A Uniform Algebraic Approach to Cut Elimination via Semi-completeness

Hiroakira Ono

Abstract This is an attempt to present a uniform algebraic framework for semantical approach to cut elimination. The basic idea came from a paper by S. Maehara of 1991. Using the notion of *quasi-homomorphisms* essentially due to Maehara, an algebraic condition called *semi-completeness* of a given sequent system is introduced. It is shown that for a given sequent system **S**, semi-completeness for the system $\mathbf{S}^-$ implies cut elimination for **S**, where $\mathbf{S}^-$ is obtained from **S** by deleting cut rule. In the present paper it is confirmed that many of existing semantical proofs of cut elimination using either Kripke semantics or algebraic one will fall into our algebraic framework. In fact, semi-completeness is considered to be an intelligible algebraic criterion of cut elimination which is applicable to both single- and multiple-succedent sequent systems for wide variety of nonclassical logics, including modal logics and substructural logics. For modal logics and intuitionistic logic, connections of quasi-homomorphisms with *downward saturations*, the conditions which are used in the constructions of *canonical* Kripke models, will be clarified. On the other hand, for substructural logics, quasi-homomorphisms will be discussed in relation to *quasi-embeddings* which are crucial in algebraic approaches to cut elimination. In the last three sections, semi-completeness arguments are extended so as to cover semantical proofs of cut elimination for nonclassical *predicate logics*. This can be carried out by generalizing quasi-homomorphisms on *expanded algebraic structures*. In this way, semi-completeness will provide a unified view of comprehending various semantical approaches to cut elimination.

1 Maehara's Approach Revisited

In his paper Maehara (1991), S. Maehara gave a *semi-algebraic* proof of cut elimination for both classical and intuitionistic sequent systems for simple type theory.[1]

H. Ono (✉)
Japan Advanced Institute of Science and Technology, Nomi, Japan
e-mail: ono@jaist.ac.jp

[1]For further exposition of Maehara's semantical proof of cut elimination for second order calculi, see a recent paper by Arai (2017).

S.C.-M. Yang et al. (eds.), *Philosophical Logic: Current Trends in Asia*,
Logic in Asia: Studia Logica Library, DOI 10.1007/978-981-10-6355-8_2

Based on his idea, we present here a uniform algebraic framework for semantical approaches to cut elimination. To clarify the essence of his work and to show its generality, we introduce the notion of *semi-completeness* of a given sequent system, which means the completeness with respect to *quasi-homomorphisms* instead of homomorphisms (or valuations).[2]

In the next section, the notion of semi-completeness is introduced and it is shown that semi-completeness for a cut-free system implies cut elimination. To see how our idea will work, we will discuss semi-completeness of sequent systems for both modal logic **S4** and intuitionistic logic in Sects. 3 and 4, respectively, comparing it with standard semantical proof using Kripke semantics of cut elimination. It will be pointed out that required properties of quasi-homomorphisms follow exactly from conditions of *downward saturation* in *canonical Kripke models*. Then, by taking the *complex algebras* of Kripke frames, standard semantical methods for cut elimination using Kripke frames can be naturally incorporated into our algebraic framework. On the other hand, semi-completeness of sequent systems for basic substructural logics will be discussed in relation to *quasi-embeddings* in their algebraic proof of cut elimination in our joint paper Belardinelli et al. (2004) with F. Belardinelli and P. Jipsen. As a matter of fact, quasi-embeddings which came also from Maehara (1991) can be regarded as an alternative way of presenting quasi-homomorphisms.

The last three sections will be devoted to show how we can extend these arguments about semi-completeness to predicate logics. For substructural predicate logics, this was essentially done in Belardinelli et al. (2004) in terms of quasi-embeddings. On the other hand, there may be some difficulties in transforming semantical proofs of cut elimination using Kripke semantics into algebraic ones, when a predicate logic under consideration is complete with respect to *Kripke frames with varying domains*, like intuitionistic predicate logic and modal predicate logic for **S4**. To overcome this we will introduce *expanded algebraic structures* and *general quasi-homomorphisms* on them in Sect. 8. Then, semi-completeness with respect to expanded algebraic structures is shown to work well in these cases.

In this way, semi-completeness arguments using (general) quasi-homomorphisms are shown to cover many of existing standard semantical methods of cut elimination for various sequent systems, and thus we can conclude that they will provide a uniform algebraic framework for semantical proofs of cut elimination. This will answer a question which we posed in my previous paper Ono (2015). We note here that in Lahav and Avron (2014), authors introduced a semantical framework using Kripke-type semantics, in order to discuss related problems. It is noticed in Sect. 6 in Lahav and Avron (2014) that some obstacles may happen in treating single-succedent sequent systems like systems for substructural logics.

[2]See Footnote 3.

2 Semi-completeness

We will explain Maehara's idea in Maehara (1991) firstly in the context of modal propositional logics. We take $\wedge$, $\vee$, $\neg$ and $\Box$ for logical connectives of our language, and use $\cap, \cup, '$ and $\Box$ for expressing corresponding operations of modal algebras. In the following, Ω_M denotes the set of all formulas of this language. We say a nonempty subset Z of Ω_M is *subformula-closed* when for any subformula β of α if $\alpha \in Z$ then $\beta \in Z$.

Definition 1 (*partial homomorphisms on modal algebras*) Let $\mathbf{A} = \langle A, \cap, \cup, ', \Box \rangle$ be any modal algebra. A mapping f from a subformula-closed subset Z of Ω_M to A is a *partial homomorphism* on $\mathbf{A}$ if it satisfies the following conditions.

1. $f(\alpha \wedge \beta) = f(\alpha) \cap f(\beta)$ for $\alpha \wedge \beta \in Z$,
2. $f(\alpha \vee \beta) = f(\alpha) \cup f(\beta)$ for $\alpha \vee \beta \in Z$,
3. $f(\neg\alpha) = f(\alpha)'$ for $\neg\alpha \in Z$,
4. $f(\Box\alpha) = \Box f(\alpha)$ for $\Box\alpha \in Z$.

When $Z = \Omega_M$, the mapping f is simply called a *homomorphism* (on $\mathbf{A}$).[3]

The set Z in Definition 1 is called the *domain* of f. The following will be almost trivial. For, it is enough to take an arbitrary element in A for each propositional variable which does not belong to the domain of a given partial homomorphism.

Lemma 1 *Every partial homomorphism to an algebra* $\mathbf{A}$ *can be extended to a homomorphism to* $\mathbf{A}$.

The notion of partial homomorphisms can be generalized in the following way.

Definition 2 (*quasi-homomorphisms on modal algebras*) Let $\mathbf{A} = \langle A, \cap, \cup, ', \Box \rangle$ be any modal algebra. A *quasi-homomorphism* on $\mathbf{A}$ is a pair of mappings k and K from a subformula-closed subset Z of Ω_M to A, which satisfies the following conditions.

1. $k(\alpha) \le K(\alpha)$ for $\alpha \in Z$,
2. $k(\alpha \wedge \beta) \le k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \le K(\alpha \wedge \beta)$ for $\alpha \wedge \beta \in Z$,
3. $k(\alpha \vee \beta) \le k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \le K(\alpha \vee \beta)$ for $\alpha \vee \beta \in Z$,
4. $k(\neg\alpha) \le K(\alpha)'$ and $k(\alpha)' \le K(\neg\alpha)$ for $\neg\alpha \in Z$,
5. $k(\Box\alpha) \le \Box k(\alpha)$ and $\Box K(\alpha) \le K(\Box\alpha)$ for $\Box\alpha \in Z$.

Mappings k and K are sometimes called the *lower* and the *upper* mappings of this quasi-homomorphism.

[3]The word *valuation* in place of *homomorphism* might be suitable. But, for further generalization, here we call it homomorphism, considering Ω_M as a freely generated modal algebra. By the same reason, we use the word *quasi-homomorphism* instead of *quasi-valuation* in Definition 2 given below.

The set Z in Definition 2 is called also the *domain* of (k, K). It is easily shown that if $k(\alpha) = K(\alpha)$ for each $\alpha \in Z$ for a quasi-homomorphism (k, K) with the domain Z, then k (and hence K) is a partial homomorphism from Z to $\mathbf{A}$. Because of this, sometimes we identify a partial homomorphism f with a quasi-homomorphism (f, f). We note that we cannot expect the result corresponding to Lemma 1 for quasi-homomorphisms. By using induction on the complexity of a formula α, the following can be shown easily.

Lemma 2 *Suppose that both a partial homomorphism f and a quasi-homomorphism (k, K) are mappings from Z to an algebra $\mathbf{A}$ such that $k(p) \leq f(p) \leq K(p)$ for each propositional variable $p \in Z$. Then $k(\alpha) \leq f(\alpha) \leq K(\alpha)$ for every formula $\alpha \in Z$.*

The following result plays a crucial role in the proof of the main theorem of Maehara (1991).

Theorem 3 *For formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, if*

$$g(\alpha_1) \cap \ldots \cap g(\alpha_m) \leq g(\beta_1) \cup \ldots \cup g(\beta_n) \tag{1}$$

holds for every homomorphism g on an algebra $\mathbf{A}$, then

$$k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq K(\beta_1) \cup \ldots \cup K(\beta_n) \tag{2}$$

holds for every quasi-homomorphism (k, K) from any domain Z to $\mathbf{A}$ such that all formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ belong to Z.

Proof For any given quasi-homomorphism (k, K), take a partial homomorphism f from Z to $\mathbf{A}$ satisfying $k(p) \leq f(p) \leq K(p)$ for each variable $p \in Z$. (For instance, let $f(p) = k(p)$ for each $p \in Z$). By Lemma 2, the inequality $k(\gamma) \leq f(\gamma) \leq K(\gamma)$ holds for every formula $\gamma \in Z$, and thus in particular for each α_i and β_j. Take an arbitrary homomorphism f^* to $\mathbf{A}$ which is an extension of f, whose existence is assured by Lemma 1. By our assumption,

$$f^*(\alpha_1) \cap \ldots \cap f^*(\alpha_m) \leq f^*(\beta_1) \cup \ldots \cup f^*(\beta_n)$$

holds. Thus $k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq f^*(\alpha_1) \cap \ldots \cap f^*(\alpha_m) \leq f^*(\beta_1) \cup \ldots \cup f^*(\beta_n) \leq K(\beta_1) \cup \ldots \cup K(\beta_n)$ holds. □

Note that in (2) we understand that $1 \leq K(\beta_1) \cup \ldots \cup K(\beta_n)$ when $m = 0$ and that $k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq 0$ when $n = 0$. We follow the same conventions for g in (1).

For a given sequent system $\mathbf{S}$ for a normal modal logic $\mathbf{M}$, by an $\mathbf{M}$-algebra we mean any modal algebra such that any formula α which is provable in $\mathbf{S}$ is valid in it. By standard argument using Lindenbaum algebras, we have that a formula α is provable in $\mathbf{M}$ if and only if it is valid in any $\mathbf{M}$-algebra. In particular, soundness of a modal logic $\mathbf{M}$ can be expressed as follows.

Lemma 4 *Suppose that* **S** *is a sequent system for a given modal logic* **M**. *If a sequent* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ *is provable in* **S**, *then* $g(\alpha_1) \cap \ldots \cap g(\alpha_m) \leq g(\beta_1) \cup \ldots \cup g(\beta_n)$ *holds for any* **M**-*algebra* **A** *and any homomorphism* g *on* **A**.

Combining this with Theorem 3, we have the following.

Corollary 5 *Suppose that* **S** *is a sequent system for a given modal logic* **M**. *If a sequent* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ *is provable in* **S**, *then*

$$k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq K(\beta_1) \cup \ldots \cup K(\beta_n)$$

holds for every quasi-homomorphism (k, K) *from any domain* Z *to* **A** *such that all formulas* $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ *belong to* Z.

Suppose that sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$ are both provable in **S**. But it may happen that $\alpha \Rightarrow \gamma$ is not provable in **S** if cut rule is not admissible in **S**. Corresponding to this, though $k(\alpha) \leq K(\beta)$ and $k(\beta) \leq K(\gamma)$ hold for a given (k, K) to an algebra **A** by the above corollary, we are not sure whether $k(\alpha) \leq K(\gamma)$ holds or not. When the converse of Corollary 5 holds for a sequent system **S**, we say that **S** is *semi-complete*.

Definition 3 (*semi-completeness for modal logics*) A sequent system **S** is *semi-complete* with respect to a class $\mathscr{C}$ of modal algebras, when for all formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, if the inequality

$$k(\alpha_1) \cap \ldots \cap k(\alpha_m) \leq K(\beta_1) \cup \ldots \cup K(\beta_n) \tag{3}$$

holds for each **M**-algebra $\mathbf{A} \in \mathscr{C}$ and each quasi-homomorphism (k, K) on **A** with a domain Z such that $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in Z$, then the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is provable in **S**.

When **S** is semi-complete with respect to a singleton set $\{\mathbf{A}\}$, we say simply that **S** is semi-complete with respect to the algebra **A**. By Corollary 5 with Definition 3, we have immediately the following.

Lemma 6 *Let* **S** *be a sequent system for a modal logic* **M** *and* $\mathbf{S}^-$ *be the sequent system obtained from* **S** *by deleting the cut rule. If* $\mathbf{S}^-$ *is semi-complete with respect to a class* $\mathscr{C}$ *of* **M**-*algebras, then cut elimination holds for* **S**.

Sometimes the following contrapositive form of semi-completeness may be more convenient. That is, a sequent system **S** is *semi-complete* with respect to $\mathscr{C}$, whenever

> *if a sequent* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ *is not provable in* **S**, *then there exists* (k, K) *on some modal algebra* $\mathbf{A} \in \mathscr{C}$ *such that*
>
> $$k(\alpha_1) \cap \ldots \cap k(\alpha_m) \not\leq K(\beta_1) \cup \ldots \cup K(\beta_n). \tag{4}$$

If such a homomorphism g is found as (k, K) in (4), or more precisely, if (g, g) satisfies (4) for a mapping g, the condition implies completeness of **S** with respect to $\mathscr{C}$. As a matter of fact, semi-completeness with respect to $\mathscr{C}$ means completeness of S with respect to quasi-homomorphisms on algebras in $\mathscr{C}$. The main reason why we focus on quasi-homomorphims in the context of cut elimination comes from the fact that in many cases it is much easier to find a necessary quasi-homomorphim than to give a required homomorphism in a definite form.

3 Quasi-homomorphisms and Downward Saturations

To explain how Lemma 6 on semi-completeness can be applied, we take here a sequent system **GS4** for the modal logic **S4** as an example. The following proof is obtained essentially by *algebraizing* a semantical proof of cut elimination for **GS4** using Kripke frames, which is shown e.g. in Ono (2015). Our sequent system **GS4** is obtained from the system **LK** for classical logic by adding the following two rules $(\Box \Rightarrow)$ and $(\Rightarrow \Box)$ for $\Box$;

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box \Rightarrow) \qquad\qquad \frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}\ (\Rightarrow \Box)$$

To simplify our arguments, each sequent is supposed to be of the form $\Sigma \Rightarrow \Theta$ where Σ and Θ are finite (possibly empty) sets of formulas. Thus, our system has neither exchange rules nor contraction rules. Let **GS4**$^-$ be the system **GS4** without cut rule. In the following, we will show the semi-completeness of **GS4**$^-$ in a stronger form. That is, we will give *uniformly* an **S4**-algebra **B** and a quasi-homomorphism (k, K) from the set Ω_M of all modal formulas to **B** (hence both k and K are total mappings) such that for every sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$, if it is not provable in **GS4**$^-$ then the corresponding inequality (3) does not hold for the quasi-homomorphism (k, K) on this algebra **B**. For this purpose, we refer our semantical proof of cut elimination for **GS4** in Ono (2015), which is briefly sketched below.

We say that a pair (Σ, Θ) of (possibly infinite) subsets Σ and Θ of Ω_M is **GS4**$^-$-*consistent* (in Ω_M) if the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is not provable in **GS4**$^-$ for any $\alpha_1, \ldots, \alpha_m \in \Sigma$ and any $\beta_1, \ldots, \beta_n \in \Theta$. A **GS4**$^-$-consistent pair (Σ, Θ) is *maximal* **GS4**$^-$*-consistent* (in Ω_M), if it is **GS4**$^-$-consistent but neither $(\Sigma \cup \{\gamma\}, \Theta)$ nor $(\Sigma, \Theta \cup \{\gamma\})$ is **GS4**$^-$-consistent for any $\gamma \in \Omega_M \backslash (\Sigma \cup \Theta)$. It can be shown e.g. by using Zorn's lemma that each **GS4**$^-$-consistent pair (Σ_0, Θ_0) can be extended to a maximal **GS4**$^-$-consistent pair (Σ, Θ). As **GS4**$^-$ does not have cut rule, the union of $\Sigma \cup \Theta$ is not always equal to Ω_M for a maximal **GS4**$^-$-consistent pair (Σ, Θ).

Let V be the set of all maximal **GS4**$^-$-consistent pairs. Clearly, the power set $\wp(V)$ of V forms a Boolean algebra with respect to set theoretical operations $\cap, \cup$ and $'$. Further, we introduce a unary operation $\Box^\circ$ on $\wp(V)$, by defining $\Box^\circ S =$

$\{(\Sigma, \Theta) \in V : \textit{for every } (\Gamma, \Delta) \in V, \textit{ if } \Sigma_\Box \subseteq \Gamma_\Box \textit{ then } (\Gamma, \Delta) \in S\}$ for each subset S of V. Here $\Lambda_\Box$ denotes the set $\{\alpha : \Box\alpha \in \Lambda\}$ for every subset Λ of Ω_M. It is easily verified that the algebra $\mathbf{B} = \langle \wp(V), \Box^\circ \rangle$ forms an **S4**-algebra. We can show the following basic result. See e.g. Ono (2015) for the details.

Lemma 7 (downward saturation in modal logics) *The following holds for every maximal* **GS4**$^-$*-consistent pair* (Σ, Θ).

(1a) *if* $\beta \wedge \gamma \in \Sigma$ *then both* β *and* γ *are in* Σ,
(1b) *if* $\beta \wedge \gamma \in \Theta$ *then either* β *or* γ *are in* Θ,
(2a) *if* $\beta \vee \gamma \in \Sigma$ *then either* β *or* γ *are in* Σ,
(2b) *if* $\beta \wedge \gamma \in \Theta$ *then both* β *and* γ *are in* Θ,
(3a) *if* $\neg\beta \in \Sigma$ *then* β *is in* Θ,
(3b) *if* $\neg\beta \in \Theta$ *then* β *is in* Σ,
(4a) *if* $\Box\beta \in \Sigma$ *then* $\beta \in \Gamma$ *for each* $(\Gamma, \Delta) \in V$ *such that* $\Sigma_\Box \subseteq \Gamma_\Box$,
(4b) *if* $\Box\beta \in \Theta$ *then* $\beta \in \Delta$ *for some* $(\Gamma, \Delta) \in V$ *such that* $\Sigma_\Box \subseteq \Gamma_\Box$.

Next, we introduce mappings k and K on $\mathbf{B}$ by $k(\alpha) = \{(\Sigma, \Theta) \in V : \alpha \in \Sigma\}$ and $K(\alpha) = \{(\Sigma, \Theta) \in V : \alpha \notin \Theta\}$ for each $\alpha \in \Omega_M$. As our proof of the following lemma shows, every condition for downward saturation is reflected exactly by a condition for quasi-homomorphisms through mappings k and K.

Lemma 8 *The pair* (k, K) *is a quasi-homomorphism from* Ω_M *to* $\mathbf{B}$.

Proof First, we show that $k(\alpha) \subseteq K(\alpha)$. If $(\Sigma, \Theta) \in k(\alpha)$, then $\alpha \in \Sigma$ and hence α cannot be a member of Θ because of the **GS4**$^-$-consistency of (Σ, Θ). Thus, $(\Sigma, \Theta) \in K(\alpha)$. The fact that k and K satisfy remaining conditions of quasi-homomorphisms can be derived by Lemma 7. To see this, let us give only proofs of the following two conditions $k(\Box\alpha) \subseteq \Box^\circ k(\alpha)$ and $\Box^\circ K(\alpha) \subseteq K(\Box\alpha)$. Suppose that $(\Sigma, \Theta) \in k(\Box\alpha)$. This means that $\Box\alpha \in \Sigma$. To show that $(\Sigma, \Theta) \in \Box^\circ k(\alpha)$, take any $(\Gamma, \Delta) \in V$ such that $\Sigma_\Box \subseteq \Gamma_\Box$. By Lemma 7 (4a), $\alpha \in \Gamma$. This means that $(\Gamma, \Delta) \in k(\alpha)$. Thus, $(\Sigma, \Theta) \in \Box^\circ k(\alpha)$. Next suppose that $(\Sigma, \Theta) \notin K(\Box\alpha)$. Then $\Box\alpha \in \Theta$. By Lemma 7 (4b), $\alpha \in \Delta$ holds for some $(\Gamma, \Delta) \in V$ such that $\Sigma_\Box \subseteq \Gamma_\Box$. That is, $(\Gamma, \Delta) \notin K(\alpha)$ for some (Γ, Δ) with $\Sigma_\Box \subseteq \Gamma_\Box$. Hence, $(\Sigma, \Theta) \notin \Box^\circ K(\alpha)$. □

Lemma 9 *The sequent system* **GS4**$^-$ *is semi-complete with respect to the* **S4**-*algebra* $\mathbf{B}$.

Proof Let $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ be any sequent which is not provable in **GS4**$^-$, i.e., the pair $(\{\alpha_1, \ldots, \alpha_m\}, \{\beta_1, \ldots, \beta_n\})$ is **GS4**$^-$-consistent. Then there exists a maximal **GS4**$^-$-consistent pair (Σ, Θ) such that $\alpha_i \in \Sigma$ for each $i \leq m$ and $\beta_j \in \Theta$ for each $j \leq n$. This implies that $(\Sigma, \Theta) \in k(\alpha_1) \cap \ldots \cap k(\alpha_m)$, while $(\Sigma, \Theta) \notin K(\beta_1) \cup \ldots \cup K(\beta_n)$. Thus, the inequality (3) does not hold. □

Then we have the following from Lemma 6 together with Lemma 9.

Corollary 10 *Cut elimination holds for* **GS4**.

Semi-completeness of sequent systems for other modal logics, including **GK** and **GKT** discussed in Ono (2015), can be shown in the same way. The above argument looks quite similar to standard proof of completeness of modal logics using canonical frames. In fact, if we replace **GS4**$^-$ by **GS4** in the above proof, the algebra **B** is exactly the complex algebra of the canonical frame for **S4** and the mapping K, which is equal to k, is the *canonical valuation* on it.

To derive cut elimination, it is not necessary to show the semi-completeness with respect to a fixed algebra in a uniform way. That is, it is enough to give such an algebra and a quasi-homomorphism, *depending on* each unprovable sequent. This can be done by modifying the above proofs slightly, as shown below.

For a given sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$, let $\Omega^\sharp$ be the set of all subformulas of formulas in this sequent. We will restrict our attention only to formulas in $\Omega^\sharp$. Let $V^\sharp$ be the set of all maximal **GS4**$^-$-consistent pairs in $\Omega^\sharp$. As $\Omega^\sharp$ is finite, so is $V^\sharp$. We can carry on the rest of arguments in the same as before, only by replacing Ω by $\Omega^\sharp$ and V by $V^\sharp$. (See Sect. 1.5 of Ono (2015) for the details.) In this way, we can get a *finite* **S4**-algebra $\mathbf{B}^\sharp$ and a quasi-homomorphism $(k^\sharp, K^\sharp)$ such that the inequality (3) does not hold for these formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$. Hence, we have the following in place of Lemma 9.

Lemma 11 *The sequent system* **GS4**$^-$ *is semi-complete with respect to the set of all finite* **S4**-*algebras.*

This implies the finite model property of the modal logic **S4**. In fact, if a formula α is not provable in **S4**, then the sequent $\Rightarrow \alpha$ is not provable in **GS4**$^-$. By Lemma 11 together with Theorem 3, there exist a finite **S4**-algebra **A** and a homomorphism f on **A** such that $1 \not\leq f(\alpha)$. This means the finite model property of **S4**.

4 A Semi-complete Sequent System for Intuitionistic Logic

The above proof of cut elimination via semi-completeness can be applied to sequent systems for some other logics which are Kripke complete. To see this, in this section we will outline a proof of cut elimination for a sequent system **LJ**$'$ for intuitionistic logic, which is sometimes called **G3im**. The system **LJ**$'$ is a multiple-succedent system obtained from the sequent system **LK** for classical logic by restricting the right implication rule to the following form;

$$\frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)$$

We assume that our language consists of primitive logical connectives $\wedge, \vee, \to$ and a constant symbol 0. As usual, the negation $\neg\alpha$ of α is defined by $\alpha \to 0$. For convenience sake, the constant 0 is regarded as a subformula of any formula. Let Ω_I be the set of all formulas in this language. As before, we suppose that each sequent is of the form $\Sigma \Rightarrow \Theta$ with finite sets Σ and Θ of formulas. For this language, we need to modify the definition of quasi-homomorphisms slightly, as shown below.

Definition 4 (*quasi-homomorphisms on Heyting algebras*) Let $\mathbf{A} = \langle A, \cap, \cup, \rightarrow, 0_A \rangle$ be any Heyting algebra.[4] Here, 0_A denotes the smallest element of A. A *quasi-homomorphism* on $\mathbf{A}$ is a pair of mappings k and K from a subformula-closed subset Z of Ω_I to A which satisfy the following conditions.

1. $k(\alpha) \leq K(\alpha) \quad \text{for } \alpha \in Z$,
2. $k(\alpha \wedge \beta) \leq k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \leq K(\alpha \wedge \beta) \quad \text{for } \alpha \wedge \beta \in Z$,
3. $k(\alpha \vee \beta) \leq k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \leq K(\alpha \vee \beta) \quad \text{for } \alpha \vee \beta \in Z$,
4. $k(0) = 0_A$,
5. $k(\alpha \rightarrow \beta) \leq K(\alpha) \rightarrow k(\beta)$ and $k(\alpha) \rightarrow K(\beta) \leq K(\alpha \rightarrow \beta) \quad \text{for } \alpha \rightarrow \beta \in Z$.

Now, it is easy to see that statements which correspond to Theorem 3 and Corollary 5 hold also for the sequent system $\mathbf{LJ}'$ and any Heyting algebra $\mathbf{A}$. The semi-completeness of $\mathbf{LJ}'^-$, i.e. $\mathbf{LJ}'$ without cut rule, can be shown similarly, again with the help of Kripke frames as follows. Let W be the set of all maximal $\mathbf{LJ}'^-$-consistent pairs. We introduce a binary relation $\leq$ of W by the condition that $(\Sigma, \Theta) \leq (\Gamma, \Delta)$ if and only if $\Sigma \subseteq \Gamma$. Clearly, $\leq$ is a preorder on W. Now, let $\mathscr{U}(W)$ is the set of all upward closed subsets of W with respect to $\leq$. By standard arguments, we can show that the algebra $\mathbf{C}$ defined by $\langle \mathscr{U}(W), \cap, \cup, \rightarrow, \emptyset \rangle$ forms a Heyting algebra. Here, for $S, T \in \mathscr{U}(W)$, the set $S \rightarrow T$ is defined by $\{(\Sigma, \Theta) :$ *if* $(\Gamma, \Delta) \in S$ *then* $(\Gamma, \Delta) \in T$ *for every* (Γ, Δ) *such that* $(\Sigma, \Theta) \leq (\Gamma, \Delta)\}$. The set $S \rightarrow T$ can be shown to be a member of $\mathscr{U}(W)$. Corresponding to Lemma 7, we have the following.

Lemma 12 (downward saturation in intuitionistic logic) *The following holds for every* (Σ, Θ) *of* W.

(1a) *if* $\beta \wedge \gamma \in \Sigma$ *then both* β *and* γ *are in* Σ,
(1b) *if* $\beta \wedge \gamma \in \Theta$ *then either* β *or* γ *are in* Θ,
(2a) *if* $\beta \vee \gamma \in \Sigma$ *then either* β *or* γ *are in* Σ,
(2b) *if* $\beta \wedge \gamma \in \Theta$ *then both* β *and* γ *are in* Θ,
(3) $0 \notin \Sigma$,
(4a) *if* $\beta \rightarrow \gamma \in \Sigma$, *then either* $\beta \in \Delta$ *or* $\gamma \in \Gamma$ *for each* $(\Gamma, \Delta) \in W$ *such that* $\Sigma \subseteq \Gamma$,
(4b) *if* $\beta \rightarrow \gamma \in \Theta$, *then* $\beta \in \Gamma$ *and* $\gamma \in \Delta$ *for some* $(\Gamma, \Delta) \in W$ *such that* $\Sigma \subseteq \Gamma$.

Proof We will give here a proof of (4a) and (4b). Suppose that $\beta \rightarrow \gamma \in \Sigma$. Moreover, suppose that there exists $(\Gamma, \Delta) \in W$ such that $\Sigma \subseteq \Gamma$ such that $\beta \notin \Delta$ or $\gamma \notin \Gamma$. Then, both $(\Gamma, \Delta \cup \{\beta\})$ and $(\{\gamma\} \cup \Gamma, \Delta)$ are $\mathbf{LJ}'^-$-inconsistent, which means that both $\Gamma \Rightarrow \Delta, \beta$ and $\gamma, \Gamma \Rightarrow \Delta$ are provable in $\mathbf{LJ}'^-$. Hence, $\beta \rightarrow \gamma, \Gamma \Rightarrow \Delta$ must be provable in it. But as (Γ, Δ) is $\mathbf{LJ}'^-$-consistent, $\beta \rightarrow \gamma$ cannot be a member of Γ. On the other hand, $\beta \rightarrow \gamma \in \Sigma \subseteq \Gamma$. This is a contradiction.

[4]By abuse of symbols, we use $\rightarrow$ for both logical connective and algebraic operation.

Suppose next that $\beta \to \gamma \in \Theta$. Then, $(\Sigma \cup \{\beta\}, \{\gamma\})$ must be $\mathbf{LJ}'^{-}$-consistent. For, otherwise $\Sigma \Rightarrow \beta \to \gamma$ is provable by using the right implication rule of $\mathbf{LJ}'$ and hence (Σ, Θ) is $\mathbf{LJ}'^{-}$-inconsistent, which contradicts our assumption. Now, take a maximal $\mathbf{LJ}'^{-}$-consistent pair (Γ, Δ) which is an extension of $(\Sigma \cup \{\beta\}, \{\gamma\})$. Then $\Sigma \subseteq \Gamma$, $\beta \in \Gamma$ and $\gamma \in \Delta$. □

As before, we define k and K by $k(\alpha) = \{(\Sigma, \Theta) \in W : \alpha \in \Sigma\}$ and $K(\alpha) = \{(\Sigma, \Theta) \in W : \alpha \notin \Theta\}$. Then, by using Lemma 12, it is easily seen that the following holds.

Lemma 13 *The pair (k, K) is a quasi-homomorphism on the Heyting algebra* $\mathbf{C}$.

In addition, we can show a result similar to Lemma 11. Thus, we have the following.

Theorem 14 *The sequent system $\mathbf{LJ}'^{-}$ is semi-complete with respect not only to the Heyting algebra $\mathbf{C}$, but also to set of all finite Heyting algebras. Thus, cut elimination holds for $\mathbf{LJ}'$.*

5 Quasi-homomorphisms and Quasi-embeddings

We have shown that semantical proofs of cut elimination in Ono (2015) using Kripke frames can be transformed into algebraic proofs of cut elimination through semi-completeness. This was carried out quite similarly to the proof of algebraic completeness of a given modal logic $\mathbf{M}$ by using the complex algebra of canonical frame of $\mathbf{M}$ when $\mathbf{M}$ is canonical.

In this section, we will explain that algebraic proofs of cut elimination developed in Belardinelli et al. (2004) can be treated also within the present framework, and in fact they can be regarded as algebraic proofs of cut elimination via semi-completeness of sequent systems. As shown below, required quasi-homomorphisms in our proofs can be derived directly from *quasi-embedding* in Theorem 5.3 of Belardinelli et al. (2004). This will make an interesting contrast with proofs in previous sections, since arguments developed in Belardinelli et al. (2004) has a close resemblance to *MacNeille completions* while arguments in the previous sections to *canonical extensions* (see e.g. Galatos et al. (2007) for the details).

We will show semi-completeness of the sequent system $\mathbf{FL_{ew}}$ for substructural logic with exchange rule and weakening rule, as an example, by following an algebraic proof of cut elimination for $\mathbf{FL_{ew}}$ described in details in Belardinelli et al. (2004).[5] We are going to outline the proof of semi-completeness of $\mathbf{FL_{ew}}$, by giving a brief sketch of the corresponding proof in Belardinelli et al. (2004), with a slight modification of expressions and statements, when necessary, in order to adjust

[5] The very basic idea of algebraic proofs of cut elimination in Belardinelli et al. (2004) came partly from Maehara (1991), as mentioned in it.

them to the present paper. Semi-completeness for other sequent systems discussed in Belardinelli et al. (2004) can be also obtained in the same way. Thus, our algebraic approach can cover both single- and multiple-succedent sequent systems for various nonclassical logics, including modal logics and substructural logics. (See Galatos et al. (2007) for general information on substructural logics.)

The language for the sequent system $\mathbf{FL_{ew}}$ is obtained from the language for $\mathbf{LJ}'$ in the previous section by adding a logical connective *fusion* $\cdot$ and a constant symbol 1. Each sequent of $\mathbf{FL_{ew}}$ is of the form $\Sigma \Rightarrow \theta$ with a *multiset* Σ of formulas and a formula θ. Both Σ and θ may possibly be empty.

The sequent system $\mathbf{FL_{ew}}$ is obtained from the single succedent sequent system **LJ** for intuitionistic logic, by deleting contraction rule but having the following rules for fusion:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \delta}{\alpha \cdot \beta, \Gamma \Rightarrow \delta}\ (\cdot \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta}\ (\Rightarrow \cdot)$$

For constant symbols *zero* 0 and *unit* 1, we assume the following initial sequents and rules: (1) $0 \Rightarrow$, (2) $\Rightarrow 1$,

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow 0}\ (\Rightarrow 0) \qquad \frac{\Gamma \Rightarrow \delta}{1, \Gamma \Rightarrow \delta}\ (1 \Rightarrow)$$

In the following, Ω_S denotes the set of all formulas in this language. Again for convenience sake, both constant symbols 0 and 1 are regarded as subformulas of every formula.

Definition 5 ($\mathbf{FL_{ew}}$-*algebras*) An algebra $\mathbf{A} = \langle A, \cap, \cup, \cdot, \rightarrow, 1\rangle$ is a *commutative residuated lattice* if it satisfies the following[6]:

1. $\langle A, \cap, \cup\rangle$ is a lattice,
2. $\langle A, \cdot, 1\rangle$ is a commutative monoid with the unit element 1,
3. $x \cdot y \leq z$ iff $x \leq (y \rightarrow z)$, for any $x, y, z \in A$.

An algebra $\mathbf{A} = \langle A, \cap, \cup, \cdot, \rightarrow, 0, 1\rangle$ is an $\mathbf{FL_{ew}}$-*algebra*, if $\langle A, \cap, \cup, \cdot, \rightarrow, 1\rangle$ is a commutative residuated lattice such that $0 \leq x \leq 1$ holds for every $x \in A$.

It is easy to see that for each sequent, it is provable in the sequent system $\mathbf{FL_{ew}}$ if and only if it is valid in every $\mathbf{FL_{ew}}$-algebra. Similarly to Definition 4, we define quasi-homomorphisms to an arbitrary $\mathbf{FL_{ew}}$-algebra as follows.

Definition 6 (*quasi-homomorphisms on residuated lattices*)Let $\mathbf{A} = \langle A, \cap, \cup; , \rightarrow, 0_A, 1_A\rangle$ be any $\mathbf{FL_{ew}}$-algebra. A *quasi-homomorphism* from Ω_S to $\mathbf{A}$ is a pair of mappings k and K from a nonempty, subformula-closed subset Z of Ω_S to A which satisfy the following conditions.

[6] Again, we use the same symbol $\cdot$ for both fusion and monoid operation.

1. $k(\alpha) \leq K(\alpha)$ for $\alpha \in Z$,
2. $k(\alpha \wedge \beta) \leq k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \leq K(\alpha \wedge \beta)$ for $\alpha \wedge \beta \in Z$,
3. $k(\alpha \vee \beta) \leq k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \leq K(\alpha \vee \beta)$ for $\alpha \vee \beta \in Z$,
4. $k(\alpha \cdot \beta) \leq k(\alpha) \cdot k(\beta)$ and $K(\alpha) \cdot K(\beta) \leq K(\alpha \cdot \beta)$ for $\alpha \cdot \beta \in Z$,
5. $k(0) = 0_A$ and $K(1) = 1_A$,
6. $k(\alpha \to \beta) \leq K(\alpha) \to k(\beta)$ and $k(\alpha) \to K(\beta) \leq K(\alpha \to \beta)$ for $\alpha \to \beta \in Z$.

Because of our language Ω_S and of the interpretation of commas in antecedents of sequents by the monoid operation $\cdot$, it is necessary to modify the definition of semi-completeness in the following way (when the system is a single-succedent one).

Definition 7 (*semi-completeness for substructural logics*) A sequent system **S** for a substructural logic is *semi-complete* with respect to a class $\mathscr{C}$ of residuated lattices, when for all formulas $\alpha_1, \ldots, \alpha_m, \beta$, if the inequality

$$k(\alpha_1) \cdot \ldots \cdot k(\alpha_m) \leq K(\beta) \tag{5}$$

holds for each residuated lattice $\mathbf{A} \in \mathscr{C}$ and each quasi-homomorphism (k, K) on $\mathbf{A}$ with a domain Z such that $\alpha_1, \ldots, \alpha_m, \beta \in Z$, then the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is provable in **S**.

In the following, U denotes the set of all (finite, possibly empty) multisets of elements of Ω_S. The empty multiset is denoted by ε, and the multiset-union of Σ and Γ is denoted by $\Sigma \cdot \Gamma$, or also by Σ, Γ, when no confusions may occur for every Σ and $\Gamma \in U$. Clearly, $\mathbf{U} = \langle U, \cdot, \varepsilon \rangle$ is a commutative monoid with the unit element ε. From this $\mathbf{U}$, we can construct a special $\mathbf{FL_{ew}}$-algebra $\mathbf{C_U}$ as shown in Belardinelli et al. (2004). We describe it briefly in the following. Let us take any *closure operator* C on $\wp(U)$, i.e. a unary function C which satisfies that for all $X, Y \in \wp(U)$,

1. $X \subseteq C(X)$,
2. $CC(X) \subseteq C(X)$,
3. $X \subseteq Y$ implies $C(X) \subseteq C(Y)$,
4. $C(X) * C(Y) \subseteq C(X * Y)$.

Here, $*$ is defined by $S * T = \{\Sigma \cdot \Gamma : \Sigma \in S \text{ and } \Gamma \in T\}$ for $S, T \in \wp(U)$. A subset X of U is *C-closed* if $C(X) = X$. Let $C(\wp(U))$ denote the set of all C-closed subsets. Define operations $\sqcup$, $\otimes$ and $\mapsto$ on $C(\wp(U))$ as follows. For all C-closed sets X and Y:

- $X \sqcup Y = C(X \cup Y)$,
- $X \otimes Y = C(X * Y)$,
- $X \mapsto Y = \{\Gamma \in U : \{\Gamma\} * X \subseteq Y\}$.

Lemma 3.2 in Belardinelli et al. (2004) says that the algebra $\mathbf{C_U} = \langle C(\wp(U)), \cap, \sqcup, \otimes, \mapsto, O, C(\{\varepsilon\}) \rangle$ is a commutative residuated lattice with the greatest element U and the least element $C(\emptyset)$. Moreover, it is a *complete* algebra, i.e. both infinite meet

$\bigcap_{i \in I} X_i$ and infinite join $\bigsqcup_{i \in I} X_i$ always exist for an arbitrary subset $\{X_i : i \in I\}$ of $C(\wp(U))$.

Now, we will introduce a special closure operator C^* on $\wp(U)$. For each Σ and θ, where Σ is a member of U and θ is either a formula or empty, $\Sigma \lhd \theta$ means that the sequent $\Sigma \Rightarrow \theta$ is provable in $\mathbf{FL}^-_{\mathbf{ew}}$, i.e. $\mathbf{FL}_{\mathbf{ew}}$ without cut rule, and $[\Sigma; \theta]$ denotes the set $\{\Gamma \in U : \Sigma, \Gamma \lhd \theta\}$. For each $X \subseteq U$, a subset $C^*(X)$ of U is defined to be the intersection of all subsets Y of U of the form $[\Sigma; \theta]$ such that $X \subseteq Y$. Lemma 5.2 in Belardinelli et al. (2004) says that this C^* is in fact a closure operator such that $C(\{0\}) = C^*(\emptyset)$ and $C^*(\{\varepsilon\}) = U$. Now let G be the set $C^*(\wp(U))$.

Lemma 15 *The algebra* $\mathbf{G} = \langle G, \cap, \sqcup, \otimes, \mapsto, C^*(\emptyset), U\rangle$ *is an* $\mathbf{FL}_{\mathbf{ew}}$*-algebra.*

We define k and K by $k(\alpha) = C^*(\{\alpha\}) = \{\Sigma \in U : if\ \Gamma, \alpha \lhd \theta\ then\ \Gamma, \Sigma \lhd \theta$ *for all* $\Gamma, \theta\}$, and $K(\alpha) = [\varepsilon; \alpha] = \{\Sigma \in U : \Sigma \lhd \alpha\}$. Obviously, both k and K are mappings from Ω_S to G. The following lemma can be shown in parallel with *quasi-embedding theorem*, i.e. Theorem 5.3 in Belardinelli et al. (2004).

Lemma 16 *The pair* (k, K) *is a quasi-homomorphism from* Ω_S *to the* $\mathbf{FL}_{\mathbf{ew}}$*-algebra* $\mathbf{G}$.

Proof To avoid confusions, it should be remarked first that *the quasi-embedding k in* Belardinelli et al. (2004) *is expressed here by K which will be shown to be the upper quasi-homomorphism*. To confirm that (k, K) satisfies each of conditions for quasi-homo-morphisms, in the proof of Theorem 5.3 of Belardinelli et al. (2004) we replace first a, b by formulas α, β, respectively, and next we take $k(\alpha), k(\beta)$ (with k in the sense of the present paper) for U, V, respectively. Then, by Theorem 5.3 of Belardinelli et al. (2004), we have that $\alpha \wedge \beta \in k(\alpha) \cap k(\beta)$. Since $k(\alpha) \cap k(\beta)$ is a C^*-closed set and $k(\gamma)$ $(= C^*(\{\gamma\}))$ is the smallest C^*-closed set containing $\{\gamma\}$ for each γ, it follows that $k(\alpha \wedge \beta) = C^*(\{\alpha \wedge \beta\}) \subseteq k(\alpha) \cap k(\beta)$. Thus, the second conditions of quasi-homomorphisms are satisfied. Similarly for the third and fourth. To see that the sixth condition is satisfied, it is enough to take $K(\alpha), k(\beta)$ for U, V for the left inequality and to take $k(\alpha), K(\beta)$ for U, V for the right inequality. □

Theorem 17 *The sequent system* $\mathbf{FL}^-_{\mathbf{ew}}$ *is semi-complete with respect to the* $\mathbf{FL}_{\mathbf{ew}}$*-algebra* $\mathbf{G}$. *Hence, cut elimination holds for* $\mathbf{FL}_{\mathbf{ew}}$.

Proof Suppose that $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ be any sequent which is not provable in $\mathbf{FL}^-_{\mathbf{ew}}$, and moreover that $k(\alpha_1) \otimes \ldots \otimes k(\alpha_m) \subseteq K(\beta)$ holds. Since the singleton multiset $\langle \alpha_i \rangle$ belongs to $k(\alpha_i)$ for each i by the definition of k, the multiset union $\langle \alpha_1 \rangle \cdots \langle \alpha_m \rangle$ of all $\langle \alpha_i \rangle$ $(1 \leq i \leq m)$, which is equal to the multiset $\langle \alpha_1, \ldots, \alpha_m \rangle$, belongs to $k(\alpha_1) \otimes \ldots \otimes k(\alpha_m)$. Hence, $\langle \alpha_1, \ldots, \alpha_m \rangle \in K(\beta)$. This means that $\alpha_1, \ldots, \alpha_m \lhd \beta$. But, then $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ must be provable in $\mathbf{FL}^-_{\mathbf{ew}}$, which is a contradiction. Thus, $k(\alpha_1) \otimes \ldots \otimes k(\alpha_m) \nsubseteq K(\beta)$. □

Similarly, we can prove semi-completeness and cut elimination for the sequent system $\mathbf{FL}_{\mathbf{e}}$, the system obtained from $\mathbf{FL}_{\mathbf{ew}}$ by deleting weakening rule, which is for intuitionistic linear logic, and also for the single succedent sequent system **LJ** for intuitionistic logic.

Semi-completeness of sequent systems for substructural logics with respect to finite algebras also holds, though to these logics we cannot apply the method given in the previous sections. The original idea is due to Lafont (1997); Okada and Terui (1999), and a proof in the present context was described in details in Sect. 7 of Belardinelli et al. (2004), which we will sketch briefly below.

We will modify the proof in the above in the following way. Suppose that a sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$ is given. We introduce a new relation $\Sigma \lhd^\natural \theta$ for each $\Sigma \in U$ and each θ which is either a formula or empty, as follows. Recall that $\Sigma \lhd \theta$ denotes the provability of the sequent $\Sigma \Rightarrow \theta$ in $\mathbf{FL}^-_{\mathbf{ew}}$.

- For any sequent $\Sigma \Rightarrow \theta$ which *appears in the proof-search tree of the sequent* $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$, define that $\Sigma \lhd^\natural \theta$ holds if and only if $\Sigma \lhd \theta$ holds. When $\Sigma \Rightarrow \theta$ does not appear in it, we define that $\Sigma \lhd^\natural \theta$ holds always.

Using this relation $\lhd^\natural$, we define a closure operator $C^\natural$ in the same way as before. Then, it can be shown that $C^\natural(X) = U$ for all $X \subseteq U$ except finitely many X. Because of this, the algebra $\mathbf{G}^\natural$ which is constructed similarly to $\mathbf{G}$ becomes finite, and the inequality (5) fails in $\mathbf{G}^\natural$. Note that $\mathbf{G}^\natural$ is obtained dependently on a given sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta$. Thus, we have the following. The corresponding statement can be shown also for $\mathbf{FL}^-_{\mathbf{e}}$ and $\mathbf{LJ}^-$.

Lemma 18 *The sequent system* $\mathbf{FL}^-_{\mathbf{ew}}$ *is semi-complete with respect to the set of all finite* $\mathbf{FL}_{\mathbf{ew}}$*-algebras.*

Remark 19 Algebras and quasi-homomorphisms in the proof of semi-completeness of this section look totally different from those in the previous two sections. As we mentioned already at the beginning of this section, The construction of the algebra $\mathbf{G}$ in this section has a strong resemblance to MacNeille completions, while the constructions of algebras in the previous ones are to canonical extensions. Also, quasi-homomorphisms in this section are defined by using the notion of *provability*, while those in the previous sections are defined by using *consistency*, i.e. the negation of provability.

But these differences are on the surface, and as a matter of fact, there are close connections between them, as we will explain below. For comparison, we consider the case for *multiple-succeedent sequent systems*. Then, quasi-homomorphisms (k, K) defined just above Lemma 16 will be modified into the following form:

- $k(\alpha) = \{(\Sigma, \Theta) \in U^2 : \textit{if } \Gamma, \alpha \lhd \Theta, \Delta \textit{ then } \Gamma, \Sigma \lhd \Theta, \Delta \textit{ for all } (\Gamma, \Delta) \in U^2\}$,
- $K(\alpha) = \{(\Sigma, \Theta) \in U^2 : \Sigma \lhd \alpha, \Theta\}$

In the following, we assume moreover that the following two conditions hold in a given sequent system $\mathbf{S}$.

1. Structural rules, i.e. weakening, contraction and exchange rules, are admissible.
2. Every consistent pair can be extended to a maximal consistent pair.

Obviously, each sequent system for intuitionistic logic and for classical modal logics satisfies these conditions. With the help of weakening rules, we can see that every inconsistent pair (Σ, Θ), i.e., each pair (Σ, Θ) for which $\Sigma \lhd \Theta$ holds, belongs always to both $k(\alpha)$ and $K(\alpha)$ for every α. Thus, it will be enough to pay attention only to *consistent pairs* in U^2.

We will focus attention further on the set M of all maximal consistent pairs, which is a subset of U^2. Now, for each formula α, let $k_M(\alpha)$ and $K_M(\alpha)$ be subsets of M obtained from $k(\alpha)$ and $K(\alpha)$, respectively, defined in the above by replacing U^2 by M. Then the following relations can be proved, which say that $k_M(\alpha)$ and $K_M(\alpha)$ thus obtained are equal exactly to those k and K, respectively, introduced in Sect. 3 and also in Sect. 4.

- $(\Sigma, \Theta) \in k_M(\alpha)$ if and only if $\alpha \in \Sigma$,
- $(\Sigma, \Theta) \in K_M(\alpha)$ if and only if $\alpha \notin \Theta$.

To show this, first suppose that $(\Sigma, \Theta) \in k_M(\alpha)$. If $\alpha \notin \Sigma$ then $\Sigma, \alpha \lhd \Theta$ holds as (Σ, Θ) is maximal consistent. Then, $\Sigma, \alpha \lhd \Theta, \Theta$ and hence $\Sigma, \Sigma \lhd \Theta, \Theta$ follows by the definition of $k_M(\alpha)$. But, the last relation implies $\Sigma \lhd \Theta$ by exchange and contraction rules, which contradicts the consistency of (Σ, Θ). Thus, α must be a member of Σ. Conversely, suppose that $\alpha \in \Sigma$. Then $\Gamma, \alpha \lhd \Theta, \Delta$ implies $\Gamma, \Sigma \lhd \Theta, \Delta$ by weakening rules. Thus, $(\Sigma, \Theta) \in k_M(\alpha)$. The second relation follows immediately from the fact that $\Sigma \lhd \alpha, \Theta$ if and only if $\alpha \notin \Theta$, when (Σ, Θ) is maximal consistent. □

6 Semi-completeness of Sequent Systems for Substructural Predicate Logics

We will discuss now how to extend semi-completeness results obtained so far to the case of predicate logics. First, we will discuss cut elimination for substructural predicate logics which are predicate extensions of sequent systems for substructural propositional logics in the previous section. Here, by the *predicate extension* of a given sequent system **S** for a propositional logic, we mean the sequent system obtained from **S** by adding the following rules for quantifiers $\forall$ and $\exists$.[7]

$$\frac{\alpha[t/x], \Gamma \Rightarrow \Delta}{\forall x\alpha, \Gamma \Rightarrow \Delta}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha[z/x], \Sigma}{\Gamma \Rightarrow \forall x\alpha, \Sigma}\ (\Rightarrow \forall)$$

$$\frac{\alpha[z/x], \Gamma \Rightarrow \Delta}{\exists x\alpha, \Gamma \Rightarrow \Delta}\ (\exists \Rightarrow) \qquad \frac{\Gamma \Rightarrow \alpha[t/x], \Sigma}{\Gamma \Rightarrow \exists x\alpha, \Sigma}\ (\Rightarrow \exists)$$

[7]When **S** is a single-succedent system, Δ contains at most one formula and Σ is empty.

Here, t is a term, x and z are individual variables, and $\alpha[z/x]$ ($\alpha[t/x]$) are the formula obtained from α by replacing all free occurrences of x in α by z (by t, respectively). Moreover, in applications of ($\Rightarrow \forall$) and ($\exists \Rightarrow$), the variable z should not occur as a free variable in the lower sequent.

In his paper Maehara (1991), the author has dealt essentially with the case for classical predicate logic. In the following, we will show how to extend our arguments about sequent systems for substructural propositional logics in Sect. 5 to those for their predicate extensions. As a matter of fact, this was done already in Belardinelli et al. (2004) for the predicate extension $\mathbf{QFL_{ew}}$ of the sequent system $\mathbf{FL_{ew}}$. So, it suffices for us to translate the arguments using the notion of quasi-embeddings in Belardinelli et al. (2004) into our framework.

For the simplicity's sake, we suppose that our language $\mathscr{L}$ has neither function symbols nor constant symbols. In our algebraic semantics for predicate logics, quantifiers $\forall$ and $\exists$ are interpreted by infinite meets and infinite joins, respectively, in a given algebra $\mathbf{A}$. To guarantee the existence of these infinite meets and infinite joins, we should take a *complete* $\mathbf{FL_{ew}}$-*algebra* $\mathbf{A}$, i.e. an $\mathbf{FL_{ew}}$-algebra in which both $\bigcap_{i \in I} a_i$ and $\bigcup_{i \in I} a_i$ exist for an arbitrary (possibly infinite) subset $\{a_i : i \in I\}$ of A.

Definition 8 (*algebraic structures*) A pair $\langle \mathbf{A}, D \rangle$ is an *algebraic structure* for the commutative, integral substructural predicate logic, if $\mathbf{A}$ is a complete $\mathbf{FL_{ew}}$-algebra and D is an nonempty set.

The above set D is called an *individual domain*. To define our interpretation, we introduce first a new constant symbol $\hat{d}$ for each $d \in D$ and take the language $\mathscr{L}[D]$ obtained from $\mathscr{L}$ by adding every $\hat{d}$ for $d \in D$. Let $\Omega_{QS[D]}$ be the set of all first-order sentences in the language $\mathscr{L}[D]$. Here a *first-order sentence* means a first-order formula containing no free variables. For convenience's sake, every formula of the form $\alpha[\hat{d}/x]$ for $d \in D$ is regarded as a subformula of both $\forall x\alpha$ and $\exists x\alpha$.

Definition 9 (*partial homomorphisms on algebraic structures*) For a given algebraic structure $\langle \mathbf{A}, D \rangle$ for the commutative, integral substructural predicate logic, a mapping f from a subformula-closed subset Z of $\Omega_{QS[D]}$ to A (the underlying set of $\mathbf{A}$) is a *partial homomorphism* on $\langle \mathbf{A}, D \rangle$ if it satisfies the following conditions.

1. $f(\alpha \wedge \beta) = f(\alpha) \cap f(\beta)$ for $\alpha \wedge \beta \in Z$,
2. $f(\alpha \vee \beta) = f(\alpha) \cup f(\beta)$ for $\alpha \vee \beta \in Z$,
3. $f(\alpha \cdot \beta) = f(\alpha) \cdot f(\beta)$ for $\alpha \cdot \beta \in Z$,
4. $f(\alpha \to \beta) = f(\alpha) \to f(\beta)$ for $\alpha \to \beta \in Z$,
5. $f(0) = 0_{\mathbf{A}}$ and $f(1) = 1_{\mathbf{A}}$,
6. $f(\forall x\alpha) = \bigcap\{f(\alpha[\hat{d}/x]) : d \in D\}$ for $\forall x\alpha \in Z$,
7. $f(\exists x\alpha) = \bigcup\{f(\alpha[\hat{d}/x]) : d \in D\}$ for $\exists x\alpha \in Z$.

A triple $\langle \mathbf{A}, D; f \rangle$ with an algebraic structure $\langle \mathbf{A}, D \rangle$ and a partial homomorphism f on it is called an *algebraic model*.

Like Lemma 1, we can show that every partial homomorphism to an algebraic structure $\langle \mathbf{A}, D\rangle$ can be extended to a homomorphism to $\langle \mathbf{A}, D\rangle$. When free variables $x_1, \dots, x_n$ are all free variables in a given formula β, we say that β is *true* in an algebraic model $\langle \mathbf{A}, D; f\rangle$ whenever $f(\beta[\hat{d}_1/x_1, \dots, \hat{d}_n/x_n]) = 1$ for all $d_1, \dots, d_n \in D$. A formula β is *valid* in an algebraic structure $\langle \mathbf{A}, D\rangle$ if it is true in every algebraic model $\langle \mathbf{A}, D; f\rangle$ for any partial homomorphism f on it. Similarly, the validity of a given sequent containing free variables can be defined. It can be shown e.g. in Ono (1993) that the completeness of $\mathbf{QFL_{ew}}$ holds with respect to the class of all algebraic structures for the commutative, integral substructural predicate logic.

Similarly we can define a *quasi-homomorphism (k,K) on an algebraic structure* $\langle \mathbf{A}, D\rangle$ as a pair of mappings k and K from a nonempty, subformula-closed subset Z of $\Omega_{QS[D]}$ to A by extending Definition 6 as follows.

7. $k(\forall x\alpha) \subseteq \bigcap\{k(\alpha[\hat{d}/x]) : d \in D\}$ and $\bigcap\{K(\alpha[\hat{d}/x]) : d \in D\} \subseteq K(\forall x\alpha)$ for $\forall x\alpha \in Z$,
8. $k(\exists x\alpha) \subseteq \bigcup\{k(\alpha[\hat{d}/x]) : d \in D\}$ and $\bigcup\{K(\alpha[\hat{d}/x]) : d \in D\} \subseteq K(\exists x\alpha)$ for $\exists x\alpha \in Z$.

Semi-completeness of a sequent system for a substructural predicate logic is defined in the same way as Definition 7, simply by replacing "residuated lattices" by "algebraic structures (which are pairs of complete residuated lattices and individual domains)". Now, if we define mappings k and K in the same way as in Sect. 5, we can show that the pair (k, K) satisfies also above conditions 7 and 8, as shown essentially in Lemma 6. 1 of Belardinelli et al. (2004) which extends quasi-embedding theorem to predicate logics. Thus. we have the following.

Theorem 20 *The sequent system* $\mathbf{QFL^-_{ew}}$ *is semi-complete. Hence, cut elimination holds for* $\mathbf{QFL_{ew}}$.

The method can be applied easily for other predicate extensions of sequent systems which were discussed in Belardinelli et al. (2004).

7 A Semantical Proof of Cut Elimination for Intuitionistic Predicate Logic

In this section we show first that proofs developed in our Sects. 2 and 3, i.e. proofs of cut elimination based on Kripke frames, can be also extended to predicate extensions. On the other hand, since Kripke frames used in these proofs are those with *varying domains*, constructing algebraic structures using complex algebras will not work. Thus, there seem to be a certain obstacle of getting an alternative algebraic proof from our semantical proof obtained in this section. Nevertheless, as we show in the next section, the proof in the present section can be naturally transformed into semi-completeness arguments if we generalize notions of both algebraic structures and quasi-homomorphisms.

We will consider here the predicate extension $\mathbf{QLJ}'$ of $\mathbf{LJ}'$ for intuitionistic logic. $\mathbf{QLJ}'$ is a multiple-succedent sequent system, *but in which* $(\Rightarrow \forall)$ of $\mathbf{QLJ}'$ *is restricted to a single-succedent form.*[8] Let $\mathbf{QLJ}'^-$ be the system obtained from $\mathbf{QLJ}'$ by deleting cut rule. Again, just for the simplicity's sake, we assume that our language $\mathscr{L}'$ has neither function symbols nor constant symbols. Let us take an increasing sequence of sets $\{E_m\}$ $(m \geq 0)$ of *new constant symbols* such that (1) $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ and (2) both the set E_0 and the difference $E_{n+1} \backslash E_n$ are countably infinite for every $n \geq 0$. The union $\bigcup_m E_m$ is denoted by E. The set of all first-order sentences in the language $\mathscr{L}'[E_m]$ ($\mathscr{L}'[E]$), i.e. $\mathscr{L}'$ with individuals in E_m (in E) as new constant symbols, is denoted by Ω_m (and $\Omega_{QI[E]}$, respectively).

We will give first a semantical proof of cut elimination for $\mathbf{QLJ}'$, and then show how to incorporate it into our algebraic framework. Our semantical proof of cut elimination will proceed similarly to those in Sect. 4. As before, we can define the notions of $\mathbf{QLJ}'^-$*-consistency in* Ω_m and *maximal* $\mathbf{QLJ}'^-$*-consistency in* Ω_m for each $m \geq 0$, a pair (Σ, Θ) of sets Σ and Θ of first-order sentences of Ω_m. Let Y be the set of all triples of the form $\langle \Sigma, \Theta, m \rangle$ such that (Σ, Θ) is maximal $\mathbf{QLJ}'^-$-consistent in Ω_m for some $m \geq 0$. Next we introduce a binary relation $\leq$ on Y by the condition that $\langle \Sigma, \Theta, m \rangle \leq \langle \Gamma, \Delta, n \rangle$ if and only if $m \leq n$ and $\Sigma \subseteq \Gamma$. Clearly, the relation $\leq$ is a preorder. Similarly to Lemma 12, we have the following.

Lemma 21 (downward saturation in intuitionistic predicate logic) *The following holds for any triple* $\langle \Sigma, \Theta, m \rangle$ *in* Y.

(1a) *if* $\beta \wedge \gamma \in \Sigma$ *then both* β *and* γ *are in* Σ,
(1b) *if* $\beta \wedge \gamma \in \Theta$ *then either* β *or* γ *are in* Θ,
(2a) *if* $\beta \vee \gamma \in \Sigma$ *then either* β *or* γ *are in* Σ,
(2b) *if* $\beta \wedge \gamma \in \Theta$ *then both* β *and* γ *are in* Θ,
(3) $0 \notin \Sigma$,
(4a) *if* $\beta \to \gamma \in \Sigma$, *then either* $\beta \in \Delta$ *or* $\gamma \in \Gamma$ *for each* $(\Gamma, \Delta) \in V$ *such that* $\Sigma \subseteq \Gamma$,
(4b) *if* $\beta \to \gamma \in \Theta$, *then* $\beta \in \Gamma$ *and* $\gamma \in \Delta$ *for some* $(\Gamma, \Delta) \in V$ *such that* $\Sigma \subseteq \Gamma$.
(5a) *if* $\forall x \beta \in \Sigma$, *then* $\beta[\hat{d}/x] \in \Gamma$ *for each* $d \in E_n$ *and each* $\langle \Gamma, \Delta, n \rangle \in Y$ *such that* $\langle \Sigma, \Theta, m \rangle \leq \langle \Gamma, \Delta, n \rangle$,
(6b) *if* $\exists x \beta \in \Theta$, *then* $\beta[\hat{d}/x] \in \Theta$ *for each* $d \in E_m$.

But, the above properties are not enough to derive cut elimination result. We need further requirements in the following.

Definition 10 (*downward Henkin saturation*) A triple $\langle \Sigma, \Theta, m \rangle \in Y$ is *downward Henkin saturated* if it satisfies that

(5b) if $\forall x \beta \in \Theta$, then $\beta[\hat{d}/x] \in \Delta$ for some $d \in E_n$ and some $\langle \Gamma, \Delta, n \rangle \in Y$ such that $\langle \Sigma, \Theta, m \rangle \leq \langle \Gamma, \Delta, n \rangle$,

---

[8] Otherwise, the *axiom of constant domain* $\forall x(\alpha \vee \beta) \to (\forall x \alpha \vee \beta)$, which is not provable in intuitionistic predicate logic, becomes provable in the system, where x does not occur in β as a free variable.

(6a) if $\exists x\beta \in \Sigma$, then $\beta[\hat{d}/x] \in \Sigma$ for some $d \in E_m$.

We can show the following lemma by applying standard arguments but to the sequent system without cut rule $\mathbf{QLJ}'^{-}$.

Lemma 22 *If a pair* (Σ, Θ) *of sets of first-order sentences of* Ω_m *is* $\mathbf{QLJ}'^{-}$*-consistent in* Ω_m *then there exists a downward Henkin saturated triple* $\langle \Gamma, \Delta, m+1\rangle$ *such that* $\Sigma \subseteq \Gamma$ *and* $\Theta \subseteq \Delta$.

Let Y^+ be the set of all downward Henkin saturated triples. The pair $\langle Y^+, \leq\rangle$ is a preordered set. Define a mapping D from Y^+ to the set E by $D(\langle \Sigma, \Theta, m\rangle) = E_m$, which says that the set of individual domain for the state $\langle \Sigma, \Theta, m\rangle$ is E_m. Obviously, the triple $\langle Y^+, \leq, D\rangle$ determines a *canonical Kripke frame for intuitionistic logic with varying domains*. Note here that $\langle \Sigma, \Theta, m\rangle \leq \langle \Gamma, \Delta, n\rangle$ implies that $D(\langle \Sigma, \Theta, m\rangle) = E_m \subseteq E_n = D(\langle \Gamma, \Delta, n\rangle)$ as $m \leq n$.

We introduce a valuation v on this Kripke frame as follows: For each n-ary predicate symbol P and each member $\langle \Sigma, \Theta, m\rangle$ of Y^+, $v(P(\hat{d}_1, \ldots, \hat{d}_n), \langle \Sigma, \Theta, m\rangle)$ is *true* if and only if $P(\hat{d}_1, \ldots, \hat{d}_n) \in \Sigma$ for all $d_1, \ldots, d_n \in E_m$. As usual, v can be extended inductively to a relation $\models$ which expresses the *truth at a state of* Y^+ in a model $\langle Y^+, \leq, D, W\rangle$. Then the following lemma holds.

Lemma 23 (partial truth lemma) *For each* $\langle \Sigma, \Theta, m\rangle \in Y^+$ *and each sentence* $\alpha \in \Omega_m$*,*

1. *if* $\alpha \in \Sigma$ *then* $\langle \Sigma, \Theta, m\rangle \models \alpha$*,*
2. *if* $\alpha \in \Theta$ *then* $\langle \Sigma, \Theta, m\rangle \not\models \alpha$*.*

Proof This is proved by induction on the complexity of the sentence α (cf. e.g. proof of Lemma 11 in Ono (2015)). We will show this when α is of the form $\forall x\beta$. First, suppose that $\forall x\beta \in \Sigma$. By Lemma 21 (5a), $\beta[\hat{d}/x] \in \Gamma$ for each $d \in E_n$ and each $\langle \Gamma, \Delta, n\rangle \in Y^+$ such that $\langle \Sigma, \Theta, m\rangle \leq \langle \Gamma, \Delta, n\rangle$. By using the hypothesis of induction, we have that for any $\langle \Gamma, \Delta, n\rangle$, if $\langle \Sigma, \Theta, m\rangle \leq \langle \Gamma, \Delta, n\rangle$ then $\langle \Gamma, \Delta, n\rangle \models \beta[\hat{d}/x]$ for each $d \in D(\langle \Gamma, \Delta, n\rangle)$. Thus, $\langle \Sigma, \Theta, m\rangle \models \forall x\beta$ by the definition of $\models$ of Kripke models for intuitionistic logic.

Next suppose that $\forall x\beta \in \Theta$. Since $\langle \Sigma, \Theta, m\rangle$ is downward Henkin saturated, $\beta[\hat{d}/x] \in \Delta$ for some $d \in E_n$ and some $\langle \Gamma, \Delta, n\rangle \in Y$ such that $\langle \Sigma, \Theta, m\rangle \leq \langle \Gamma, \Delta, n\rangle$. By using the hypothesis of induction, $\langle \Gamma, \Delta, n\rangle \not\models \beta[\hat{d}/x]$ for some $d \in D(\langle \Gamma, \Delta, n\rangle)$ and some $\langle \Gamma, \Delta, n\rangle \in Y^+$ such that $\langle \Sigma, \Theta, m\rangle \leq \langle \Gamma, \Delta, n\rangle$. This implies that $\langle \Sigma, \Theta, m\rangle \not\models \forall x\beta$. □

Suppose that an arbitrary sequent $\Pi \Rightarrow \Lambda$ is given, where both Π and Λ are finite sets of first-order formulas in the language $\mathscr{L}'$. If they contain some free variables, then we replace distinct free variables among them by distinct constant symbols of E in all formulas in Π and Λ. Let Π' and Λ' are finite sets of first-order sentences in $\Omega_{QI[E]}$ thus obtained from Π and Λ. As they are finite, we can assume that all formulas in them belong to Ω_m for some m. It is clear that $\Pi \Rightarrow \Lambda$ is provable in $\mathbf{QLJ}'^{-}$ if and only if $\Pi' \Rightarrow \Lambda'$ is provable in $\mathbf{QLJ}'^{-}$.

Now suppose that $\Pi' \Rightarrow \Lambda'$ is not provable in $\mathbf{QLJ}'^{-}$, i.e. the pair (Π', Λ') is $\mathbf{QLJ}'^{-}$-consistent in Ω_m. By Lemma 22, there exists a downward Henkin saturated triple $\langle \Gamma, \Delta, m+1 \rangle$ in Y^+ such that $\Pi' \subseteq \Gamma$ and $\Lambda' \subseteq \Delta$. By Lemma 23, $\langle \Gamma, \Delta, m+1 \rangle \models \alpha$ holds for each formula $\alpha \in \Pi'$ while $\langle \Gamma, \Delta, m+1 \rangle \not\models \beta$ holds for each $\beta \in \Lambda'$. Therefore, $\Pi' \Rightarrow \Lambda'$ is false in this model. By the soundness of intuitionistic predicate logic, $\Pi' \Rightarrow \Lambda'$ is not provable in $\mathbf{QLJ}'$. By taking a contraposition, we have the following.

Theorem 24 (cut elimination) *If a sequent $\Gamma \Rightarrow \Delta$ is provable in* $\mathbf{QLJ}'$*, it is provable in* $\mathbf{QLJ}'$ *without using cut rule.*

8 General Quasi-homomorphisms on Expanded Algebraic Structures

While a semantical proof of cut elimination for the sequent system $\mathbf{QLJ}'$ for intuitionistic predicate logic is given in the previous section, there seems to be a certain difficulty in transforming the present proof into algebraic one in such a way as we have done in Sect. 4. To overcome the difficulty, we will expand the notion of algebraic structures and interpretations on them. This idea was inspired by Part 3 Sect. 5 of the work by Dragalin (1988). (See the chapter on algebraic semantics in Gabbay et al. (2009) for the details. Also see Chap. 1 of Goldblatt (2011) in which closely related topics are discussed but in a slightly different context.) As before, an *algebraic structure for intuitionistic predicate logic* is defined to be a pair $\langle \mathbf{A}, D \rangle$ of a complete Heyting algebra $\mathbf{A}$ and a nonempty set D.

Definition 11 (*definiteness functions and expanded algebraic structures*) For a given algebraic structure $\langle \mathbf{A}, D \rangle$ for intuitionistic predicate logic, a mapping ϕ from D to A is called a *definiteness function* (or, simply a *d-function*) on it, if it satisfies $\bigcup\{\phi(d) : d \in D\} = 1_{\mathbf{A}}$. Any triple $\langle \mathbf{A}, D, \phi \rangle$ with an algebraic structure $\langle \mathbf{A}, D \rangle$ with a d-function ϕ on it is called an *expanded algebraic structure* for intuitionistic predicate logic.

In Dragalin (1988), $\phi(d)$ is denoted by $\|d\|$ and is read as *measure of definiteness* of an individual object d. Each expanded algebraic structure can be regarded as a simplified form of *Heyting-valued structures* in Gabbay et al. (2009), where ϕ is read as *measure of existence*.

Definition 12 (*partial general homomorphisms*) For a given expanded algebraic structure $\langle \mathbf{A}, D, \phi \rangle$, a mapping f from a subformula-closed subset Z of Ω_{QI} to A is a *partial general homomorphism* on $\langle \mathbf{A}, D, \phi \rangle$ if it satisfies the following conditions.

1. $f(\alpha \wedge \beta) = f(\alpha) \cap f(\beta)$ for $\alpha \wedge \beta \in Z$,
2. $f(\alpha \vee \beta) = f(\alpha) \cup f(\beta)$ for $\alpha \vee \beta \in Z$,
3. $f(\alpha \to \beta) = f(\alpha) \to f(\beta)$ for $\alpha \to \beta \in Z$,

4. $f(0) = 0_{\mathbf{A}}$ and $f(1) = 1_{\mathbf{A}}$,
5. $f(\forall x\alpha) = \bigcap\{\phi(d) \to f(\alpha[\hat{d}/x]) : d \in D\}$ for $\forall x\alpha \in Z$,
6. $f(\exists x\alpha) = \bigcup\{\phi(d) \wedge f(\alpha[\hat{d}/x]) : d \in D\}$ for $\exists x\alpha \in Z$.

The pair (ϕ, f) of a d-function ϕ and a partial general homomorphism f can be considered also as an interpretation on an algebraic structure $\langle \mathbf{A}, D\rangle$. Because of this, the quadruple $\langle \mathbf{A}, D, \phi; f\rangle$ is sometimes called an *general algebraic model* over an algebraic structure $\langle \mathbf{A}, D\rangle$ for intuitionistic predicate logic. When $\phi(d) = 1_{\mathbf{A}}$ holds for any $d \in D$ for a given d-function in the above, any partial general homomorphism is a partial homomorphism and hence a general algebraic model can be identified with an algebraic model of the previous section.

For a formula β such that $x_1, \ldots, x_n$ are all free variables in it, we say that β is *true in a general algebraic model* $\langle \mathbf{A}, D, \phi; f\rangle$ whenever $(\phi(d_1) \cap \ldots \cap \phi(d_n)) \to f(\beta[\hat{d}_1/x_1, \ldots, \hat{d}_n/x_n]) = 1$, or equivalently $(\phi(d_1) \cap \ldots \cap \phi(d_n)) \leq f(\beta[\hat{d}_1/x_1, \ldots, \hat{d}_n/x_n])$, holds for all $d_1, \ldots, d_n \in D$. A formula β is *g-valid* if it is true in every general algebraic model, which is equivalent to say that β is true for any partial general homomorphism on every expanded algebraic structure. The *g-validity* can be defined similarly for an arbitrary sequent.

Theorem 25 (Completeness of $\mathbf{QLJ}'$ with respect to general algebraic models) *A sequent is g-valid if and only if it is provable in* $\mathbf{QLJ}'$.

Proof The only-if part follows from the completeness with respect to algebraic models, as every algebraic structure can be regarded as a particular expanded algebraic structure whose d-function ϕ takes always the value 1_A for all $d \in D$. To see that the converse direction holds, it suffices to check that the g-validity of the lower sequent follows from the g-validity of the upper sequent for each rule of $\mathbf{QLJ}'$ for quantifiers. This is almost obvious for $(\Rightarrow \forall)$ and $(\exists \Rightarrow)$.

We consider the rule $(\forall \Rightarrow)$. Suppose that $\forall x\alpha, \Gamma \Rightarrow \Delta$ is inferred from $\alpha[y/x], \Gamma \Rightarrow \Delta$. Let γ and δ be the conjunction of formulas in Γ and the disjunction of formulas in Δ, respectively. Take an arbitrary element $d \in D$. For simplicity's sake, we assume that no free variables appear in $\alpha[\hat{d}/x]$, γ and δ. From our assumption that $\alpha[y/x], \Gamma \Rightarrow \Delta$ is g-valid, $\phi(d) \to ((f(\alpha[\hat{d}/x]) \cap f(\gamma)) \to f(\delta)) = 1$, or equivalently, $\phi(d) \cap f(\alpha[\hat{d}/x]) \leq f(\gamma) \to f(\delta)$ holds for any general algebraic model $\langle \mathbf{A}, D, \phi; f\rangle$. On the other hand, it holds that $\phi(d) \cap f(\forall x\alpha) = \phi(d) \cap \bigcap\{\phi(e) \to f(\alpha[\hat{e}/x]) : e \in D\} \leq \phi(d) \cap f(\alpha[\hat{d}/x])$. Combining these two inequalities, we have that $\phi(d) \cap f(\forall x\alpha) \leq f(\gamma) \to f(\delta)$, and hence that $\phi(d) \leq (f(\forall x\alpha) \cap f(\gamma)) \to f(\delta)$ for every $d \in D$. Therefore, $1 = \bigcup\{\phi(d) : d \in D\} \leq (f(\forall x\alpha) \cap f(\gamma)) \to f(\delta)$. Thus, the lower sequent is true in the general algebraic model $\langle \mathbf{A}, D, \phi; f\rangle$.

Next, consider the rule $(\Rightarrow \exists)$, and suppose that $\Gamma \Rightarrow \Delta, \exists x\alpha$ is inferred from $\Gamma \Rightarrow \Delta, \alpha[y/x]$. Suppose that $\Gamma \Rightarrow \Delta, \alpha[y/x]$ is g-valid. Thus, $\phi(d) \cap f(\gamma) \leq f(\delta) \cup f(\alpha[\hat{d}/x])$ holds for any general algebraic model $\langle \mathbf{A}, D, \phi; f\rangle$, by using the same abbreviation and assumption in the above. Then, $\phi(d) \cap f(\gamma) \leq f(\delta) \cup (\phi(d) \cap f(\alpha[\hat{d}/x])) \leq f(\delta) \cup \bigcup\{\phi(e) \cap f(\alpha[\hat{e}/x]) : e \in D\}$. Hence, $1 = \bigcup\{\phi(d) : d \in D\} \leq f(\gamma) \to [f(\delta) \cup \bigcup\{\phi(e) \cap f(\alpha[\hat{e}/x]) : e \in D\}] = f(\gamma) \to (f(\delta) \cup f(\exists x\alpha))$.

Thus, the lower sequent is true in this general algebraic model. □

Definition 12 will suggest us how to extend the notion of quasi-homomorphisms on expanded algebraic structures.

Definition 13 (*general quasi-homomorphisms*) For a given expanded algebraic structure $\langle \mathbf{A}, D, \phi \rangle$, a pair (k, K) of mappings k and K from a nonempty, subformula-closed subset Z of $\Omega_{QI[E]}$ to A is a *general quasi-homomorphism* on $\langle \mathbf{A}, D, \phi \rangle$ if they satisfy the following conditions.

1. $k(\alpha) \leq K(\alpha)$ for $\alpha \in Z$,
2. $k(\alpha \wedge \beta) \leq k(\alpha) \cap k(\beta)$ and $K(\alpha) \cap K(\beta) \leq K(\alpha \wedge \beta)$ for $\alpha \wedge \beta \in Z$,
3. $k(\alpha \vee \beta) \leq k(\alpha) \cup k(\beta)$ and $K(\alpha) \cup K(\beta) \leq K(\alpha \vee \beta)$ for $\alpha \vee \beta \in Z$,
4. $k(0) = 0_A$,
5. $k(\alpha \to \beta) \leq K(\alpha) \to k(\beta)$ and $k(\alpha) \to K(\beta) \leq K(\alpha \to \beta)$ for $\alpha \to \beta \in Z$.
6. $k(\forall x\alpha) \leq \bigcap\{\phi(d) \to k(\alpha[\hat{d}/x]) : d \in D\}$ and $\bigcap\{\phi(d) \to K(\alpha[\hat{d}/x]) : d \in D\} \leq K(\forall x\alpha)$ for $\forall x\alpha \in Z$,
7. $k(\exists x\alpha) \leq \bigcup\{\phi(d) \cap k(\alpha[\hat{d}/x]) : d \in D\}$ and $\bigcup\{\phi(d) \cap K(\alpha[\hat{d}/x]) : d \in D\} \leq K(\exists x\alpha)$ for $\exists x\alpha \in Z$.

These inequalities in the above were discussed also in Part 3 Sect. 5 of Dragalin (1988), in which these triples of ϕ, k and K satisfying the above conditions are called *semivaluations*. Using these conditions, the author of Dragalin (1988) has derived cut elimination for $\mathbf{QLJ}'$ semantically.

It is obvious that Theorem 3 holds also between general homomorphisms and general quasi-homomorphisms. Thus the next lemma follows from Theorem 25, which corresponds to Lemma 4. Suppose that any sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ of the original language $\mathscr{L}'$ is given. Let $\mathbf{x} = x_1, \ldots, x_k$ be a fixed enumeration of distinct variables in this sequent. Suppose also that an expanded algebraic structure $\langle \mathbf{A}, D, \phi \rangle$ is given. Expressions $\alpha_i[\hat{\mathbf{d}}]$ and $\beta_i[\hat{\mathbf{d}}]$ denote formulas obtained from α_i and β_j, respectively, by replacing each free occurrence of x_h by a constant symbol $\hat{d}_h$ for $1 \leq h \leq k$, for any k-tuple $\mathbf{d}$ $(= d_1, \ldots, d_k)$ of elements of D.

Lemma 26 *For any sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ of the language $\mathscr{L}'$, if it is provable in $\mathbf{QLJ}'$ then*

$$(\phi(d_1) \cap \ldots \cap \phi(d_k)) \cap k(\alpha_1[\hat{\mathbf{d}}]) \cap \ldots \cap k(\alpha_m[\hat{\mathbf{d}}]) \leq K(\beta_1[\hat{\mathbf{d}}]) \cup \ldots \cup K(\beta_n[\hat{\mathbf{d}}])$$

holds for any expanded algebraic structure $\langle \mathbf{A}, D, \phi \rangle$ for intuitionistic predicate logic, any general quasi-homomorphism (k, K) on it and any k-tuple $\mathbf{d}$ $(= d_1, \ldots, d_k)$ of elements of D.

The semi-completeness with respect to a class of expanded algebraic structures can be defined as before, The following proof of semi-completeness of $\mathbf{QLJ}'^{-}$, i.e. $\mathbf{QLJ}'$ without cut rule, goes essentially in the same way as that in Dragalin (1988). But here we will clarify exact correspondences between downward Henkin saturated triples and general quasi-homomorphisms.

Theorem 27 (semi-completeness of $\mathbf{QLJ}'^{-}$ with respect to expanded algebraic structures) *For all formulas $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$, if the inequality*

$$(\phi(d_1) \cap \ldots \cap \phi(d_k)) \cap k(\alpha_1[\hat{\mathbf{d}}]) \cap \ldots \cap k(\alpha_m[\hat{\mathbf{d}}]) \leq K(\beta_1[\hat{\mathbf{d}}]) \cup \ldots \cup K(\beta_n[\hat{\mathbf{d}}])$$

holds for any expanded algebraic structure $\langle \mathbf{A}, D, \phi \rangle$ for intuitionistic predicate logic, any k-tuple $\mathbf{d}$ $(= d_1, \ldots, d_k)$ of elements of D and any general quasi-homomorphism (k, K) on $\langle \mathbf{A}, D, \phi \rangle$ with a domain Z such that $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in Z$, then the sequent $\alpha_1, \ldots, \alpha_m \Rightarrow \beta_1, \ldots, \beta_n$ is provable in $\mathbf{QLJ}'^{-}$.

As we have already had a semantical proof of cut elimination for $\mathbf{QLJ}'$ in the previous section, our proof of semi-completeness of $\mathbf{QLJ}'^{-}$ will go in the same way as the proof of semi-completeness of $\mathbf{LJ}'^{-}$ given in Sect. 4. Recall that we have introduced a preorder $\leq$ on the set Y^+ of all downward Henkin saturated triples. Now, let $\mathcal{U}(Y^+)$ be the set of all upward closed subsets of Y^+ with respect to $\leq$. Then the algebra $\mathbf{H}$ defined by $\langle \mathcal{U}(Y^+), \cap, \cup, \rightarrow, \emptyset \rangle$ is a complete Heyting algebra ordered by the set inclusion with the greatest element Y^+. Define a mapping ϕ from the set E of all new constant symbols to $\mathcal{U}(Y^+)$ by $\phi(d) = \{\langle \Sigma, \Theta, m \rangle : d \in E_m\}$ for each $d \in E$. It is easy to see that each $\phi(d)$ is upward closed. For each $\langle \Gamma, \Delta, n \rangle \in Y^+$, we can take an element $e \in E_n$, as E_n is always non-empty. Then, $\langle \Gamma, \Delta, n \rangle \in \phi(e)$ and hence $\bigcup \{\phi(d) : d \in E\} = Y^+ (= 1_{\mathbf{H}})$. Therefore, $\langle \mathbf{H}, E, \phi \rangle$ forms an expanded algebraic structure. Just as arguments in Sect. 4, define k and K by $k(\alpha) = \{\langle \Sigma, \Theta, m \rangle \in Y^+ : \alpha \in \Sigma\}$ and $K(\alpha) = \{\langle \Sigma, \Theta, m \rangle \in Y^+ : \alpha \notin \Theta\}$ for any $\alpha \in \Omega_{QI}$.

Lemma 28 *The pair (k, K) is a general quasi-homomorphism on the expanded algebraic structure $\langle \mathbf{H}, E, \phi \rangle$.*

Proof Conditions from 1 to 5 in Definition 13 can be shown similarly to Lemma 13 using downward saturation. For, the condition 6, suppose that $\langle \Sigma, \Theta, m \rangle \in k(\forall x\alpha)$. This means that $\forall x\alpha \in \Sigma$. Then by Lemma 21 (5a), we have that $\alpha[\hat{d}/x] \in \Gamma$ for any $d \in E_n$ and any $\langle \Gamma, \Delta, n \rangle \in Y^+$ such that $\langle \Sigma, \Theta, m \rangle \leq \langle \Gamma, \Delta, n \rangle$. In other words, for any $\langle \Gamma, \Delta, n \rangle \in Y^+$ such that $\langle \Sigma, \Theta, m \rangle \leq \langle \Gamma, \Delta, n \rangle$, if $\langle \Gamma, \Delta, n \rangle \in \phi(d)$ then $\alpha[\hat{d}/x] \in \Gamma$ for each $d \in E$. Thus, $\langle \Sigma, \Theta, m \rangle \in \bigcap \{\phi(d) \rightarrow k(\alpha[\hat{d}/x]) : d \in E\}$ by using also the hypothesis of induction. Similarly, $\bigcap \{\phi(d) \rightarrow K(\alpha[\hat{d}/x]) : d \in E\} \subseteq K(\forall x\alpha)$ follows from the first condition (5b) of downward Henkin saturation. Consider next the second inclusion in the condition 7. Taking the contraposition, assume that $\langle \Sigma, \Theta, m \rangle \notin K(\exists x\alpha)$. From the definition of K, it follows that $\exists x\alpha \in \Theta$. By Lemma 21 (6b), $\alpha[\hat{d}/x] \in \Theta$ for each $d \in E_m$. That is, if $\langle \Sigma, \Theta, m \rangle \in \phi(d)$ then $\langle \Sigma, \Theta, m \rangle \notin K(\alpha[\hat{d}/x]$ for any d. Thus $\langle \Sigma, \Theta, m \rangle \notin \bigcup \{\phi(d) \cap K(\alpha[\hat{d}/x]) : d \in E\}$ □

In this way, we can translate a semantical proof of cut elimination of $\mathbf{QLJ}'$ given in the previous section into an algebraic proof via semi-completeness. Thus cut elimination theorem for $\mathbf{QLJ}'$ follows Lemma 26 and Theorem 27.

Cut elimination for predicate extensions of sequent systems for modal logics can be shown similarly. We will explain it briefly below. An *algebraic structures for modal predicate logics* is a pair $\langle \mathbf{A}, D \rangle$ of a complete Boolean algebra with a modality $\mathbf{A}$ and a non-empty set D.

Definition 14 (*expanded algebraic structures for modal predicate logics*) A triple $\langle \mathbf{A}, D, \phi \rangle$ is an *expanded algebraic structures for modal predicate logics*, if $\langle \mathbf{A}, D \rangle$ is an algebraic structures for modal predicate logics and ϕ is a mapping from D to A satisfying (1) $\bigcup\{\phi(d) : d \in D\} = 1_{\mathbf{A}}$ and (2) $\phi(d) \leq \Box\phi(d)$ for all $d \in D$. The mapping ϕ is called a definiteness function on an algebraic structure $\langle \mathbf{A}, D \rangle$ for modal predicate logics.

We can introduce *general quasi-homomorphisms on expanded algebraic structures for modal predicate logics* similarly to Definition 13. In this case, each expanded algebraic structure $\langle \mathbf{A}, D, \phi \rangle$ for modal predicate logics is a simplified form of *modal valued structures* in Gabbay et al. (2009). The additional condition that $\phi(d) \leq \Box\phi(d)$ for all $d \in D$ is necessary to show that the g-validity is preserved by such rules as $(\Rightarrow \Box)$ in which all formulas in the antecedent of the lower sequent must be *boxed*. Conversely, this condition is verified in our *canonical Kripke frame for modal predicate logics with varying domains*, which can be defined similarly as the canonical Kripke frame for intuitionistic predicate logics with varying domains discussed above. In fact, this condition is shown to correspond to the statement that aRb implies $D(a) \subseteq D(b)$ for all a, b, which is of course true. In this way, we can give an alternative algebraic proof of cut elimination for sequent systems of various modal predicate logics, though we omit the details here.

Theorem 29 *The predicate extension of each of sequent systems* **GK**, **GKT** *and* **GS4** *is semi-complete with respect to corresponding expanded algebraic structures for modal predicate logics. Hence, cut elimination holds for each of them.*

Acknowledgements The author would like to express many thanks to C.-M. Yang and K. Sano for their helpful comments and suggestions, and also to T. Kowalski for valuable discussions.

References

Arai, T. (2017). Cut-eliminability in second order logic calculi, from https://arxiv.org/abs/1701.00929, submitted on 2017.

Belardinelli, F., Jipsen, P., & Ono, H. (2004). Algebraic aspects of cut elimination. *Studia Logica*, *77*, 209–240.

Dragalin, A. G. (1988). *Mathematical intuitionism: introduction to proof theory*: American Mathematical Society.

Gabbay, D. M., Shehtman, V., & Shehtman, D. (2009). *Quantification in nonclassical logic I* (Vol. 153)., Studies in Logic and the Foundations of Mathematics: Elsevier.

Galatos, N., Jipsen, P., Kowalski, T., & Ono, H. (2007). *Residuated lattices: an algebraic glimpse at substructural logics* (Vol. 151)., Studies in Logic and the Foundations of Mathematics: Elsevier.

Goldblatt, R. (2011). *Quantifiers, propositions and identity: admissible semantics for quantified modal and substructural logics*., Lecture Notes in Logic, Association for Symbolic Logic: Cambridge University Press.

Lafont, Y. (1997). The finite model property for various fragments of linear logic. *Journal of Symbolic Logic*, *62*, 1202–1208.

Lahav, O., & Avron, A. (2014). A unified semantic framework for fully-structural propositional sequent systems. *ACM Transactions on Computational Logic*, *165*, 26–48.

Maehara, S. (1991). Lattice-valued representation of the cut elimination theorem. *Tsukuba Journal of Mathematics*, *15*, 509–521.

Okada, M., & Terui, K. (1999). The finite model property for various fragments of intuitionistic linear logic. *Journal of Symbolic Logic*, *64*, 790–802.

Ono, H. (1993). Semantics for substructural logics. In K. Dosen & P. Schroeder-Heister (Eds.), *Substructural logics* (Vol. 2, pp. 259–291)., Studies in Logic and Computation: Oxford University Press.

Ono, H. (2015). Semantical approach to cut elimination and subformula property in modal logic. In C.-M. Yang, D.-M. Deng, & H. Lin (Eds.), *Structural Analysis of Non-Classical Logics, The Proceedings of the Second Taiwan Philosophical Logic Colloquium* (pp. 1–15)., Logic in Asia: Springer.

Springer

http://www.springer.com/978-981-10-6354-1

Philosophical Logic: Current Trends in Asia
Proceedings of AWPL-TPLC 2016
Yang, S.C.-M.; Lee, K.Y.; Ono, H. (Eds.)
2017, XVI, 296 p. 7 illus., Hardcover
ISBN: 978-981-10-6354-1