Chapter 2
Evolution Inclusions with *m*-Dissipative Operator

Abstract This chapter deals with a nonlinear delay differential inclusion of evolution type involving m-dissipative operator and source term of multivalued type in a Banach space. Under rather mild conditions, the R_δ-structure of C^0-solution set is studied on compact intervals, which is then used to obtain the R_δ-property on noncompact intervals. Secondly, the result about the structure is furthermore employed to show the existence of C^0-solutions for the inclusion (mentioned above) subject to nonlocal condition defined on right half-line. No nonexpansive condition on nonlocal function is needed. As samples of applications, we consider a partial differential inclusion with time delay and then with nonlocal condition at the end of the chapter.

2.1 Introduction

It is worth mentioning that for differential inclusions on noncompact intervals, governed by a nonlinear multivalued operator (specially, an m-dissipative operator), the research of topological structure of solution sets is much more delicate and the related results are still very rare. Furthermore, much of the previous research on differential inclusions in infinite dimensional spaces was done provided the nonlinearity (a multivalued function), with compact values, is upper semicontinuous with respect to solution variable. This condition turns out to be restrictive to some extent and is not satisfied usually in practical applications (see, e.g., Vrabie [192, Example 5.1, Example 5.2] and [189]). To make things more applicable, an appropriate alterative is that the nonlinearity, with closed and convex values, is weakly upper semicontinuous with respect to solution variable.

Throughout this section, X is a real Banach space with norm $|\cdot|$, X^* denotes the topological dual of Banach space X. Denote by $|\cdot|_0$ the sup-norm of $C([-\tau, 0], X)$. Note that $X \times C([-\tau, 0], X)$, endowed with the norm

$$|(x, v)|_\tau := \max\{|x|, |v|_0\}, \quad (x, v) \in X \times C([-\tau, 0], X),$$

is a Banach space.

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We consider the Cauchy problem of nonlinear delay differential inclusion of evolution type

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}^+, \\ f(t) \in F(t, u(t), u_t), & t \in \mathbb{R}^+, \\ u(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{2.1}$$

Here $A : D(A) \subset X \to P(X)$ is an m-dissipative operator (possible multivalued and/or nonlinear), the forcing source $F : \mathbb{R}^+ \times \overline{D(A)} \times C([-\tau, 0], \overline{D(A)}) \to P(X)$ is a multivalued function with convex, closed values, and $\phi \in C([-\tau, 0], \overline{D(A)})$. $u_t \in C([-\tau, 0], \overline{D(A)})$ is defined by $u_t(s) = u(t + s)$ ($s \in [-\tau, 0]$) for every $u \in \widetilde{C}([-\tau, \infty), \overline{D(A)})$ and $t \in \mathbb{R}^+$.

Here, we are interested in studying the topological characterization of the solution set for the Cauchy problem (2.1) in some Fréchet spaces. We first investigate the existence of C^0-solutions and R_δ-structure of the solution set for the Cauchy problem (2.1) on compact intervals, then proceed to study the R_δ-structure of the solution set for the Cauchy problem (2.1). In the proof of the latter result, the key tool is the inverse limit method.

As an application of the information about the structure, we shall deal with the C^0-solutions for the nonlocal Cauchy problem of nonlinear delay evolution inclusion of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}^+, \\ f(t) \in F(t, u(t), u_t), & t \in \mathbb{R}^+, \\ u(t) = g(u)(t), & t \in [-\tau, 0], \end{cases} \tag{2.2}$$

where A and F are defined the same as those in the problem (2.1), and

$$g : \widetilde{C}_b([-\tau, \infty), \overline{D(A)}) \to C([-\tau, 0], \overline{D(A)})$$

is a function to be specified later. As can be seen, g constitutes a nonlocal condition. It is also noted that the nonlocal function g depends on history states, that is, it takes history values. We emphasize that in the proof of our main result, no nonexpansive condition on nonlocal function g will be required.

The consideration for nonlocal initial condition g is stimulated by the observation that this type of conditions is more realistic than usual ones in treating physical problems, see, e.g., [5, 8, 82, 110, 192, 194, 197] for more detailed information about the importance of nonlocal initial conditions in applications. Some typical examples for g are

- $g(u)(t) = u(t + \omega)$ for each $t \in [-\tau, 0]$ (Periodicity condition);
- $g(u)(t) = -u(t + \omega)$ for each $t \in [-\tau, 0]$ (Anti-periodicity condition);
- $g(u)(t) = \displaystyle\int_\tau^\infty k(\theta)u^{\frac{1}{3}}(t + \theta)d\theta$ for each $t \in [-\tau, 0]$ with $k \in L^1(\mathbb{R}^+, \mathbb{R}^+)$ and $\displaystyle\int_\tau^\infty k(\theta)d\theta = 1$ (Mean value condition);

- $g(u)(t) = \sum_{i=1}^{n} \alpha_i u^{\frac{1}{3}}(t_i + t)$ for each $t \in [-\tau, 0]$, where $\sum_{i=1}^{n} |\alpha_i| \leq 1$ and $\tau < t_1 < t_2 < \cdots < t_n < \infty$ are constants (Multi-point discrete mean condition).

Remark 2.1 The final case on g above can be seen as a generalization of the nonlocal function introduced in Deng [82], where the nonlocal function is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

It is noted that by using an interplay of compactness arguments and invariance techniques, Vrabie [192] obtained an existence result of C^0-solutions to the nonlocal Cauchy problem (2.2). Similar arguments are also used to solve other nonlocal problems, we refer the reader to Paicu and Vrabie [162], Vrabie [191], Wang and Zhu [197] and references therein. However, there exists a limitation among these results, that is, it is assumed that the nonlocal function is nonexpansive. Thus, there naturally arises a question: "Is there any chance to solve this problem without this condition?". The results in Sect. 2.4 in fact gives an affirmative answer to this question and close this gap.

Remark 2.2 Let us mention that the lack of nonexpansive condition on nonlocal function prevents us from using the well-known tools such as Banach and Schauder fixed point theorems to show the existence of C^0-solutions to the nonlocal Cauchy problem (2.2). This difficulty leads us to study the topological structure of the solution set to the Cauchy problem (2.1), before applying a fixed point theorem for multivalued mappings with non-convex values.

This chapter is organized as follows. Section 2.2 gives some properties of m-dissipative operators and the definition of C^0-Solutions. Section 2.3 is devoted to the existence of C^0-solutions and R_δ-structure of the solution set for the Cauchy problem (2.1) on compact intervals. In Sect. 2.3.2, we obtain the R_δ-structure of the solution set for the Cauchy problem (2.1) on noncompact intervals by the inverse limit method. Section 2.4 is concerned with the existence of C^0-solutions to the nonlocal Cauchy problem (2.2) defined on right half-line. Finally, as an illustration of the developed theory, we apply it to the examples of partial differential inclusions defined on right half-line.

The results in this chapter are taken from Chen, Wang and Zhou [71].

2.2 The m-Dissipative Operators and C^0-Solution

Given a multivalued operator $A : D(A) \subset X \to P(X)$ with the domain $D(A)$, we let $R(A) = \bigcup_{x \in D(A)} Ax$ stand for the range of A.

Let $x, y \in X$ and $h \in \mathbb{R} \setminus \{0\}$. We put

$$[x, y]_h = \frac{|x + hy| - |x|}{h}$$

and then note that there exists the limit

$$[x, y]_+ = \lim_{h \to 0^+} [x, y]_h.$$

Furthermore, for each $x, y \in X$ and $\alpha > 0$,

$$[\alpha x, y]_+ = [x, y]_+, \quad |[x, y]_+| \leq |y|.$$

Recall that $A : D(A) \subset X \to P(X)$ is m-dissipative if $R(I - \lambda A) = X$ for all $\lambda > 0$ and A is dissipative, i.e.,

$$[x_1 - x_2, y_2 - y_1]_+ \geq 0 \text{ for all } (x_i, y_i) \in \mathrm{Gra}(A), \; i = 1, 2.$$

Consider the following evolution inclusion

$$u'(t) \in Au(t) + f(t), \tag{2.3}$$

where A is m-dissipative. By a C^0-solution of (2.3) on $[a, b]$, it will be understood an element $u \in C([a, b], X)$, $u(t) \in \overline{D(A)}$ for each $t \in [a, b]$ and u verifies

$$|u(t) - x| \leq |u(s) - x| + \int_s^t [u(\sigma) - x, f(\sigma) - y]_+ d\sigma$$

for each $(x, y) \in \mathrm{Gra}(A)$ and $a \leq s \leq t \leq b$.

From [139, Theorems 3.5.1 and 3.6.1] it follows that for each $x \in \overline{D(A)}$ and $f \in L^1([a, b], X)$, there exists an unique C^0-solution to (2.3) on $[a, b]$ which satisfies $u(a) = x$. Moreover, as proved in [31, Theorem 2.1], if $f, g \in L^1([a, b], X)$ and u, v are two C^0-solutions to (2.3) corresponding to f and g, respectively, then

$$|u(t) - v(t)| \leq |u(s) - v(s)| + \int_s^t [u(\sigma) - v(\sigma), f(\sigma) - g(\sigma)]_+ d\sigma$$

for all $a \leq s \leq t \leq b$. In particular, we see

$$|u(t) - v(t)| \leq |u(s) - v(s)| + \int_s^t |f(\sigma) - g(\sigma)| d\sigma$$

for all $a \leq s \leq t \leq b$.

Let $x \in \overline{D(A)}$, $c \in [a, b)$ and $f \in L^1([a, b], X)$. We denote by $u(\cdot, c, x, f)$ the unique C^0-solution $v : [c, b] \to \overline{D(A)}$ of (2.3) on $[c, b]$ which satisfies $v(c) = x$. Define

$$S(t) : \overline{D(A)} \to \overline{D(A)} \text{ with } S(t)x = u(t, 0, x, 0) \text{ for each } t \geq 0, \; x \in \overline{D(A)}.$$

Then it follows readily that $\{S(t)\}_{t\geq 0}$ is a semigroup of contractions on $\overline{D(A)}$ (see, e.g., Barbu [31] for more details). We say that this semigroup is generated by A.

The semigroup $\{S(t)\}_{t\geq 0}$ is called compact if $S(t)$ is a compact operator for each $t > 0$.

Definition 2.1 An m-dissipative operator $A : D(A) \subset X \to P(X)$ is called of compact type if for each $a < b$ and each sequence $\{(f_n, u_n)\}$ in $L^1([a, b], X) \times C([a, b], X)$ such that u_n is a C^0-solution on $[a, b]$ of the evolution inclusion

$$u_n'(t) \in Au_n(t) + f_n(t), \quad n = 1, 2, \ldots,$$

$f_n \rightharpoonup f$ in $L^1([a, b], X)$ and $u_n \to u$ in $C([a, b], X)$, then it follows that u is a C^0-solution on $[a, b]$ of the limit problem

$$u'(t) \in Au(t) + f(t).$$

Lemma 2.1 [189, Corollary 2.3.1]) *Let X^* be uniformly convex and A an m-dissipative operator generating a compact semigroup. Then A is of compact type.*

The following compactness result is due to Baras [30]. See also Vrabie [189, Theorem 2.3.3].

Lemma 2.2 *Let A be an m-dissipative operator generating a compact semigroup. Suppose in addition that B is a bounded set in $\overline{D(A)}$ and $\mathscr{F}$ is uniformly integrable in $L^1([a, b], X)$. Then for each $c \in (a, b)$, the C^0-solution set*

$$\{u(\cdot, a, x, f) : x \in B, f \in \mathscr{F}\}$$

is relatively compact in $C([c, b], X)$. If, in addition, B is relatively compact, then the C^0-solution set is relatively compact in $C([a, b], X)$.

Next, for each $\phi \in C([-\tau, 0], \overline{D(A)})$ and $f \in L^1([0, b], X)$, we define the mapping $S_{\phi,b} : L^1([0, b], X) \to C([-\tau, b], \overline{D(A)})$ by setting

$$S_{\phi,b}(f)(t) = \begin{cases} \phi(t), & t \in [-\tau, 0), \\ u(t, 0, \phi(0), f), & t \in [0, b]. \end{cases}$$

Clearly, $S_{\phi,b}(f)$ is the unique C^0-solution for the evolution inclusion with time delay of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in [0, b], \\ u(t) = \phi(t), & t \in [-\tau, 0]. \end{cases}$$

As an immediate consequence of Lemmas 2.1 and 2.2, we obtain the following result.

Lemma 2.3 *Let X^* be uniformly convex and A an m-dissipative operator generating a compact semigroup. Then the following results hold:*

(i) *if $\mathscr{F}$ is uniformly integrable in $L^1([0,b],X)$ and $\mathcal{B} \subset C([-\tau,0],\overline{D(A)})$ is relatively compact, then $S_{\mathcal{B},b}(\mathscr{F})$ is relatively compact in $C([-\tau,b],X)$;*

(ii) *for each sequence $\{(f_n,u_n)\}$ in $L^1([0,b],X)\times C([-\tau,b],X)$ such that $u_n = S_{\phi,b}(f_n)$, $n \geq 1$, f_n converges weakly to f and u_n converges to u, it follows that $u = S_{\phi,b}(f)$.*

2.3 Topological Structure of Solution Set

We introduce the following assumptions:

(H_0) $A : D(A) \subset X \to P(X)$ is an m-dissipative operator with $0 \in A0$ and A generates a compact semigroup. In addition, $\overline{D(A)}$ is convex and X^* is uniformly convex.

(H_1) $F : \mathbb{R}^+ \times \overline{D(A)} \times C([-\tau,0],\overline{D(A)}) \to P_{cl,cv}(X)$ is a multivalued function for which $F(t,\cdot,\cdot)$ is weakly u.s.c. for a.e. $t \in \mathbb{R}^+$ and $F(\cdot,x,v)$ has a strongly measurable selection for each $(x,v) \in \overline{D(A)} \times C([-\tau,0],\overline{D(A)})$.

(H_2) There exists $L \in L^1_{loc}(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$|F(t,x,v)| = \sup\{|f| : f \in F(t,x,v)\} \leq L(t)\,(1+|x|+|v|_0)$$

for a.e $t \in \mathbb{R}^+$ and each $(x,v) \in \overline{D(A)} \times C([-\tau,0],\overline{D(A)})$.

Define a multivalued mapping $Sel_F : C([-\tau,\infty),\overline{D(A)}) \to P(L^1_{loc}(\mathbb{R}^+,X))$ by setting

$$Sel_F(u) = \{f \in L^1_{loc}(\mathbb{R}^+,X) : f(t) \in F(t,u(t),u_t) \text{ for a.e. } t \in \mathbb{R}^+\}$$

for each $u \in C([-\tau,\infty),\overline{D(A)})$.

Remark 2.3 Let us note that if $u \in C([-\tau,T],\overline{D(A)})$, then Sel_F will be seen as a multivalued mapping from $C([-\tau,T],\overline{D(A)})$ into $L^1([0,T],X)$.

2.3.1 Compact Intervals Case

For the sake of convenience, put $J_\tau = [-\tau,0] \cup J$ with $J = [0,T]$. Let us consider the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in J, \\ f(t) \in F(t,u(t),u_t), & t \in J, \\ u(t) = \phi(t), & t \in [-\tau,0]. \end{cases} \tag{2.4}$$

The following lemma provides an useful property of Sel_F.

Lemma 2.4 *Let* (H_1) *and* (H_2) *be satisfied and let* X *be reflexive. Then* Sel_F *is weakly u.s.c. with nonempty, convex and weakly compact values.*

Proof Let us first show that $Sel_F(u) \neq \emptyset$ for each $u \in C(J_\tau, \overline{D(A)})$. For this purpose we assume that $u \in C(J_\tau, \overline{D(A)})$ and $\{(u_n, v_n)\}$ is a sequence of step functions from J to $\overline{D(A)} \times C([-\tau, 0], \overline{D(A)})$ such that

$$\sup_{t \in J} |u_n(t) - u(t)| \to 0, \quad \sup_{t \in J} |v_n(t) - u_t|_0 \to 0 \ \text{ as } n \to \infty.$$

By (H_1) we see readily that for each n, $F(\cdot, u_n(\cdot), v_n(\cdot))$ admits a strongly measurable selection $f_n(\cdot)$. Furthermore, it follows from (H_2) that $\{f_n\}$ is integrably bounded in $L^1(J, X)$. Making use of Lemma 1.24 we then see that $\{f_n\}$ is relatively weakly compact in $L^1(J, X)$. Hence, we may assume, by passing to a subsequence if necessary, that $f_n \rightharpoonup f$ in $L^1(J, X)$. An application of Mazur's theorem enables us to find that there exists a sequence $\{\tilde{f}_n\} \subset L^1(J, X)$ such that $\tilde{f}_n \in \text{co}\{f_k : k \geq n\}$ for each $n \geq 1$ and $\tilde{f}_n \to f$ in $L^1(J, X)$. Hence, $\tilde{f}_{n_k}(t) \to f(t)$ in X for a.e. $t \in J$ with some subsequence $\{\tilde{f}_{n_k}\}$ of $\{\tilde{f}_n\}$.

Denote by E the set of all $t \in J$ such that $\tilde{f}_{n_k}(t) \to f(t)$ in X and $f_n(t) \in F(t, u_n(t), v_n(t))$ for all $n \geq 1$. Let $x^* \in X^*$, $\varepsilon > 0$, and $t \in E$ be fixed. From (H_1), it follows immediately that $(x^* \circ F)(t, \cdot, \cdot) : X \to P(\mathbb{R})$ is u.s.c. with compact convex values, so $\varepsilon - \delta$ u.s.c. with compact convex values. Accordingly, we have

$$\begin{aligned} x^*(\tilde{f}_{n_k}(t)) \in \text{co}\{x^*(f_k(t)) : k \geq n\} &\subset x^*(F(t, u_n(t), v_n(t))) \\ &\subset x^*(F(t, u(t), u_t)) + (-\varepsilon, \varepsilon) \end{aligned}$$

with k large enough. Therefore, we obtain that $x^*(\tilde{f}(t)) \in x^*(F(t, u(t), u_t))$ for each $x^* \in X^*$ and $t \in E$. Since F has convex and closed values, we conclude that $f(t) \in F(t, u(t), u_t)$ for each $t \in E$, which implies that $f \in Sel_F(u)$.

In the sequel, let $\{u_n\}$ be a sequence converging to $u \in C(J_\tau, \overline{D(A)})$ and $f_n \in Sel_F(u_n)$, $n \geq 1$. Using the same argument as above, we obtain that $\{f_n\}$ is relatively weakly compact, and there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $f \in Sel_F(u)$ such that $f_{n_k} \rightharpoonup f$ in $L^1(J, X)$. This, together with Lemma 1.7 (ii), shows that Sel_F is weakly u.s.c. Also, from the arguments above it is easy to see that Sel_F has weakly compact values. Moreover, it is readily checked that Sel_F has convex values. The proof is complete.

In order to study the topological structure of solution set for the Cauchy problem (2.4), we first establish the following existence result.

Theorem 2.1 *Let* (H_0)–(H_2) *be satisfied. Then the Cauchy problem (2.4) has at least one* C^0*-solution for each* $\phi \in C([-\tau, 0], \overline{D(A)})$.

Proof Let $\phi \in C([-\tau, 0], \overline{D(A)})$. Consider the set

$$K_T = \{u \in C(J_\tau, \overline{D(A)}) : u(t) = \phi(t) \text{ for } t \in [-\tau, 0] \text{ and } |u(t)| \le x_\phi(t) \text{ for all } t \in J\},$$

where $x_\phi \in C(J, \mathbb{R}^+)$ is the unique continuous solution of the integral equation in the form

$$x_\phi(t) = |\phi|_0 + \int_0^t L(\sigma)\left(1 + 2x_\phi(\sigma)\right) d\sigma, \quad t \in J. \tag{2.5}$$

We seek for solutions in K_T. To the end, let us define a multivalued mapping W^ϕ on K_T by setting

$$W^\phi(u) = S_{\phi,T}(Sel_F(u)), \quad u \in K_T.$$

It is clear that we obtain the result if we show that the map W^ϕ admits a fixed point in K_T. Below, we shall omit the subscript "T" and write only S_ϕ instead of $S_{\phi,T}$ if there is no danger of confusion.

Observe that for every $u \in K_T$, $Sel_F(u) \neq \emptyset$ due to Lemma 2.4 and hence $W^\phi(u) \subset C(J_\tau, \overline{D(A)})$. Also, $\{v|_{[-\tau,0]} : v \in W^\phi(u)\} = \{\phi\}$ for all $u \in K_T$. Moreover, taking $f \in Sel_F(u)$ with $u \in K_T$, it follows from (H_2) that for every $t \in J$,

$$\begin{aligned}
|S_\phi(f)(t)| \le & |\phi(0)| + \int_0^t |f(\sigma)| d\sigma \\
\le & |\phi(0)| + \int_0^t L(\sigma)\,(1 + |u(\sigma)| + |u_\sigma|_0)\, \mathrm{d}\sigma \\
\le & |\phi|_0 + \int_0^t L(\sigma)\left(1 + 2x_\phi(\sigma)\right) \mathrm{d}\sigma \\
= & x_\phi(t).
\end{aligned}$$

Here, we have tacitly used the condition $0 \in A0$ and the fact $|u_t|_0 \le x_\phi(t)$ for every $t \in J$ and $u \in K_T$. Hence, it is proved that $W^\phi(u) \subset K_T$ for every $u \in K_T$.

We process to verify that W^ϕ is u.s.c. on K_T. Due to Lemma 1.9, it suffices to prove that W^ϕ is quasi-compact and closed. By (H_2) we obtain that for all $f \in \mathscr{F} := Sel_F(K_T)$,

$$|f(t)| \le L(t)(1 + 2x_\phi(T)) \text{ for a.e. } t \in J, \tag{2.6}$$

which implies that $\mathscr{F}$ is integrably bounded and thus uniformly integrable. From this and Lemma 2.3 (i) we see that $W^\phi(K_T)(= S_\phi(\mathscr{F}))$ is relatively compact in $C(J_\tau, X)$. This in particular implies that W^ϕ is quasi-compact.

Let $\{(u_n, v_n)\}$ be a sequence in $\mathrm{Gra}(W^\phi)$ such that $(u_n, v_n) \to (u, v)$ in $C(J_\tau, X) \times C(J_\tau, X)$. Since $v_n \in W^\phi(u_n)$, there exists a sequence $\{f_n\} \subset L^1(J, X)$ satisfying $f_n \in Sel_F(u_n)$ and $v_n = S_\phi(f_n)$. Therefore, noticing that Sel_F is weakly u.s.c. with convex, weakly compact values due to Lemma 2.4, an application of Lemma 1.7 (ii) yields that there exists $f \in Sel_F(u)$ and a subsequence of $\{f_n\}$, still denoted by $\{f_n\}$, such that $f_n \rightharpoonup f$ in $L^1(J, X)$. From this and Lemma 2.3 (ii) we see that $v = S_\phi(f)$ and then $v \in W^\phi(u)$. It follows that W^ϕ is closed.

Consider the set

$$\mathscr{K}_T = \overline{\text{co}}(\, W^{\phi}(K_T)),$$

the closed convex hull of $W^{\phi}(K_T)$. Clearly, $\mathscr{K}_T$ is a compact, convex set in $C(J_\tau, X)$ and $W^{\phi}(\mathscr{K}_T) \subset \mathscr{K}_T$.

Below, we shall prove that W^{ϕ} has a fixed point in $\mathscr{K}_T$. Due to Theorem 1.17, it suffices to show that W^{ϕ} has compact, contractible values. Given $u \in \mathscr{K}_T$, it is easy to see that $W^{\phi}(u)$ is compact because of the closedness and qusi-compactness of W^{ϕ}. Fix $f^* \in Sel_F(u)$ and put $u^* = S_\phi(f^*)$. Define a function $H : [0, 1] \times W^{\phi}(u) \to W^{\phi}(u)$ by setting

$$H(\lambda, v)(t) = \begin{cases} v(t), & t \in [-\tau, \lambda T], \\ u(t, \lambda T, v(\lambda T), f^*), & t \in (\lambda T, T] \end{cases}$$

for each $(\lambda, v) \in [0, 1] \times W^{\phi}(u)$, where $u(\cdot, \lambda T, v(\lambda T), f^*)$, as prescribed in Sect. 2.2, is the unique C^0-solution of the evolution inclusion in the form

$$\begin{cases} u'(t) \in Au(t) + f^*(t), \quad t \in [\lambda T, T], \\ u(\lambda T) = v(\lambda T). \end{cases}$$

What followed is to show that $H(\lambda, v) \in W^{\phi}(u)$ for each $(\lambda, v) \in [0, 1] \times W^{\phi}(u)$. Note that for each $v \in W^{\phi}(u)$, there exists $\tilde{f} \in Sel_F(u)$ such that $v = S_\phi(\tilde{f})$. Put

$$\hat{f}(t) = \tilde{f}(t)\chi_{[0,\lambda T]}(t) + f^*(t)\chi_{(\lambda T,T]}(t) \text{ for each } t \in J.$$

It is clear that $\hat{f} \in Sel_F(u)$. Also, it is readily checked that $S_\phi(\hat{f})(t) = v(t)$ for all $t \in [-\tau, \lambda T]$ and $S_\phi(\hat{f})(t) = u(t, \lambda T, v(\lambda T), f^*)$ for all $t \in (\lambda T, T]$, which gives $S_\phi(\hat{f}) = H(\lambda, v)$ and hence $H(\lambda, v) \in W^{\phi}(u)$.

To show that $W^{\phi}(u)$ is contractible, we first note that

$$H(0, v) = u^* \text{ and } H(1, v) = v \text{ for every } v \in W^{\phi}(u).$$

It remains to show that H is continuous. Given $(\lambda_i, v_i) \in [0, 1] \times W^{\phi}(u)$, $i = 1, 2$, with $\lambda_1 \le \lambda_2$, we can choose $f_i \in Sel_F(u)$ such that $H(\lambda_i, v_i) = S_\phi(f_i)$ and $f_i(t) = f^*(t)$ for all $t \in [\lambda_i T, T]$. Then, we have that for $-\tau \le s \le t \le T$,

$$\begin{aligned} |H(\lambda_1, v_1)(t) - H(\lambda_2, v_2)(t)| \le & |H(\lambda_1, v_1)(s) - H(\lambda_2, v_2)(s)| \\ & + \int_s^t |f_1(\sigma) - f_2(\sigma)| d\sigma. \end{aligned}$$

Noticing (2.6) and the fact $f_1(t) = f_2(t)$ for $t \in [\lambda_2 T, T]$, we see that for all $t \in [\lambda_1 T, T]$,

$$
\begin{aligned}
&|H(\lambda_1, v_1)(t) - H(\lambda_2, v_2)(t)| \\
\leq &|H(\lambda_1, v_1)(\lambda_1 T) - H(\lambda_2, v_2)(\lambda_1 T)| + \int_{\lambda_1 T}^{\lambda_2 T} |f_1(\sigma) - f_2(\sigma)| d\sigma \\
\leq &|H(\lambda_1, v_1)(\lambda_1 T) - H(\lambda_2, v_2)(\lambda_1 T)| + \big(2 + 4x_\phi(T)\big) \int_{\lambda_1 T}^{\lambda_2 T} L(\sigma) d\sigma,
\end{aligned}
$$

which combining with the fact that $H(\lambda_i, v_i)(t) = v_i(t)$ for all $t \in [-\tau, \lambda_i T]$ yields

$$
\sup_{t \in J_\tau} |H(\lambda_1, v_1)(t) - H(\lambda_2, v_2)(t)| \leq \|v_1 - v_2\| + \big(2 + 4x_\phi(T)\big) \int_{\lambda_1 T}^{\lambda_2 T} L(\sigma) d\sigma.
$$

The continuity of H follows immediately.

Finally, an application of Theorem 1.17 yields that W^ϕ has at least one fixed point, which is a C^0-solution of the Cauchy problem (2.4). This completes the proof.

In the sequel,we denote by $\Sigma_{\phi,T}^F$ the solution set of the Cauchy problem (2.4), i.e.,

$$
\begin{aligned}
\Sigma_{\phi,T}^F = \{u \in C(J_\tau, \overline{D(A)}) : & u \text{ is the} C^0\text{-solution of (2.4)} \\
& \text{satisfying } u(t) = \phi(t) \text{ for } t \in [-\tau, 0]\},
\end{aligned}
$$

and, by $\hat{K}_T$ the set

$$
\hat{K}_T = \{u \in C(J_\tau, \overline{D(A)}) : u(t) = \phi(t), \quad t \in [-\tau, 0]\}.
$$

Let $\mathrm{Fix}(W^\phi)$ be the fixed point set of W^ϕ acting on K_T, where K_T and W^ϕ were introduced in Theorem 2.1. We present the following characterization.

Lemma 2.5 *Let the hypotheses in Theorem 2.1 hold. Then* $\Sigma_{\phi,T}^F = \mathrm{Fix}(W^\phi)$ *and* $\Sigma_{\phi,T}^F$ *is compact in* $C(J_\tau, X)$ *for each* $\phi \in C([-\tau, 0], \overline{D(A)})$.

Proof Let $\phi \in C([-\tau, 0], \overline{D(A)})$ and let x_ϕ be the unique continuous solution of (2.5). Along the same line with the proof of Theorem 2.1, we define a mapping $\hat{W}^\phi$ on $\hat{K}_T$ by

$$
\hat{W}^\phi(u) = S_\phi(Sel_F(u)), \quad u \in \hat{K}_T,
$$

which is regarded as an extension of W^ϕ. Observe that $\Sigma_{\phi,T}^F = \mathrm{Fix}(\hat{W}^\phi)$. Below, it will be sufficient to show that $u \in K_T$ whenever $u \in \mathrm{Fix}(\hat{W}^\phi)$. Taking $u \in \mathrm{Fix}(\hat{W}^\phi)$, it follows that there exists $f \in Sel_F(u)$ such that $u = S_\phi(f)$. Then, noticing (H_2) and the condition $0 \in A0$ and using the same arguments as in the proof of Theorem 2.1 one can show

$$
|u_t|_0 \leq |\phi|_0 + \int_0^t L(\sigma)\,(1 + 2|u_\sigma|_0)\,\mathrm{d}\sigma, \quad t \in J.
$$

With the aid of the generalized Gronwall-Bellman's inequality we obtain that for each $t \in J$,

$$
\begin{aligned}
|u_t|_0 \leq & |\phi|_0 + \int_0^t L(\sigma)d\sigma \\
& + 2\int_0^t L(s)\left(|\phi|_0 + \int_0^s L(\sigma)d\sigma\right)\exp\left(2\int_s^t L(\sigma)d\sigma\right)ds \\
= & x_\phi(t),
\end{aligned}
$$

which implies that $u \in K_T$. Based on the considerations above, we have $\Sigma^F_{\phi,T} = \mathrm{Fix}(W^\phi)$.

Moreover, as in the proof of Theorem 2.1, $\mathscr{K}_T$ is compact in $C(J_\tau, X)$ and W^ϕ is closed, from this we see that $\mathrm{Fix}(W^\phi)$ is a compact set in $\mathscr{K}_T$, so is $\Sigma^F_{\phi,T}$. The proof is complete.

We present the following approximation result.

Lemma 2.6 *Put $\mathscr{D} = \overline{D(A)} \times C([-\tau, 0], \overline{D(A)})$. Suppose that F satisfies the hypotheses (H_1) and (H_2). Then there exists a sequence of multivalued functions $\{F_n\}$ with $F_n : J \times \mathscr{D} \to P_{cl,cv}(X)$ such that*

(i) *$F(t, x, v) \subset F_{n+1}(t, x, v) \subset F_n(t, x, v) \subset \overline{\mathrm{co}}(F(t, B_{3^{1-n}}(x, v) \cap \mathscr{D}))$, $n \geq 1$, for each $t \in J$, $(x, v) \in \mathscr{D}$;*
(ii) *$|F_n(t, x, v)| \leq L(t)(3 + |x| + |v|_0)$, $n \geq 1$, for a.e. $t \in J$ and each $(x, v) \in \mathscr{D}$;*
(iii) *there exists $\mathscr{T} \subset J$ with $\mathrm{mes}(\mathscr{T}) = 0$ such that for each $x^* \in X^*$, $\varepsilon > 0$ and $(t, x, v) \in J\backslash\mathscr{T} \times \mathscr{D}$, there exists $N > 0$ such that for all $n \geq N$,*

$$
x^*(F_n(t, x, v)) \subset x^*(F(t, x, v)) + (-\varepsilon, \varepsilon);
$$

(iv) *$F_n(t, \cdot) : \mathscr{D} \to P_{cl,cv}(X)$ is continuous for a.e. $t \in J$ with respect to Hausdorff metric for each $n \geq 1$;*
(v) *for each $n \geq 1$, there exists a selection $G_n : J \times \mathscr{D} \to X$ of F_n such that $G_n(\cdot, x, v)$ is strongly measurable for each $(x, v) \in \mathscr{D}$ and for any compact subset $\mathscr{D}' \subset \mathscr{D}$ there exist constants $C_V > 0$ and $\delta > 0$ for which the estimate*

$$
|G_n(t, x_1, v_1) - G_n(t, x_2, v_2)| \leq C_V L(t)(|x_1 - x_2| + |v_1 - v_2|_0) \tag{2.7}
$$

holds for a.e. $t \in J$ and each $(x_1, v_1), (x_2, v_2) \in V$ with $V := (\mathscr{D}' + B_\delta(0)) \cap \mathscr{D}$;
(vi) *F_n verifies the condition (H_1) with F_n instead of F for each $n \geq 1$, provided that X is reflexive.*

Proof Put $r_n = 3^{-n}$, $n \geq 1$. For each $n \geq 1$, let $\{B_{r_n}(x, v)\}_{(x,v)\in\mathscr{D}}$ be an open cover of $\mathscr{D}$. Therefore, there exists a locally finite refinement $\{V_{j,n}\}_{j\in I_n}$ of $\{B_{r_n}(x, v)\}_{(x,v)\in\mathscr{D}}$. For each $j \in I_n$, we can choose $y_{j,n} := (x_{j,n}, v_{j,n}) \in \mathscr{D}$ such that $V_{j,n} \subset B_{r_n}(y_{j,n})$.

Now let $\{p_{j,n}(x,v)\}_{j\in I_n}$ be a locally Lipschitz partition of unity subordinated to the open cover $\{V_{j,n}\}_{j\in I_n}$. For each $n \geq 1$, define

$$F_n(t,x,v) = \overline{\sum_{j\in I_n} p_{j,n}(x,v)\overline{\mathrm{co}}(F(t, B_{2r_n}(y_{j,n}) \cap \mathscr{D}))}, \quad (t,x,v) \in J \times \mathscr{D},$$

and

$$G_n(t,x,v) = \sum_{j\in I_n} p_{j,n}(x,v) g_{j,n}(t), \quad (t,x,v) \in J \times \mathscr{D},$$

where $g_{j,n}(\cdot)$ is a strongly measurable selection of $F(\cdot, y_{j,n})$ for each $j \in I_n$.

With the preparation above at hand, the assertions (i), (iv) and (v) can be proved by the same kind of manipulations as in [106, Theorem 3.5] (see also [80, Lemma 2.2]). The assertion (ii) is an immediate consequence of (i) and (H_2).

We process to prove the assertion (iii). Let $\mathscr{T}$ be the set of all $t \in J$ such that both $F(t,\cdot,\cdot) : \mathscr{D} \to P_{cl,cv}(X)$ is weakly u.s.c. and $F(t,x,v)$ verifies the condition (H_2) for all (t,x,v) with $(x,v) \in \mathscr{D}$. Given $y = (x,v) \in \mathscr{D}$, we put $I_n^y = \{j \in I_n : p_{j,n}(y) > 0\}$, which is a finite set due to the local finiteness of the cover $\{V_{j,n}\}_{j\in I_n}$. It is readily checked that

$$j \in I_n^y \text{ implies } y \in B_{r_n}(y_{j,n}), \quad F_n(t,y) = \overline{\sum_{j\in I_n^y} p_{j,n}(y)\overline{\mathrm{co}}(F(t, B_{2r_n}(y_{j,n}) \cap \mathscr{D}))} \tag{2.8}$$

and hence $|z-y|_\tau < 3r_n$ for each $j \in I_n^y$ and $z \in B_{2r_n}(y_{j,n})$, which gives $B_{2r_n}(y_{j,n}) \subset B_{3r_n}(y)$.

Let $x^* \in X^*$, $\varepsilon > 0$ and $t \in \mathscr{T}$ be fixed. From (H_1) it follows immediately that $(x^* \circ F)(t,\cdot,\cdot) : \mathscr{D} \to 2^{\mathbb{R}}$ is u.s.c. and thus ε-δ u.s.c. That is, there exists $\delta > 0$ such that for all $z \in B_\delta(y) \cap \mathscr{D}$,

$$x^*(F(t,z)) \subset x^*(F(t,y)) + \left(-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right). \tag{2.9}$$

Selecting N large enough so that $n \geq N$ implies $3r_n \leq \delta$, we conclude from (2.9) that

$$x^*(F(t, B_{2r_n}(y_{j,n}) \cap \mathscr{D})) \subset x^*(F(t,y)) + \left(-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right) \text{ for each } n \geq N \text{ and } j \in I_n^y. \tag{2.10}$$

On the other hand, since $x^*(F(t,y))$ is convex due to (H_1), we obtain

$$\begin{aligned}\overline{\mathrm{co}}\left(x^*(F(t,y)) + \left(-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right)\right) &= \overline{co\left(x^*(F(t,y))\right) + \left(-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right)} \\ &\subset x^*(F(t,y)) + \left(-\frac{2\varepsilon}{3}, \frac{2\varepsilon}{3}\right),\end{aligned}$$

whence (2.10) gives

$$x^*\left(\overline{\mathrm{co}}(F(t, B_{2r_n}(y_{j,n}) \cap \mathscr{D}))\right) = \overline{\mathrm{co}}(x^*(F(t, B_{2r_n}(y_{j,n}) \cap \mathscr{D}))) \subset x^*(F(t, y)) + \left(-\frac{2\varepsilon}{3}, \frac{2\varepsilon}{3}\right)$$

for each $n \geq N$ and $j \in I_n^y$. We thus use (2.8) to obtain that for all $n \geq N$,

$$x^*(F_n(t, y)) \subset \overline{\mathrm{co}}\Big(x^*(F(t, y)) + \Big(-\frac{2\varepsilon}{3}, \frac{2\varepsilon}{3}\Big)\Big) \subset x^*(F(t, y)) + (-\varepsilon, \varepsilon).$$

This proves the assertion (iii).

It remains to verify the assertion (vi). Let $n \geq 1$ be fixed and $\mathscr{T}'$ the set of all $t \in J$ such that both $F_n(t, \cdot, \cdot) : \mathscr{D} \to P_{cl,cv}(X)$ is continuous with respect to Hausdorff metric and $F_n(t, x, v)$ verifies the inequality in the assertion (ii) for all (t, x, v) with $(x, v) \in \mathscr{D}$. Clearly, $J \setminus \mathscr{T}'$ has null measure and $F_n(t, \cdot, \cdot)$ is ε-δ u.s.c. for each $t \in \mathscr{T}'$. From the reflexivity of X it follows that $F_n(t, \cdot, \cdot)$ has weakly compact values for each $t \in \mathscr{T}'$. Therefore, we conclude from Lemma 1.7 (i) that $F_n(t, \cdot, \cdot)$ is weakly u.s.c. for a.e $t \in J$. Also, it is clear that $F_n(\cdot, x, v)$ has a strongly measurable selection $G_n(\cdot, x, v)$ for each $(x, v) \in \mathscr{D}$, and thereby the assertion is established.

Remark 2.4 It is assumed in Lemma 2.6 that for a.e. $t \in J$, $F(t, \cdot, \cdot)$ is weakly u.s.c. rather than u.s.c. Such condition is more easily verified usually in practical applications (see Sect. 5 below and [192, Sect. 5]). The latter condition can be found in some situations of previous research such as [1, 14, 130].

The following result is the main result in this subsection.

Theorem 2.2 *Let the hypotheses in Theorem 2.1 be satisfied. Then $\Sigma_{\phi,T}^F$ is an R_δ-set for each $\phi \in C([-\tau, 0], \overline{D(A)})$.*

Proof Assume that $\{F_n\}$ is the approximate sequence established in Lemma 2.6. For each $n \geq 1$, consider the approximate problem of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in J, \\ f(t) \in F_n(t, u(t), u_t), & t \in J, \\ u(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{2.11}$$

where $\phi \in C([-\tau, 0], \overline{D(A)})$. Let $\Sigma_{\phi,T}^{F_n}$ be the solution set of (2.11).

Noticing Lemma 2.6 (ii) and (vi) and performing similar arguments as in Theorem 2.1 and Lemma 2.5, we infer that $\Sigma_{\phi,T}^{F_n}$ is nonempty and compact in $C(J_\tau, X)$. Moreover, by Lemma 2.6 (i) we have

$$\Sigma_{\phi,T}^F \subset \cdots \subset \Sigma_{\phi,T}^{F_n} \cdots \subset \Sigma_{\phi,T}^{F_2} \subset \Sigma_{\phi,T}^{F_1}.$$

We claim that $\Sigma^F_{\phi,T} = \bigcap_{n\geq 1} \Sigma^{F_n}_{\phi,T}$. Note first that $\Sigma^F_{\phi,T} \subset \bigcap_{n=1} \Sigma^{F_n}_{\phi,T}$. To prove the reverse inclusion, we take $u \in \bigcap_{n=1} \Sigma^{F_n}_{\phi,T}$. Therefore, there exists a sequence $\{f_n\} \subset L^1(J, X)$ such that $f_n \in Sel_{F_n}(u)$, $u = S_\phi(f_n)$, and for all $n \geq 1$,

$$|f_n(t)| \leq L(t)(3 + 2|u_t|_0) \ \text{ for a.e. } t \in J$$

in view of Lemma 2.6 (ii). From which together with the fact that X is reflexive it follows that $\{f_n\}$ is relatively weakly compact in $L^1(J, X)$ due to Lemma 1.24. Thus, there exists a subsequence of $\{f_n\}$, still denoted by $\{f_n\}$, such that f_n converges weakly to $f \in L^1(J, X)$. An application of Mazur's theorem yields that there exists a sequence $\{\tilde{f}_n\} \subset L^1(J, X)$ such that $\tilde{f}_n \in \text{co}\{f_k : k \geq n\}$ for each $n \geq 1$ and $\tilde{f}_n \to f$ in $L^1(J, X)$ as $n \to \infty$. Passing to a subsequence if necessary, we may assume that for a.e. $t \in J$, $\tilde{f}_n(t) \to f(t)$ in X. Denote by $\mathscr{T}_c$ the set of all $t \in J$ such that $\tilde{f}_n(t) \to f(t)$ in X and $f_n(t) \in F_n(t, u(t), u_t)$ for all $n \geq 1$. Clearly, $J \backslash \mathscr{T}_c$ has null measure.

Now by Lemma 2.6 (iii) we have that there exists $E \subset J$ with $\text{mes}(E) = 0$ such that for each $t \in (J \setminus E) \cap \mathscr{T}_c$, $\varepsilon > 0$ and $x^* \in X^*$,

$$x^*(\tilde{f}_n(t)) \in \text{co}\{x^*(f_k(t)) : k \geq n\} \subset x^*(F_n(t, u(t), u_t) \subset x^*(F(t, u(t), u_t)) + (-\varepsilon, \varepsilon)$$

with n large enough. Here we use Lemma 2.6 (i) and the result that F_n has convex values for each $n \geq 1$. Passing to the limit in the inclusion above for $n \to \infty$ and taking into account the arbitrariness of ε, we get that $x^*(f(t)) \in x^*(F(t, u(t), u_t))$ for each $x^* \in X^*$ and $t \in (J \setminus E) \cap \mathscr{T}_c$. Since x^* is arbitrary and F has convex and closed values, we conclude that $f(t) \in F(t, u(t), u_t)$ for each $t \in (J \setminus E) \cap \mathscr{T}_c$, which implies that $f \in Sel_F(u)$. Moreover, noticing $f_n \rightharpoonup f$ in $L^1(J, X)$, we deduce, in view of Lemma 2.3 (ii), that $S_\phi(f) = u$. This proves that $u \in \Sigma^F_{\phi,T}$, as desired.

Finally, in order to show that $\Sigma^F_{\phi,T}$ is an R_δ-set, it suffices to verify that $\Sigma^{F_n}_{\phi,T}$ is contractible for each $n \geq 1$. Let G_n be the selection of F_n which is established in Lemma 2.6 (v). Observe, thanks to Lemma 2.6 (v), that $G_n(t, \cdot, \cdot)$ is continuous for a.e. $t \in J$. Also, $\mathcal{D}_n := \{(u(t), u_t) : t \in J, u \in \Sigma^{F_n}_{\phi,T}\}$ is a relatively compact set in $X \times C([-\tau, 0], X)$, since $\Sigma^{F_n}_{\phi,T}$ is compact in $C(J_\tau, X)$. Therefore, we conclude, again by Lemma 2.6 (v), that there exists a neighborhood U of $\overline{\mathcal{D}}_n$ and a constant $C_U > 0$ such that (2.7) is satisfied with C_V replaced by C_U. Furthermore, it is easy to see that G_n verifies

$$|G_n(t, x, v)| \leq L(t)(3 + |x| + |v|_0) \tag{2.12}$$

for a.e. $t \in J$ and each $(x, v) \in \overline{D(A)} \times C([-\tau, 0], \overline{D(A)})$.

Now, performing a trivial variant of an argument from Theorem 2.1, we obtain the existence of C^0-solutions of the Cauchy problem of the form

$$\begin{cases} v'(t) \in Av(t) + G_n(t, v, v_t), & t \in [s, T], \\ v(s+\theta) = \varphi(\theta), & \theta \in [-\tau, 0] \end{cases} \tag{2.13}$$

for each $s \in J$ and $\varphi \in C([-\tau, 0], \overline{D(A)})$. Moreover, we point out that the C^0-solution to (2.13) is unique. Indeed, if v_1 and v_2 are two solutions of (2.13) corresponding to $s \in J$ and $\varphi \in C([-\tau, 0], \overline{D(A)})$, then there exists a neighborhood U' related to v_1 and v_2 and $w := v_1 - v_2$ satisfies

$$\begin{aligned} |w(t)| \leq & \int_s^t |G_n(t, v_1(\sigma), v_{1\sigma}) - G_n(t, v_2(\sigma), v_{2\sigma})| d\sigma \\ \leq & C_{U'} \int_s^t L(\sigma)(|w(\sigma)| + |w_\sigma|_0) d\sigma \end{aligned}$$

for every $t \in [s, T]$. We here used the result (v) of Lemma 2.6. Therefore, by Lemma 1.4 we see that $v_1 \equiv v_2$, as desired.

We denote by $v(\cdot, s, \varphi)$ the unique C^0-solution of (2.13) corresponding to $s \in J$ and $\varphi \in C([-\tau, 0], \overline{D(A)})$. Define a function $\hat{H} : [0, 1] \times \Sigma_{\phi,T}^{F_n} \to \Sigma_{\phi,T}^{F_n}$ by setting

$$\hat{H}(\lambda, u)(t) = \begin{cases} u(t), & t \in [-\tau, \lambda T], \\ v(t, \lambda T, u_{\lambda T}), & t \in (\lambda T, T] \end{cases}$$

for each $(\lambda, u) \in [0, 1] \times \Sigma_{\phi,T}^{F_n}$. In a manner similar to the proof in Theorem 2.1 we can show that $\hat{H}(\lambda, u) \in \Sigma_{\phi,T}^{F_n}$ for each $(\lambda, u) \in [0, 1] \times \Sigma_{\phi,T}^{F_n}$, and $\hat{H}(0, u) = v(\cdot, 0, \phi)$ and $\hat{H}(1, u) = u$ for each $u \in \Sigma_{\phi,T}^{F_n}$.

Below is to show that $\hat{H}$ is continuous. Let us consider a sequence $\{(\lambda_k, u_k)\} \subset [0, 1] \times \Sigma_{\phi,T}^{F_n}$ with $(\lambda_k, u_k) \to (\lambda, u)$ in $[0, 1] \times C(J_\tau, X)$ as $k \to \infty$. Set

$$\rho_k(t) = |\hat{H}(\lambda, u)(t) - \hat{H}(\lambda_k, u_k)(t)| \ \text{ for } t \in J_\tau.$$

We are going to show that $\sup\limits_{t \in J_\tau} \rho_k(t) \to 0$ as $k \to \infty$. Without loss of generality we assume that $\lambda_k \leq \lambda$ for all $k \geq 1$, since the remaining cases can be treated in a similar way. For simplicity in presentation, we put $\hat{v}_k = \hat{H}(\lambda_k, u_k)$, $k \geq 1$, and $\hat{v} = \hat{H}(\lambda, u)$.

From Lemma 2.6 (v) it follows that for each $t \in [\lambda T, T]$,

$$\begin{aligned} \rho_k(t) = & |\hat{v}(t) - \hat{v}_k(t)| \\ \leq & |\hat{v}(\lambda T) - \hat{v}_k(\lambda T)| + \int_{\lambda T}^t |G_n(\sigma, \hat{v}(\sigma), \hat{v}_\sigma) - G_n(\sigma, \hat{v}_k(\sigma), \hat{v}_{k\sigma})| d\sigma \\ \leq & \rho_k(\lambda T) + C_U \int_{\lambda T}^t L(\sigma) \left(\rho_k(\sigma) + \sup_{\theta \in [\sigma - \tau, \sigma]} \rho_k(\theta) \right) d\sigma. \end{aligned}$$

Then an application of Lemma 1.4 yields

$$\rho_k(t) \le \sup_{\theta\in[\lambda T-\tau,\lambda T]} \rho_k(\theta)\exp\left(2C_U\int_{\lambda T}^{t} L(\sigma)d\sigma\right), \quad \lambda T \le t \le T. \tag{2.14}$$

Also, noticing that $\Sigma_{\phi,T}^{F_n}$ is compact, we can find a constant $M > 0$ for which the estimates

$$|u(t)| \le M \ \text{ and } |u_t|_0 \le M$$

hold for all $u \in \Sigma_{\phi,T}^{F_n}$ and $t \in J$, which together with (2.12) imply that for every $\lambda_k T \le t \le \lambda T$,

$$\begin{aligned}\rho_k(t) \le& |\hat{v}(\lambda_k T) - \hat{v}_k(\lambda_k T)| + \int_{\lambda_k T}^{t} |G_n(\sigma, \hat{v}(\sigma), \hat{v}_\sigma) - G_n(\sigma, \hat{v}_k(\sigma), \hat{v}_{k\sigma})| d\sigma \\ \le& |u(\lambda_k T) - u_k(\lambda_k T)| + (6+4M)\int_{\lambda_k T}^{t} L(\sigma)d\sigma.\end{aligned} \tag{2.15}$$

Then, note that

$$\rho_k(t) = |u(t) - u_k(t)| \ \text{ for } t \in [-\tau, \lambda_k T], \tag{2.16}$$

which, together with (2.15), yields

$$\sup_{\theta\in[\lambda T-\tau,\lambda T]} \rho_k(\theta) \le \|u - u_k\| + (6+4M)\int_{\lambda_k T}^{\lambda T} L(\sigma)d\sigma. \tag{2.17}$$

Recalling (2.14)–(2.17), we end up with

$$\begin{aligned}\rho_k(t) \le& 2\|u - u_k\| + (6+4M)\int_{\lambda_k T}^{\lambda T} L(\sigma)d\sigma \\ &+ \left(\|u - u_k\| + (6+4M)\int_{\lambda_k T}^{\lambda T} L(\sigma)d\sigma\right)\exp\left(2C_U\int_{\lambda T}^{T} L(\sigma)d\sigma\right)\end{aligned}$$

for every $t \in J_\tau$. The right-hand side of the inequality above can be made small when k is large independently of $t \in J_\tau$. Accordingly, our result follows. Therefore, we conclude that $\Sigma_{\phi,T}^{F_n}$ is contractible, and thus $\Sigma_{\phi,T}^{F}$ is an R_δ-set. This proof is complete.

2.3.2 *Noncompact Intervals Case*

Throughout this subsection, let $\tilde{J}_\tau = [-\tau, 0] \cup \mathbb{R}^+$. We first present the following result.

Lemma 2.7 *Let X be reflexive. Suppose further that F satisfies the hypotheses (H_1) and (H_2). Then $Sel_F(u) \neq \emptyset$ for each $u \in \widetilde{C}(\tilde{J}_\tau, \overline{D(A)})$.*

Proof Let $u \in \widetilde{C}(\tilde{J}_\tau, \overline{D(A)})$. By Lemma 2.4, one can choose $f_m \in Sel_{F|_{[0,m]}}(u|_{[-\tau,m]})$ for each $m \in \mathbb{N} \setminus \{0\}$, where $F|_{[0,m]}$ is the restriction of F to $[0, m]$, it is to say

$$F|_{[0,m]}(t, x, v) = F(t, x, v) \text{ on } [0, m] \times \overline{D(A)} \times C([-\tau, 0], \overline{D(A)}).$$

Consider the function $f : \mathbb{R}^+ \to X$ defined as

$$f(t) = \sum_{m=1}^{\infty} \chi_{[m-1,m)}(t) f_m(t), \quad t \in \mathbb{R}^+,$$

where $\chi_{[m-1,m)}$ denotes the characteristic function of interval $[m-1, m)$. It is not difficult to see that $f \in Sel_F(u)$ and it is locally integrable. This gives desired result.

Assume that $\{C([a, m], X), \pi^p_{a,m}, \mathbb{N}(a)\}$ and $\{L^1([0, m], X), \dot{\pi}^p_m, \mathbb{N} \setminus \{0\}\}$ are the inverse systems established in Sect. 1.2.4. Given $\phi \in C([-\tau, 0], \overline{D(A)})$, we have that the family $\{id, S_{\phi,m}\}$ is a mapping from $\{L^1([0, m], X), \dot{\pi}^p_m, \mathbb{N} \setminus \{0\}\}$ into $\{C([-\tau, m], X), \pi^p_{-\tau,m}, \mathbb{N} \setminus \{0\}\}$. Indeed, this can be seen from the observation

$$\pi^p_{-\tau,m}(S_{\phi,p}(f)) = S_{\phi,m}(\dot{\pi}^p_m(f)) \text{ for all } f \in L^1(0, p, X) \text{ and } m \le p.$$

So the family $\{id, S_{\phi,m}\}$ induces a limit mapping $S_{\phi,\infty} : L^1_{loc}(\mathbb{R}^+, X) \to \widetilde{C}(\tilde{J}_\tau, X)$ such that $S_{\phi,\infty}(f)|_{[-\tau,m]} = S_{\phi,m}(f|_{[0,m]})$ for each $f \in L^1_{loc}(\mathbb{R}^+, X)$ and $m \in \mathbb{N} \setminus \{0\}$.

In this subsection, by a C^0-solution of the Cauchy problem (2.1), we mean a continuous function $u : \tilde{J}_\tau \to \overline{D(A)}$ which satisfies $u(t) = \phi(t)$ for all $t \in [-\tau, 0]$ and is a C^0-solution in the sense of *Benilan* to $u'(t) + Au(t) \ni f(t)$, where $f \in L^1_{loc}(\mathbb{R}^+, X)$ and $f(t) \in F(t, u(t), u_t)$ for a.e. $t \in \mathbb{R}^+$.

Let $\Sigma^F_{\phi,\infty}$ stand for the set of all C^0-solutions to the Cauchy problem (2.1). We are in the position to present our main result in this subsection.

Theorem 2.3 *Assume that the hypotheses (H_0)-(H_2) are satisfied. Then $\Sigma^F_{\phi,\infty}$ is an R_δ-set for each $\phi \in C([-\tau, 0], \overline{D(A)})$.*

Proof Assume that $\phi \in C([-\tau, 0], \overline{D(A)})$. For every $m \in \mathbb{N} \setminus \{0\}$, let $W^\phi_m : \hat{K}_m \to 2^{\hat{K}_m}$ be a multivalued mapping defined by

$$W^\phi_m(u) = S_{\phi,m}(Sel_{F|_{[0,m]}}(u)) \text{ for each } u \in \hat{K}_m,$$

where

$$\hat{K}_m := \{u \in C([-\tau, m], \overline{D(A)}) : u(t) = \phi(t), \quad t \in [-\tau, 0]\}.$$

Applying Theorem 2.1 and Lemma 2.5 to $F|_{[0,m]}$ we obtain that $\mathrm{Fix}(W^\phi_m) = \Sigma^{F|_{[0,m]}}_{\phi,m}$ and $\mathrm{Fix}(W^\phi_m)$ is nonempty and compact. Also, it is seen, thanks to Theorem 2.2, that

$\mathrm{Fix}(W_m^\phi)$ is an R_δ-set. Moreover, one finds that $\{\hat{K}_m, \pi_{-\tau,m}^p, \mathbb{N}\setminus\{0\}\}$ is an inverse system and

$$\begin{aligned}\widetilde{K} :=&\{u \in \widetilde{C}(\tilde{J}_\tau, \overline{D(A)}) : u(t) = \phi(t) \text{ for all } t \in [-\tau, 0]\}\\ =&\lim_{\leftarrow}\{\hat{K}_m, \pi_{-\tau,m}^p, \mathbb{N}\setminus\{0\}\}.\end{aligned}$$

In order to apply Theorem 1.19, we first show that the family $\{id, W_m^\phi\}$ is a mapping from $\{\hat{K}_m, \pi_{-\tau,m}^p, \mathbb{N}\setminus\{0\}\}$ into itself. Let $p, m \in \mathbb{N}$ with $p \geq m$ and $u \in \hat{K}_p$. We claim that

$$Sel_{F|_{[0,m]}}(u|_{[-\tau,m]}) = \{f|_{[0,m]} : f \in Sel_{F|_{[0,p]}}(u)\}. \tag{2.18}$$

The case $p = m$ is obvious. For the case $p > m$ it is readily checked that $\{f|_{[0,m]} : f \in Sel_{F|_{[0,p]}}(u)\} \subset Sel_{F|_{[0,m]}}(u|_{[-\tau,m]})$. It remains to prove the reverse inclusion. Let $f \in Sel_{F|_{[0,m]}}(u|_{[-\tau,m]})$. Choose $g \in Sel_{F|_{[0,p]}}(u)$ and put

$$\hat{f}(t) = f(t)\chi_{[0,m]}(t) + g(t)\chi_{(m,p]}(t), \quad t \in [0, p].$$

We then obtain that $\hat{f} \in Sel_{F|_{[0,p]}}(u)$, which gives $Sel_{F|_{[0,m]}}(u|_{[-\tau,m]}) \subset \{f|_{[0,m]} : f \in Sel_{F|_{[0,p]}}(u)\}$, as desired.

Now, by using (2.18) and the fact $\pi_{-\tau,m}^p(S_{\phi,p}(f)) = S_{\phi,m}(\dot{\pi}_m^p(f))$ for every $f \in L^1(0, p, X)$, we have

$$\begin{aligned}\pi_{-\tau,m}^p(W_p^\phi(u)) =&\pi_{-\tau,m}^p(S_{\phi,p}(Sel_{F|_{[0,p]}}(u)))\\ =&\{S_{\phi,m}(\dot{\pi}_m^p(f)) : f \in Sel_{F|_{[0,p]}}(u)\}\\ =&\{S_{\phi,m}(f) : f \in Sel_{F|_{[0,m]}}(u|_{[-\tau,m]})\}\\ =&W_m^\phi(\pi_{-\tau,m}^p(u)).\end{aligned}$$

Hence, $\{id, W_m^\phi\}$ induces a limit mapping $W_\infty^\phi : \widetilde{K} \to 2^{\widetilde{K}}$, defined by

$$\begin{aligned}W_\infty^\phi(u) = \{w \in \widetilde{K} :&w|_{[-\tau,m]} = S_{\phi,m}(f|_{[0,m]}) \text{ for every } m \in \mathbb{N}\setminus\{0\},\\ &f \in L^1_{loc}(\mathbb{R}^+, X) \text{ and } f(t) \in F(t, u(t), u_t) \text{ for a.e. } t \in \mathbb{R}^+\}\end{aligned}$$

for each $u \in \widetilde{K}$. Here we used Lemma 2.7. Moreover, it readily follows that

$$W_\infty^\phi(u) = S_{\phi,\infty}(Sel_F(u)) \text{ for every } u \in \widetilde{K}.$$

Now, applying Theorem 1.19 yields that $\mathrm{Fix}(W_\infty^\phi)$ is an R_δ-set, which together with relation $\Sigma_{\phi,\infty}^F = \mathrm{Fix}(W_\infty^\phi)$ implies that $\Sigma_{\phi,\infty}^F$ is an R_δ-set. Thus, the proof is complete.

2.4 Nonlocal Cauchy Problem

We are concerned with the existence of C^0-solutions to the nonlocal Cauchy problem (2.2) defined on right half-line.

The next lemma, which gives the convergence property of Sel_F in the case when $J = \mathbb{R}^+$, plays an important role in the sequel.

Lemma 2.8 *Let X be reflexive and F verify the hypotheses (H_1) and (H_2). If $\{u_n\} \subset \widetilde{C}(\tilde{J}_\tau, \overline{D(A)})$ with $u_n \to u_0$ in $\widetilde{C}(\tilde{J}_\tau, X)$ and $f_n \in Sel_F(u_n)$, then there exists $f \in Sel_F(u_0)$ and a subsequence $\{f_{n'}\}$ of $\{f_n\}$ such that $f_{n'} \rightharpoonup f$ in $L^1([0, m], X)$ for each $m \in \mathbb{N} \setminus \{0\}$.*

Proof Observe that $u_n \to u_0$ in $C([-\tau, m], X)$ for each $m \in \mathbb{N} \setminus \{0\}$. Also, from Lemma 2.4 it follows that $Sel_{F|_{[0,1]}}$ is weakly u.s.c. with convex and weakly compact values. Since $f_n|_{[0,1]} \in Sel_{F|_{[0,1]}}(u_n|_{[-\tau,1]})$, we see, in view of Lemma 1.7 (ii), that there exists a subsequence of $\{f_n\}$, say $\{f_{n,1}\}$, and $\hat{f}_1 \in Sel_{F|_{[0,1]}}(u_0|_{[-\tau,1]})$ such that $f_{n,1}|_{[0,1]}$ converges weakly to $\hat{f}_1$ in $L^1([0, 1], X)$. Similarly, we can select a subsequence $\{f_{n,2}\}$ of $\{f_{n,1}\}$ and $\hat{f}_2 \in Sel_{F|_{[0,2]}}(u_0|_{[-\tau,2]})$ such that $f_{n,2}|_{[0,2]} \rightharpoonup \hat{f}_2$ in $L^1([0, 2], X)$. Proceeding in this manner, we can choose a family of subsequences $\{f_{n,m}\}$, $m \geq 1$, of $\{f_n\}$ and a sequence $\{\hat{f}_m\}$ such that $f_{n,m}|_{[0,m]} \rightharpoonup \hat{f}_m$ in $L^1([0, m], X)$. Note that $\hat{f}_m \in Sel_{F|_{[0,m]}}(u_0|_{[-\tau,m]})$. Write

$$\hat{f}(t) = \sum_{m=1}^{\infty} \chi_{[m-1,m)}(t)\hat{f}_m(t), \quad t \in \mathbb{R}^+.$$

It is clear that $\hat{f} \in Sel_F(u_0)$. Moreover, we see that the diagonal sequence $\{f_{n,n}\}$, as a subsequence of $\{f_n\}$, verifies $f_{n,n}|_{[0,m]} \rightharpoonup \hat{f}|_{[0,m]}$ in $L^1([0, m], X)$ for each $m \in \mathbb{N} \setminus \{0\}$. The lemma is proved.

To present our main result, we also need the following conditions.
(H_3) There exists $r > 0$ such that $[x, f]_+ \leq 0$ for each $x \in \overline{D(A)}$ with $|x| = r$, $t \in \mathbb{R}^+$, $v \in C([-\tau, 0], \overline{D(A)})$ with $|v|_0 \leq r$ and $f \in F(t, x, v)$.
(H_4) $g : \widetilde{C}_b(\tilde{J}_\tau, \overline{D(A)}) \to C([-\tau, 0], \overline{D(A)})$ verifies

(i) the restriction of g to Ω_r is continuous and $|g(u)|_0 \leq r$ for each $u \in \Omega_r$, where $\Omega_r = \{u \in \widetilde{C}_b(\tilde{J}_\tau, \overline{D(A)}) : |u(t)| \leq r \text{ for all } t \in \tilde{J}_\tau\}$, and
(ii) for each subset $\mathscr{U} \subset \Omega_r$ which restricted to $[\delta, \infty)$ is relatively compact in $\widetilde{C}([\delta, \infty), X)$ for each $\delta \in (0, \infty)$, $g(\mathscr{U})$ is relatively compact in $C([-\tau, 0], X)$, where r is given by (H_3).

Remark 2.5 (a) Let us mention that the condition (ii) above on g is quite general. In particular, we claim that the condition (ii) is satisfied when the condition (i) above and the following condition are fulfilled: (H_g) There exists $\delta' \in (0, \infty)$ such that for every $u, w \in \Omega_r$ satisfying $u(t) = w(t)$ $(t \in [\delta', \infty))$, $g(u) = g(w)$. To illustrate it, let us define a linear operator $\Lambda : \widetilde{C}([\delta', \infty), X) \to \widetilde{C}(\tilde{J}_\tau, X)$ by

$$\Lambda(u)=\begin{cases} u(t), & t\in(\delta',\infty),\\ u(\delta'), & t\in[-\tau,\delta'].\end{cases}$$

Then it is clear that Λ is bounded and hence $\hat{g}:=g\circ\Lambda$ is a continuous function from $\Omega_r|_{[\delta',\infty)}$ to $C([-\tau,0],\overline{D(A)})$. Moreover, if $\mathscr{U}\subset\Omega_r$ and $\mathscr{U}_{[\delta,\infty)}$ is relatively compact in $\widetilde{C}([\delta,\infty),X)$ for each $\delta\in(0,\infty)$, then we see that $\hat{g}(\overline{\mathscr{U}|_{[\delta',\infty)}})$ is compact in $C([-\tau,0],X)$. From this and (H_g) it follows that $g(\mathscr{U})(=\hat{g}(\mathscr{U}|_{[\delta',\infty)})\subset\hat{g}(\overline{\mathscr{U}|_{[\delta',\infty)}}))$ is relatively compact in $C([-\tau,0],X)$.

(b) Note that the condition (H_g), which was used in some situations of previous research (cf. e.g., Wang [194] et al. and references therein), covers the multipoint discrete mean condition mentioned in the Introduction.

For some $\widetilde{r}>0$, denote $Q_{\widetilde{r}}:=\{w\in C([-\tau,0],\overline{D(A)}):|w|_0\le\widetilde{r}\}$ below.

Lemma 2.9 *Let $\widetilde{r}>0$ be fixed. Under the hypotheses (H_0)-(H_2), the multivalued mapping $\Gamma:Q_{\widetilde{r}}\to P(\widetilde{C}(\tilde{J}_\tau,X))$, defined by $\Gamma(\phi)=\Sigma^F_{\phi,\infty}$ for each $\phi\in Q_{\widetilde{r}}$, is an R_δ-mapping.*

Proof As proved in Theorem 2.3, $\Gamma(\phi)$ is an R_δ-set for each $\phi\in Q_r$. It suffices to verify the upper semi-continuity of Γ.

We first show that Γ is quasi-compact. Let $\mathcal{A}\subset Q_{\widetilde{r}}$ be a compact set and

$$\mathscr{F}_{\widetilde{r}}=\{f\in L^1_{loc}(\mathbb{R}^+,X):|f(t)|\le L(t)(1+2x_{\widetilde{r}}(t))\ \text{ for a.e. } t\in\mathbb{R}^+\},\tag{2.19}$$

where $x_{\widetilde{r}}$ is the unique continuous solution of

$$x_{\widetilde{r}}(t)=\widetilde{r}+\int_0^t L(\sigma)\,(1+2x_{\widetilde{r}}(\sigma))\,\mathrm{d}\sigma,\quad t\in\mathbb{R}^+.$$

An argument similar to that in Lemma 2.5 enables us to obtain that $|v_t|_0\le x_{\widetilde{r}}(t)$ for each $t\in\mathbb{R}^+$ and $v\in\Gamma(Q_{\widetilde{r}})$. From this and the fact that

$$\Gamma(\phi)\subset S_{\phi,\infty}(Sel_F(\Gamma(\phi)))\ \text{ for each } \phi\in Q_{\widetilde{r}},$$

we deduce, thanks to (H_2), that $\Gamma(\phi)\in S_{\phi,\infty}(\mathscr{F}_{\widetilde{r}})$ and hence $\Gamma(\mathcal{A})\subset S_{\mathcal{A},\infty}(\mathscr{F}_{\widetilde{r}})$. Also, it is easy to see that $\mathscr{F}_{\widetilde{r}}|_{[0,m]}$ is uniformly integrable in $L^1([0,m],X)$ for each $m\in\mathbb{N}\setminus\{0\}$. Applying Lemma 2.3 (i) gives that $\Gamma(\mathcal{A})|_{[-\tau,m]}(\subset S_{\mathcal{A},m}(\mathscr{F}_{\widetilde{r}}|_{[0,m]}))$ is a relatively compact set in $C([-\tau,m],X)$ for each $m\in\mathbb{N}\setminus\{0\}$. Therefore, by Lemma 1.18 we see that $\Gamma(\mathcal{A})$ is relatively compact in $\widetilde{C}(\tilde{J}_\tau,X)$, as desired.

What followed is to show that Γ is closed. Let $\{(\phi_n,u_n)\}$ be a sequence in $\mathrm{Gra}(\Gamma)$, which converges to $(\phi,u)\in C([-\tau,0],X)\times\widetilde{C}(\tilde{J}_\tau,X)$. It is known that there exists a sequence $\{f_n\}\subset\mathscr{F}_{\widetilde{r}}$ such that $f_n\in Sel_F(u_n)$ and $S_{\phi_n,\infty}(f_n)=u_n$. Then an application of Lemma 2.8 yields that there exists $f\in Sel_F(u)$ and a subsequence of $\{f_n\}$, still denoted by $\{f_n\}$, such that $f_n|_{[0,m]}\rightharpoonup f|_{[0,m]}$ in $L^1([0,m],X)$ for every $m\in\mathbb{N}\setminus\{0\}$. Recalling Lemma 2.1 and the representation of $S_{\phi,m}$, we see that $S_{\phi,m}(f|_{[0,m]}) = u|_{[-\tau,m]}$ for every $m \in \mathbb{N} \setminus \{0\}$, which gives $u = S_{\phi,\infty}(f)$. Thus, it follows that $u \in \Gamma(\phi)$. An application of Lemma 1.9 then completes this proof.

Theorem 2.4 *Suppose that the hypotheses* (H_0)-(H_4) *are satisfied. Then the nonlocal Cauchy problem (2.2) admits at least one* C^0*-solution.*

Proof Since $\overline{D(A)}$ is convex, it follows from Theorem 1.4 that there exists a continuous extension $\widetilde{id}$ of identity mapping $id : \overline{D(A)} \to \overline{D(A)}$ satisfying $\widetilde{id}(X) \subset \overline{D(A)}$. Let $\rho : X \times C([-\tau, 0], X) \to \overline{B_r(0, 0)}$ be defined by

$$\rho(x, v) = \begin{cases} (x, v), & \text{if } (x, v) \in \overline{B_r(0, 0)}, \\ r|(x, v)|_\tau^{-1}(x, v), & \text{in rest.} \end{cases}$$

Then we define the multi-value function $F_\rho : \mathbb{R}^+ \times X \times C([-\tau, 0], X) \to P_{cl,cv}(X)$ by

$$F_\rho(t, x, v) = F(t, \rho(\widetilde{id}(x), \widetilde{id}(v))), \quad (t, x, v) \in \mathbb{R}^+ \times X \times C([-\tau, 0], X),$$

where $\widetilde{id}(v)(s) = \widetilde{id}(v(s))$ for each $s \in [-\tau, 0]$.

Since both ρ and $\widetilde{id}$ are continuous, it follows that F_ρ verifies the condition (H_1). Clearly it satisfies the condition (H_2) (with a modified $L(\cdot)$). Moreover, from (H_3) one has

$$[x, f]_+ \le 0 \tag{2.20}$$

for each $x \in \overline{D(A)}$ with $|x| \ge r$, $t \in \mathbb{R}^+$, $v \in C([-\tau, 0], \overline{D(A)})$ and $f \in F_\rho(t, x, v)$.

In the sequel, let $\Sigma_{\phi,\infty}^{F_\rho}$ be the set of all C^0-solutions to the Cauchy problem of the form

$$\begin{cases} u'(t) \in Au(t) + f(t), & t \in \mathbb{R}^+, \\ f(t) \in F_\rho(t, u(t), u_t), & t \in \mathbb{R}^+, \\ u(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{2.21}$$

Define the multivalued mapping $\Gamma_\rho : Q_r \to P(\widetilde{C}(\tilde{J}_\tau, X))$ by

$$\Gamma_\rho(\phi) = \Sigma_{\phi,\infty}^{F_\rho} \text{ for each } \phi \in Q_r.$$

Then based on the considerations above with Lemma 2.9 we deduce that Γ_ρ is an R_δ-mapping. Moreover, we claim that $\Gamma_\rho(Q_r) \subset \Omega_r$. In fact, if this is not the case, then we can assume that there exist $\phi \in Q_r$, $u \in \Gamma_\rho(\phi)$ and $t_0 > 0$ such that $u(t_0) > r$. Therefore, it can find $h \in (0, t_0]$ such that $|u(t)| \ge r$ on $[t_0 - h, t_0]$ and $|u(t_0 - h)| = r$, since u is continuous and $|u(0)| \le r$. We thus use (2.20) to obtain

$$r < |u(t_0| = |u(t_0 - h)| + \int_{t_0-h}^{t_0} [u(\sigma), f(\sigma)]_+ d\sigma \le |u(t_0 - h)| = r,$$

where $f \in Sel_F(u)$ such that $u = S_{\phi,\infty}(f)$, which is a contradiction.

Put

$$Q_r^i = \overline{\mathrm{co}}(\,g(\Omega_r^i)) \;\text{ and }\; \Omega_r^{i+1} = \overline{\mathrm{co}}(\,\varGamma_\rho(Q_r^i)), \quad i = 0, 1,$$

where $\Omega_r^0 := \Omega_r$. Then, by (H_4) (i) we have $Q_r^0 \subset Q_r$, which together with the result $\varGamma_\rho(Q_r) \subset \Omega_r$ implies that $\Omega_r^1 \subset \Omega_r^0$. From this it follows that $\varGamma_\rho(Q_r^1) \subset \Omega_r^2 \subset \Omega_r^1$. Therefore, the following composition is well-defined:

$$\varGamma_\rho \circ g : \quad \Omega_r^1 \xrightarrow{g} Q_r^1 \xrightarrow{\varGamma_\rho} \Omega_r^1.$$

We seek for solutions in Ω_r^1. To do this, we show that the multivalued mapping $\varGamma_\rho \circ g$ has a fixed point in Ω_r^1. Observe that Ω_r^1 and Q_r^1 being respectively convex subset of $\widetilde{C}(\tilde{J}_\tau, X)$ and $C([-\tau, 0], X)$, are AR-spaces. Also, (H_4) (i) implies that g is an R_δ-mapping.

Next, we verify that the set $\Omega_r^1|_{[\delta,\infty)}$ is relatively compact in $\widetilde{C}([\delta, \infty), X)$ for each $\delta > 0$. Assume that $\delta > 0$ and $m \in \mathbb{N}(\delta)$. Let $\mathscr{F}_r$ be defined by (2.19) with r instead of $\widetilde{r}$. As

$$\varGamma_\rho(Q_r^0)|_{[0,m]} \subset \{u(\cdot, 0, x, f) \in C([0, m], X) : x \in \overline{D(A)} \text{ with } |x| \le r, \;\; f \in \mathscr{F}_r|_{[0,m]}\}$$

and Lemma 2.2, we find that $\varGamma_\rho(Q_r^0)|_{[\delta,m]}$ is relatively compact in $C([\delta, m], X)$. Moreover, using Theorem 1.1 we obtain that $\mathrm{co}(\varGamma_\rho(Q_r^0)|_{[\delta,m]})$ is relatively compact and hence $\overline{\mathrm{co}}(\varGamma_\rho(Q_r^0)|_{[\delta,m]})$ is compact. Now, noticing $\overline{\mathrm{co}}(\varGamma_\rho(Q_r^0)|_{[\delta,m]}) \supset \Omega_r^1|_{[\delta,m]}$ it follows that $\Omega_r^1|_{[\delta,m]}$ is relatively compact, which together with the arbitrariness of m and Lemma 1.18 yields that $\Omega_r^1|_{[\delta,\infty)}$ is relatively compact in $\widetilde{C}([\delta, \infty), X)$. Hence $g(\Omega_r^1)$ is relatively compact in $C([-\tau, 0], X)$ by the arbitrariness of $\delta > 0$ and (H_4) (ii). We thus see, again using Theorem 1.1, that Q_r^1 is compact.

Since $\varGamma_\rho$ is u.s.c. with compact values, we obtain the compactness of $\varGamma_\rho(Q_r^1)$ due to Lemma 1.11. Therefore, we conclude from the result $\varGamma_\rho(g(\Omega_r^1)) \subset \varGamma_\rho(Q_r^1)$ and Theorem 1.16 that there exists a fixed point u of $\varGamma_\rho \circ g$ in Ω_r^1. Moreover, it is readily checked that $u(t) \in \overline{D(A)}$ and $\max\{|u(t)|, |u_t|_0\} \le r$ for each $t \in \mathbb{R}^+$. From this we see $F_\rho(t, u(t), u_t) = F(t, u(t), u_t)$ for every $t \in \mathbb{R}^+$, which implies that u is a C^0-solution of the nonlocal Cauchy problem (2.2). The proof is complete.

2.5 Applications

As samples of applications, we consider a system of partial differential inclusions defined on right half-line in this section. The topological characterization of solution set to the system considering a time delay condition is discussed. Then, for the system subject to a nonlocal condition, we establish the existence of C^0-solutions in the absence of nonexpansive condition on nonlocal function. These examples do not aim at generality but indicate how our theorems can be applied to concrete problems.

Our examples are inspired directly from the work of [192, Example 5.1] (see also [189]).

Example 2.1 Let Ω be a nonempty bounded open set in $\mathbb{R}^n$ with C^2-boundary $\partial\Omega$, $p \in [2, \infty)$ and $\lambda > 0$. Consider the system of partial differential inclusions in the form

$$\begin{cases} \dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{n} \dfrac{\partial}{\partial \xi_i}\left(\left|\dfrac{\partial u}{\partial \xi_i}\right|^{p-2} \dfrac{\partial u}{\partial \xi_i}\right) + \lambda |u|^{p-2} u \in F(t, \xi, u(t, \xi), u_t(\xi)), \\ \qquad\qquad\qquad\qquad\qquad\qquad (t, \xi) \in \mathbb{R}^+ \times \Omega, \\ - \displaystyle\sum_{i=1}^{n} \left|\dfrac{\partial u}{\partial \xi_i}\right|^{p-2} \dfrac{\partial u}{\partial \xi_i} \cos(\overrightarrow{n}, \overrightarrow{e_i}) \in \beta(u(t, \xi)), \qquad (t, \xi) \in \mathbb{R}^+ \times \partial\Omega \end{cases} \tag{2.22}$$

subject to a initial history

$$u(t, \xi) = \phi(t, \xi), \quad (t, \xi) \in [-\tau, 0] \times \Omega, \tag{2.23}$$

where the partial derivatives are taken in the sense of distributions over Ω, $\overrightarrow{n}$ is the outward normal of $\partial\Omega$, $\{\overrightarrow{e_1}, \cdots, \overrightarrow{e_n}\}$ is the canonical base in $\mathbb{R}^n$, $\beta : D(\beta) \subset \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone operator with $0 \in D(\beta)$, $0 \in \beta(0)$, and

$$F(t, \xi, u, v) = [f_1(t, \xi, u, v) + h(\xi), f_2(t, \xi, u, v) + h(\xi)]$$

is a closed interval for each $(t, \xi, u, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R} \times C([-\tau, 0], L^2(\Omega, \mathbb{R}))$, in which $h \in L^2(\Omega, \mathbb{R})$ and $f_i : \mathbb{R}^+ \times \Omega \times \mathbb{R} \times C([-\tau, 0], L^2(\Omega, \mathbb{R})) \to \mathbb{R}$ are given functions such that $f_1(t, \xi, u, v) \leq f_2(t, \xi, u, v)$ for each $(t, \xi, u, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R} \times C([-\tau, 0], L^2(\Omega, \mathbb{R}))$, f_1 is l.s.c., and f_2 is u.s.c.

Here, our objective is to investigate the topological characterization of solution set to the system (2.22)–(2.23).

Take $X = L^2(\Omega, \mathbb{R})$ and denote its norm by $|\cdot|$ and inner product by $(\cdot, \cdot)$. Assume that f_1, f_2 verify the following hypothesis:
[(A_1)] there exist $L_1, L_2 \in L^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$|f_i(t, \xi, u, v)| \leq L_1(t)\,(|u| + |v|_0) + L_2(t), \quad i = 1, 2$$

for each $(t, \xi, u, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R} \times C([-\tau, 0], X)$.

Before stating our main results, we first present the following lemma, which can be seen from [192, Lemma 5.1].

Lemma 2.10 *Suppose that (A_1) is satisfied. Define a multivalued function $F : \mathbb{R}^+ \times X \times C([-\tau, 0], X) \to P(X)$ as*

$$F(t, u, v) = \{x \in X : x(\xi) \in [f_1(t, \xi, u(\xi), v) + h(\xi), f_2(t, \xi, u(\xi), v) + h(\xi)] \text{ a.e.}\}$$

for each $(t, u, v) \in \mathbb{R}^+ \times X \times C([-\tau, 0], X)$. *Then* F *has nonempty, convex and closed values,* $F(\cdot, u, v)$ *has a strongly measurable selection for every* $(u, v) \in X \times C([-\tau, 0], X)$*, and* $F(t, \cdot, \cdot)$ *is weakly u.s.c. for each* $t \in \mathbb{R}^+$. *Moreover,*

$$|F(t, u, v)| \leq \max\{L_1(t), \mathrm{mes}^{\frac{1}{2}}(\Omega)L_1(t), \mathrm{mes}^{\frac{1}{2}}(\Omega)L_2(t) + |h|\}(1 + |u| + |v|_0)$$

for a.e. $t \in \mathbb{R}^+$*, each* $u \in X$ *and* $v \in C([-\tau, 0], X)$.

Theorem 2.5 *Under the hypothesis* (A_1)*, the set of all* C^0*-solutions to the system (2.22)–(2.23) is an* R_δ*-set for each* $\phi \in C([-\tau, 0], X)$.

Proof Let $A : D(A) \subset X \to X$ be defined by

$$\begin{aligned} Au &= \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\Big(\Big|\frac{\partial u}{\partial \xi_i}\Big|^{p-2}\frac{\partial u}{\partial \xi_i}\Big), \\ D(A) &= \Big\{u \in W^{1,p}(\Omega) : \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i}\Big(\Big|\frac{\partial u}{\partial \xi_i}\Big|^{p-2}\frac{\partial u}{\partial \xi_i}\Big) \in X, \text{ and} \\ &\qquad -\sum_{i=1}^{n} \Big|\frac{\partial u}{\partial \xi_i}\Big|^{p-2}\frac{\partial u}{\partial \xi_i}\cos(\overrightarrow{n}, \overrightarrow{e_i}) \in \beta(u(\xi)) \text{ a.e. } \xi \in \partial\Omega\Big\}. \end{aligned}$$

From [189, Example 1.5.4] and [49, Théorème 1.10, p.43] we see that A is an m-dissipative operator with $0 \in A0$ and $\overline{D(A)} = X$. In addition, as in [189, Example 2.2.4 and Corollary 2.3.2], A generates a compact semigroup of nonexpansive mappings on X, which implies that the hypothesis (H_0) holds. Also, by Lemma 2.10 one finds that F verifies conditions (H_1) and (H_2) with $J = \mathbb{R}^+$ and $L(t) = \max\{L_1(t), \mathrm{mes}^{\frac{1}{2}}(\Omega)L_1(t), \mathrm{mes}^{\frac{1}{2}}(\Omega)L_2(t) + |h|\}$. Therefore, applying Theorem 2.3 gives the result as desired.

Next, we consider the system (2.22) equipped with a nonlocal condition as follows:

$$u(t, \xi) = \int_{\tau}^{\infty} \mathscr{N}(u(t + \theta, \xi))d\mu(\theta), \quad (t, \xi) \in [-\tau, 0] \times \Omega, \tag{2.24}$$

where μ is a σ-finite and complete measure on $[\tau, \infty)$ such that

$$\mu([\tau, \infty)) = 1 \text{ and } \lim_{s \to \tau^+} \mu([\tau, s]) = 0.$$

We assume that $\mathscr{N} : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying either for some $C_1, C_2 \geq 0$ and $b \in [0, 1)$,

$$|\mathscr{N}(y)| \leq C_1 + C_2|y|^b \text{ for all } y \in \mathbb{R}, \tag{2.25}$$

or

$$|\mathscr{N}(y)| \le |y| \text{ for all } y \in \mathbb{R}. \tag{2.26}$$

It can define a Nemytskiĭ operator $\mathscr{N}$ from X into itself by $\mathscr{N}(x)(\xi) = \mathscr{N}(x(\xi))$ for each $x \in X$. Moreover, one finds that $\mathscr{N}$ is continuous on X.

Remark 2.6 If (2.25) is satisfied, then a direct computation upon Hölder's inequality yields that for each $x \in X$,

$$|\mathscr{N}(x)| \le C_1 \mathrm{mes}^{\frac{1}{2}}(\Omega) + C_2 \mathrm{mes}^{\frac{1-b}{2}}(\Omega)|x|^b.$$

Write, for each $l > 0$,

$$\Phi(l) = \max\{C_1 \mathrm{mes}^{\frac{1}{2}}(\Omega) + C_2 \mathrm{mes}^{\frac{1-b}{2}}(\Omega) l^b, l\}.$$

Theorem 2.6 *Let (A_1) and (2.25) or (2.26) hold. Suppose further that the following hypothesis is satisfied.*
(A_2) There exists $c > 0$ such that for every $(t, \xi, u, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R} \times C([-\tau, 0], X)$,

$$\max\{u f_i(t, \xi, u, v) : i = 1, 2\} \le -cu^2.$$

Then the system (2.22) and (2.24) has at least one C^0-solution.

Proof Let $r > 0$ be such that $r \ge c^{-1}|h|$ and $\Phi(r) \le r$. Take $(t, u, v) \in \mathbb{R}^+ \times X \times C([-\tau, 0], X)$ with $|u| = r$ and $f \in F(t, u, v)$. Noticing (A_2) and using an argument similar to that in [192, Theorem 5.1] we obtain

$$[u, f]_+ \le |u|^{-1} \int_\Omega \left(-c|u(\xi)|^2 + |u(\xi)h(\xi)|\right) d\xi \le -cr + |h| \le 0,$$

which yields that (H_3) remains true.

Next, let us define a mapping $g : \widetilde{C}_b(\tilde{J}_\tau, X) \to C([-\tau, 0], X)$ as

$$g(u)(t) = \int_\tau^\infty \mathscr{N}(u(t+\theta)) d\mu(\theta), \quad u \in \widetilde{C}_b(\tilde{J}_\tau, X), \ t \in [-\tau, 0].$$

Taking $u \in \widetilde{C}_b(\tilde{J}_\tau, X)$, we have, in view of $\mu[\tau, \infty) = 1$, that

$$\begin{aligned} |g(u)(\cdot)|_0 \le & \sup_{t\in[-\tau,0]} \left(\int_\tau^\infty |\mathscr{N}(u(t+\theta))|^2 \, d\mu(\theta)\right)^{\frac{1}{2}} \\ \le & \Phi\left(\sup_{t\in\mathbb{R}^+} |u(t)|\right), \end{aligned}$$

which implies that $|g(u)|_0 \le r$ for all $u \in \Omega_r$. Also, by means of Lebesgue's dominated convergence theorem it is not difficult to see that $g(u)(\cdot)$ is continuous on $[-\tau, 0]$.

We process to show that g is continuous on Ω_r. Given $\varepsilon > 0$. Let $\{u_n\}$ be a sequence in Ω_r such that u_n converges to $u \in \widetilde{C}(\tilde{J}_\tau, X)$. Since μ is σ-finite, we can choose $m' \in \mathbb{N} \setminus \{0\}$ such that $\mu[m', \infty) \leq \frac{\varepsilon}{4\Phi(r)}$. Therefore, we have that for each $t \in [-\tau, 0]$,

$$|g(u_n)(t) - g(u)(t)| \leq \int_\tau^{m'} |\mathscr{N}(u_n(t+\theta)) - \mathscr{N}(u(t+\theta))| d\mu(\theta) + \frac{\varepsilon}{2}. \quad (2.27)$$

On the other hand, noticing that

$$u_n|_{[-\tau, m']} \to u|_{[-\tau, m']} \text{ in } C([-\tau, m'], X) \text{ and } \mathscr{N} \text{ is continuous on } X,$$

we conclude that $\mathscr{N}$ is uniformly continuous on $\overline{\{u_n(t) : n \geq 1, t \in [-\tau, m']\}}$, which implies that

$$\mathscr{N}(u_n|_{[0,m']}) \to \mathscr{N}(u|_{[0,m']}) \text{ in } C([0, m'], X)$$

as $n \to \infty$. So, there exists $N > 0$ such that for all $n \geq N$,

$$\sup_{t \in [-\tau, 0]} \int_\tau^{m'} |\mathscr{N}(u_n(t+\theta)) - \mathscr{N}(u(t+\theta))| d\mu(\theta) \leq \frac{\varepsilon}{2}.$$

This together with (2.27) proves the desired result.

Assume that $\mathscr{U} \subset \Omega_r$ and $\mathscr{U}|_{[\delta, \infty)}$ is relatively compact in $\widetilde{C}([\delta, \infty), X)$ for each $\delta > 0$. To prove that $g(\mathscr{U})$ is relatively compact in $C([-\tau, 0], X)$, it suffices to show that $g(\mathscr{U})$ is totally bounded. Given $\varepsilon > 0$, it follows that there exists $\delta_0 > 0$ such that $\mu([\tau, \tau + \delta_0]) \leq \frac{\varepsilon}{2\Phi(r)}$.

Next, to construct a finite ε-net of $g(\mathscr{U})$, we need to define an operator

$$g_{\delta_0} : \widetilde{C}_b([\delta_0, \infty), X) \to C([-\tau, 0], X)$$

as

$$g_{\delta_0}(u)(t) = \int_{\tau+\delta_0}^{\infty} \mathscr{N}(u(t+\theta)) d\mu(\theta), \quad u \in \widetilde{C}_b([\delta, \infty), X),\ t \in [-\tau, 0].$$

The same idea as above can be used to prove that g_{δ_0} is continuous on the set

$$\{u \in \widetilde{C}_b([\delta_0, \infty), X) : |u(t)| \leq r \text{ for all } t \in [\delta_0, \infty)\}.$$

Since $\mathscr{U}|_{[\delta_0, \infty)}$ is relatively compact, we obtain that $g_{\delta_0}(\mathscr{U}|_{[\delta_0, \infty)})$ is relatively compact in $C([-\tau, 0], X)$ and thus it admits a finite $\frac{\varepsilon}{2}$-net, denoted by $\mathscr{V}_\varepsilon = \{v_1, \cdots, v_k\}$.

We claim that $\mathscr{V}_\varepsilon$ is a finite ε-net of $g(\mathscr{U})$. Indeed, given $v \in g(\mathscr{U})$, we have that there exists $u \in \mathscr{U}$ such that $v = g(u)$. We choose $v_i \in \mathscr{V}_\varepsilon$ such that

$$|v_i - g_{\delta_0}(u|_{[\delta_0,\infty)})|_0 \leq \frac{\varepsilon}{2}. \tag{2.28}$$

Here we are using the fact $g_{\delta_0}(u|_{[\delta_0,\infty)}) \in g_{\delta_0}(\mathscr{U}|_{[\delta_0,\infty)})$. Also, a direct computation gives

$$\begin{aligned}|g(u) - g_{\delta_0}(u|_{[\delta_0,\infty)})|_0 \leq & \sup_{t\in[-\tau,0]} \int_{\tau}^{\tau+\delta_0} |\mathscr{N}(u(t+\theta))| d\mu(\theta) \\ \leq & \frac{\varepsilon}{2},\end{aligned}$$

which, together with (2.28), implies that $|v - v_i|_0 \leq \varepsilon$, as desired. Therefore, the desired result follows from Theorem 2.4.

At the end of this chapter, we leave two problems for further research.

(1) Is Theorem 2.4 true for the case when either A is a linear operator generating a C_0-semigroup or A is replaced with a family of linear operators generating an evolution system? More specially, is it true for a linear operator A whose resolvent satisfies the estimate of growth $-\gamma$ $(-1 < \gamma < 0)$ in a sector of the complex plane? Let us note that such operator, generating a semigroup of growth $1 + \gamma$, is called an almost sectorial operator (see e.g., Wang et al. [195]) for more details).
(2) Is Theorem 2.3 true under the weaker condition that the semigroup generated by A is only equicontinuous?

We believe it is possible to find some interesting positive answers.

Remark 2.7 It is noted that if A is a linear operator generating a C_0-semigroup, A is replaced with a family of linear operators generating an evolution system, or A is an almost sectorial operator, then in treating the nonlocal Cauchy problem (2.2), it is inappropriate to impose the invariance condition (H_3) on F.

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