Chapter 2
Basics of Quandles

Abstract We study the basics of quandles. In Sect. 2.1, we define quandles and examine some properties. Although the reader who first sees the definition may think it incomprehensible, we give many examples of quandles, and observe that quandles are somehow compatible with geometry in some sense. After that, in Sect. 2.2 we see that any quandle is characterized by (a union of) "homogeneous quandles" (Theorems 2.23 and 2.24). In Sect. 2.5, we give some comments on quandles.

Keywords Quandle · Homogenous set · Central extension · Knot quandle

2.1 Definitions and Examples of Quandles

We start by introducing the definition of quandles, and see basic notation.

Definition 2.1 A *quandle* is a set X with a binary operation $\lhd : X \times X \to X$ satisfying the following three conditions:
(QI) The identity $a \lhd a = a$ holds for any $a \in X$.
(QII) The map $(\bullet \lhd b) : X \to X$ defined by $a \mapsto a \lhd b$ is bijective for any $b \in X$.
(QIII) The distributive identity $(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$ holds for any $a, b, c \in X$.

By $a \lhd^N b$, we mean, $(\cdots(a \lhd b) \lhd b \cdots) \lhd b$, the N-times on the right operation with b. Analogously, we denote the inverse mapping of $\bullet \lhd b : X \to X$ by $\bullet \lhd^{-1} b$.

A quandle X is *of type* n, if there exists $n \in \mathbb{Z} \cup \{\infty\}$ which is the minimum number satisfying $x \lhd^n y = x$ for any $x, y \in X$. That is,

$$\mathrm{Type}(X) = \min\{\, n \mid x \lhd^n y = x \ \text{ for any } x, y \in X \,\} \in \mathbb{N} \cup \{\infty\}.$$

We should pay attention to that, if X is of finite order, then $\mathrm{Type}(X) < \infty$ (Why?). The concept of type is useful to capture illustrations for quandles, as seen in Examples 2.4–2.10.

Next, we discuss homomorphisms among quandles, and give some remarks.

Definition 2.2 (I) A map $f : X \to Y$ between quandles is called a *homomorphism* if $f(a \lhd b) = f(a) \lhd f(b)$ for any $a, b \in X$.

T. Nosaka, *Quandles and Topological Pairs*,
SpringerBriefs in Mathematics, https://doi.org/10.1007/978-981-10-6793-8_2

(II) We write $\mathrm{Hom}_{\mathrm{Qnd}}(X, Y)$ for the set of quandle homomorphisms from X to Y:

$$\mathrm{Hom}_{\mathrm{Qnd}}(X, Y) := \{\ f : X \longrightarrow Y \mid {}^{\forall}x, y \in X,\ f(x \lhd y) = f(x) \lhd f(y)\ \}.$$

(III) A subset A of a quandle X is called *a subquandle (of X)* if A also forms a quandle under the operation $\lhd$.

Remark 2.3 (1) There are many papers on quandles, which denote the binary operation by $*$, $\star$, or $\rhd$ instead, and present the axiom (QIII) by left distribution. When reading papers on quandle, such differences require careful attention.
(2) The axiom (QIII) implies that the bijective map $\bullet \lhd c : X \to X$ for any $c \in X$ is a quandle homomorphism.
(3) Besides, for any quandles X and Y, every constant map $X \to Y$ is a quandle homomorphism. We often deal with these maps as something trivial.
(4) There are many cases that subsets $A, B, C \subset X$ satisfy $(A \lhd B) \lhd C \neq (A \lhd C) \lhd (B \lhd C)$.
(5) In seminar, it is useful to abbreviate quandle as "$qd'l$".

Next, we establish some examples of quandles, and observe a wide variety of quandles. The reader may skip the check for the quandle axioms hold since these checks are basic and direct calculations. As observed below, a quandle is a set consisting of, figuratively speaking, 'operations centered at $y \in X$ itself'. The hasty reader may skip some examples, but should understand Theorems 2.23 and 2.24. Actually, the theorems unify all the examples, and are keys in this book.

Example 2.4 (Trivial quandle) Any set X is with the operation $x \lhd y = x$ for any $x, y \in X$ is a quandle called the trivial quandle.

Example 2.5 (Dihedral quandle D_m) Consider the following situation:

$$X = D_m = \mathbb{Z}/m, \quad x \lhd y := 2y - x \qquad \text{for any } x, y \in X.$$

Then X is a quandle of type 2. Figuratively speaking, this operation $\bullet \lhd y$ is the reflection at y, where we identify X with the vertices of the regular m-sided polygon.

Example 2.6 (Alexander quandle) Let $\mathbb{Z}[T^{\pm 1}]$ be the Laurent polynomial ring. Then, any $\mathbb{Z}[T^{\pm 1}]$-module M is made into a quandle by the operation

$$x \lhd y := y + T(x - y) \tag{2.1}$$

for $x, y \in M$. We call this quandle an *Alexander quandle*, and later study it in detail (see Sect. B.3). The right operation $(\bullet \lhd y)$ with $y \in M$ can be geometrically compared to the T-multiple with center y; see Fig. 2.1. Furthermore, we can easily see $\mathrm{Type}(M) = \min\{N | T^N = 1\}$. (This example is generalized in Sect. 3.2.3).

In the following example, we can illustratively verify the distributive law (QIII) .

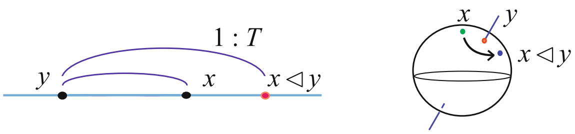


Fig. 2.1 Topological descriptions of the Alexander and spherical quandle operations

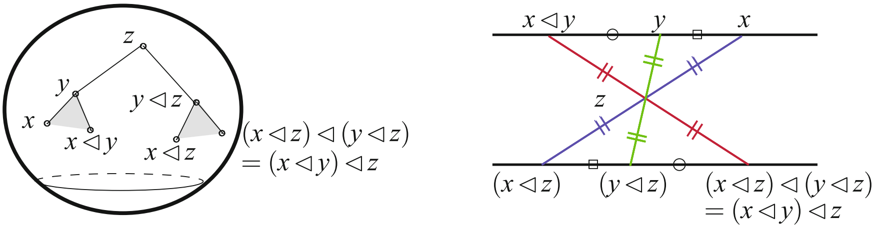


Fig. 2.2 Topological descriptions of the distributive law on S^2 and symmetric spaces (cf. Congruence transformation)

Example 2.7 (A quandle on the 2-sphere S^2) Let X be the 2-sphere S^2, and fix $\theta \in \mathbb{R}$. For two points $x, y \in S^2$, we define a map $(\bullet \triangleleft y) : S^2 \to S^2$ by the θ-rotation centered at y. So we can easily see $\mathrm{Type}(X) < \infty$ iff $\theta / 2\pi \in \mathbb{Q}$. To check the rest (QIII) can be done by observing the congruent transformation as in Fig. 2.2.

More generally, the concept of quandles contains the symmetric space in differential geometry.

Example 2.8 (Symmetric space) Let X be a symmetric space, i.e., a C^∞-manifold equipped with a Riemannian metric such that each point $y \in X$ admits an isometry $s_y : X \to X$ that reverses every geodesic line $\gamma : (\mathbb{R}, 0) \to (X, y)$, meaning that $s_y \circ \gamma(t) = \gamma(-t)$. Then, X has a quandle structure of type 2 defined by $x \triangleleft y := s_y(x)$ (see, e.g., [Loos, Joy, E3]). (Incidentally, similar Riemannian manifolds with quandle structure of type > 2 are called *generalized symmetric space*; see [Hel, K80]).

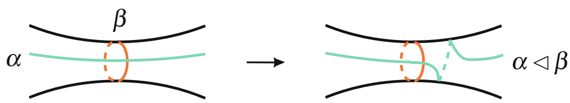
While the examples above are C^∞-manifolds, we can find further examples over commutative fields. Here are two quandles, in analogy of Lie groups of type B, C, and D (see [Tak] for details of the scheme structure).

Example 2.9 (Spherical quandle) Let F be a commutative field of characteristic $\neq 2$. Let $\langle , \rangle : F^{n+1} \otimes F^{n+1} \to F$ be the standard symmetric bilinear form. Consider

$$S^n_F := \{\, x \in F^{n+1} \mid \langle x, x \rangle = 1 \,\}.$$

We define the operation $x \triangleleft y$ to be $2\langle x, y \rangle y - x \in S^n_F$. This pair $(S^n_F, \triangleleft)$ is a quandle of type 2, and is referred to as *the spherical quandle* (over F). This operation $\bullet \triangleleft y$ can be interpreted as a $180°$-rotation centered at y.

Fig. 2.3 The Dehn twist



Example 2.10 (Symplectic quandle) Let F be a commutative field, and fix $r \in F \setminus 0$. Let Σ_g be the closed oriented surface of genus g. Let $\langle , \rangle$ denote the standard symplectic 2-form $H^1(\Sigma_g; F) \times H^1(\Sigma_g; F) \to F$. Letting X be the first cohomology with F-coefficients outside 0, and we define a binary operation

$$X = H^1(\Sigma_g; F) \setminus \{0\} = F^{2g} \setminus \{0\}, \qquad x \lhd y := r\langle x, y\rangle y + x \in X$$

Then this set X is made into a quandle, and is called *a symplectic quandle (over F)*. The operation $\bullet \lhd y : X \to X$ is usually called *the transvection* of y; see, e.g., [Jac, MR]. The quandle X is of type $p =$Char(F) since $x \lhd^N y = N r\langle x, y\rangle y + x$.

Furthermore, we can get a quandle from the closed surface Σ_g of genus g:

Example 2.11 (Dehn quandle) Consider the sets, $\mathscr{D}_g$ and $\mathscr{D}_g^{\mathrm{ns}}$, defined to be

$$\mathscr{D}_g := \{ \text{ isotopy classes of (unoriented) simple closed curves } \gamma \text{ in } \Sigma_g \}, \tag{2.2}$$

$$\mathscr{D}_g^{\mathrm{ns}} := \{ \text{ isotopy classes of non-separating simple closed curves } \gamma \text{ in } \Sigma_g \}.$$

Then, for α and $\beta \in \mathscr{D}_g$, we can consider $\tau_\beta(\alpha)$ that is called the (positive) Dehn twist of α along β; see Fig. 2.3.[1] Then, we define $\alpha \lhd \beta \in \mathscr{D}_g$ by $\tau_\beta(\alpha)$; The pair $(\mathscr{D}_g, \lhd)$ is a quandle, and called *the Dehn quandle*, according to [Zab]. Further, the subset $\mathscr{D}_g^{\mathrm{ns}} \subset \mathscr{D}_g$ is a subquandle. As seen in [Zab] or in Sect. 9.2, the Dehn quandle $\mathscr{D}_g$ is applicable to study 4-dimensional Lefschetz fibrations.

As is seen above, quandle structures can be roughly summarized to "operations centered at y on homogenous sets". In the next section, we will justify this imagery in mathematical terms.

Besides, given a group G, there exist many ways to construct quandles as follows.

Example 2.12 (Core quandle) A group $X = G$ as the quandle operation: $a \lhd b = ba^{-1}b$. This quandle is referred to as *the core quandle*, and is of type 2.

For instance, every Lie group G has a symmetric space as in Example 2.8; see [Hel]. The associated quandle structure is known to be this core quandle on G.

[1]The explicit definition is as follows: Regard every element α and β in $\mathscr{D}_g$ as an embedding of the anulus $S^1 \times [0, 1]$. Let $\tau_\beta(\alpha)$ be a map from Σ_g to itself which is the identity outside of Im(β) and inside Im(β) we have $f(s,t) = (se^{\sqrt{-1}2\pi t}, t)$. Then $\tau_\beta(\alpha)$ is a Dehn twist of α about the curve β; see also [FM] for details.

Example 2.13 (Conjugacy quandle) Let $X = G$. Then, the *conjugacy quandle* is defined to be X with quandle operation $a \lhd b = b^{-1}ab$. This quandle is *conjugacy quandle* written in $\mathrm{Conj}(G)$. In addition, any subset of G that is closed under conjugation is also a subquandle.

Example 2.14 More generally, for any $n \in \mathbb{Z}$, $X = G$ has a quandle operation $a \lhd b = b^{-n}ab^{n}$.

Example 2.15 (Coxeter quandle) If G is a Coxeter group (see Example B.16 for the definition), let X be the subset consisting of elements conjugate to the generators of G. Namely, X is the set of the reflections in G. Then, X can be regarded as a quandle of type 2.

Example 2.16 (Free quandle) Let I be a set of indices, and F_I be the free group of basis x_i with $i \in I$. Then *the free quandle* of X, denoted by Q_I^{free}, is the conjugacy class of x_i's ($i \in I$) with the conjugacy quandle operation, i.e., $Q_I^{\mathrm{free}} = \cup_{i \in I,\ g \in F_I} g^{-1}x_i g \subset F_I$.

2.2 Characterization Theorem of Quandles from Groups

The purpose is to show Theorems 2.23–2.24. These theorems indicate that any quandle is characterized by (a union of) the homogeneous quandles in Definition 2.17. Roughly speaking, quandle structures turn to be 'good' operations defined on homogenous spaces. To begin with,

Definition 2.17 *(Quandles on homogenous sets, Joyce [Joy] and Matveev [Mat])*
Let G denote a group, and H denote a subgroup of G. Let $\rho : G \to G$ be a group isomorphism such that $\rho(h) = h$ for any $h \in H$. Then, the left quotient $H\backslash G$ becomes a quandle with operation $[Hx] \lhd [Hy] := [H\rho(xy^{-1})y]$. It is easy to check the well-definedness.

For example, if ρ is the map $g \mapsto z_0^{-1} g z_0$ for some $z_0 \in G$ and if $z_0 \in G$ commutes with any elements of H, then the quandle operation on $H\backslash G$ is given by

$$[Hx] \lhd [Hy] = [Hz_0^{-1}xy^{-1}z_0 y] \in H\backslash G, \tag{2.3}$$

for any $x, y \in G$. We later denote this quandle by the triple (G, H, z_0).

Example 2.18 Some quandles in the preceding section are expressed by such triples (G, H, z_0) as follows:

- The dihedral quandle $X = \mathbb{Z}/m$. Here G is the dihedral group $\mathbb{Z}/m \rtimes \mathbb{Z}/2$, and $H = \{0\} \rtimes \mathbb{Z}/2 \ni (0, 1) = z_0$.
- The quandle on the 2-sphere S^2. Here $G = SO(3) \supset SO(2)$, $z_0 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

• The spherical quandle S^n_F. Here G is the orthogonal group $O(n+1; F)$, and $H := O(n; F)$ with $z_0 = \mathrm{diag}(1, \ldots, 1, -1)$.
• The symplectic quandle X; Here G is the symplectic group $Sp(g; F)$, with a certain subgroup H which contains $Sp(g-1; F)$; see Lemma B.2 for details.

In addition, to state the theorems, we introduce a group $\mathrm{As}(X)$:

Definition 2.19 Given a quandle X, we define *the associated group* $\mathrm{As}(X)$ to be the abstract group defined by generators e_x labeled by $x \in X$ modulo the relations $e_x \cdot e_y = e_y \cdot e_{x \lhd y}$ with $x, \; y \in X$: In other words, $\mathrm{As}(X)$ has the presentation

$$\mathrm{As}(X) = \langle \, e_x \;\; (x \in X) \;\mid\; e^{-1}_{x \lhd y} \cdot e^{-1}_y \cdot e_x \cdot e_y \;\; (x, y \in X) \, \rangle.$$

Note that every quandle homomorphism $X \to Y$ canonically induces a group homomorphism $\mathrm{As}(X) \to \mathrm{As}(Y)$. Hence, we can regard $\mathrm{As}(\bullet)$ as a functor from the category of quandles to that of groups. Further, we can easily show the adjointness by a routine work regarding adjoint functors: More precisely,

Lemma 2.20 *The functor* As *is the left adjoint to the functor* Conj*; see Example 2.13. Namely, for any quandle X and any group G, there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{gr}}(\mathrm{As}(X), \; G) \simeq \mathrm{Hom}_{\mathrm{Qnd}}(X, \; \mathrm{Conj}(G)).$$

In particular, by adjointness, we can define small (co)limits in the category of quandles (see [Mac, Chapter V.5] for the details).

Next, we introduce connectivity and state Theorem 2.23. Define a right action

$$X \curvearrowleft \mathrm{As}(X) \quad \text{by} \quad x \cdot e_y := x \lhd y$$

for $x, y \in X$. One can easily check the well-definedness.

Definition 2.21 Let X be a quandle. The *connected components* of X are the orbits of the action of $\mathrm{As}(X)$ on X. We denote the orbits by $O(X)$.

Further, a quandle X is said to be *connected* if the action of $\mathrm{As}(X)$ on X is transitive, i.e., $|O(X)| = 1$. In other words, X is connected iff every pair $(x, y) \in X^2$ admits $a_1, \ldots, a_n \in X$ such that $y = (\cdots (x \lhd^{\varepsilon_1} a_1) \lhd^{\varepsilon_2} \cdots) \lhd^{\varepsilon_n} a_n$ for some $\varepsilon_j \in \{\pm 1\}$.

Remark 2.22 For any quandle epimorphism $f : X \to Y$, if X is connected, so is Y.

Then, any connected quandle is reduced to nothing but Definition 2.17 as follows:

Theorem 2.23 ([Joy, Mat]) *Let X be a connected quandle and fix $x_0 \in X$. Let H be the stabilizer of x_0, i.e., $H = \mathrm{Stab}(x_0) = \{h \in \mathrm{As}(X) \mid x_0 \cdot h = x_0\}$. Equip the quotient $H \backslash \mathrm{As}(X)$ with a quandle operation from the triple $(\mathrm{As}(X), H, e_{x_0})$.*

Then, the map

$$E : \mathrm{As}(X) \longrightarrow X; \qquad \phi \longmapsto x_0 \cdot \phi$$

induces a quandle isomorphism $(\mathrm{As}(X), H, e_{x_0}) \cong X$.

Proof. Notice that, from a set-theoretical viewpoint, the map E descends to a bijection between the left quotient $H\backslash \mathrm{As}(X)$ and the quandle X. Hence, it is enough to show that E is a quandle homomorphism. Indeed, compute:

$$E(\phi \lhd \psi) = E(z^{-1}\phi\psi^{-1}z\psi) = ((((x_0 z^{-1})\phi)\psi^{-1})z)\psi = (((x_0\phi)\psi^{-1})\lhd x_0)\psi$$

$$= (((x_0\phi)\psi^{-1})\psi)\lhd(x_0\psi) = (x_0\phi)\lhd(x_0\psi) = E(\phi)\lhd E(\psi) \in X,$$

for any $\phi, \psi \in \mathrm{As}(X)$. Hence, this completes the proof. □

Remark. Such a representation as a triple (G, H, z) of X is not unique. For example, we may replace $\mathrm{As}(X)$ by "the inner automorphism group $\mathrm{Inn}(X)$" in Sect. 2.3.

Exercise 1 Describe explicitly quandle structures on all the regular polyhedra, which should be subquandles of the spherical quandle on S^2; see [HSV, Example 8.8] for the answer (cf. Platonic solid).

Exercise 2 Consider the $(3 \times 3 \times 3)$-rubic cube. As is known, every situation on the rubic cube (with certain orientation and boundary conditions) can be solved. In other words, some group transitively acts on every situation of the cube. So, describe explicitly an appropriate quandle structure of type 4 on the rubic cube.

Finally, some adjustments are needed to represent the non-connected case. Given a group G and an index set I, we fix elements z_i of G, and subgroups H_i of G with respect to each index $i \in I$. Assume that, for each $i \in I$, any element h_i in H_i satisfies the commutativity $z_i h_i = h_i z_i$. Then, we can define a quandle, denoted by

$$(H_i\backslash G;\ z_i \mid i \in I),$$

to be the disjoint union of the left quotients

$$X := \sqcup_{i\in I}(H_i\backslash G)$$

with the quandle operation

$$[H_i x]\lhd[H_j y] = [H_i z_i^{-1} x y^{-1} z_j y], \quad \text{for any } x, y \in G. \tag{2.4}$$

It is left to the reader to check the well-definedness and the quandle axioms.

Theorem 2.24 ([Joy, Mat]) *Every quandle X is representable as $(H_i\backslash G;\ z_i \mid i \in I)$.*

Proof. Let G be $\mathrm{As}(X)$, and I be the orbit set of the action $X \curvearrowleft \mathrm{As}(X)$. Decompose X as $\sqcup_{i\in I} X_i$ orbitwise. For each $i \in I$, we fix an element $x_i \in X_i$, and denote $e_{x_i} \in \mathrm{As}(X)$ by z_i, and let H_i be the stabilizer of x_i. Then, we have a quandle $(H_i\backslash G;\ z_i \mid i \in I)$. Define a map $E_i : G \to X_i$ by $\phi \mapsto x_i \cdot \phi$. Note the bijection between X_i and the left quotient $H_i\backslash \mathrm{As}(X)$ for any $i \in I$. In a similar way to the proof of Theorem 2.23, the disjoint union $\sqcup_{i\in I} E_i$ induces a quandle isomorphism $X \cong (H_i\backslash G;\ z_i \mid i \in I)$. □

2.3 Quandles and Centrally Extended Groups

As seen in the preceding theorems, it is important to determine $\mathrm{As}(X)$. In this section, we propose an outline to compute $\mathrm{As}(X)$. Here, we should emphasize a close relation between central extensions of groups and quandles

We start by introducing inner automorphism groups.

Definition 2.25 *The inner automorphism group*, $\mathrm{Inn}(X)$, is defined as the subgroup of the automorphism group generated by $(\bullet \lhd x)$, where x runs over every elements of X (here we should recall Remark 2.3). Concisely, the group is formulated by

$$\mathrm{Inn}(X) := \langle \ (\bullet \lhd x) \ \rangle_{x \in X} \ \subset \ \mathrm{Bij}(X, X).$$

For a general quandle X, it is not always easy to describe its group of inner automorphisms (or even its full automorphism group). However, in some familiar cases we can do so. For example,

Example 2.26 We consider a group $X = G$ to be the conjugacy quandle in Example 2.13. Then, by definitions, the inner automorphism group $\mathrm{Inn}(X)$ is exactly the usual one in group theory; i.e., $\mathrm{Inn}(X)$ is isomorphic to G/Z, where Z is the center of G.

In contrast to Lemma 2.20, quandle homomorphisms do not always yield group homomorphisms on $\mathrm{Inn}(\bullet)$; find such non-faithful examples as an exercise.

However, we will see that the group $\mathrm{Inn}(X)$ is useful to analyze $\mathrm{As}(X)$ in detail (see **Summary** below and Theorem 2.29). To see this, it is worth noting the equality

$$e_{x \cdot g} = g^{-1} e_x g \in \mathrm{As}(X) \qquad \text{for} \quad x \in X, \quad g \in \mathrm{As}(X), \tag{2.5}$$

which is shown by induction on the length of g. Regard the action of $\mathrm{As}(X)$ as a group epimorphism ψ_X from $\mathrm{As}(X)$ to $\mathrm{Inn}(X)$. Thus, we have a group extension

$$0 \longrightarrow \mathrm{Ker}(\psi_X) \longrightarrow \mathrm{As}(X) \xrightarrow{\psi_X} \mathrm{Inn}(X) \longrightarrow 0 \quad \text{(central extension)}. \tag{2.6}$$

Here, we should notice that this kernel of ψ_X is contained in the center (Indeed, apply $g \in \mathrm{Ker}(\psi_X)$ to (2.5)). Furthermore, recalling the 5-exact sequence associated with (2.6) [see [Bro, Wei1] for the proof], we immediately have an exact sequence

$$H_2^{\mathrm{gr}}(\mathrm{As}(X)) \to H_2^{\mathrm{gr}}(\mathrm{Inn}(X)) \to \mathrm{Ker}(\psi_X) \to H_1^{\mathrm{gr}}(\mathrm{As}(X)) \to H_1^{\mathrm{gr}}(\mathrm{Inn}(X)) \to 0. \tag{2.7}$$

(See Sect. 7.1 for the definition of group homology $H_n^{\mathrm{gr}}(G)$, though we need not it here.)

Summary. In order to analyse $\mathrm{As}(X)$, we compute the group homologies $H_1^{\mathrm{gr}}(\mathrm{As}(X))$ and $H_2^{\mathrm{gr}}(\mathrm{As}(X))$ and determine the group $\mathrm{Inn}(X)$.

For this purpose, we begin by studying $H_1(\mathrm{As}(X))$:

Lemma 2.27 *With respect to an element, i, in the orbit (i.e., $i \in O(X)$), define a group homomorphism*

$$\varepsilon_i : \mathrm{As}(X) \longrightarrow \mathbb{Z} \quad \text{by} \begin{cases} \varepsilon_i(e_x) = 1 \in \mathbb{Z}, & \text{if } x \in X_i, \\ \varepsilon_i(e_x) = 0 \in \mathbb{Z}, & \text{if } x \in X \setminus X_i. \end{cases} \tag{2.8}$$

Then, the direct sum $\oplus_{i \in O(X)} \varepsilon_i$ yields the abelianization $H_1(\mathrm{As}(X)) \cong \mathbb{Z}^{\oplus O(X)}$.

If X is connected, i.e., $|O(X)| = 1$, then the epimorphism ε_i splits. In particular,

$$\mathrm{As}(X)_{\mathrm{ab}} \cong \mathbb{Z}, \quad \text{and} \quad \mathrm{As}(X) \cong \mathrm{Ker}(\varepsilon_i) \rtimes \mathbb{Z}.$$

Proof. The equality (2.5) means that $\varepsilon_i(e_x) = \varepsilon_i(e_y) \in \mathbb{Z}$ if and only if x and y are contained in the same orbit. Hence, the sum $\oplus_{i \in O(X)} \varepsilon_i$ is the maximum map among abelian groups, that is, the abelianization.

The latter statement is clear, since $\mathbb{Z}$ is free and is generated by $\varepsilon_i(e_x)$'s. □

Since we later use this lemma, the reader should keep it in mind. Next, we will see that the concept of type is important for studying the centrality of $\mathrm{As}(X)$:

Lemma 2.28 *Let X be a connected quandle of type $t < \infty$. Then, for any $x, y \in X$, the identity $(e_x)^t = (e_y)^t$ holds in the central kernel $\mathrm{Ker}(\psi_X)$.*

In particular, (2.7) implies that $H_1(\mathrm{Inn}(X))$ is annihilated by t.

Proof. Since $b \lhd^t x = b$ in X by definition, $(e_x)^t$ lies in the kernel $\mathrm{Ker}(\psi_X)$. Further, the connectivity admits $g \in \mathrm{As}(X)$ such that $x \cdot g = y$. Hence, it follows from (2.5) and centrality that $(e_x)^t = g^{-1}(e_x)^t g = (e_{x \cdot g})^t = (e_y)^t$ as desired. □

Finally, we state a theorem on $H_2^{\mathrm{gr}}(\mathrm{As}(X))$ as a useful estimate:

Theorem 2.29 *For any connected quandle X of type t_X (possibly, X could be of infinite order), the second group homology $H_2^{\mathrm{gr}}(\mathrm{As}(X))$ is annihilated by t_X.*

In particular, for any prime $\ell \in \mathbb{Z}$ which is relatively prime to t_X, the ℓ-localization of the sequence (2.6) gives the isomorphism

$$\mathrm{Ker}(\psi_X)_{(\ell)} \cong \mathbb{Z}_{(\ell)} \oplus H_2^{\mathrm{gr}}(\mathrm{Inn}(X))_{(\ell)}.$$

The proof will appear in Sect. 6.2. In conclusion, metaphorically speaking, $\mathrm{As}(X)$ turns out to be the 'universal central extension' of $\mathrm{Inn}(X)$ up to t_X-torsion; hence, this theorem emphasizes importance of the concept of types. Furthermore, as seen in Sect. B.3, one can sometimes determine $\mathrm{As}(X)$ by using this theorem.

2.4 Link Quandles and Their Properties

In this section, we turn to knot theory from the homological view of quandle.

In knot theory (or the geometrization theorem), the fundamental *group* of the complementary space is quite important. Analogously, this section studies fundamental *quandles* of links, and sees that the concept goes well with the knot theory.

Since this section uses some knowledge of knot theory, we roughly review some notation (see Fig. 2.4 and Appendix A for more details). *A link* is an oriented C^∞-embedding of $L : \sqcup^m S^1 \hookrightarrow S^3$. *A link diagram* is a pair of a transverse immersion $D : \sqcup S^1 \to \mathbb{R}^2$, "with information of signs of crossings". Every link is represented by some link diagram D, via a canonical projection $S^3 \setminus \{\infty\} = \mathbb{R}^3 \to \mathbb{R}^2$; see Reidemeister theorem A.7. Then, $\pi_1(S^3 \setminus L)$ has a finite group presentation, which is described by the arcs and crossings of D; see Wirtinger presentation in Theorem A.8. Furthermore, when we use some theorem in this section, we cite it from Appendix A case by case.

Next, using homogenous quandles defined in (2.4), we algebraically introduce

Definition 2.30 Let L be a link embedded in S^3 with $\#L$-components, and I be $\{1, \dots, \#L\}$. For $\ell \in I$, we fix a meridian longitude pair $\mathfrak{m}_\ell,\ \mathfrak{l}_\ell \in \pi_1(S^3 \setminus L)$ as in Sect. A. Then, *the link quandle (of L)*, Q_L, is defined to be the quandle arising from $(H_\ell \backslash G; z_\ell \mid \ell \in I)$, where we set

$$G := \pi_1(S^3 \setminus L), \qquad H_\ell := \langle \mathfrak{m}_\ell,\ \mathfrak{l}_\ell \rangle, \qquad z_\ell := \mathfrak{m}_\ell.$$

When $\#L = 1$, we often call Q_L *the knot quandle (of L).*

We can easily check that Q_L is independent of the choices of $(\mathfrak{m}_\ell,\ \mathfrak{l}_\ell)$, up to quandle isomorphisms. Further, we note that the link quandle Q_L recovers the fundamental group $\pi_1(S^3 \setminus L)$:

Theorem 2.31 ([Joy, Mat]) *For any link L, the associated group* $\mathrm{As}(Q_L)$ *is isomorphic to the fundamental group $\pi_1(S^3 \setminus L)$.*

Proof. Let us consider a homomorphism $\mathrm{As}(Q_L) \to \pi_1(S^3 \setminus L)$ which takes $e_{H_\ell x}$ to $x^{-1}\mathfrak{m}_\ell x$. Thus, we shall construct the inverse mapping hereafter.

For this, we prepare notation. Fix a link diagram D. For each ℓ, we consider the path $\mathscr{P}_\ell$ along the longitude $\mathfrak{l}_\ell$ as illustrated in Fig. 2.5. Furthermore, let $\alpha_1, \alpha_2, \dots, \alpha_{N_\ell}, \alpha_{N_\ell+1} = \alpha_1$ be the over-path on this $\mathscr{P}_\ell$, and let β_k be the path that

Fig. 2.4 Examples; a link diagram with three arcs, and a meridian-longitude pair

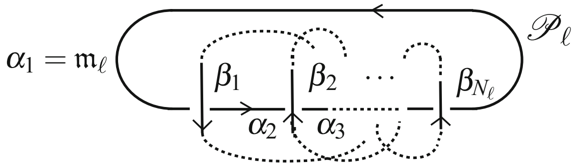


Fig. 2.5 Le Trompe-l'œil; The arcs α_i's and β_i around the ℓ-th link component (where we ignore under arcs). The reader should keep this figure in mind, since we use it later in several times.

divides α_k and α_{k+1}. Here α_1 corresponds to the meridian $\mathfrak{m}_\ell$. Denote by $\varepsilon_k \in \{\pm 1\}$ the sign of the crossing between α_k and β_k.

Next, consider the Wirtinger presentation of $\pi_1(S^3 \setminus L)$ (see Theorem A.8), which are generated by α_i's. With abuse of notation, we regard a meridian circulating around the arc β_i as an element of $\pi_1(S^3 \setminus L)$, and denote it by β_i as well. Then, the longitude $\mathfrak{l}_\ell$ is expressed as

$$\mathfrak{l}_\ell := \beta_1^{\varepsilon_1} \beta_2^{\varepsilon_2} \cdots \beta_{N_\ell}^{\varepsilon_{N_\ell}} \in \pi_1(S^3 \setminus L). \tag{2.9}$$

In the situation above, the correspondence $\alpha_i \mapsto e_{(H_\ell \beta_1^{\varepsilon_1} \beta_2^{\varepsilon_2} \cdots \beta_{i-1}^{\varepsilon_{i-1}})}$ gives rise to a group homomorphism $\pi_1(S^3 \setminus L) \to \mathrm{As}(Q_L)$; check the well-definedness. By construction, this is the desired inverse mapping. □

As a consequence of this theorem, it is sensible to ask whether the link quandle has more useful information than $\pi_1(S^3 \setminus L)$ or not.

As a typical case, we will observe that quandle is qualitatively useful in the knot case, i.e., $\#L = 1$. To see this, we should notice that, if L is a tame knot K, i.e., $\#L = 1$, the link quandle Q_K is connected. Further, notice from Theorem A.3 that the subgroup $H_\ell = \langle \mathfrak{m}_\ell, \mathfrak{l}_\ell \rangle$ is isomorphic to $\mathbb{Z}^2$ and $H_\ell \neq \pi_1(S^3 \setminus K)$ unless K is the unknot.[2] Hence, Q_K is a single point if and only if K is the unknot.

Next, we show a corollary, which concludes that the knot quandle is a complete knot-invariant (up to $K \sim (-K)^*$); see Appendix A for the definition of the mirror image $(-K)^*$ with opposite orientation.

Corollary 2.32 ([Joy, Mat]) *Let K and K' be two oriented knots in the 3-sphere. Then, K' is ambient isotopic to either K or $(-K)^*$ if and only if there exists a quandle isomorphism between the knot quandles Q_K and $Q_{K'}$.*

Proof. Since the "if" part is clear, we show the converse. Neither of K and K' may be the unknot. Then, by Theorem 2.31, the assumption implies a quandle isomorphism $(\pi_1(S^3 \setminus K), H_\ell, \mathfrak{m}) \cong (\pi_1(S^3 \setminus K'), H'_\ell, \mathfrak{m}')$. Namely, this is exactly the condition in the classifying theorem A.2, immediately leading to $K' \simeq K$ or $K' \simeq -K^*$. □

In comparison, there are many pairs of nonisotopic knots which have the same fundamental group (however, there is a classifying theorem of prime knots, only using the fundamental groups; see [BZ, Lic]).

[2] For links with $\#L > 1$, H_ℓ is not always $\mathbb{Z}^2$ in $\pi_1(S^3 \setminus L)$; e.g., consider a union of unknots.

That being said, it is true that the knot quandle Q_K is a complete knot invariant, but it is essentially difficult to directly analyze the algebraic system Q_K, in general. So, in this book, in order to study the knot K, we often relatively deal with a homomorphism $Q_K \to X$ for some quandle X. Such homomorphisms will be studied in Sect. 3.1, as X-colorings (see Proposition 3.7). Here, it is worth noticing from Remark 2.22 that X shall be assumed to be connected.

Furthermore, in advance of such studies, we now observe that every finitely generated quandle is immanently useful to some knot. Here a quandle X is *finitely generated* if there are a finite index-set I and a surjective quandle homomorphism from the free quandle Q_I^{free} to X.

Proposition 2.33 *Let X be a connected quandle. Then, there are a knot K and a quandle epimorphism $f : Q_K \to X$, if and only if X is finitely generated.*

Proof. Since the "only if" part is clear, we show the converse. Here the key is Theorem A.4. Fix $x_0 \in X$. Since $\mathrm{As}(Q_I^{\mathrm{free}}) \cong F_I$, the epimorphism $F_I \to \mathrm{As}(X)$ implies that $\mathrm{As}(X)$ is finitely generated. Since $\mathrm{As}(X)$ is generated by $\{g^{-1}e_{x_0}g\}_{g\in \mathrm{As}(X)}$ by connectivity, Theorem A.4 ensures a knot K with meridian $\mathfrak{m}$ and a group epimorphism $f : \pi_1(S^3 \setminus K) \to \mathrm{As}(X)$ with $f(\mathfrak{m}) = e_{x_0}$. Since $X \cong (\mathrm{As}(X), \mathrm{Stab}(x_0), e_{x_0})$ by Theorem 2.23, we obtain a quandle epimorphism $Q_K \to X$ as desired. $\square$

Conclusion. It is sensible for studying the knot quandle to relatively consider other quandles X which are connected and finitely generated, and to analyse the group $\mathrm{As}(X)$. However, for the sake of knot invariants, we need to develop methods to get something quantitative from quandles, as in Chaps. 3–8.

2.5 Appendix: Historical and Topological Comments on Quandles

In this appendix, we comment a history of quandles, and give a topological interpretation of the link quandle, although there are other surveys written by J.S. Carter [Car] and S. Kamada [Kam1] (see also the dissertation [Joy3]).

Historical comments on quandles.

It is often said that the original model of quandle is first introduced by Takasaki [T] in 1943. Since the paper was written in old characters of Japanese and in the WW2, the concept was not widely known in the world. The idea was rediscovered and generalized in (unpublished) 1959 correspondence between John Conway and Gavin Wraith. It is heard that the modern definition of quandles first appears.

These efforts surfaced again in the 1980's with applications to knot theory; by Joyce [Joy] (where the term *quandle* was coined), by Matveev [Mat] (under the name *distributive groupoids*), and by Brieskorn [Bri] (where they were called *automorphic* *sets*). A detailed overview of racks [3] may be seen in the paper by Rourke, Fenn and Sanderson [FR] [FRS1]–[FRS3]. Further, quandles have been intensively studied by several authors and under various names, for example, as "crossed G-sets" by Freyd and Yetter [FY], as "crystals" by Kauffman [Kau].

We further mention the work [T] of Takasaki. His motivation was to find a distributive algebraic structure to capture the notion of a reflection in the context of finite geometry, in comparison with the symmetric space in differential geometry. The algebraic structure would later come to be known as a quandle of type 2, and he called it *Kei*. In origin, the term *Kei*, written in 圭 (Pinyin: guī), is a Chinese character that means a jade tablet of a triangular shape which is used officials in ancient China when addressing the emperor in court. Inspired by it, Japanese mathematicians until the 19th century used the term *Kei* to briefly express an isosceles triangle in the 2-dimensional plane $\mathbb{R}^2$. So, it is believed that Takasaki employed the term.

Topological construction of the link quandle.
Changing the subject, we roughly explain a topological interpretation of the link quandle (see [Joy, Sect. 12] for details). Loosely speaking, the quandle is compatible with topological pairs of codimension two.

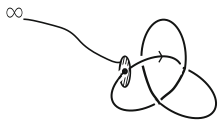
In general, let M be a connected oriented C^∞-manifold with basepoint ∞, and $L \subset M$ an oriented submanifold of codimension 2. Let N denote an open tubular neighborhood of L. *The fundamental quandle*, $Q(M, N)$, is defined to be the set of homotopy classes of continuous maps

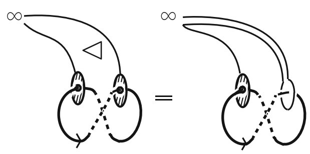
$$\mu : ([0,1],\ \{0\},\ \{1\}) \longrightarrow (M \setminus N,\ \partial\overline{N},\ \infty).$$

Here, the homotopies between such maps are required to have their bottom boundaries on ∂N and their top boundaries fixed at the base point. Namely,

$$^{\exists}H : [0,1] \times [0,1] \to M, \ \text{ s.t. } \ H(s,0) \subset \partial N, \ \ H(s,1) = \{\infty\}, \ H(i,t) = \mu_i(t).$$

Given two such maps μ and ν, then there is uniquely an oriented meridian $\mathfrak{m}_\nu \subset N$ that passes through the initial point of ν. Then, the quandle operation is defined as the path composition $\mu \lhd \nu := \mu\nu^{-1}\mathfrak{m}_\nu\nu$; see the right hand side of the figure below. This operation is known to be compatible with pushforward in some sense: actually, Joyce discussed van Kampen theorem on fundamental quandles; see [Joy, Sect. 13] (for unoriented knots, Kamada and Oshiro [KO] studied "symmetric quandles", and the associated link quandles; see [Kam1]).





[3] A *rack* is a is a set X with a binary operation $\lhd : X \times X \to X$ satisfying only the two conditions (QII) and Q(III). In particular, any quandle is a rack.

We end this appendix by considering three examples.

First, consider the case where M is S^3 and N is a link embedded in S^3. Then, it can be verified that $Q(M, N)$ is isomorphic to the link quandle Q_L. Indeed, $\pi_1(S^3 \smallsetminus L)$ canonically acts on $Q(M, N)$ by considering the canonical connection between a loop and a path, and $Q(M, N)$ is represented as $(H_\ell \backslash G; z_\ell \mid \ell \in I)$ as in Definition 2.30.

Next, let M be the 2-sphere S^2 and N be the set consisting of n-points on S^2. It is shown [N13] that the Hopf fibration $S^3 \to S^2$ induces a quandle isomorphism between the link quandle of the (n, n)-torus link $T_{n,n}$ and the fundamental quandle $Q(M, N)$. In particular, $\mathrm{As}(Q(M, N))$ turns out to be $\pi_1(S^3 \smallsetminus T_{n,n}) \cong \pi_1(S^2 \smallsetminus N) \times \mathbb{Z}$. In general, the group $\mathrm{As}(Q(M, N))$ is not always the fundamental group $\pi_1(M \smallsetminus N)$.

Exercise 3 Show that, if L is the Hopf link, then Q_L is the trivial quandle of order 2, and that if L is the (n, n)-torus link with $n > 2$, then Q_L is of infinite order.

Exercise 4 ([NP]) Let L be the trefoil knot 3_1. Then, the knot quandle Q_L is isomorphic to the Dehn quandle $\mathscr{D}_g$ with $g = 1$. (cf. the fact of D. Quillen that the moduli space $\mathscr{M}_{1,1}$ is homotopic to $S^3 \smallsetminus L$).

Finally, we consider an embedding $K : S^2 \times D^2 \to S^4$, which is called *a 2-knot*. So we get the fundamental quandle $Q(S^4, \mathrm{Im}(K))$. However, since the quandle often is of finite order, it is natural to anticipate that invariants arising from quandle seem something weak. For example, if K is "the 2-twist spin of the $(p, 2)$-torus knot", $Q(S^4, \mathrm{Im}(K))$ is isomorphic to the dihedral quandle of order p.

Exercise 5 Look over the meaning of the twisting spun knot, and show the quandle isomorphism $Q(S^4, \mathrm{Im}(K)) \cong D_p$.

http://www.springer.com/978-981-10-6792-1

Quandles and Topological Pairs
Symmetry, Knots, and Cohomology
Nosaka, T.
2017, IX, 136 p. 25 illus., 11 illus. in color., Softcover
ISBN: 978-981-10-6792-1