

# Contents

<b>1</b>	<b>Introduction to Physics of Fluctuation</b>	<b>1</b>
1.1	Background: Physics of Thermal Fluctuation	1
1.2	Toward Physics of Athermal Fluctuation	2
1.3	Organization of This Thesis	3
	References	6
 <b>Part I Review on Stochastic Theory for Fluctuating Thermal Systems</b>		
<b>2</b>	<b>Markovian Stochastic Processes</b>	<b>11</b>
2.1	Master Equations	11
2.2	Ordinary Differential Equation Without Jumps	12
2.3	Ordinary Differential Equation with Jumps	13
2.4	Poisson Noise	15
2.4.1	Symmetric Poisson Noise	17
2.4.2	Discrete Compound Poisson Noise	18
2.4.3	Continuous Compound Poisson Noise	19
2.5	Gaussian Noise	20
2.6	White Noise	21
2.7	General Master Equation	22
2.8	Kramers–Moyal Expansion	23
2.9	Cumulant Generating Function for the White Noise	24
2.10	Cumulant Generating Functional	25
	References	26
<b>3</b>	<b>Kinetic Theory for Dilute Gas</b>	<b>27</b>
3.1	Pseudo-Liouville Equation for a Simple Collision	27
3.2	Pseudo-Liouville Equation for Many-Body Hardcore Systems	29
3.2.1	Setup	29

3.2.2	Hardcore Potential and Collision Rule . . . . .	30
3.2.3	Pseudo-Liouville Equation . . . . .	32
3.3	BBGKY Hierarchy and Boltzmann Equation . . . . .	34
3.3.1	BBGKY Hierarchy . . . . .	34
3.3.2	Boltzmann Equation . . . . .	36
3.3.3	Boltzmann Lorentz Equation . . . . .	37
3.4	Example: Rayleigh Particle . . . . .	37
3.5	Further Remarks . . . . .	39
	References . . . . .	39
<b>4</b>	<b>Langevin Equation and Its Microscopic Derivation . . . . .</b>	<b>41</b>
4.1	Langevin Equation . . . . .	41
4.1.1	Fokker–Planck Equation . . . . .	42
4.1.2	Path Integral Representation . . . . .	43
4.1.3	Detailed Fluctuation Theorem . . . . .	44
4.2	Microscopic Derivation . . . . .	44
4.2.1	Idea of the System Size Expansion . . . . .	45
4.2.2	Mathematical and Explicit Assumptions . . . . .	46
4.2.3	Rayleigh Particle . . . . .	49
4.3	What Is not Revealed in van Kampen’s Formulation? . . . . .	50
4.3.1	Nonequilibrium Rayleigh Particle . . . . .	51
4.3.2	Granular Motor . . . . .	52
4.3.3	Where Does the Microscopic Irreversibility Go? . . . . .	53
4.3.4	Toward the Minimal Model of Athermal Fluctuating Systems . . . . .	53
	References . . . . .	54
<b>5</b>	<b>Stochastic Calculus for the Single-Trajectory Analysis . . . . .</b>	<b>55</b>
5.1	Introduction to Multiplicative Noises . . . . .	55
5.2	The Itô-Type Stochastic Differential Equation . . . . .	56
5.3	The Differential Rule for the Poisson Processes . . . . .	57
5.4	Gaussian Stochastic Calculus . . . . .	58
5.4.1	Special Characters of the Gaussian Noise (Itô Rule) . . . . .	58
5.4.2	Itô Formula . . . . .	59
5.4.3	Ordinary Differential Rule and the Stratonovich Integral . . . . .	60
5.4.4	Fokker–Planck Equation . . . . .	62
5.4.5	Relation to the Stochastic Liouville Equation . . . . .	63
5.5	Stratonovich-Type Stochastic Differential Equation . . . . .	63
5.5.1	Definition . . . . .	64
5.5.2	Differential Rule . . . . .	64
5.5.3	Fokker–Planck Equation . . . . .	65
5.5.4	Relation to the Stochastic Liouville Equation . . . . .	65

5.5.5	Relation to the Wong-Zakai Theory . . . . .	66
5.6	Marcus-Type Stochastic Differential Equation . . . . .	67
5.6.1	Definition . . . . .	67
5.6.2	Relation to the Wong-Zakai Theory . . . . .	68
5.6.3	Differential Rule . . . . .	69
5.6.4	Unsolved Problems in Conventional Approaches . . . . .	69
	References. . . . .	70
<b>6</b>	<b>Stochastic Energetics for Langevin Dynamics . . . . .</b>	<b>73</b>
6.1	Thermodynamics from Many-Particle to Single-Particle Systems . . . . .	73
6.2	Langevin Equation for Various Thermal Systems. . . . .	74
6.2.1	Colloidal Systems . . . . .	74
6.2.2	Electrical Systems . . . . .	75
6.3	First Law of Thermodynamics . . . . .	76
6.3.1	Underdamped Langevin Equation . . . . .	76
6.3.2	Helmholtz Free Energy . . . . .	78
6.3.3	Overdamped Langevin Equation . . . . .	79
6.4	Second Law of Thermodynamics . . . . .	80
6.4.1	Detailed Fluctuation Theorem . . . . .	80
6.4.2	Crooks Theorem . . . . .	82
6.4.3	Various Nonequilibrium Equalities . . . . .	83
	References. . . . .	84
 <b>Part II Statistical Mechanics for Fluctuating Athermal Systems</b>		
<b>7</b>	<b>Microscopic Derivation of Linear Non-Gaussian Langevin Equation. . . . .</b>	<b>89</b>
7.1	Introduction . . . . .	89
7.2	Linear Non-Gaussian Langevin Equation and Its Mathematical Properties . . . . .	90
7.2.1	Master Equation and Its Exact Solution. . . . .	91
7.2.2	Lack of the Detailed Balance . . . . .	91
7.3	Microscopic Derivation. . . . .	92
7.3.1	Asymptotic Derivation of the Non-Gaussian Langevin Equation . . . . .	92
7.3.2	Inverse Formula. . . . .	94
7.3.3	Nonlinear Temperature . . . . .	95
7.3.4	Violation of the CLT. . . . .	95
7.4	Example: Granular Motor Under Viscous Friction . . . . .	96
7.4.1	Setup. . . . .	96
7.4.2	Reduction to the Non-Gaussian Langevin Equation . . . . .	97
7.4.3	Inverse Formula for the Granular Velocity Distribution . . . . .	99

7.4.4	Numerical Validation. . . . .	100
7.5	Concluding Remarks. . . . .	101
	References. . . . .	102
<b>8</b>	<b>Analytical Solution to Nonlinear Non-Gaussian Langevin Equation. . . . .</b>	<b>103</b>
8.1	Introduction . . . . .	103
8.2	Microscopic Derivation of Nonlinear Non-Gaussian Langevin Equation. . . . .	104
8.2.1	Setup. . . . .	104
8.2.2	Derivation of Non-Gaussian Langevin Equations Under Nonlinear Frictions . . . . .	106
8.2.3	Weak Friction Cases: Reduction to the Gaussian Langevin Equation . . . . .	109
8.2.4	Asymptotic Connection from the Non-Gaussian to the Gaussian Theory . . . . .	111
8.3	Asymptotic Solution for Nonlinear Non-Gaussian Langevin Equation . . . . .	112
8.3.1	Setup. . . . .	113
8.3.2	Asymptotic Solution for Strong Friction . . . . .	113
8.3.3	First-Order Approximation: The Independent-Kick Model . . . . .	115
8.3.4	Toy Model 1: Coulombic Friction. . . . .	117
8.3.5	Toy Model 2: The Cubic Friction . . . . .	119
8.3.6	Higher Order Corrections: Multiple-Kicks. . . . .	122
8.4	Example: Granular Motor Under Dry Friction . . . . .	126
8.4.1	Setup. . . . .	126
8.4.2	Reduction to the Non-Gaussian Langevin Equation. . . . .	128
8.4.3	First-Order Asymptotic Solution . . . . .	129
8.5	Concluding Remarks. . . . .	131
	References. . . . .	131
<b>9</b>	<b>Stochastic Energetics for Non-Gaussian Stochastic Dynamics . . . . .</b>	<b>133</b>
9.1	Introduction . . . . .	133
9.2	Basic Concepts and Notations . . . . .	134
9.3	Itô Type SDEs and the Ordinary Chain Rule . . . . .	135
9.3.1	Products for Smooth $\delta$ -Functions . . . . .	135
9.3.2	Itô Type SDE for Smooth State-Dependent Poisson Noise . . . . .	137
9.3.3	Reformulation of Itô Type SDE for Non-smooth Limit . . . . .	139
9.3.4	Stratonovich and $*$ Integrals for the Itô Type SDE . . . . .	139
9.3.5	Ordinary Chain Rule for General Markov Processes . . . . .	140

9.3.6	Remark on the Gaussian Limit: Relation to the Stratonovich Integral . . . . .	141
9.3.7	Remark on the Marcus SDE . . . . .	142
9.4	Stochastic Thermodynamics for Non-Gaussian State-Dependent Noises . . . . .	145
9.4.1	Review of First Law of Thermodynamics for the Gaussian Langevin Equation . . . . .	146
9.4.2	First Law of Thermodynamics for State-Dependent Non-Gaussian Noise . . . . .	146
9.4.3	Example 1: The BGK-type Kinetic Model and Its Complementarity Relation . . . . .	148
9.4.4	Example 2: Non-Gaussian Langevin Equation. . . . .	150
9.5	Concluding Remarks. . . . .	152
	References. . . . .	153
<b>10</b>	<b>Energy Transport Between Athermal Systems . . . . .</b>	<b>155</b>
10.1	Introduction . . . . .	155
10.2	Model. . . . .	156
10.3	Main Results. . . . .	157
10.3.1	Generalized Fourier Law . . . . .	157
10.3.2	Generalized Heat Fluctuation Theorem . . . . .	159
10.3.3	Generalized Zeroth Law of Thermodynamics . . . . .	160
10.3.4	Example: Athermal Energy Transport Between Granular Motors . . . . .	161
10.4	Derivations of the Main Results . . . . .	163
10.4.1	Generalized Fourier Law . . . . .	163
10.4.2	Generalized Heat Fluctuation Theorem . . . . .	165
10.5	Concluding Remarks. . . . .	169
	References. . . . .	169
<b>11</b>	<b>Energy Pumping from Athermal Systems . . . . .</b>	<b>171</b>
11.1	Introduction . . . . .	171
11.2	Model. . . . .	172
11.3	Main Results. . . . .	174
11.3.1	Work Along Quasi-static Processes. . . . .	174
11.3.2	Power Along Slow Operational Processes . . . . .	176
11.4	Derivations of the Main Results . . . . .	178
11.4.1	Work Along Quasi-static Processes. . . . .	179
11.4.2	Power Along Slow Operational Processes . . . . .	180
11.5	Concluding Remarks. . . . .	182
	References. . . . .	182
<b>12</b>	<b>Conclusion . . . . .</b>	<b>185</b>
	References. . . . .	187

**Appendix A: Technical Notes** . . . . . 189

**Curriculum Vitae** . . . . . 221

Statistical Mechanics for Athermal Fluctuation

Non-Gaussian Noise in Physics

Kanazawa, K.

2017, XVI, 222 p. 53 illus., 46 illus. in color., Hardcover

ISBN: 978-981-10-6330-5