.$$

The σ-algebra of Borel subsets of $\mathscr{C}_0(\mathbb{R}, \mathbb{R})$ is taken as the σ-algebra of events $\mathscr{F}$, while $\mathbb{P}$ is the corresponding Wiener measure.[2] Essentially, the canonical noise system here is represented by the sample paths of the Wiener process.

2.3 Endnotes

Bunke's monograph [20], which is in German, is a classical reference on RODEs and contains most of the relevant literature before the 1970s. Bobrowski [18], which is in Polish, is similar in content. Sufficient conditions guaranteeing the existence and uniqueness of solutions of RODEs can be found in Arnold [4] and Bunke [20].

Properties of absolutely continuous and weakly differentiable functions are discussed in Evans and Gariepy [47], Gordon [52], Leoni [98] and Ziemer [146]. Existence and uniqueness theorems under classical and Carathéodory conditions are discussed extensively in Coddington and Levinson [32]. See also Carathéodory [29] and, for more general conditions, Goodman [51] and Biles and Binding [17]. See also Jentzen and Neuenkirch [79].

Monotonicity conditions on differential equations and the positivity of solutions are discussed in [95, 121, 131, 137].

Kac and Krasovski [143] considered RODEs driven by a finite Markov chain, while Arnold and Kloeden [5] analysed 2-dimensional RODEs driven by telegraphic noise. RODEs with fractional Brownian motion were investigated in Garrido–Atienza, Kloeden and Neuenkirch [49]. The recent book [108] by Neckel and Rupp focuses on modeling with RODEs.

[1] Essentially, $\{W_t, t \geq 0\}$, and $\{W_{-t}, t \leq 0\}$ are two independent Wiener processes.

[2] No other topological properties of the space $\mathscr{C}_0(\mathbb{R}, \mathbb{R})$ are used here apart from those defining the Borel sets.

A systematic treatment of the random dynamical system theory and RODEs in the form (2.21) is given in Arnold [4]. This theory will be briefly reviewed in Chap. 4.

The discretisation of RODEs near a saddle point is investigated in Arnold and Kloeden [6]. A delay differential equation with the randomness in the delay under discretisation was investigated by Caraballo, Kloeden and Real [26].

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