Chapter 2
Reverse Order Law

The problem of finding the generalized inverse of a product of matrices led to the investigation of the so-called "reverse order law". The reverse order law for many types of generalized inverses has been the subject of intensive research over the years. In the 1960s, Greville was the first to study it by considering the reverse order law for the Moore-Penrose inverse and gave a necessary and sufficient condition for the reverse order law

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}, \tag{2.1}$$

to hold for matrices A and B. This was followed by further research on this subject branching in several directions:

- Products with more than two matrices were considered;
- Different classes of generalized inverses were studied;
- Different settings were considered (operator algebras, C^*-algebras, rings etc.)

Over the years this topic has been the subject of interest in various investigations. In this chapter we will set as our primary goal a chronological and systematic presentation, thus taking into account both the time of publication and the level of generalization, of all the published results covering this topic and to point to some problems that are still open and the difficulties that one is faced with when attempting to solve them. Such an approach is intended to give the reader a clear picture of the current status of the research concerning this topic and also some guidelines for future research that they might be interested in doing.

We will discuss the reverse order laws for K-inverses when $K \in \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, \{1, 3, 4\}\}$ and present all recently published results on this subject as well as some simple examples and open problems.

When we are talking about the reverse order law for the K-inverse, where $K \subseteq \{1, 2, 3, 4\}$, we are actually considering the following inclusions:

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$$BK \cdot AK \subseteq (AB)K,$$
$$(AB)K \subseteq BK \cdot AK,$$
$$(AB)K = BK \cdot AK.$$

The reverse order law problem for each of the above mentioned types of generalized inverses will receive special attention, and we will describe necessary and sufficient conditions in various settings such as that of matrices, algebras of bounded linear operators, C^*-algebras and, when possible, in general rings. Beside presenting to the reader some results that have large application, primarily in solving different types of matrix and operator equations, they will have the opportunity to familiarize themselves with the techniques that are used to generalize results obtained in the case of matrices to more general settings such as those of algebras of bounded linear operators, C^*-algebras or rings.

2.1 Reverse Order Laws for {1}-Inverses

In this section, we address the question of when the reverse order laws for $\{1\}$-inverses is valid. It is interesting that although the reverse order law has been considered for many types of generalized inverses and from various aspects too, there are only a few papers which are concerned with this problem for the $\{1\}$-inverse.

In his article, Rao [1] proves that if A and B are complex matrices such that AB is defined, and if either A is of full column rank or B is of full row rank, then

$$B\{1\}A\{1\} \subseteq (AB)\{1\}. \tag{2.2}$$

After this, Pringle and Rayner [2], state incorrectly that any of the two conditions from the Rao's result (i.e., if A is of full column rank or B is of full row rank) imply that

$$(AB)\{1\} = B\{1\}A\{1\} \tag{2.3}$$

which is noted in 1994 by Werner [3], who gives a simple counterexample to this assertion and proves that for given matrices A and B of appropriate sizes, (2.2) holds if and only if

$$\mathscr{N}(A) \subseteq \mathscr{R}(B) \text{ or } \mathscr{R}(B) \subseteq \mathscr{N}(A),$$

where $\mathscr{N}(A)$ and $\mathscr{R}(B)$ are the null space of A and the range of B, respectively. It can easily be seen that Werner's proof, when suitably modified, carries over to operators on Hilbert spaces.

Theorem 2.1 *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $B \in \mathscr{B}(\mathscr{L}, \mathscr{H})$ be regular operators such that the product AB is also regular. Then $B\{1\}A\{1\} \subseteq (AB)\{1\}$ if and only if*

$$\mathcal{N}(A) \subseteq \mathcal{R}(B) \text{ or } AB = 0. \tag{2.4}$$

Also he proves that (2.3) holds in particular in each of the following cases:

(i) A and B are both of full column rank
(ii) A and B are both of full row rank
(iii) A is nonsingular and/or B is nonsingular

but in general, the more difficult problem of finding equivalent descriptions of the condition (2.3) still remains open. The next paper on this topic was by M. Wei [4] where, using P-SVD of matrices A and B, some equivalents of (2.2) are derived and compared with the conditions given by Werner and finally certain necessary and sufficient conditions for (2.3) to hold are given.

Theorem 2.2 ([4]) *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. The following conditions are equivalent:*

(i) $(AB)\{1\} = B\{1\}A\{1\}$
(ii) *One of the following conditions hold:*
   (a) $\mathrm{r}(AB) = 0$, $n \geq \min\{m + \mathrm{r}(B), p + \mathrm{r}(A)\}$,
   (b) $\mathrm{r}(A) + \mathrm{r}(B) - \mathrm{r}(AB) = n$ *and* $(\mathrm{r}(A) = m$ *or* $\mathrm{r}(B) = p)$
(iii) *One of the following conditions hold:*
   (a) $\mathcal{R}(B) \subseteq \mathcal{N}(A)$, $n \geq \min\{m + \mathrm{r}(B), p + \mathrm{r}(A)\}$,
   (b) $\mathcal{N}(A) \subseteq \mathcal{R}(B)$ *and* $(\mathrm{r}(A) = m$ *or* $\mathrm{r}(B) = p)$.

Let us now take a look at the following few examples.

Example 2.1 If $m = n$ and $A = I$, then for any $B \in \mathbb{C}^{n\times p}$ we have $(AB)\{1\} = B\{1\}$ and $A\{1\} = \{I\}$, so $(AB)\{1\} = B\{1\}A\{1\}$, which can also be concluded from Theorem 2.2,(iii)b.

Example 2.2 Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then evidently $AB = 0$ and from Theorem 2.1 (or using the fact that $(AB)\{1\} = \mathbb{C}^{2\times 2}$) we have that $B\{1\}A\{1\} \subseteq (AB)\{1\}$. On the other hand, since

$$A\{1\} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & 1 - a_3 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{C} \right\}$$

and

$$B\{1\} = \left\{ \begin{bmatrix} 1 & b_1 \\ b_2 & b_3 \end{bmatrix} : b_1, b_2, b_3 \in \mathbb{C} \right\},$$

we can check that $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \in (AB)\{1\}$ can not be written as a product $B^{(1)}A^{(1)}$ for some $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$. This means that $(AB)\{1\} \neq B\{1\}A\{1\}$ which can be checked also using Theorem 2.2.

Also, the inclusion (2.2) in the case of the product of more than two matrices was considered by M. Wei [4] by applying the multiple product singular value decomposition.

Theorem 2.3 ([5]) *Let* $A_i \in \mathbb{C}^{m_i \times m_{i+1}}$, $i = \overline{1,n}$, $n \geq 3$. *The following conditions are equivalent:*

(i) $A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_n)\{1\}$

(ii) *One of the following conditions hold:*

 (a) $\mathrm{r}(A_1 \cdots A_n) = 0$

 (b) $\mathrm{r}(A_1 \cdots A_i) + \mathrm{r}(A_{i+1}) - \mathrm{r}(A_1 \cdots A_{i+1}) = m_{i+1}$, $i = \overline{1, n-1}$

(iii) *One of the following conditions hold:*

 (a) $\mathscr{R}(A_{i+1} \cdots A_n) \subseteq \mathscr{N}(A_1 \cdots A_i)$, *for some* $i \in \{1, \ldots, n-1\}$

 (b) $\mathscr{N}(A_1 \cdots A_i) \subseteq \mathscr{R}(A_{i+1})$, $i = \overline{1, n-1}$.

Recently, the previous result was generalized by Nikolov-Radenković [6] for bounded linear operators on Hilbert spaces. We will give a proof of this result.

Theorem 2.4 ([6]) *Let* $A_i \in \mathscr{B}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{1,n}$, *be such that* A_i, $i = \overline{1,n}$ *and* $A_1 A_2 \cdots A_j$, $j = \overline{2,n}$, *are regular operators. The following conditions are equivalent:*

(i) $A_n\{1\} \cdot A_{n-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_n)\{1\}$

(ii) $A_1 A_2 \cdots A_n = 0$ *or* $\mathscr{N}(A_1 \cdots A_{j-1}) \subseteq \mathscr{R}(A_j)$, *for* $j = \overline{2,n}$

(iii) $A_1 A_2 \cdots A_n = 0$ *or* $A_k\{1\} \cdot A_{k-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_k)\{1\}$, *for* $k = \overline{2,n}$.

Proof (ii) $\Rightarrow$ (iii) : If $A_1 A_2 \cdots A_n = 0$, it is evident that (iii) holds. Suppose that $A_1 A_2 \cdots A_n \neq 0$ and

$$\mathscr{N}(A_1 \cdots A_{j-1}) \subseteq \mathscr{R}(A_j), \text{ for } j = \overline{2,n}. \tag{2.5}$$

We will prove by induction on k that

$$A_k\{1\} \cdot A_{k-1}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_k)\{1\} \tag{2.6}$$

holds for $k = \overline{2,n}$. From (2.5) it follows that $\mathscr{N}(A_1) \subseteq \mathscr{R}(A_2)$ which by (2.4) implies that (2.6) holds for $k = 2$. Suppose that (2.6) holds for $k = l - 1$, where $l \in \{2, 3, \ldots, n\}$, i.e.,

$$A_{l-1}\{1\} \cdot A_{l-2}\{1\} \cdots A_1\{1\} \subseteq (A_1 A_2 \cdots A_{l-1})\{1\}. \tag{2.7}$$

We prove that it must also hold for $k = l$. Since (2.5) holds for $j = l$, by (2.4) we have

$$A_l\{1\} \cdot (A_1 A_2 \cdots A_{l-1})\{1\} \subseteq (A_1 A_2 \cdots A_{l-1} A_l)\{1\}, \tag{2.8}$$

which by (2.7) implies that (2.6) holds for $k = l$. Hence, by induction it follows that (2.6) holds for $k = \overline{2, n}$.
(iii) $\Rightarrow$ (i) : This is evident.
(i) $\Rightarrow$ (ii) : Suppose that $A_1 A_2 \cdots A_n \neq 0$ and that $n > 2$ since the assertion in the case $n = 2$ follows by Theorem 2.1. Let $j \in \{3, 4, \ldots, n\}$ and $i \in \{1, 2, \ldots, j-2\}$ be arbitrary. Then for arbitrary $A_i^{(1)} \in A_i\{1\}$ and $A_j^{(1)} \in A_j\{1\}$, we have that

$$\begin{aligned}&A_1 A_2 \cdots A_n A_n^{(1)} \cdots A_{j+1}^{(1)} \cdot (A_j^{(1)} + Y(I_{\mathscr{H}_j} - A_j A_j^{(1)}))A_{j-1}^{(1)} \cdots A_{i+1}^{(1)}\\&(A_i^{(1)} + (I_{\mathscr{H}_{i+1}} - A_i^{(1)} A_i)X) \cdot A_{i-1}^{(1)} \cdots A_1^{(1)} A_1 \cdots A_n = A_1 \cdots A_n\end{aligned} \tag{2.9}$$

holds for every $X \in \mathscr{B}(\mathscr{H}_i, \mathscr{H}_{i+1})$ and every $Y \in \mathscr{B}(\mathscr{H}_{j+1}, \mathscr{H}_j)$. Substituting $X = 0$ in (2.9), we get

$$\begin{aligned}&A_1 A_2 \cdots A_n A_n^{(1)} \cdots A_{j+1}^{(1)} \cdot (A_j^{(1)} + Y(I_{\mathscr{H}_j} - A_j A_j^{(1)}))A_{j-1}^{(1)} \cdots A_{i+1}^{(1)} A_i^{(1)} \cdot\\&A_{i-1}^{(1)} \cdots A_1^{(1)} A_1 \cdots A_n = A_1 \cdots A_n.\end{aligned} \tag{2.10}$$

Subtracting (2.10) from (2.9), we get that

$$\begin{aligned}&A_1 A_2 \cdots A_n A_n^{(1)} \cdots A_{j+1}^{(1)} \cdot (A_j^{(1)} + Y(I_{\mathscr{H}_j} - A_j A_j^{(1)}))A_{j-1}^{(1)} \cdot\\&\cdots A_{i+1}^{(1)}(I_{\mathscr{H}_{i+1}} - A_i^{(1)} A_i)X A_{i-1}^{(1)} \cdots A_1^{(1)} A_1 \cdots A_n = 0.\end{aligned} \tag{2.11}$$

Substituting $Y = 0$ in (2.11), we get

$$\begin{aligned}&A_1 A_2 \cdots A_n A_n^{(1)} \cdots A_{j+1}^{(1)} A_j^{(1)} A_{j-1}^{(1)} \cdots A_{i+1}^{(1)} (I_{\mathscr{H}_{i+1}} - A_i^{(1)} A_i)X \cdot\\&A_{i-1}^{(1)} \cdots A_1^{(1)} A_1 \cdots A_n = 0.\end{aligned} \tag{2.12}$$

Finally, from (2.12) and (2.11), we get that

$$\begin{aligned}&A_1 A_2 \cdots A_n A_n^{(1)} \cdots A_{j+1}^{(1)} \cdot Y(I_{\mathscr{H}_j} - A_j A_j^{(1)})A_{j-1}^{(1)} \cdots A_{i+1}^{(1)} \cdot\\&(I_{\mathscr{H}_{i+1}} - A_i^{(1)} A_i)X A_{i-1}^{(1)} \cdots A_1^{(1)} A_1 \cdots A_n = 0\end{aligned} \tag{2.13}$$

holds for arbitrary $X \in \mathscr{B}(\mathscr{H}_i, \mathscr{H}_{i+1})$ and $Y \in \mathscr{B}(\mathscr{H}_{j+1}, \mathscr{H}_j)$.

Now, it follows that either

$$A_1 A_2 \cdots A_n A_n^{(1)} \cdots A_{j+1}^{(1)} = 0 \tag{2.14}$$

or

$$(I_{\mathscr{H}_j} - A_j A_j^{(1)})A_{j-1}^{(1)} \cdots A_{i+1}^{(1)}(I_{\mathscr{H}_{i+1}} - A_i^{(1)} A_i) = 0 \tag{2.15}$$

or

$$A_{i-1}^{(1)} \cdots A_1^{(1)} A_1 \cdots A_n = 0. \tag{2.16}$$

It is easy to see that (2.14) and (2.16) imply $A_1A_2\cdots A_n=0$ which is not the case, so (2.15) must hold. Hence, for arbitrary $j\in\{3,4,\dots,n\}$ and $i\in\{1,2,\dots,j-2\}$, we have that

$$(I_{\mathscr{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_{i+1}^{(1)}A_i^{(1)}A_i=(I_{\mathscr{H}_j}-A_jA_j^{(1)})A_{j-1}^{(1)}\cdots A_{i+1}^{(1)}. \tag{2.17}$$

Let $j\in\{2,3,\dots,n\}$, be arbitrary. Then by (i) it follows that

$$\begin{aligned}&A_1A_2\cdots A_nA_n^{(1)}\cdots A_{j+1}^{(1)}\cdot\\&(A_j^{(1)}+Y(I_{\mathscr{H}_j}-A_jA_j^{(1)}))(A_{j-1}^{(1)}+(I_{\mathscr{H}_j}-A_{j-1}^{(1)}A_{j-1})X)\cdot\\&A_{j-2}^{(1)}\cdots A_1^{(1)}A_1\cdots A_n=A_1\cdots A_n\end{aligned} \tag{2.18}$$

holds for arbitrary $X\in\mathscr{B}(\mathscr{H}_{j-1},\mathscr{H}_j)$ and $Y\in\mathscr{B}(\mathscr{H}_j,\mathscr{H}_{j+1})$. Analogously to the previous part of the proof, we get that for arbitrary $j\in\{2,3,\dots,n\}$

$$(I_{\mathscr{H}_j}-A_jA_j^{(1)})(I_{\mathscr{H}_j}-A_{j-1}^{(1)}A_{j-1})=0. \tag{2.19}$$

Taking $j=2$ in (2.19), we conclude that

$$\mathscr{N}(A_1)\subseteq\mathscr{R}(A_2). \tag{2.20}$$

Now, choose arbitrary $j\in\{3,4,\dots,n\}$. Using (2.17) and (2.19), we have

$$\begin{aligned}&A_jA_j^{(1)}(I-A_{j-1}^{(1)}\cdots A_2^{(1)}A_1^{(1)}A_1A_2A_3\cdots A_{j-1})\\&=I-A_{j-1}^{(1)}\cdots A_2^{(1)}A_1^{(1)}A_1A_2A_3\cdots A_{j-1},\end{aligned} \tag{2.21}$$

which implies that $\mathscr{N}(A_1A_2\cdots A_{j-1})\subseteq\mathscr{R}(A_j)$. □

In [7] some necessary and sufficient conditions are given under which, in the matrix case, for some $(AB)^{(1)}\in(AB)\{1\}$ satisfying some special conditions there exist $A^{(1)}\in A\{1\}$ and $B^{(1)}\in B\{1\}$ such that $(AB)^{(1)}=B^{(1)}A^{(1)}$:

Theorem 2.5 ([7]) *Let $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{n\times p}$ and let $(AB)^{(1)}\in(AB)\{1\}$ be arbitrarily given, and let*

$$t((AB)^{(1)})=\dim(\mathscr{R}((AB)^{(1)})\cap\mathscr{N}(B))\ \textit{and}$$

$$v((AB)^{(1)})=\dim(\mathscr{R}([(AB)^{(1)}]^*)\cap\mathscr{N}(A^*)).$$

Then $(AB)^{(1)}\in B\{1\}A\{1\}$ if and only if:

$$\mathrm{r}((AB)^{(1)})-t((AB)^{(1)})-v((AB)^{(1)})\geq\mathrm{r}(A)+\mathrm{r}(B)-n.$$

The reverse order law

$$(AB)\{1\} \subseteq B\{1\}A\{1\} \tag{2.22}$$

in the setting of matrices was completely solved in 1998 in [8], where using P-SVD of matrices A and B it was proved that (2.22) holds if and only if

$$\dim \mathcal{N}(A) - \dim (\mathcal{N}(A) \cap \mathcal{R}(B)) \geq \min \left\{\dim \mathcal{N}(A^*), \dim \mathcal{N}(B)\right\}.$$

In [9], some other necessary and sufficient conditions for (2.22) to hold were presented without using SVD or P-SVD of matrices A and B, potentially allowing for our purely algebraic proof to be generalized to more general settings:

Theorem 2.6 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. The following conditions are equivalent:*

(i) $(AB)\{1\} \subseteq B\{1\}A\{1\}$,
(ii) $\mathrm{r}(A) + \mathrm{r}(B) - n \leq \mathrm{r}(AB) - \min\{m - \mathrm{r}(A), p - \mathrm{r}(B)\}$.

Proof Since (i) is equivalent with

$$(B^*A^*)\{1\} \subseteq A^*\{1\}B^*\{1\}, \tag{2.23}$$

without loss of generality we can suppose throughout the proof that

$$\min\{m - r(A), p - r(B)\} = m - r(A). \tag{2.24}$$

Indeed, if this is not the case we can have (2.24) by simply replacing A with B^* and B with A^*, given that $m - \mathrm{r(A)} = \dim \mathcal{N}(\mathrm{A}^*)$ and $p - \mathrm{r(B)} = \dim \mathcal{N}(\mathrm{B})$. Now, assuming (2.24), we need to prove that (i) is equivalent with $m - \mathrm{r}(AB) \leq n - \mathrm{r}(B)$.

Evidently, (i) is equivalent with the fact that for any $(AB)^{(1)} \in (AB)\{1\}$ there exist $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$ such that

$$(AB)^{(1)} = B^{(1)}A^{(1)}.$$

Using Lemma 1.1 from [9] (or the more general version of it—Lemma 3.5) as well as appropriate notations used therein, (i) is equivalent with the fact that for any $(A_1B_1)^{(1)} \in (A_1B_1)\{1\}$, $Z_2 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{R}(B^*))$, $Z_3 \in \mathcal{B}(\mathcal{R}(A), \mathcal{N}(B))$ and $Z_4 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{N}(B))$ there exist matrices $Y_2 \in \mathcal{B}(\mathcal{N}(B^*), \mathcal{R}(B^*))$, $Y_3 \in \mathcal{B}(\mathcal{R}(B), \mathcal{N}(B))$ and $Y_4 \in \mathcal{B}(\mathcal{N}(B^*), \mathcal{N}(B))$ and $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{N}(B^*) \end{bmatrix}$ satisfying

$$A_1X_1 + A_2X_3 = I_{\mathcal{R}(A)}, \tag{2.25}$$

such that

$$\begin{bmatrix} (A_1B_1)^{(1)} & Z_2 \end{bmatrix} = \begin{bmatrix} B_1^{-1} & Y_2 \end{bmatrix} X \tag{2.26}$$

$$\begin{bmatrix} Z_3 & Z_4 \end{bmatrix} = \begin{bmatrix} Y_3 & Y_4 \end{bmatrix} X. \tag{2.27}$$

In general for fixed Y_2 the Eq. (2.26) is solvable for X and we have that the set of solutions is given by

$$\begin{aligned} S = &\left\{ \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} (A_1B_1)^{(1)} & Z_2 \end{bmatrix} + \left(I - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} B_1^{-1} & Y_2 \end{bmatrix} \right) W : \right. \\ &\left. W \in \mathbb{C}^{n\times m} \right\} \\ &= \left\{ \begin{bmatrix} B_1(A_1B_1)^{(1)} - B_1Y_2W_3 & B_1Z_2 - B_1Y_2W_4 \\ W_3 & W_4 \end{bmatrix} : \right. \\ &\left. \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix} \right\}, \end{aligned} \tag{2.28}$$

given that obviously $\begin{bmatrix} B_1 \\ 0 \end{bmatrix} \in \begin{bmatrix} B_1^{-1} & Y_2 \end{bmatrix}\{1\}$.

Thus (i) is equivalent with the existence of at least one $X \in S \cap A\{1\}$ for which the Eq. (2.27) is solvable for $\begin{bmatrix} Y_3 & Y_4 \end{bmatrix}$. The solvability of Eq. (2.27) is equivalent with

$$\begin{bmatrix} Z_3 & Z_4 \end{bmatrix} (I - X^{(1)}X) = 0, \tag{2.29}$$

for some (any) $X^{(1)} \in X\{1\}$.

Hence (i) is equivalent with the existence of $X \in S \cap A\{1\}$ for which (2.29) holds. Write $X = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$, where

$$K_1 = \begin{bmatrix} B_1(A_1B_1)^{(1)} - B_1Y_2W_3 \\ W_3 \end{bmatrix} \text{ and } K_2 = \begin{bmatrix} B_1Z_2 - B_1Y_2W_4 \\ W_4 \end{bmatrix}.$$

Using Lemma 2.3 [10], we have that one inner inverse of X is given by $X^{(1)} = \begin{bmatrix} K_1^{(1)} - K_1^{(1)}K_2T^{(1)}R_{K_1} \\ T^{(1)}R_{K_1} \end{bmatrix}$, where $T = R_{K_1}K_2$ and $R_{K_1} = I - K_1K_1^{(1)}$. Thus (2.29) is equivalent with

$$(Z_4 - Z_3K_1^{(1)}K_2)(I - T^{(1)}T) = 0, \;\; Z_3(I - K_1^{(1)}K_1) = 0, \tag{2.30}$$

for some $T^{(1)} \in T\{1\}$.

(i) $\Rightarrow$ (ii): Taking $(A_1B_1)^{(1)} = (A_1B_1)^\dagger$, $Z_2 = 0$, $Z_3 = 0$ and a left invertible Z_4 (such Z_4 exists since $\dim \mathscr{N}(A^*) \le \dim \mathscr{N}(B)$), by (2.30) we get that T must be left invertible. Since $T = P_{\mathscr{N}(K_1K_1^{(1)}),\mathscr{R}(K_1)}K_2$, we get that $\mathscr{N}(T) = \{0\}$ if and only if $\mathscr{N}(K_2) = \{0\}$ and $\mathscr{R}(K_1) \cap \mathscr{R}(K_2) = \{0\}$. The first condition, $\mathscr{N}(K_2) = \{0\}$ is satisfied if and only if $\mathscr{N}(W_4) = \{0\}$, which is possible only if $\dim \mathscr{N}(A^*) \le \dim \mathscr{N}(B^*)$. The second condition $\mathscr{R}(K_1) \cap \mathscr{R}(K_2) = \{0\}$ (in the case when K_1

and K_2 are left invertible) is equivalent with $\mathcal{N}(X) = \{0\}$, i.e.,

$$\mathcal{N}\left(\begin{bmatrix} B_1(A_1B_1)^\dagger & 0 \end{bmatrix}\right) \cap \mathcal{N}\left(\begin{bmatrix} W_3 & W_4 \end{bmatrix}\right) = \{0\}. \tag{2.31}$$

Thus $\mathcal{N}((A_1B_1)^\dagger) \cap \mathcal{N}(W_3) = \{0\}$, so the condition (2.31) is equivalent with $\mathcal{R}(W_3 \mid_{\mathcal{N}((A_1B_1)^\dagger)}) \cap \mathcal{R}(W_4) = \{0\}$, which is possible only when $\dim \mathcal{N}((A_1B_1)^\dagger) + \dim \mathcal{N}(A^*) \leq \dim \mathcal{N}(B^*)$. Since $\dim \mathcal{N}((A_1B_1)^\dagger) = \mathrm{r}(A) - \mathrm{r}(AB)$, we get that $m - \mathrm{r}(AB) \leq \mathrm{n} - \mathrm{r}(B)$.

(ii) $\Rightarrow$ (i): Suppose $(A_1B_1)^{(1)} \in (A_1B_1)\{1\}$, $Z_2 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{R}(B^*))$, $Z_3 \in \mathcal{B}(\mathcal{R}(A), \mathcal{N}(B))$ and $Z_4 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{N}(B))$ are given. We will show that there exists a left-invertible matrix $X \in S \cap A\{1\}$. Let $Y_2 = (A_1B_1)^{(1)}A_2$ and $W_3 = X_3'$, where $\begin{bmatrix} X_1' \\ X_3' \end{bmatrix} : \mathcal{R}(A) \to \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix}$ is an arbitrary right inverse of $\begin{bmatrix} A_1 & A_2 \end{bmatrix}$. Using Theorem 2.7 from [11], we will show that there is some W_4 such that

$$X' = \begin{bmatrix} B_1(A_1B_1)^{(1)} & B_1Z_2 \\ W_3 & W_4 \end{bmatrix} \tag{2.32}$$

is left invertible. It is easy to check that the first column of X' is left-invertible. Thus it remains to check the inequality

$$n(X_0) \leq d(W_3) + \dim\,(\mathcal{R}(W_3^*) \cap \mathcal{R}((B_1(A_1B_1)^{(1)})^* \mid_{\mathcal{N}((B_1Z_2)^*)}) \tag{2.33}$$

where $X_0 = \begin{bmatrix} B_1Z_2 & B_1(A_1B_1)^{(1)} \\ 0 & W_3 \end{bmatrix}$.
Note that

$$\begin{aligned} n(X_0^*) &= n(W_3^*) + n((B_1(A_1B_1)^{(1)})^* \mid_{\mathcal{N}((B_1Z_2)^*)}) \\ &\quad + \dim\,(\mathcal{R}(W_3^*) \cap \mathcal{R}((B_1(A_1B_1)^{(1)})^* \mid_{\mathcal{N}((B_1Z_2)^*)})) \end{aligned}$$

and since $n(X_0) = m - n + n(X_0^*)$, that (2.33) is equivalent with

$$m - n + n((B_1(A_1B_1)^{(1)})^* \mid_{\mathcal{N}((B_1Z_2)^*)}) \leq 0. \tag{2.34}$$

As $n((B_1(A_1B_1)^{(1)})^*) \leq \mathrm{r}(B) - \mathrm{r}(AB)$ and $\mathrm{r}(B) - \mathrm{r}(AB) - n + m \leq 0$, we get that (2.34) holds for any $(A_1B_1)^{(1)} \in (A_1B_1)\{1\}$ and any $Z_2 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{R}(B^*))$.

Now by Theorem 2.7 from [11] there is some W_4 such that X' given by (2.32) is left invertible. It is easy to see that

$$X = \begin{bmatrix} B_1(A_1B_1)^{(1)} - B_1Y_2W_3 & B_1Z_2 - B_1Y_2W_4 \\ W_3 & W_4 \end{bmatrix}$$

is left invertible as well. $\square$

Let us take a look at the following examples.

Example 2.3 We will show that in the case when (ii) of Theorem 2.6 is not satisfied, which means that $(AB)\{1\} \not\subseteq B\{1\}A\{1\}$, we can find a general form of the inner inverses of AB, $(AB)^{(1)}$ for which there does not exist $A^{(1)}$ and $B^{(1)}$ such that $(AB)^{(1)} = B^{(1)}A^{(1)}$. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then evidently $AB = 0$ and consequently $(AB)\{1\} = \mathbb{C}^{2\times 2}$. Since

$$A\{1\} = \left\{ \begin{bmatrix} 1 & a_1 \\ a_2 & a_3 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{C} \right\}$$

and

$$B\{1\} = \left\{ \begin{bmatrix} b_1 & 1 \\ b_2 & b_3 \end{bmatrix} : b_1, b_2, b_3 \in \mathbb{C} \right\}$$

we can check that none of the matrices of the form $\begin{bmatrix} 0 & 0 \\ 0 & w \end{bmatrix}$, where $w \neq 0$, can be written as a product $B^{(1)}A^{(1)}$ for some $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$.

Example 2.4 If $A \in \mathbb{C}^{m\times n}$ is left invertible and $B \in \mathbb{C}^{n\times p}$, then by Theorem 2.1 we have that $B\{1\}A\{1\} \subseteq (AB)\{1\}$. In this case $(AB)\{1\} = B\{1\}A\{1\}$ if and only if

$$\mathrm{r}(B) \leq \mathrm{r}(AB) - \min\{m - n, p - \mathrm{r}(B)\}.$$

The last inequality is satisfied if and only if $m \leq n$ or $\mathrm{r}(B) = p$. Since $A \in \mathbb{C}^{m\times n}$ is left invertible, we have $n \leq m$, so we can conclude that $(AB)\{1\} = B\{1\}A\{1\}$ if and only if A is invertible ($m = n$) or B is left invertible.

In spite of the many results obtained by various authors the problem of settling the reverse order law (2.22) for operators acting on separable Hilbert spaces remained open until 2015. This was finally completely resolved by Pavlović et al. [12] and this was using some radically new approaches involving some of the previous research on completions of operator matrices. These results will be presented in the Chap. 3.

2.2 Reverse Order Laws for {1, 2}-Inverses

In this section, we address all the known results so far results on the the reverse order laws for {1, 2}- generalized inverses. Shinozaki and Sibuya [7] proved that for matrices A, B such that the product AB is defined

$$(AB)\{1, 2\} \subseteq B\{1, 2\}A\{1, 2\} \tag{2.35}$$

always hold. To verify Shinozaki and Sibuya's result in the case of regular bounded linear operators on Hilbert spaces we will consider suitable representations of given regular operators $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ to first prove the lemma given below. More precisely if $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $B \in \mathscr{B}(\mathscr{L}, \mathscr{H})$ are arbitrary regular operators, then using the following decompositions of the Hilbert spaces $\mathscr{L}$, $\mathscr{H}$ and $\mathscr{K}$,

$$\mathscr{L} = \mathscr{R}(B^*) \oplus \mathscr{N}(B),\ \mathscr{H} = \mathscr{R}(B) \oplus \mathscr{N}(B^*),\ \mathscr{K} = \mathscr{R}(A) \oplus \mathscr{N}(A^*),$$

we have that the corresponding decompositions of A and B are given by

$$\begin{aligned} A &= \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix}, \\ B &= \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix}, \end{aligned} \tag{2.36}$$

where B_1 is an invertible operator and $\begin{bmatrix} A_1 & A_2 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix} \to \mathscr{R}(A)$ is a right invertible operator. In that case AB is given by

$$AB = \begin{bmatrix} A_1B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix}.$$

Now, using decompositions given above, we have the following result.

Lemma 2.1 *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $B \in \mathscr{B}(\mathscr{L}, \mathscr{H})$ be regular operators given by (2.36). Then*

(i) *an arbitrary* $\{1, 2\}$*-inverse of A is given by:*

$$A^{(1,2)} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix},$$

where X_1 and X_3 satisfy

$$A_1X_1 + A_2X_3 = I_{\mathscr{R}(A)},$$

and X_2 and X_4 are of the form

$$\begin{aligned} X_2 &= X_1A_1Z_1 + X_1A_2Z_2, \\ X_4 &= X_3A_1Z_1 + X_3A_2Z_2, \end{aligned}$$

for some operators $Z_1 \in \mathscr{B}(\mathscr{N}(A^*), \mathscr{R}(B))$ and $Z_2 \in \mathscr{B}(\mathscr{N}(A^*), \mathscr{N}(B^*))$.

(ii) *an arbitrary* $\{1, 2\}$*-inverse of B is given by:*

$$B^{(1,2)} = \begin{bmatrix} B_1^{-1} & U \\ V & VB_1U \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix},$$

where $U \in \mathscr{B}(\mathscr{N}(B^*), \mathscr{R}(B^*))$ and $V \in \mathscr{B}(\mathscr{R}(B), \mathscr{N}(B))$.

(iii) *if AB is regular, then an arbitrary* $\{1,2\}$*-inverse of AB is given by:*

$$(AB)^{(1,2)} = \begin{bmatrix} (A_1B_1)^{(1,2)} & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix},$$

where $(A_1B_1)^{(1,2)} \in (A_1B_1)\{1,2\}$ and Y_i, $i = \overline{2,4}$ satisfy the following system of the equations:

$$\begin{aligned} Y_2 &= (A_1B_1)^{(1,2)}A_1B_1Y_2, \\ Y_3 &= Y_3A_1B_1(A_1B_1)^{(1,2)}, \\ Y_4 &= Y_3A_1B_1Y_2. \end{aligned} \tag{2.37}$$

Proof (i) Without loss of generality, we can suppose that an arbitrary $\{1,2\}$-inverse of A is given by:

$$A^{(1,2)} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix}.$$

From $AXA = A$ and $XAX = X$, we get that $X \in A\{1,2\}$ if and only if for X_i, $i = \overline{1,4}$ the following equations

$$(A_1X_1 + A_2X_3)A_i = A_i, \quad i = 1,2 \tag{2.38}$$

$$X_j(A_1X_1 + A_2X_3) = X_j, \quad j = 1,3 \tag{2.39}$$

$$X_1(A_1X_2 + A_2X_4) = X_2, \quad X_3(A_1X_2 + A_2X_4) = X_4, \tag{2.40}$$

are satisfied. Since $S = \begin{bmatrix} A_1 & A_2 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix} \to \mathscr{R}(A)$ is a right invertible operator, there exists $S_r^{-1} : \mathscr{R}(A) \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix}$ such that $\begin{bmatrix} A_1 & A_2 \end{bmatrix} S_r^{-1} = I_{\mathscr{R}(A)}$. Notice that (2.38) is equivalent to

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \end{bmatrix}. \tag{2.41}$$

Multiplying (2.41) by S_r^{-1} from the right, we get that (2.41) is equivalent with $\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} = I_{\mathscr{R}(A)}$, i.e.,

$$A_1X_1 + A_2X_3 = I_{\mathscr{R}(A)}. \tag{2.42}$$

Note, that for X_1 and X_3 which satisfy (2.42), (2.39) also holds. Condition (2.40) is equivalent to

$$\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_2 \\ X_4 \end{bmatrix}$$

i.e.,

$$(I - P)\begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = 0,$$

where $P = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \end{bmatrix}$. Since P is a projection,

$$\begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = P\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} X_2 \\ X_4 \end{bmatrix} = \begin{bmatrix} X_1 A_1 Z_1 + X_1 A_2 Z_2 \\ X_3 A_1 Z_1 + X_3 A_2 Z_2 \end{bmatrix},$$

where Z_1 and Z_2 are operators from appropriate spaces.

(ii) Suppose that an arbitrary {1, 2}-inverse of B is given by:

$$B^{(1,2)} = \begin{bmatrix} S & U \\ V & W \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix}.$$

From $BB^{(1,2)}B = B$ it follows that $B_1 S B_1 = B_1$ and since B_1 is invertible, $S = B_1^{-1}$. From $B^{(1,2)}BB^{(1,2)} = B^{(1,2)}$ we easily get $W = VB_1U$, where U and V are operators from appropriate spaces.

(iii) Let an arbitrary {1, 2}-inverse of AB be given by:

$$(AB)^{(1,2)} = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix}.$$

From $AB(AB)^{(1,2)}AB = AB$, we get

$$A_1 B_1 Y_1 A_1 B_1 = A_1 B_1, \tag{2.43}$$

and from $(AB)^{(1,2)}AB(AB)^{(1,2)} = (AB)^{(1,2)}$, we get

$$Y_1 A_1 B_1 Y_1 = Y_1, \tag{2.44}$$

$$Y_1 A_1 B_1 Y_2 = Y_2, \tag{2.45}$$

$$Y_3 A_1 B_1 Y_1 = Y_3, \tag{2.46}$$

$$Y_3 A_1 B_1 Y_2 = Y_4. \tag{2.47}$$

Now, by (2.43) and (2.44), we get that $Y_1 \in (A_1B_1)\{1, 2\}$. Substituting $Y_1 = (A_1B_1)^{(1,2)}$ in (2.45), (2.46) and (2.47), we get (2.37). □

Finally, we will give the proof of the result of Shinozaki and Sibuya in the case of regular bounded linear operators on Hilbert spaces the product of which is also regular.

Theorem 2.7 *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $B \in \mathscr{B}(\mathscr{L}, \mathscr{H})$ be regular operators such that AB is regular. Then*

$$(AB)\{1,2\} \subseteq B\{1,2\} \cdot A\{1,2\}.$$

Proof Take an arbitrary $(AB)^{(1,2)} \in (AB)\{1,2\}$. We will show that there exist $A^{(1,2)} \in A\{1,2\}$ and $B^{(1,2)} \in B\{1,2\}$ such that $(AB)^{(1,2)} = B^{(1,2)}A^{(1,2)}$. Without loss of generality, we can suppose that A and B are given by (2.36). By Lemma 2.1, we have that

$$(AB)^{(1,2)} = \begin{bmatrix} (A_1B_1)^{(1,2)} & Y_2 \\ Y_3 & Y_4 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix},$$

for $(A_1B_1)^{(1,2)} \in (A_1B_1)\{1,2\}$ and some Y_i, $i = \overline{2,4}$ which satisfy system (2.37). Since $\begin{bmatrix} A_1 & A_2 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix} \to \mathscr{R}(A)$ is a right invertible operator, there exists (not unique in general) $\begin{bmatrix} X_1' \\ X_3' \end{bmatrix} : \mathscr{R}(A) \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix}$ such that $A_1X_1' + A_2X_3' = I_{\mathscr{R}(A)}$. Since B_1 is invertible, we have that $(A_1B_1)(A_1B_1)^{(1,2)}A_1X_1' = A_1X_1'$. Let $X_3 = X_3'$ and $X_1 = B_1(A_1B_1)^{(1,2)}A_1X_1'$. Obviously, $A_1X_1 + A_2X_3 = I_{\mathscr{R}(A)}$. Now, let

$$C = \begin{bmatrix} X_1 & X_1A_1B_1Y_2 \\ X_3 & X_3A_1B_1Y_2 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix},$$
$$D = \begin{bmatrix} B_1^{-1} & U \\ Y_3A_1 & Y_3A_1B_1U \end{bmatrix} : \begin{bmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{bmatrix},$$

where $U = (A_1B_1)^{(1,2)}A_2$. We will show that $C \in A\{1,2\}$, $D \in B\{1,2\}$ and that $(AB)^{(1,2)} = DC$. Using Lemma 2.1, we can check that $C \in A\{1,2\}$ and $D \in B\{1,2\}$. To prove that $(AB)^{(1,2)} = DC$, it suffices to show that the following system of the equations is satisfied:

$$\begin{aligned}
(A_1B_1)^{(1,2)} &= B_1^{-1}X_1 + UX_3, \\
Y_2 &= B_1^{-1}X_1A_1B_1Y_2 + UX_3A_1B_1Y_2, \\
Y_3 &= Y_3A_1X_1 + Y_3A_1B_1UX_3, \\
Y_4 &= Y_3A_1X_1A_1B_1Y_2 + Y_3A_1B_1UX_3A_1B_1Y_2.
\end{aligned}$$

The first equation is satisfied, since $X_1 = B_1(A_1B_1)^{(1,2)}(I - A_2X_3)$, while the other three equations are satisfied by virtue of (2.37). □

The reverse inclusion of (2.35) was considered by De Pierro and M. Wei [8]. Using the product singular value decomposition of matrices they investigated when

$$B\{1,2\}A\{1,2\} \subseteq (AB)\{1,2\} \tag{2.48}$$

is satisfied and proved the following:

Theorem 2.8 ([8]) *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. The following conditions are equivalent:*

(i) $B\{1,2\}A\{1,2\} \subseteq (AB)\{1,2\}$*;*
(ii) $A = 0$ *or* $B = 0$ *or* $\mathrm{r}(B) = n$ *or* $\mathrm{r}(A) = n$*.*

The following two examples are in order.

Example 2.5 In the case when (ii) of Theorem 2.8 is not satisfied, which means that $B\{1,2\}A\{1,2\} \nsubseteq (AB)\{1,2\}$, we can find particular reflexive inverses of A and B, $A^{(1,2)}$ and $B^{(1,2)}$ such that $B^{(1,2)}A^{(1,2)} \in (AB)\{1,2\}$. Let $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then evidently $AB = 0$ and consequently $(AB)\{1,2\} = \{0\}$. But for $A^{(1,2)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B^{(1,2)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have that $B^{(1,2)}A^{(1,2)} = 0 \in (AB)\{1,2\}$.

Example 2.6 If $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, then $B\{1,2\} \cdot A\{1,2\} \subseteq (AB)\{1,2\}$ implies $B\{1\} \cdot A\{1\} \subseteq (AB)\{1\}$ (see Theorems. 2.1 and 2.8).

Using a completely different approaches, Cvetković-Ilić and Nikolov [13] improved the results from [8] and verified Shinozaki and Sibuya's results in the case of regular bounded linear operators on Hilbert spaces the product of which is also regular. Notice that all the results stated in the sequel can be generalized to the C^*-algebra case.

Theorem 2.9 ([13]) *Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L},\mathcal{H})$ be such that A, B and AB are regular operators. The following conditions are equivalent:*

(i) $B\{1,2\} \cdot A\{1,2\} \subseteq (AB)\{1,2\}$,
(ii) $A = 0$ *or* $B = 0$ *or* $A \in \mathcal{B}_l^{-1}(\mathcal{H},\mathcal{K})$ *or* $B \in \mathcal{B}_r^{-1}(\mathcal{H},\mathcal{K})$.

Proof (i) $\Rightarrow$ (ii) : If (i) holds, then evidently $B^\dagger A^\dagger \in (AB)\{1,2\}$, so

$$ABB^\dagger A^\dagger AB = AB \tag{2.49}$$

and

$$B^\dagger A^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger. \tag{2.50}$$

Since, for any $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$, $A^\dagger + (I - A^\dagger A)XAA^\dagger \in A\{1,2\}$, we get

$$ABB^{\dagger}(A^{\dagger} + (I - A^{\dagger}A)XAA^{\dagger})AB = AB,$$

which using (2.49) further implies that

$$ABB^{\dagger}(I - A^{\dagger}A)XAB = 0, \tag{2.51}$$

for any $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$.
Similarly, for any $Y \in \mathscr{B}(\mathscr{H}, \mathscr{L})$, $B^{\dagger} + B^{\dagger}BY(I - BB^{\dagger}) \in B\{1, 2\}$, so

$$AB(B^{\dagger} + B^{\dagger}BY(I - BB^{\dagger}))A^{\dagger}AB = AB,$$

which using (2.49) implies that

$$ABY(I - BB^{\dagger})A^{\dagger}AB = 0, \tag{2.52}$$

for any $Y \in \mathscr{B}(\mathscr{H}, \mathscr{L})$.
Since, for any $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ and $Y \in \mathscr{B}(\mathscr{H}, \mathscr{L})$, we have that

$$AB(B^{\dagger} + B^{\dagger}BY(I - BB^{\dagger}))(A^{\dagger} + (I - A^{\dagger}A)XAA^{\dagger})AB = AB, \tag{2.53}$$

using (2.49), (2.51) and (2.52), we get that

$$ABY(I - BB^{\dagger})(I - A^{\dagger}A)XAB = 0,$$

for any $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ and $Y \in \mathscr{B}(\mathscr{H}, \mathscr{L})$. Now,

$$AB = 0 \quad \text{or} \quad (I - BB^{\dagger})(I - A^{\dagger}A) = 0. \tag{2.54}$$

Since, for any $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$,

$$\begin{aligned}&B^{\dagger}(A^{\dagger} + (I - A^{\dagger}A)XAA^{\dagger})ABB^{\dagger}(A^{\dagger} + (I - A^{\dagger}A)XAA^{\dagger})\\&= B^{\dagger}(A^{\dagger} + (I - A^{\dagger}A)XAA^{\dagger})\end{aligned}$$

using (2.50) we get that

$$\begin{aligned}&B^{\dagger}A^{\dagger}ABB^{\dagger}(I - A^{\dagger}A)XAA^{\dagger} + B^{\dagger}(I - A^{\dagger}A)XABB^{\dagger}A^{\dagger}\\&+ B^{\dagger}(I - A^{\dagger}A)XABB^{\dagger}(I - A^{\dagger}A)XAA^{\dagger} = B^{\dagger}(I - A^{\dagger}A)XAA^{\dagger}.\end{aligned} \tag{2.55}$$

Now by (2.54), we get that the first and the third term on the left-hand side of (2.55) are zero, so

$$B^{\dagger}(I - A^{\dagger}A)X(ABB^{\dagger}A^{\dagger} - AA^{\dagger}) = 0,$$

for any $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. Hence,

$$B^\dagger = B^\dagger A^\dagger A \quad \text{or} \quad ABB^\dagger A^\dagger = AA^\dagger.$$

Now, following (2.54) we have two cases:
Case 1. If $AB = 0$, then if $B^\dagger = B^\dagger A^\dagger A$ it follows that $B = 0$. If $ABB^\dagger A^\dagger = AA^\dagger$, we easily get that $A = 0$.
Case 2. If $(I - BB^\dagger)(I - A^\dagger A) = 0$, then if $B^\dagger = B^\dagger A^\dagger A$, it follows that $A^\dagger A = I$, i.e., A is left invertible. If $ABB^\dagger A^\dagger = AA^\dagger$, then multiplying $(I - BB^\dagger)(I - A^\dagger A) = 0$, by A from the left, we get

$$A = ABB^\dagger.$$

Now, given that $A^\dagger A$ and $BB^\dagger$ commute, we have that $BB^\dagger = I$, i.e., B is right invertible.

(ii) $\Rightarrow$ (i) : If A or B is zero, it is evident that (i) holds. Now, suppose that B is right invertible and let $B^{(1,2)} \in B\{1, 2\}$ be arbitrary. Evidently, $B^{(1,2)}$ is a right inverse of B, i.e., $BB^{(1,2)} = I$. Then, for arbitrary $A^{(1,2)} \in A\{1, 2\}$,

$$ABB^{(1,2)}A^{(1,2)}AB = AA^{(1,2)}AB = AB$$

and

$$B^{(1,2)}A^{(1,2)}ABB^{(1,2)}A^{(1,2)} = B^{(1,2)}A^{(1,2)}AA^{(1,2)} = B^{(1,2)}A^{(1,2)}.$$

If A is a left invertible operator, for any $A^{(1,2)} \in A\{1, 2\}$ we have that $A^{(1,2)}A = I$. Then, for arbitrary $A^{(1,2)} \in A\{1, 2\}$ and $B^{(1,2)} \in B\{1, 2\}$,

$$ABB^{(1,2)}A^{(1,2)}AB = ABB^{(1,2)}B = AB$$

and

$$B^{(1,2)}A^{(1,2)}ABB^{(1,2)}A^{(1,2)} = B^{(1,2)}BB^{(1,2)}A^{(1,2)} = B^{(1,2)}A^{(1,2)}.$$

$\square$

It is interesting to note that by the first part of the proof of Theorem 2.9, we can conclude that

$$B\{1, 2\} \cdot A\{1, 2\} \subseteq (AB)\{1\}$$

if and only if

$$AB = 0 \quad \text{or} \quad (I - BB^\dagger)(I - A^\dagger A) = 0$$

i.e.

$$AB = 0 \quad \text{or} \quad \mathcal{N}(A) \subseteq \mathcal{R}(B),$$

which is equivalent with $B\{1\} \cdot A\{1\} \subseteq (AB)\{1\}$ (see [3, 14]).
The proof that (2.48) is always satisfied in the case of a multiple product of regular operators is very similar to the one given in [5] (see [Theorem 4.1, [5]]) for the matrix case. We give it here for completeness' sake.

Theorem 2.10 *Let* $A_i \in \mathscr{B}(\mathscr{H}_{i+1}, \mathscr{H}_i)$*, be such that* A_i*,* $i = \overline{1,n}$ *and* $A_1 A_2 \cdots A_j$*,* $j = \overline{2,n}$*, are regular operators. Then*

$$(A_1 A_2 \cdots A_n)\{1,2\} \subseteq A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\}. \tag{2.56}$$

Proof Suppose that $A_i \in \mathscr{B}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{1,n}$ and $A_1 A_2 \cdots A_j$, $j = \overline{2,n}$, are regular operators. We will prove that (2.56) holds by induction on n. For $n = 2$, the assertion holds by virtue of Theorem 2.7. Now suppose this is true for $2 \leq k \leq n$. For $k = n+1$, let $A_1 A_2 \cdots A_k = B$. Using again Theorem 2.7, we obtain $(BA_{n+1})\{1,2\} \subseteq A_{n+1}\{1,2\}B\{1,2\}$. From the induction hypothesis,

$$(A_1 \cdots A_n)\{1,2\} \subseteq A_n\{1,2\} \cdots A_1\{1,2\},$$

so we get

$$\begin{aligned}(A_1 \cdots A_n A_{n+1})\{1,2\} &\subseteq A_{n+1}\{1,2\}(A_1 \cdots A_n)\{1,2\} \\ &\subseteq A_{n+1}\{1,2\}A_n\{1,2\} \cdots A_1\{1,2\}.\end{aligned}$$

□

The reverse inclusion of (2.56) in the case of matrices was considered by M. Wei [4] who, applying the multiple product singular value decomposition (P-SVD), gave necessary and sufficient conditions for

$$A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1 A_2 \cdots A_n)\{1,2\}.$$

Theorem 2.11 ([4]) *Suppose that* $A_i \in \mathbb{C}^{s_i \times s_{i+1}}$, $i = 1, 2, \ldots n$. *Then the following conditions are equivalent:*

(i) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1 A_2 \cdots A_n)\{1,2\}$*;*
(ii) *One of the following conditions is satisfied:*

  (a) $\mathrm{r}(\mathrm{A}_1 \cdots \mathrm{A}_n) > 0$ *and for each* $j \in \{1, \ldots, n-1\}$*,* A_j *is of full column rank*
  (b) $\mathrm{r}(\mathrm{A}_1 \cdots \mathrm{A}_n) > 0$ *and for each* $j \in \{2, \ldots, n\}$*,* A_j *is of full row rank*
  (c) $\mathrm{r}(\mathrm{A}_1 \cdots \mathrm{A}_n) > 0$ *and there exists an integer* $q \in \{2, ..., n-1\}$ *such that for each* $j \in \{1, \ldots, q-1\}$*,* A_j *is of full column rank and for each* $j \in \{q, \ldots, n\}$*,* A_j *is of full row rank*
  (d) *There exists an integer* $q \in \{1, \ldots, n\}$*, such that* $\mathrm{r}(A_q) = 0$.

The generalization of the previous result for the case of bounded regular operators on Hilbert spaces is given in [6] as follows:

Theorem 2.12 ([6]) *Let* $A_i \in \mathscr{B}(\mathscr{H}_{i+1}, \mathscr{H}_i)$*, be such that* A_i*,* $i = \overline{1,n}$ *and* $A_1 A_2 \cdots A_j$*,* $j = \overline{2,n}$ *are regular operators. The following conditions are equivalent:*

(i) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} = (A_1 A_2 \cdots A_n)\{1,2\}$,

(ii) $A_n\{1,2\} \cdot A_{n-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1A_2 \cdots A_n)\{1,2\}$,
(iii) *There exists an integer* i, $1 \leq i \leq n$, *such that* $A_i = 0$,
or
$A_1A_2 \cdots A_n \neq 0$ *and* $A_i \in \mathscr{B}_r^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, *for* $i = \overline{2, n}$,
or
$A_1A_2 \cdots A_n \neq 0$ *and* $A_i \in \mathscr{B}_l^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, *for* $i = \overline{1, n-1}$,
or
$A_1A_2 \cdots A_n \neq 0$ *and there exists an integer* k, $2 \leq k \leq n-1$, *such that* $A_i \in \mathscr{B}_l^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, *for* $i = \overline{1, k-1}$, *and* $A_i \in \mathscr{B}_r^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, *for* $i = \overline{k+1, n}$.

Proof (i) $\Leftrightarrow$ (ii) : Follows from Theorem 2.10.

(ii) $\Rightarrow$ (iii) : We prove this by induction on n. For $n = 2$, this follows from Theorem 2.9. Assume that (ii) $\Rightarrow$ (iii) holds for $n = k-1$; we will prove that the implication still holds for $n = k$. Suppose that

$$A_k\{1,2\} \cdot A_{k-1}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1A_2 \cdots A_k)\{1,2\}. \tag{2.57}$$

By virtue of Theorem 2.10, we have

$$A_k\{1,2\} \cdot (A_1A_2 \cdots A_{k-1})\{1,2\} \subseteq (A_1A_2 \cdots A_k)\{1,2\}. \tag{2.58}$$

which by Theorem 2.9 implies that at least one of the following cases must hold true:

$$\begin{aligned}&A_1A_2 \cdots A_{k-1} = 0 \ \text{ or } \ A_k = 0 \text{ or } \ A_1A_2 \cdots A_{k-1} \in \mathscr{B}_l^{-1}(\mathscr{H}_k, \mathscr{H}_1)\\ &\text{or} \quad A_k \in \mathscr{B}_r^{-1}(\mathscr{H}_{k+1}, \mathscr{H}_k).\end{aligned}$$

Now, we will consider all these cases:
Case 1. $A_1A_2 \cdots A_{k-1} = 0$. Then $A_1A_2 \cdots A_{k-1}A_k = 0$ which by (ii) implies

$$A_k\{1,2\} \cdot A_{k-1}\{1,2\} \cdots A_1\{1,2\} = \{0\}. \tag{2.59}$$

Let $A_i^{(1,2)} \in A_i\{1,2\}$, $i = \overline{1, k-1}$ be arbitrary. Then from (2.59) we have

$$A_k^\dagger A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} = 0. \tag{2.60}$$

Since for any $Z \in \mathscr{B}(\mathscr{H}_k, \mathscr{H}_{k+1})$, $A_k^\dagger + A_k^\dagger A_k Z(I_{\mathscr{H}_k} - A_kA_k^\dagger) \in A_k\{1,2\}$, we get

$$(A_k^\dagger + A_k^\dagger A_k Z(I_{\mathscr{H}_k} - A_kA_k^\dagger))A_{k-1}^{(1,2)}A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} = 0,$$

which by (2.60) gives that $A_k^\dagger A_k Z A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} = 0$. Now,

$$A_k = 0 \quad \text{or} \quad A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} = 0. \tag{2.61}$$

If $A_k = 0$, then (iii) holds. Suppose that $A_k \neq 0$. Then $A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} = 0$ for arbitrary $A_i^{(1,2)} \in A_i\{1, 2\}$, $i = \overline{1, k-1}$, implying

$$A_{k-1}\{1, 2\}A_{k-2}\{1, 2\} \cdots A_1\{1, 2\} = \{0\} \subseteq (A_1 A_2 \cdots A_{k-2} A_{k-1})\{1, 2\}. \quad (2.62)$$

By the induction hypothesis, from (2.62) it follows that at least one of the following conditions is satisfied:

(1) There exists $i \in \{1, 2, \ldots, k-1\}$ such that $A_i = 0$,
(2) $A_i \in \mathscr{B}_l^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{1, k-2}$,
(3) $A_i \in \mathscr{B}_r^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{2, k-1}$,
(4) There exists $i \in \{1, 2, \ldots, k-1\}$ such that $A_j \in \mathscr{B}_l^{-1}(\mathscr{H}_{j+1}, \mathscr{H}_j)$ for $j = \overline{1, i-1}$ and $A_j \in \mathscr{B}_r^{-1}(\mathscr{H}_{j+1}, \mathscr{H}_j)$, $j = \overline{i+1, k-1}$.

If (1) holds, then (iii) is satisfied. Suppose that (2) is true. Since $A_1 A_2 \cdots A_{k-1} = 0$ we get that $A_{k-1} = 0$ so (iii) holds. If (3) holds, then from $A_1 A_2 \cdots A_{k-1} = 0$ we get that $A_1 = 0$. Suppose that (4) holds. Multiplying $A_1 A_2 \cdots A_{k-1} = 0$ by $A_{i-1}^{\dagger} A_{i-2}^{\dagger} \cdots A_1^{\dagger}$ from the left, we get

$$A_i A_{i+1} \cdots A_{k-1} = 0. \quad (2.63)$$

Multiplying (2.63) by $A_{k-1}^{\dagger} A_{k-2}^{\dagger} \cdots A_{i+1}^{\dagger}$ from the right we get $A_i = 0$. Hence, (iii) is satisfied.
Case 2. If $A_k = 0$, then (iii) obviously holds.
Case 3. Suppose that $A_1 A_2 \cdots A_{k-1} \in \mathscr{B}_l^{-1}(\mathscr{H}_k, \mathscr{H}_1)$. Then $A_{k-1} \in \mathscr{B}_l^{-1}(\mathscr{H}_k, \mathscr{H}_{k-1})$. From Theorem 2.9, we have

$$(A_{k-1} A_k)\{1, 2\} \subseteq A_k\{1, 2\}A_{k-1}\{1, 2\},$$

so it follows that

$$\begin{aligned} &(A_{k-1} A_k)\{1, 2\} \cdot A_{k-2}\{1, 2\} \cdots A_1\{1, 2\} \\ &\subseteq A_k\{1, 2\} \cdot A_{k-1}\{1, 2\} \cdots A_1\{1, 2\} \subseteq (A_1 A_2 \cdots A_k)\{1, 2\}. \end{aligned} \quad (2.64)$$

By the induction hypothesis, from (2.64) it follows that at least one of the following conditions is true:

(1′) There exists $i \in \{1, 2, \ldots, k-2\}$ such that $A_i = 0$ or $A_{k-1} A_k = 0$,
(2′) $A_i \in \mathscr{B}_l^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{1, k-2}$,
(3′) $A_i \in \mathscr{B}_r^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{2, k-2}$, and $A_{k-1} A_k \in \mathscr{B}_r^{-1}(\mathscr{H}_{k+1}, \mathscr{H}_{k-1})$,
(4′) There exists $i \in \{1, 2, \ldots, k-1\}$ such that $A_j \in \mathscr{B}_l^{-1}(\mathscr{H}_{j+1}, \mathscr{H}_j)$ for $j = \overline{1, i-1}$ and $A_j \in \mathscr{B}_r^{-1}(\mathscr{H}_{j+1}, \mathscr{H}_j)$, $j = \overline{i+1, k-2}$ and $A_{k-1} A_k \in \mathscr{B}_r^{-1}(\mathscr{H}_{k+1}, \mathscr{H}_{k-1})$.

As before, we can check that in all these cases, (iii) is satisfied.
Case 4. Suppose that $A_k \in \mathscr{B}_r^{-1}(\mathscr{H}_{k+1}, \mathscr{H}_k)$. Then $A_k A_k^{(1,2)} = I_{\mathscr{H}_k}$ for arbitrary $A_k^{(1,2)} \in A_k\{1,2\}$. Let $A_i^{(1,2)} \in A_i\{1,2\}$, $i = \overline{1,k}$ be arbitrary. Then

$$A_1 A_2 \cdots A_k A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_k = A_1 A_2 \cdots A_k \tag{2.65}$$

and

$$\begin{aligned} &A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_k A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)} \\ &= A_k^{(1,2)} A_{k-1}^{(1,2)} \cdots A_1^{(1,2)}. \end{aligned} \tag{2.66}$$

Multiplying (2.65) by $A_k^{(1,2)}$ from the right and (2.66) by A_k from the left, we get

$$A_1 A_2 \cdots A_{k-1} A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_{k-1} = A_1 A_2 \cdots A_{k-1} \tag{2.67}$$

and

$$\begin{aligned} &A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} A_1 A_2 \cdots A_{k-1} A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)} \\ &= A_{k-1}^{(1,2)} A_{k-2}^{(1,2)} \cdots A_1^{(1,2)}. \end{aligned} \tag{2.68}$$

Evidently,

$$A_{k-1}\{1,2\} \cdot A_{k-2}\{1,2\} \cdots A_1\{1,2\} \subseteq (A_1 A_2 \cdots A_{k-1})\{1,2\}. \tag{2.69}$$

By the induction hypothesis, from (2.69) it follows that at least one of the following conditions is true:

(1″) There exists $i \in \{1, 2, \ldots, k-1\}$ such that $A_i = 0$,
(2″) $A_i \in \mathscr{B}_l^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{1, k-2}$,
(3″) $A_i \in \mathscr{B}_r^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$, $i = \overline{2, k-1}$,
(4″) There exists $i \in \{1, 2, \ldots, k-1\}$ such that $A_j \in \mathscr{B}_l^{-1}(\mathscr{H}_{j+1}, \mathscr{H}_j)$ for $j = \overline{1, i-1}$ and $A_j \in \mathscr{B}_r^{-1}(\mathscr{H}_{j+1}, \mathscr{H}_j)$, $j = \overline{i+1, k-1}$.

It is easy to check that in all these four cases (iii) is satisfied.
(iii) $\Rightarrow$ (ii) : If $A_1 A_2 \cdots A_n = 0$, then it is evident that (ii) holds. Suppose that $A_1 A_2 \cdots A_n \neq 0$ and let $A_i^{(1,2)} \in A_i\{1,2\}$, $i = \overline{1,n}$ be arbitrary.
If $A_i \in \mathscr{B}_r^{-1}(\mathscr{H}_{i+1}, \mathscr{H}_i)$ for $i = \overline{2,n}$, then $A_i A_i^{(1,2)} = I_{\mathscr{H}_i}$ for $i = \overline{2,n}$. Now,

$$\begin{aligned}
&A_1A_2\cdots A_{n-1}A_nA_n^{(1,2)}A_{n-1}^{(1,2)}\cdots A_1^{(1,2)}\\
&= A_1A_2\cdots A_{n-2}A_{n-1}A_{n-1}^{(1,2)}A_{n-2}^{(1,2)}\cdots A_1^{(1,2)}\\
&\vdots\\
&= A_1A_2A_2^{(1,2)}A_1^{(1,2)} = A_1A_1^{(1,2)}.
\end{aligned} \tag{2.70}$$

From (2.70), it follows

$$A_1A_2\cdots A_nA_n^{(1,2)}A_{n-1}^{(1,2)}\cdots A_1^{(1,2)}A_1A_2\cdots A_n = A_1A_2\cdots A_n$$

and

$$A_n^{(1,2)}A_{n-1}^{(1,2)}\cdots A_1^{(1,2)}A_1A_2\cdots A_nA_n^{(1,2)}A_{n-1}^{(1,2)} = A_n^{(1,2)}A_{n-1}^{(1,2)}\cdots A_1^{(1,2)},$$

so $A_n^{(1,2)}A_{n-1}^{(1,2)}\cdots A_1^{(1,2)} \in (A_1A_2\cdots A_n)\{1,2\}$. Hence (ii) holds.
Analogously, if $A_i \in \mathscr{B}_l^{-1}(\mathscr{H}_{i+1},\mathscr{H}_i)$ for $i = \overline{1, n-1}$ or if there exists $k \in \{2, ..., n-1\}$ such that $A_i \in \mathscr{B}_l^{-1}(\mathscr{H}_{i+1},\mathscr{H}_i)$ for $i = \overline{1, k-1}$, and $A_i \in \mathscr{B}_r^{-1}(\mathscr{H}_{i+1},\mathscr{H}_i)$ for $i = \overline{k+1, n}$, we can prove that $A_n^{(1,2)}A_{n-1}^{(1,2)}\cdots A_1^{(1,2)} \in (A_1A_2\cdots A_n)\{1,2\}$. Thus (ii) holds. □

Since the proof of the previous result is algebraic, it can easily be generalized to C^*-algebras and rings. It is thus safe to say that the reverse order law for $\{1,2\}$-inverses has been completely solved.

2.3 Reverse Order Laws for {1, 3} and {1, 4}-Inverses

The reverse order laws for $\{1,3\}$ and $\{1,4\}$-inverses were for the first time considered by M. Wei and Guo [15] in the matrix case. They presented some equivalent conditions for

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\} \tag{2.71}$$

and

$$(AB)\{1,3\} \subseteq B\{1,3\}A\{1,3\} \tag{2.72}$$

obtained by applying the product singular value decomposition (P-SVD) of matrices. Namely, in [15] they proved that for $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$ one has $B\{1,3\}\cdot A\{1,3\} \subseteq (AB)\{1,3\}$ if and only if

$$Z_{12} = 0 \text{ and } Z_{14} = 0$$

and that that $(AB)\{1,3\} \subseteq B\{1,3\}\cdot A\{1,3\}$ holds if and only if

$$\dim(\mathscr{R}(Z_{14}) = \dim(\mathscr{R}(Z_{12}, Z_{14})), \text{ and}$$
$$0 \leq \min\{p - r_2, m - r_1\} \leq n - r_1 - r_2^2 - \mathrm{r}(Z_{14}),$$

where the submatrices Z_{12}, Z_{14} and the constants r_1, r_2, r_2^2 are described in the P-SVD of matrices A and B given in Theorem 1.1 and Corollary 1.1 of [15].

Evidently a disadvantage of the results presented in [15] lies in the fact that the necessary and sufficient conditions for (2.71) and (2.72) to be satisfied contain information about the subblocks produced by P-SVD. In other words, they are dependent on P-SVD. In order to overcome this shortcoming, two methods are employed. One of the methods use certain operator matrix representations (see [16]) and the other one is based on some maximal and minimal ranks of matrix expressions (see [17]). Using these two different methods, in both of the papers [16, 17] it is proved that

$$B\{1,3\}A\{1,3\} \subseteq (AB)\{1,3\} \Leftrightarrow \mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$$

but in the first one in the case of regular operators and in the second one in the setting of matrices. These results are more elegant because they require no information on the P-SVD. Note that in the matrix case $\mathscr{R}(A^*AB) \subseteq \mathscr{R}(B)$ is equivalent to $\mathrm{r}(B, A^*AB) = \mathrm{r}(B)$.

All these results were generalized in the paper of Cvetković-Ilić and Harte [18] where purely algebraic necessary and sufficient conditions for (2.71) in C^*-algebras are offered, extending rank conditions for matrices and range conditions for Hilbert space operators. To present the result form [18] and its proof, first we will introduce some notations and give some preliminaries.

Let $\mathscr{A}$ be a complex unital C*-algebra. Then we have the following characterization of $a\{1,3\}$, where $a \in \mathscr{A}$ is regular:

Lemma 2.2 *Let $a \in \mathscr{A}$ be regular and $b \in \mathscr{A}$. Then $b \in a\{1,3\}$ if and only if $a^\dagger ab = a^\dagger$.*

Lemma 2.2 can be expressed by saying

$$a\{1,3\} = \{a^\dagger + (1 - a^\dagger a)y : y \in \mathscr{A}\}. \tag{2.73}$$

Theorem 2.13 ([18]) *If $a, b \in \mathscr{A}$ are such that a, b, ab and $a(1 - bb^\dagger)$ are regular, then the following conditions are equivalent:*

(1′) $bb^\dagger a^*ab = a^*ab$,
(2′) $b\{1,3\} \cdot a\{1,3\} \subseteq (ab)\{1,3\}$,
(3′) $b^\dagger a^\dagger \in (ab)\{1,3\}$,
(4′) $b^\dagger a^\dagger \in (ab)\{1,2,3\}$.

Proof With $p = bb^\dagger$, $q = b^\dagger b$ and $r = aa^\dagger$, we have that $b = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{p,q}$ and $a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{r,p}$. Using Lemma 2.2 and (2.73), we see that arbitrary $b^{(1,3)} \in b\{1,3\}$ can be represented as $b^{(1,3)} = \begin{bmatrix} b^\dagger & 0 \\ u & v \end{bmatrix}_{q,p}$, for some $u \in (1-q)\mathscr{A}p$ and $v \in (1-q)\mathscr{A}(1-p)$, as well as that $a^\dagger = a^*(aa^*)^\dagger = \begin{bmatrix} a_1^* d^\dagger & 0 \\ a_2^* d^\dagger & 0 \end{bmatrix}_{p,r}$, where $d = a_1a_1^* + a_2a_2^*$. Remark that $d \in r\mathscr{A}r$ is invertible in that subalgebra, $(d)^{-1}_{r\mathscr{A}r} = d^\dagger$ and $dd^\dagger = d^\dagger d = r$. Also, again by (2.73), any $a^{(1,3)}$ has the form $a^{(1,3)} = a^\dagger + (1 - a^\dagger a)x$, for some $x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p,r}$ i.e., $a^{(1,3)} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{p,r}$, where

$$\begin{aligned}
z_1 &= a_1^* d^\dagger + (1 - a_1^* d^\dagger a_1)x_1 - a_1^* d^\dagger a_2 x_3, \\
z_2 &= (1 - a_1^* d^\dagger a_1)x_2 - a_1^* d^\dagger a_2 x_4, \\
z_3 &= a_2^* d^\dagger - a_2^* d^\dagger a_1 x_1 + (1 - a_2^* d^\dagger a_2)x_3, \\
z_4 &= -a_2^* d^\dagger a_1 x_2 + (1 - a_2^* d^\dagger a_2)x_4.
\end{aligned}$$

With these preliminaries, we turn to the four conditions of the statement; we will show $(1') \Rightarrow (2') \Rightarrow (3') \Rightarrow (1')$ and then $(1') \Rightarrow (4') \Rightarrow (3')$.

$(1') \Rightarrow (2')$: Suppose that $bb^\dagger a^*ab = a^*ab$ which is equivalent to $a_2^*a_1 = 0$, i.e., $a_1^*a_2 = 0$. For arbitrary $a^{(1,3)}$ and $b^{(1,3)}$ we have that

$$abb^{(1,3)}a^{(1,3)}ab = \begin{bmatrix} a_1 z_1 a_1 b & 0 \\ 0 & 0 \end{bmatrix}_{r,q}.$$

Let $s = a_1a_1^\dagger$. Since $d \in s\mathscr{A}s + (1-s)\mathscr{A}(1-s)$, we have that $d^\dagger \in s\mathscr{A}s + (1-s)\mathscr{A}(1-s)$. Now, $a_1^* d^\dagger a_2 \in \mathscr{A}s \cdot (s\mathscr{A}s + (1-s)\mathscr{A}(1-s)) \cdot (1-s)\mathscr{A} = \{0\}$. Hence, $a_1^* d^\dagger a_2 = 0$, i.e., $a_2^* d^\dagger a_1 = 0$.
Since,

$$\begin{aligned}
a_1 z_1 a_1 &= a_1 a_1^* d^\dagger a_1 + a_1(1 - a_1^* d^\dagger a_1)x_1 a_1 \\
&= (d - a_2 a_2^*)d^\dagger a_1 + (a_1 - (d - a_2 a_2^*)d^\dagger a_1)x_1 a_1 \\
&= a_1,
\end{aligned}$$

it follows that $abb^{(1,3)}a^{(1,3)}ab = ab$. To prove that $abb^{(1,3)}a^{(1,3)}$ is Hermitian it is sufficient to prove that a_1z_1 is Hermitian and $a_1z_2 = 0$. By computation, we get that $a_1z_1 = a_1a_1^*d^\dagger = a_1a_1^*(a_1a_1^*)^\dagger$ which is Hermitian. Also,

$$\begin{aligned}
a_1 z_2 &= (a_1 - a_1 a_1^* d^\dagger a_1)x_2 - a_1 a_1^* d^\dagger a_2 x_4 \\
&= (a_1 - (d - a_2 a_2^*)d^\dagger a_1)x_2 \\
&= a_2 a_2^* d^\dagger a_1 x_2 \\
&= 0.
\end{aligned}$$

$(2') \Rightarrow (3')$: This is evident.

$(3') \Rightarrow (1')$: From $abb^\dagger a^\dagger ab = ab$ it follows that $a_1 a_1^* d^\dagger a_1 = a_1$, i.e., $a_2 a_2^* d^\dagger a_1 = 0$. Similarly, from $(abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger$, we get that $a_1 a_1^* d^\dagger$ is Hermitian. Now, $d^\dagger a_1 a_1^* a_1 = a_1$, i.e., $a_2 a_2^* a_1 = 0$. Multiplying the last equality by $a_2^\dagger$ from the left side, we get $a_2^* a_1 = 0$ which is equivalent to the statement $(1')$.

$(4') \Rightarrow (3')$: This is obvious.

$(1') \Rightarrow (4')$: We need to prove that $b^\dagger a^\dagger abb^\dagger a^\dagger = b^\dagger a^\dagger$ which is equivalent to $b^\dagger a_1^* d^\dagger a_1 a_1^* d^\dagger = b^\dagger a_1^* d^\dagger$. The last equality follows from the fact that $d^\dagger a_1 a_1^* = s$. □

Example 2.7 Let $b \in \mathscr{A}$ be right invertible. Then for any $a \in \mathscr{A}$ such that a, ab are regular, we have $b\{1, 3\} \cdot a\{1, 3\} \subseteq (ab)\{1, 3\}$.

Example 2.8 Let $p, q \in \mathscr{A}$ be orthogonal projections. Then $q\{1, 3\} \cdot p\{1, 3\} \subseteq (pq)\{1, 3\}$ if and only if $qpq = pq$, which is equivalent with the fact that p and q commute, which is in turn equivalent with the fact that pg is an orthogonal projection.

A similar result in the case $K = \{1, 4\}$ follows from Theorem 2.13 by reversal of products:

Theorem 2.14 *If $a, b \in \mathscr{A}$ are such that a, b, ba and $(1 - a^\dagger a)b$ are regular, then the following conditions are equivalent:*

$(1'')$ $abb^* a^\dagger a = abb^*$,
$(2'')$ $b\{1, 4\} \cdot a\{1, 4\} \subseteq (ab)\{1, 4\}$,
$(3'')$ $b^\dagger a^\dagger \in (ab)\{1, 4\}$,
$(4'')$ $b^\dagger a^\dagger \in (ab)\{1, 2, 4\}$.

Example 2.9 Let $a \in \mathscr{A}$ be left invertible. Then for any $b \in \mathscr{A}$ such that b and ba are regular, we have $b\{1, 4\} \cdot a\{1, 4\} \subseteq (ab)\{1, 4\}$.

Example 2.10 Let $p, q \in \mathscr{A}$ be orthogonal projections such that pq, $(1 - p)q$ and $p(1 - q)$ are regular. Then $q\{1, 3\} \cdot p\{1, 3\} \subseteq (pq)\{1, 3\}$ if and only if $q\{1, 4\} \cdot p\{1, 4\} \subseteq (pq)\{1, 4\}$ if and only if pq is an orthogonal projection.

The inclusion $(AB)\{1, 3\} \subseteq B\{1, 3\}A\{1, 3\}$ was considered by Liu and Yang [19] in the matrix case.

Theorem 2.15 ([19]) *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times k}$. Then $(AB)\{1, 3\} \subseteq B\{1, 3\}A\{1, 3\}$ if and only if*

$$\mathrm{r}(A^*AB \;\; B) + \mathrm{r}(A) = \mathrm{r}(AB) + \min\{\mathrm{r}(A^* \;\; B), \max\{n + \mathrm{r}(A) - m, n + \mathrm{r}(B) - k\}\}.$$

For (2.72), some equivalent conditions with the one given in [15, 19] can be found in [20]:

Theorem 2.16 ([20]) *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Then the following conditions are equivalent:*

(i) $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$,
(ii) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$ *and* $\mathrm{r}(C) \geq \min\{n - \mathrm{r}(A), k - \mathrm{r}(B)\}$,

where $S = B^\dagger(I - A^\dagger A)$ *and* $C = I - A^\dagger A - S^\dagger S$.

Notice that $C = P_{\mathcal{N}(A)\cap\mathcal{N}(B^*)}$, so $\mathrm{r}(C) = \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$.

Example 2.11 Let $A \in \mathbb{C}^{n\times m}$ be left invertible and $B \in \mathbb{C}^{m\times k}$. Then $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$ if and only if $(AB)^\dagger = B^\dagger A^\dagger$ and either A is invertible or B is left-invertible.

Corollary 2.1 *Let* $A \in \mathbb{C}^{n\times m}$ *and* $B \in \mathbb{C}^{m\times k}$. *Then the following conditions are equivalent:*

(i*) $(AB)\{1,3\} \subseteq B\{1,3\} \cdot A\{1,3\}$,
(ii*) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$, *and at least one of the two conditions below holds:*

(a) $\mathrm{r}(C) \geq k - \mathrm{r}(B)$, $k - \mathrm{r}(B) < n - \mathrm{r}(A)$,
(b) $\mathrm{r}(C) \geq n - \mathrm{r}(A)$, $k - \mathrm{r}(B) \geq n - \mathrm{r}(A)$,

where $S = B^\dagger(I - A^\dagger A)$ *and* $C = I - A^\dagger A - S^\dagger S$.

Example 2.12 Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. If $m < n$ and $m < k$ then

$$(AB)\{1,3\} \not\subseteq B\{1,3\} \cdot A\{1,3\}.$$

Open question: As we can see, in contrast to the case of inclusion (2.71) of which Theorem 2.13 provides a purely algebraic characterization, none of the results we have presented do so for the inclusion (2.72). To our knowledge no such results can be found in literature so far, which leaves the formulated problem still unsolved.

The reverse order law problem for $\{1,3\}$ and $\{1,4\}$-inverses, in the matrix setting, was considered by M. Wei [4]. He obtained necessary and sufficient conditions for the following inclusions to hold:

$$A_n\{1,3\} \cdot A_{n-1}\{1,3\} \cdots A_1\{1,3\} \subseteq (A_1 A_2 \cdots A_n)\{1,3\}$$

and

$$A_n\{1,4\} \cdot A_{n-1}\{1,4\} \cdots A_1\{1,4\} \subseteq (A_1 A_2 \cdots A_n)\{1,4\}$$

by applying the multiple product singular value decomposition (P-SVD).
Using the next lemma, his results were generalized in [21] in the case of regular bounded linear operators on Hilbert spaces and new simple conditions which involve only ranges of operators were presented.

Lemma 2.3 *Let* $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ *be regular. Then*

$$X \in A\{1,3\} \Leftrightarrow A^*AX = A^*.$$

Theorem 2.17 *Let $A_i \in \mathcal{B}(\mathcal{H}_{i+1}, \mathcal{H}_i)$ be regular operators such that $A_1 A_2 \cdots A_n$ is regular. The following are equivalent:*

(i) $A_n\{1,3\} \cdot A_{n-1}\{1,3\} \cdots A_1\{1,3\} \subseteq (A_1 A_2 \cdots A_n)\{1,3\}$,
(ii) $\mathcal{R}(A_k^* A_{k-1}^* \cdots A_1^* A_1 A_2 \cdots A_n) \subseteq \mathcal{R}(A_{k+1})$, *for* $k = \overline{1, n-1}$.

Proof (i) $\Rightarrow$ (ii) : If $A_1 A_2 \cdots A_n = 0$, then

$$\mathcal{R}(A_k^* A_{k-1}^* \cdots A_1^* A_1 A_2 \cdots A_n) = \{0\} \subseteq \mathcal{R}(A_{k+1}),$$

for $k = \overline{1, n-1}$, so (ii) holds.
Assume now that $A_1 A_2 \cdots A_n \neq 0$. Let $A_i^{(1,3)} \in A_i\{1,3\}$, $i = \overline{1, n}$ be arbitrary. By Lemma 2.3 it follows that

$$(A_1 A_2 \cdots A_n)^* A_1 A_2 \cdots A_n A_n^{(1,3)} A_{n-1}^{(1,3)} \cdots A_1^{(1,3)} = (A_1 A_2 \cdots A_n)^*. \tag{2.74}$$

Let $i \in \{1, 2, \ldots, n-1\}$ be arbitrary. Since, for arbitrary $X \in \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i+1})$, we have that $A_i^{(1,3)} + (I_{\mathcal{H}_{i+1}} - A_i^{(1,3)} A_i)X \in A_i\{1,3\}$, by Lemma 2.3 we have

$$\begin{aligned} &(A_1 A_2 \cdots A_n)^* A_1 A_2 \cdots A_n \cdot \\ &A_n^{(1,3)} \cdots A_{i+1}^{(1,3)} (A_i^{(1,3)} + (I_{\mathcal{H}_{i+1}} - A_i^{(1,3)} A_i)X) A_{i-1}^{(1,3)} \cdots A_1^{(1,3)} \\ &= (A_1 A_2 \cdots A_n)^*. \end{aligned} \tag{2.75}$$

Substracting (2.74) from (2.75), we get that

$$\begin{aligned} &(A_1 A_2 \cdots A_n)^* A_1 A_2 \cdots A_n \cdot \\ &A_n^{(1,3)} \cdots A_{i+1}^{(1,3)} (I_{\mathcal{H}_{i+1}} - A_i^{(1,3)} A_i) X A_{i-1}^{(1,3)} \cdots A_1^{(1,3)} = 0. \end{aligned} \tag{2.76}$$

From (2.76) it follows that

$$A_{i-1}^{(1,3)} \cdots A_1^{(1,3)} = 0 \tag{2.77}$$

or

$$(A_1 A_2 \cdots A_n)^* A_1 A_2 \cdots A_n A_n^{(1,3)} \cdots A_{i+1}^{(1,3)} (I_{\mathcal{H}_{i+1}} - A_i^{(1,3)} A_i) = 0. \tag{2.78}$$

If (2.77) holds, then from (2.74) it follows that $A_1 A_2 \cdots A_n = 0$, which is a contradiction, so (2.78) must hold for arbitrary $i \in \{1, 2, \ldots, n-1\}$.

Condition (ii) is equivalent to

$$\begin{aligned} &(A_1 A_2 \cdots A_n)^* A_1 A_2 \cdots A_{n-k} A_{n-k+1} A_{n-k+1}^{(1,3)} \\ &= (A_1 A_2 \cdots A_n)^* A_1 A_2 \cdots A_{n-k}, \quad k = \overline{1, n-1}. \end{aligned} \tag{2.79}$$

Now, we will prove (2.79) by induction on k.

From (2.78) and (2.74) it follows that

$$\begin{aligned}
&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-1}A_nA_n^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}A_{n-1}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}A_{n-2}^{(1,3)}A_{n-2}A_{n-1}\\
&\vdots\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)}A_1A_2\cdots A_{n-2}A_{n-1}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-2}A_{n-1},
\end{aligned}$$

so (2.79) holds for $k=1$.
Assume now that (2.79) holds for $k<l\le n$, i.e.,

$$\begin{aligned}
&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k}A_{n-k+1}A_{n-k+1}^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k},\quad k=1,2,\ldots,l-1
\end{aligned}\tag{2.80}$$

and prove that (2.79) is true for $k=l$. Using (2.80), we have

$$\begin{aligned}
&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l}A_{n-l+1}A_{n-l+1}^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots\cdots A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_{n-l+1}^{(1,3)}.
\end{aligned}\tag{2.81}$$

Now, using (2.81) and (2.74), we get

$$\begin{aligned}
&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l}A_{n-l+1}A_{n-l+1}^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-l},
\end{aligned}$$

so (2.79) holds for $k=l$.
(ii) $\Rightarrow$ (i) : Let $A_i^{(1,3)}\in A_i\{1,3\}$, $i=\overline{1,n}$ be arbitrary. Condition (ii) is equivalent to

$$\begin{aligned}
&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k}A_{n-k+1}A_{n-k+1}^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-k},\quad k=1,2,\ldots,n-1.
\end{aligned}\tag{2.82}$$

Now, from (2.82) it follows

$$\begin{aligned}
&(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-1}A_nA_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-2}A_{n-1}A_{n-1}^{(1,3)}A_{n-2}^{(1,3)}\cdots A_1^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*A_1A_2\cdots A_{n-2}A_{n-2}^{(1,3)}\cdots A_1^{(1,3)}\\
&\vdots\\
&=(A_1A_2\cdots A_n)^*A_1A_1^{(1,3)}\\
&=(A_1A_2\cdots A_n)^*.
\end{aligned}$$

Hence by Lemma 2.3 it follows that

$$A_n^{(1,3)}A_{n-1}^{(1,3)}\cdots A_1^{(1,3)}\in(A_1A_2\cdots A_n)\{1,3\}.$$

□

The next result follows from Theorem 2.17 by taking adjoints:

Theorem 2.18 *Let $A_i\in\mathscr{B}(\mathscr{H}_{i+1},\mathscr{H}_i)$ be regular operators such that $A_1A_2\cdots A_n$ is regular. The following are equivalent:*

(i) $A_n\{1,4\}\cdot A_{n-1}\{1,4\}\cdots A_1\{1,4\}\subseteq(A_1A_2\cdots A_n)\{1,4\}$,
(ii) $\mathscr{R}(A_{k+1}A_{k+2}\cdots A_nA_n^*A_{n-1}^*\cdots A_1^*)\subseteq\mathscr{R}(A_k^*)$ *for* $k=\overline{1,n-1}$.

2.4 Reverse Order Laws for {1, 2, 3} and {1, 2, 4}-Inverses

The reverse order law for {1, 2, 3}-inverses for the matrix case was considered by Xiong and Zheng [22]. They presented necessary and sufficient conditions under which

$$B\{1,2,3\}A\{1,2,3\}\subseteq(AB)\{1,2,3\}\tag{2.83}$$

is satisfied. The method of the proof of their result, which will be stated below, involved expressions for maximal and minimal ranks of the generalized Schur complement.

Theorem 2.19 ([22]) *Let $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(i) $B\{1,2,3\}A\{1,2,3\}\subseteq(AB)\{1,2,3\}$;
(ii) $\mathrm{r}(B,A^*AB)=\mathrm{r}(B)$ *and* $\mathrm{r}(AB)=\min\{\mathrm{r}(A),\mathrm{r}(B)\}=\mathrm{r}(A)+\mathrm{r}(B)-\mathrm{r}\begin{pmatrix}A\\B^*\end{pmatrix}$.

This result was generalized to the C^*-algebra case by Cvetković-Ilić and Harte [18] using the following characterization of the set $a\{1,2,3\}$:

Lemma 2.4 *Let $a\in\mathscr{A}$ be regular and $b\in\mathscr{A}$. Then $b\in a\{1,2,3\}$ if and only if $a^*ab=a^*$ and $baa^\dagger=b$.*

Theorem 2.20 ([18]) *If $a, b \in \mathscr{A}$ are such that a, b, ab and $a - abb^\dagger$ are regular, then the following conditions are equivalent:*

(i) $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$,
(ii) $bb^\dagger a^* ab = a^* ab$ *and* $(bb^\dagger - (abb^\dagger)^\dagger abb^\dagger)\mathscr{A}(aa^\dagger - (ab)(ab)^\dagger) = \{0\}$.

Proof Let $p = bb^\dagger, q = b^\dagger b$ and $r = aa^\dagger$. Then $b = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}_{p,q}$ and $a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{r,p}$. We have that $b\{1,2,3\} = \left\{ \begin{bmatrix} b^\dagger & 0 \\ u & 0 \end{bmatrix}_{q,p} : u \in (1-q)\mathscr{A}p \right\}$ and $a^\dagger = a^*(aa^*)^\dagger = \begin{bmatrix} a_1^* d^\dagger & 0 \\ a_2^* d^\dagger & 0 \end{bmatrix}_{p,r}$, where $d = a_1 a_1^* + a_2 a_2^*$. Using Lemma 2.4, $a\{1,2,3\} = \left\{ \begin{bmatrix} z_1 & 0 \\ z_3 & 0 \end{bmatrix}_{p,r} : a_1^* a_1 z_1 + a_1^* a_2 z_3 = a_1^*, a_2^* a_1 z_1 + a_2^* a_2 z_3 = a_2^*, z_1 \in p\mathscr{A}r, z_3 \in (1-p)\mathscr{A}r \right\}$. Hence $x \in b\{1,2,3\} \cdot a\{1,2,3\}$ if and only if $x = \begin{bmatrix} b^\dagger z_1 & 0 \\ u z_1 & 0 \end{bmatrix}_{q,r}$ for some $u \in (1-q)\mathscr{A}p$ and some $z_1 \in p\mathscr{A}r$ such that for some $z_3 \in (1-p)\mathscr{A}r$ the following hold:

$$a_1^* a_1 z_1 + a_1^* a_2 z_3 = a_1^*, \quad a_2^* a_1 z_1 + a_2^* a_2 z_3 = a_2^*. \tag{2.84}$$

By Lemma 2.4, $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$ if and only if

$$\begin{aligned} &(ab)^*(ab)b^{(1,2,3)} \cdot a^{(1,2,3)} = (ab)^*, \\ &b^{(1,2,3)} \cdot a^{(1,2,3)}(ab)(ab)^\dagger = b^{(1,2,3)} \cdot a^{(1,2,3)} \end{aligned} \tag{2.85}$$

hold for any $a^{(1,2,3)}$ and $b^{(1,2,3)}$.
Now, using the matrix forms introduced above, we find that (2.85) is equivalent to the following equalities:

$$\begin{aligned} &(ab)^*(ab)b^\dagger z_1 = (ab)^*, \\ &z_1(ab)(ab)^\dagger = z_1, \end{aligned} \tag{2.86}$$

for any $z_1 \in p\mathscr{A}r$ which satisfies (2.84).

(ii) $\Rightarrow$ (i) : Suppose that (ii) holds. Since $bb^\dagger a^* ab = a^* ab$, is equivalent to $a_2^* a_1 = 0$, i.e., $a_1^* a_2 = 0$, we have that (2.84) is equivalent to

$$a_1^* a_1 z_1 = a_1^*, \quad a_2^* a_2 z_3 = a_2^*.$$

Now, to prove that $b\{1,2,3\}a\{1,2,3\} \subseteq (ab)\{1,2,3\}$ it is sufficient to prove that (2.86) holds for every $z_1 \in p\mathscr{A}r$ which satisfies the equation $a_1 z_1 = a_1 a_1^\dagger$. Denote the set of all such z_1 by Z. Note that $z_1 = bb^\dagger z aa^\dagger$ for some $z \in \mathscr{A}$ which is a solution

of the equation $abb^\dagger zaa^\dagger = abb^\dagger(abb^\dagger)^\dagger$. So, $Z = \{(abb^\dagger)^\dagger aa^\dagger + bb^\dagger yaa^\dagger - (abb^\dagger)^\dagger abb^\dagger yaa^\dagger : y \in \mathscr{A}\}$.
The first equality from (2.86) is satisfied because for every $z_1 \in Z$:

$$(ab)^*(ab)b^\dagger z_1 = (a_1b)^* a_1 bb^\dagger z_1 = b^* a_1^* a_1 z_1 = b^* a_1^* = (a_1b)^* = (ab)^*.$$

Now, the second equality from (2.86) is equivalent to

$$\begin{aligned}(abb^\dagger)^\dagger aa^\dagger + bb^\dagger yaa^\dagger - (abb^\dagger)^\dagger abb^\dagger yaa^\dagger &= (abb^\dagger)^\dagger (ab)(ab)^\dagger \\ + bb^\dagger y(ab)(ab)^\dagger - (abb^\dagger)^\dagger abb^\dagger y(ab)(ab)^\dagger.\end{aligned} \tag{2.87}$$

Since, $(ab)(ab)^\dagger = abb^\dagger(abb^\dagger)^\dagger$, we get that $(abb^\dagger)^\dagger(ab)(ab)^\dagger = (abb^\dagger)^\dagger = (abb^\dagger)^\dagger aa^\dagger$, so (2.87) is equivalent to

$$\Big(bb^\dagger - (abb^\dagger)^\dagger(abb^\dagger)\Big)y\Big(aa^\dagger - (ab)(ab)^\dagger\Big) = 0$$

which holds since $(bb^\dagger - (abb^\dagger)^\dagger abb^\dagger)\mathscr{A}(aa^\dagger - (ab)(ab)^\dagger) = \{0\}$.

(i) $\Rightarrow$ (ii) : If (i) holds, then $b^\dagger a^\dagger \in (ab)\{1, 2, 3\}$. Now, from $abb^\dagger a^\dagger ab = ab$, it follows that $a_1 a_1^* d^\dagger a_1 = a_1$, i.e., $a_2 a_2^* d^\dagger a_1 = 0$. Similarly, from $(abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger$, we get that $a_1 a_1^* d^\dagger$ is Hermitian. Now, $d^\dagger a_1 a_1^* a_1 = a_1$, i.e., $a_2 a_2^* a_1 = 0$. Multiplying the last equality by $a_2^\dagger$ from the left side, we get $a_2^* a_1 = 0$ which is equivalent to $bb^\dagger a^* ab = a^* ab$. Now, (2.86) holds for every $z_1 \in p\mathscr{A}r$ which satisfies the equation $a_1 z_1 = a_1 a_1^\dagger$. Hence, $\Big(bb^\dagger - (abb^\dagger)^\dagger(abb^\dagger)\Big)y\Big(aa^\dagger - (ab)(ab)^\dagger\Big) = 0$, for every $y \in \mathscr{A}$, i.e., $(bb^\dagger - (abb^\dagger)^\dagger abb^\dagger)\mathscr{A}(aa^\dagger - (ab)(ab)^\dagger) = \{0\}$. □

Note that if the algebra $\mathscr{A}$ is prime, in the sense that

$$a\mathscr{A}b = \{0\} \Longrightarrow 0 \in \{a, b\}\ ,$$

then the second half of condition (ii) of Theorem 2.20 is equivalent to

$$bb^\dagger - (abb^\dagger)^\dagger abb^\dagger = 0 \ \text{ or } \ aa^\dagger - (ab)(ab)^\dagger = 0.$$

The C*-algebra $\mathscr{A} = \mathscr{B}(\mathscr{H})$ of operators on Hilbert space is prime, in particular (Lemma 3 [23]) the matrix algebra. We thus have the following results.

Corollary 2.2 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(i) $B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$,
(ii) $BB^\dagger A^* AB = A^* AB$ *and* $\big((ABB^\dagger)^\dagger ABB^\dagger = BB^\dagger$ *or* $(AB)(AB)^\dagger = AA^\dagger\big)$.

Example 2.13 We will show that in the case when $B\{1, 2, 3\}A\{1, 2, 3\} \nsubseteq (AB)\{1, 2, 3\}$ we can find particular $A^{(1,2,3)}$ and $B^{(1,2,3)}$ such that $B^{(1,2,3)}A^{(1,2,3)} \in$

$(AB)\{1,2,3\}$: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then evidently $AB = 0$, which implies $(AB)\{1,2,3\} = \{0\}$ and by Corollary 2.2 that $B\{1,2,3\}A\{1,2,3\} \not\subseteq (AB)\{1,2,3\}$. But for $A^{(1,2,3)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^{(1,2,3)} = \begin{bmatrix} 0 & 1 \\ 0 & d \end{bmatrix}$, where d is any complex number, we have that $B^{(1,2,3)}\ A^{(1,2,3)} = 0 \in (AB)\{1,2,3\}$.

Example 2.14 Let $b \in \mathscr{A}$ be right invertible and $a \in \mathscr{A}$ be regular such that ab is regular. Then $b\{1,2,3\} \cdot a\{1,2,3\} \subseteq (ab)\{1,2,3\}$ if and only if either a is left invertible or $aa^\dagger = (ab)(ab)^\dagger$.

Also, in [24] using some block-operator matrix techniques, the authors presented some different conditions than the one presented in Corollary 2.2 for (2.83) to hold in the case of linear bounded operators on Hilbert spaces:

Theorem 2.21 ([24]) *Let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and $B \in \mathscr{B}(\mathscr{L}, \mathscr{H})$ be such that A, B, AB are regular operators and $AB \neq 0$. The following conditions are equivalent:*

(i) $B\{1,2,3\}A\{1,2,3\} \subseteq (AB)\{1,2,3\}$;
(ii) $\mathscr{R}(B) = \mathscr{R}(A^*AB) \oplus^\perp [\mathscr{R}(B) \cap \mathscr{N}(A)]$, $\mathscr{R}(AB) = \mathscr{R}(A)$.

In the matrix case, in [25], it is shown that $B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\}$ implies $B\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}$. Here we will present a completely different proof which can easily be adapted to more general cases.

Theorem 2.22 *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. The following conditions are equivalent:*

(i) $B\{1,2,3\} \cdot A\{1,2,3\} \subseteq (AB)\{1,2,3\}$,
(ii) $BB^\dagger A^*AB = A^*AB$ *and* $\big((ABB^\dagger)^\dagger ABB^\dagger = BB^\dagger$ *or* $(AB)(AB)^\dagger = AA^\dagger\big)$,
(iii) $B\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}$.

Proof (i) $\Leftrightarrow$ (ii): Follows from Corollary 2.2.

(i) $\Rightarrow$ (iii): Let $P = BB^\dagger$, $Q = B^\dagger B$ and $R = AA^\dagger$. We have that $A = A_1 + A_2$, where $A_1 = AP$ and $A_2 = A(I-P)$. To prove (iii), take arbitrary $X \in (AB)\{1,2,3\}$. We will show that there exist $Y \in B\{1,2,3\}$ and $Z \in A\{1,2,3\}$ such that $X = YZ$. Since $X \in (AB)\{1,2,3\}$, it is of the form $X = QX_1R + (I - Q)X_3R$, for some $X_1 \in \mathbb{C}^{k\times n}$ and $X_3 \in \mathbb{C}^{k\times n}$ such that $QX_1R \in (A_1B)\{1,2,3\}$ and $(I-Q)X_3A_1BX_1R = (I-Q)X_3R$.
Let $Z = BX_1R + A_2^\dagger$ and $Y = B^\dagger + (I-Q)X_3A_1$. We have $B\{1,2,3\} = \big\{B^\dagger + (I-Q)UP : U \in \mathbb{C}^{k\times m}\big\}$, so $Y \in B\{1,2,3\}$. To prove that $Z \in A\{1,2,3\}$, we can check that the first three Penrose equations are satisfied using that $A_2^*A_1 = 0$, which follows from the condition $BB^\dagger A^*AB = A^*AB$. Since $X = YZ$, it follows that $B\{1,2,3\} \cdot A\{1,2,3\} = (AB)\{1,2,3\}$.

(iii) $\Rightarrow$ (i): This is evident. $\square$

The opposite reverse order law

$$(AB)\{1, 2, 3\} \subseteq B\{1, 2, 3\} \cdot A\{1, 2, 3\} \tag{2.88}$$

on the set of matrices is considered in [25] where purely algebraic necessary and sufficient conditions for (2.88) to hold are offered.

Theorem 2.23 ([25]) *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. The following conditions are equivalent:*

(i) $(AB)\{1, 2, 3\} \subseteq B\{1, 2, 3\} \cdot A\{1, 2, 3\}$,
(ii) $(I - B^\dagger(B^\dagger(I - A^\dagger A))^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$

It is very important to remark that the results from Theorems 2.23 and 2.22 can be generalized to the setting of bounded linear operators on Hilbert spaces and to the C^*-algebra setting by imposing the additional condition of regularity of suitable elements.

Example 2.15 We will show that $(AB)\{1, 2, 3\} \subseteq B\{1, 2, 3\}A\{1, 2, 3\}$ doesn't imply that $B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$, although the reverse implication is always true. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then evidently $A^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B^\dagger = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So, we can check that $(AB)^\dagger = B^\dagger A^\dagger$, which implies by Theorem 2.23 that $(AB)\{1, 2, 3\} \subseteq B\{1, 2, 3\} \cdot A\{1, 2, 3\}$. On the other hand, neither of the conditions $(ABB^\dagger)^\dagger ABB^\dagger = BB^\dagger$ and $(AB)(AB)^\dagger = AA^\dagger$ is satisfied, so from Theorem 2.22 it follows that $B\{1, 2, 3\}A\{1, 2, 3\} \nsubseteq (AB)\{1, 2, 3\}$

By taking adjoints, we obtain analogous results for $\{1, 2, 4\}-$ generalized inverses.

Theorem 2.24 *Let $a, b \in \mathscr{A}$ be such that a, b, ab and $(1 - a^\dagger a)b$ are regular. Then the following conditions are equivalent:*

(i′) $b\{1, 2, 4\}a\{1, 2, 4\} \subseteq (ab)\{1, 2, 4\}$
(ii′) $a^\dagger abb^*a^* = bb^*a^*$ *and* $(a^\dagger a - a^\dagger ab(a^\dagger ab)^\dagger)\mathscr{A}(b^\dagger b - (ab)^\dagger(ab)) = \{0\}$.

Theorem 2.25 *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$. Then the following statements are equivalent:*

(i) $B\{1, 2, 4\}A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}$,
(ii) $ABB^*A^\dagger A = ABB^*$ *and* $\big((A^\dagger AB)(A^\dagger AB)^\dagger = A^\dagger A$ *or* $(AB)^\dagger(AB) = B^\dagger B\big)$,
(iii) $B\{1, 2, 4\}A\{1, 2, 4\} = (AB)\{1, 2, 4\}$,

Theorem 2.26 ([24]) *Let $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$ be Hilbert spaces and let $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, $B \in \mathscr{B}(\mathscr{L}, \mathscr{H})$ be such that $\mathscr{R}(A)$, $\mathscr{R}(B)$ and $\mathscr{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

(i) $B\{1, 2, 4\}A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}$.

(ii) $\mathscr{R}(A^*) = \mathscr{R}(BB^*A^*) \oplus^{\perp} [\mathscr{R}(A^*) \cap \mathscr{N}(B^*)]$, $\mathscr{N}(AB) = \mathscr{N}(B)$

Theorem 2.27 *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. The following conditions are equivalent:*

(i) $(AB)\{1,2,4\} \subseteq B\{1,2,4\} \cdot A\{1,2,4\}$,
(ii) $((AB)^{\dagger} - B^{\dagger}A^{\dagger})(I - ((I - BB^{\dagger})A^{\dagger})^{\dagger}A^{\dagger}) = 0$

2.5 Reverse Order Laws for {1, 3, 4}-Generalized Inverses

Reverse order laws for {1, 3, 4}-generalized inverses of matrices $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$ were considered by Liu and Yang [26], who gave certain necessary and sufficient conditions for

$$B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\} \tag{2.89}$$

and

$$(AB)\{1,3,4\} \subseteq B\{1,3,4\} \cdot A\{1,3,4\}. \tag{2.90}$$

Let

$$n_1 = \mathrm{r}\begin{bmatrix} B^* \\ B^*A^*A \end{bmatrix},\ n_2 = \mathrm{r}\begin{bmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{bmatrix},\ n_3 = \mathrm{r}\begin{bmatrix} A \\ ABB^* \end{bmatrix}$$

and

$$n_4 = \mathrm{r}\begin{bmatrix} AA^* & ABB^*A^* \\ B^*A^* & B^*BB^*A^* \end{bmatrix}.$$

With this notations the following is proved:

Theorem 2.28 ([26]) *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Then the following conditions are equivalent:*

(i) $B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\}$,
(ii) $\mathrm{r}(A) = \min\{n_3, n_4 + k - \mathrm{r}(B)\}$, $\mathrm{r}(B) = \min\{n_1, n_2 + m - \mathrm{r}(A)\}$.

Theorem 2.29 ([26]) *Let $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$. Then the following conditions are equivalent:*

(i) $(AB)\{1,3,4\} \subseteq B\{1,3,4\} \cdot A\{1,3,4\}$,
(ii) $\mathrm{r}(AB) = \max\{\min\{n_2, n_4\}, \min\{k - \mathrm{r}(B), n - \mathrm{r}(A)\} - m + n_1 + n_3\}$.

What the authors of [26] did not realize was that $(AB)\{1,3,4\} \subseteq B\{1,3,4\} \cdot A\{1,3,4\}$ is actually equivalent to $(AB)\{1,3,4\} = B\{1,3,4\} \cdot A\{1,3,4\}$, which was shown in [27] in the matrix case and later generalized in [28] to the C^*-algebra setting. In the following theorem the proof of this fact in C^*-algebras will be presented together with some necessary and sufficient conditions for (2.90) to hold.

Theorem 2.30 ([28]) *Let $a, b \in \mathscr{A}$ be such that $a, b, ab, a(1-bb^\dagger)$ and $(1-a^\dagger a)b$ are generalized invertible. Then the following conditions are equivalent:*

(i) $(ab)\{1, 3, 4\} \subseteq b\{1, 3, 4\} \cdot a\{1, 3, 4\}$,
(ii) $(ab)\{1, 3, 4\} = b\{1, 3, 4\} \cdot a\{1, 3, 4\}$,
(iii) $(ab)^\dagger = b^\dagger a^\dagger$, $\big(b^\dagger b-(ab)^\dagger(ab)\big)\mathscr{A}\big(aa^\dagger-(ab)(ab)^\dagger\big) = \{0\}$ *and the equation*

$$(1-b^\dagger b)z(1-aa^\dagger) = (1-b^\dagger b)x(1-s_2s_2^\dagger)f_b e_a(1-s_1^\dagger s_1)y(1-aa^\dagger), \quad (2.91)$$

is solvable for any $z \in \mathscr{A}$, where $s_1 = 1-(ab)^\dagger(ab)$, $s_2 = (1-bb^\dagger)a^\dagger$, $e_a = 1-a^\dagger a$ and $f_b = 1-bb^\dagger$.

To give a proof of Theorem 2.30, we will need some auxiliary results given as follows:

Lemma 2.5 *Let $a \in \mathscr{A}$ be generalized invertible and $b \in \mathscr{A}$. Then the following statements are equivalent:*

(1) $b \in a\{1, 3, 4\}$
(2) $a^*ab = a^*$ *and* $baa^* = a^*$.
(3) *There exists $y \in \mathscr{A}$ such that $b = a^\dagger + (1-a^\dagger a)y(1-aa^\dagger)$.*

Proof (1) $\Rightarrow$ (2) If $b \in a\{1, 3, 4\}$, then

$$a^*ab = a^*(ab)^* = (aba)^* = a^* \quad \text{and} \quad baa^* = (aba)^* = a^*.$$

(2) $\Rightarrow$ (1) If $a^*ab = a^*$ and $baa^* = a^*$, then

$$\begin{aligned} aba = aa^\dagger aba = (a^\dagger)^*a^*aba = (a^\dagger)^*a^*a = a, \\ ab = b^*a^*ab = (ab)^*(ab), \\ ba = baa^*b^* = (ba)(ba)^*, \end{aligned}$$

so, $b \in a\{1, 3, 4\}$.

(1) $\Rightarrow$ (3) If $b \in a\{1, 3, 4\}$, then $b \in a\{1, 3\}$, so $b = a^\dagger + (1-a^\dagger a)t$, for some $t \in \mathscr{A}$. Then $ba = a^\dagger a$, so $(1-a^\dagger a)ta = 0$. Put $z = (1-a^\dagger a)t$. We have that z is a solution of the equation $za = 0$, so $z = y(1-aa^\dagger)$, for some $y \in \mathscr{A}$. Now, $b = a^\dagger + (1-a^\dagger a)y(1-aa^\dagger)$.

(3) $\Rightarrow$ (1) This is evident. □

Now, let $p = bb^\dagger$, $q = b^\dagger b$ and $r = aa^\dagger$.

Remark 2.1 By Lemma 2.5, we get that $b\{1, 3, 4\} = \left\{\begin{bmatrix} b^\dagger & 0 \\ 0 & u \end{bmatrix}_{q,p} : u \in (1-q)\mathscr{A}(1-p)\right\}$ and

$$a\{1,3,4\} = \left\{ \begin{bmatrix} a_1^* d^\dagger & z_2 - a_1^* d^\dagger a_1 z_2 - a_1^* d^\dagger a_2 z_4 \\ a_2^* d^\dagger & z_4 - a_2^* d^\dagger a_1 z_2 - a_2^* d^\dagger a_2 z_4 \end{bmatrix}_{p,r} : z_2 \in p\mathscr{A}(1-r), \right.$$
$$\left. z_4 \in (1-p)\mathscr{A}(1-r) \right\}.$$

Lemma 2.6 *Let $a, b \in \mathscr{A}$ be such that $a, b, ab \in \mathscr{A}^\dagger$ and $(ab)^\dagger = b^\dagger a^\dagger$. Then*

$$(abb^\dagger)^\dagger = b(ab)^\dagger. \tag{2.92}$$

Proof of Theorem 2.30: (i) $\Leftrightarrow$ (iii) : The fact that $(ab)\{1,3,4\} \subseteq b\{1,3,4\} \cdot a\{1,3,4\}$ is equivalent to the fact that for every $(ab)^{(1,3,4)}$ there exist $a^{(1,3,4)}$ and $b^{(1,3,4)}$ such that $(ab)^{(1,3,4)} = b^{(1,3,4)} \cdot a^{(1,3,4)}$.
Since, $ab = \begin{bmatrix} a_1 b & 0 \\ 0 & 0 \end{bmatrix}_{r,q}$, by Lemma 2.5,

$$(ab)\{1,3,4\} = \left\{ \begin{bmatrix} s & \big(b^\dagger b - (a_1 b)^\dagger (a_1 b)\big) y_2 \\ y_3 (aa^\dagger - (ab)(ab)^\dagger) & y_4 \end{bmatrix}_{q,r} : \right.$$
$$\left. y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}_{q,r} \in \mathscr{A} \right\},$$

where $s = (a_1 b)^\dagger + \big(1 - (a_1 b)^\dagger (a_1 b)\big) y_1 \big(1 - (a_1 b)(a_1 b)^\dagger\big)$.
Now, using (ii) of Remark 2.1, we may conclude that $(ab)\{1,3,4\} \subseteq b\{1,3,4\} \cdot a\{1,3,4\}$ holds if and only if for arbitrary $y = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}_{q,r} \in \mathscr{A}$, there exist $u, z \in \mathscr{A}$ such that

$$(a_1 b)^\dagger + \big(1 - (a_1 b)^\dagger (a_1 b)\big) y_1 \big(1 - (a_1 b)(a_1 b)^\dagger\big) = b^\dagger a_1^* d^\dagger, \tag{2.93}$$
$$\big(b^\dagger b - (a_1 b)^\dagger (a_1 b)\big) y_2 = b^\dagger (z_2 - a_1^* d^\dagger a_1 z_2 - a_1^* d^\dagger a_2 z_4), \tag{2.94}$$
$$y_3 (aa^\dagger - (ab)(ab)^\dagger) = (1 - b^\dagger b) u a_2^* d^\dagger, \tag{2.95}$$
$$y_4 = (1 - b^\dagger b) u (z_4 - a_2^* d^\dagger a_1 z_2 - a_2^* d^\dagger a_2 z_4), \tag{2.96}$$

where $z_2 = pz(1-r)$ and $z_4 = (1-p)z(1-r)$.
The fact that the equation (2.93) holds for every $y_1 \in q\mathscr{A}r$ is equivalent to

$$(ab)^\dagger = b^\dagger a^* d^\dagger = b^\dagger a^\dagger, \ \big(b^\dagger b - (ab)^\dagger (ab)\big)\mathscr{A}\big(aa^\dagger - (ab)(ab)^\dagger\big) = \{0\}. \tag{2.97}$$

Now, using Lemma 2.6 and the fact that $(ab)^\dagger = b^\dagger a^\dagger \Rightarrow a_1^* d^\dagger a_2 = 0$, we have that the equations (2.94), (2.95) and (2.96), respectively have the forms:

$$\big(1-(ab)^\dagger(ab)\big)b^\dagger by(1-aa^\dagger)=\big(1-(ab)^\dagger(ab)\big)b^\dagger z(1-aa^\dagger), \quad (2.98)$$

$$\big(1-b^\dagger b\big)ya(1-bb^\dagger)a^\dagger=\big(1-b^\dagger b\big)u(1-bb^\dagger)a^\dagger, \quad (2.99)$$

$$\big(1-b^\dagger b\big)y(1-aa^\dagger)=\big(1-b^\dagger b\big)uf_be_az(1-aa^\dagger). \quad (2.100)$$

It is evident that the equations (2.98) and (2.99) are solvable for any $y\in\mathscr{A}$ and that solutions of these equations are respectively, given by

$$z=by+t_1-s_1^\dagger s_1t_1(1-aa^\dagger), \quad u=ya+t_2-(1-b^\dagger b)t_2s_2s_2^\dagger,$$

for $t_1,t_2\in\mathscr{A}$. Notice that $s_1\in\mathscr{A}^\dagger$ because it is a projection and $s_2\in\mathscr{A}^\dagger$ since $a(1-bb^\dagger)\in s_2\{1\}$.
Since $abb^\dagger=ab(ab)^\dagger a$, we have that

$$a(1-bb^\dagger)(1-a^\dagger a)(1-bb^\dagger)=0. \quad (2.101)$$

Also, from $a_2^*d^\dagger a_1=0$, we get that

$$(1-bb^\dagger)(1-a^\dagger a)(1-bb^\dagger)=(1-bb^\dagger)(1-a^\dagger a). \quad (2.102)$$

Now, by (2.101) and (2.102), it follows that for z and u satisfying the equations (2.98) and (2.99), respectively, the equation (2.100) is equivalent to

$$\big(1-b^\dagger b\big)y(1-aa^\dagger)=\big(1-b^\dagger b\big)t_2(1-s_2s_2^\dagger)f_be_a(1-s_1^\dagger s_1)t_1(1-aa^\dagger). \quad (2.103)$$

Now, (i) holds if and only if (2.97) holds and for arbitrary $y\in\mathscr{A}$ there exist $t_1,t_2\in\mathscr{A}$ such that the equation (2.103) is satisfied.

(i) $\Rightarrow$ (ii) : If (i) holds, then there exist $a^{(1,3,4)}$ and $b^{(1,3,4)}$ such that $(ab)^\dagger=b^{(1,3,4)}a^{(1,3,4)}$. Now, if we multiply the last equality by $b^\dagger b$ from the left and by $aa^\dagger$ from the right, we get that $(ab)^\dagger=b^\dagger bb^{(1,3,4)}a^{(1,3,4)}aa^\dagger=b^\dagger bb^\dagger a^\dagger aa^\dagger=b^\dagger a^\dagger$. Now, by Theorem 2.32, we get that (ii) holds.

(ii) $\Rightarrow$ (i) : This is evident. □

For matrices in the special case when $k=n$ it was proved in [27] that the solvability of the equation (2.91) is equivalent to the condition $n\le m$. Hence, when $n>m$ then

$$(AB)\{1,3,4\}\nsubseteq B\{1,3,4\}\cdot A\{1,3,4\}. \quad (2.104)$$

The case $n\le m$ is treated in the next result.

Theorem 2.31 ([27]) *Let $A\in\mathbb{C}^{n\times m}$ and $B\in\mathbb{C}^{m\times n}$ and $n\le m$. The following conditions are equivalent:*

(i) $(AB)\{1,3,4\}\subseteq B\{1,3,4\}\cdot A\{1,3,4\}$,
(ii) $(AB)^\dagger=B^\dagger A^\dagger$ *and* $\big(B=A^\dagger AB$ *or* $A=ABB^\dagger\big)$.

Example 2.16 We will show that $(AB)\{1,3,4\} \subseteq B\{1,3,4\} \cdot A\{1,3,4\}$ is not equivalent with $B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\}$. Also, in the case when $(AB)\{1,3,4\} \nsubseteq B\{1,3,4\} \cdot A\{1,3,4\}$, for some $(AB)^{(1,3,4)}$, we will find particular $A^{(1,3,4)}$ and $B^{(1,3,4)}$ such that $(AB)^{(1,3,4)} = B^{(1,3,4)}A^{(1,3,4)} \in (AB)\{1,2,3\}$: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then evidently $AB = 0$, which implies by Theorem 2.31 that $(AB)\{1,3,4\} \nsubseteq B\{1,3,4\} \cdot A\{1,3,4\}$, and also that $(AB)\{1,3,4\} = \mathbb{C}^{2\times 2}$. This further implies that $B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\}$. Since $A\{1,3,4\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} : c \in \mathbb{C}\right\}$, $B\{1,3,4\} = \left\{\begin{bmatrix} 0 & 1 \\ d & 0 \end{bmatrix} : d \in \mathbb{C}\right\}$ then for each bidiagonal matrix of the form $\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \in (AB)\{1,3,4\}$, there exist $A^{(1,3,4)}$ and $B^{(1,3,4)}$ such that $\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} = B^{(1,3,4)}A^{(1,3,4)}$.

Example 2.17 It can be shown that for $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times k}$, the condition $\min\{n,k\} \leq m$ is necessary for $(AB)\{1,3,4\} \subseteq B\{1,3,4\} \cdot A\{1,3,4\}$ to hold.

Recently, some algebraic conditions for $B\{1,3,4\} \cdot A\{1,3,4\} \subseteq (AB)\{1,3,4\}$ to hold in the matrix case were given in [27] and their generalization in the case of C^*-algebras in [28]. Here, we will present only the more general version in the C^*-algebra case.

Theorem 2.32 ([28]) *Let $a, b \in \mathscr{A}$ be such that a, b, ab, $a(1-bb^\dagger)$ and $(1-a^\dagger a)b$ are generalized invertible. Then the following conditions are equivalent:*

(i) $b\{1,3,4\} \cdot a\{1,3,4\} \subseteq (ab)\{1,3,4\}$,
(ii) $bb^\dagger a^*ab = a^*ab$ *and* $abb^*a^\dagger a = abb^*$,
(iii) $b^\dagger a^\dagger = (ab)^\dagger$.

Proof (i) $\Rightarrow$ (ii) Since $b^\dagger a^\dagger \in (ab)\{1,3,4\}$, from $abb^\dagger a^\dagger ab = ab$ it follows that $a_1a_1^*d^\dagger a_1 = a_1$ i.e., $a_2a_2^*d^\dagger a_1 = 0$. Similarly, from $(abb^\dagger a^\dagger)^* = abb^\dagger a^\dagger$, we get that $a_1a_1^*d^\dagger$ is Hermitian. Now, $d^\dagger a_1a_1^*a_1 = a_1$, i.e., $a_2a_2^*a_1 = 0$. Multiplying the last equality by $a_2^\dagger$ from the left side, we get $a_2^*a_1 = 0$ which is equivalent to $bb^\dagger a^*ab = a^*ab$. Similarly, we get that $abb^*a^\dagger a = abb^*$.

(ii) $\Rightarrow$ (i) The condition $bb^\dagger a^*ab = a^*ab$ is equivalent to $a_2^*a_1 = 0$, i.e., $a_1^*a_2 = 0$. Let $s = a_1a_1^\dagger$. Since $d \in s\mathscr{A}s + (1-s)\mathscr{A}(1-s)$, we have that $d^\dagger \in s\mathscr{A}s + (1-s)\mathscr{A}(1-s)$. Now, $a_1^*d^\dagger a_2 \in \mathscr{A}s \cdot (s\mathscr{A}s + (1-s)\mathscr{A}(1-s)) \cdot (1-s)\mathscr{A} = \{0\}$. Hence, $a_1^*d^\dagger a_2 = 0$, i.e., $a_2^*d^\dagger a_1 = 0$.
For arbitrary $a^{(1,3,4)}$ and $b^{(1,3,4)}$ we have that

$$abb^{(1,3,4)}a^{(1,3,4)}ab = \begin{bmatrix} a_1a_1^*d^\dagger a_1b & 0 \\ 0 & 0 \end{bmatrix}_{r,q}.$$

Since

$$a_1a_1^*d^\dagger a_1 = (d - a_2a_2^*)d^\dagger a_1 = a_1,$$

it follows that $abb^{(1,3,4)}a^{(1,3,4)}ab = ab$. To prove that $abb^{(1,3)}a^{(1,3)}$ is Hermitian it is sufficient to prove that $a_1a_1^*d^\dagger$ is Hermitian and $a_1(z_2 - a_1^*d^\dagger a_1z_2) = 0$. By computation, we get that $a_1z_1 = a_1a_1^*d^\dagger = a_1a_1^*(a_1a_1^*)^\dagger$ which is Hermitian. Also,

$$a_1z_2 - a_1a_1^*d^\dagger a_1z_2 = (a_1 - (d - a_2a_2^*)d^\dagger a_1)z_2 = 0.$$

Hence, $b\{1, 3, 4\} \cdot a\{1, 3, 4\} \subseteq (ab)\{1, 3\}$. Similarly, the condition $abb^*a^\dagger a = abb^*$ implies that $b\{1, 3, 4\} \cdot a\{1, 3, 4\} \subseteq (ab)\{1, 4\}$, so (i) holds.

(i) $\Rightarrow$ (iii) : It is sufficient to prove that $b^\dagger a^\dagger$ is an outer inverse of ab. That is equivalent to $b^\dagger a_1^*d^\dagger a_1a_1^*d^\dagger = b^\dagger a_1^*d^\dagger$ which holds since $a_1^*d^\dagger a_1a_1^* = a_1^*$.

(iii) $\Rightarrow$ (ii) : The proof of this part follows directly from the proof of the part (i) $\Rightarrow$ (ii). □

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