

Set Topology

Somnath Basu and Atreyee Bhattacharya

In the first part we discuss metric spaces and continuous maps between them. Reformulating the notion of continuity leads to the concept of topological spaces. Some important properties of such spaces are discussed. We end with the concept of topological manifolds, as a particular class of examples of topological spaces.

The second part is a tutorial on equivalence relations and quotient sets. The main aim here is to recall a few relevant definitions in this context and more importantly make the reader comfortable with these definitions by providing a series of examples. Examples are discussed keeping in mind their significance in advanced topics of Mathematics such as topology and geometry.

I Topology: A Quick Review

2.1 Equivalence relation

Let us start with the notion of an equivalence relation. This shall be used throughout in several examples that we will encounter.

Definition 2.1.1. Let X be a set. An *equivalence relation* \sim on X is an identification between elements of X satisfying:

- (a) [reflexive] $x \sim x$ for any $x \in X$;
- (b) [symmetric] if $x \sim y$ then $y \sim x$;
- (c) [transitive] if $x \sim y$ and $y \sim z$ then $x \sim z$.

The equivalence class of x , denoted by $[x]$, is the set of all elements in X that are related to x .

The set of equivalence classes will be denoted by X/\sim . We shall see a host of examples and identify X/\sim with standard objects from analysis, geometry and topology. However, the identifications, for now, are just bijections between sets. In later sections we shall revisit some of these examples and prove that the identifications can be upgraded to an equivalence of *topological spaces*.

Example 2.1.2. An equivalence relation on X can be specified by giving a subset of $X \times X$ satisfying the three properties in Definition 2.1.1.

(1) Let $X = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$. The equivalence classes are $[1] = [2]$ and $[3]$.

(2) Let $X = \mathbb{R}$ and $x \sim y$ if $x - y \in \mathbb{Z}$. Then it can be seen that X/\sim is in bijective correspondence with the unit circle S^1 in \mathbb{R}^2 . The map that sends $[t]$ to $e^{2\pi it}$ is one such bijection.

(3) Let $X = V$ be a finite dimensional vector space over \mathbb{R} . Given a subspace W , we define $v_1 \sim_W v_2$ if $v_1 - v_2 \in W$. This defines an equivalence relation and the resulting set V/\sim_W (usually denoted by V/W) can be given the structure of a vector space. It is tempting to say that V/W is in bijection with W^\perp , the orthogonal complement of W in V . However, there is no such natural bijection. In case V has a positive definite inner product then the orthogonal projection of V to W^\perp induces a linear isomorphism from V/W to W^\perp .

(4) Let $X = V - \{0\}$ with V as in (3) above. Consider the equivalence relation where $v \sim w$ if $v = \lambda w$ for some $\lambda \in \mathbb{R} - \{0\}$. The set of equivalence classes is denoted by $\mathbb{P}(V)$ and called the *space of lines* in V or *projectivization* of V .

(5) Let $X = G$ be a group. We say $g_1 \sim g_2$ if $g_1 = gg_2g^{-1}$. The equivalence classes are called the *conjugacy classes* of G . For an abelian group, the conjugacy relation is simply the reflexive relation on G .

(6) Let \mathbb{D}^n denote the closed unit disk in \mathbb{R}^n . The equivalence relation generated by identifying all points on \mathbb{D}^n of length 1 is denoted by \sim . Then it can be shown that \mathbb{D}^n/\sim is identifiable with S^n , the unit sphere in \mathbb{R}^{n+1} .

These are not surprising examples but should be kept in mind. In a later section we shall expect the unfamiliar reader to go through these armed with the notion of *quotient topology*.

2.2 From metric spaces to topology

Let us recall the familiar notion of metric spaces.

Definition 2.2.1. A set X equipped with a non-negative real valued function $d : X \times X \rightarrow \mathbb{R}$ is called a *metric space* if the following holds:

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) d is symmetric, i.e., $d(x, y) = d(y, x)$;
- (iii) d satisfies the *triangle inequality*, i.e., $d(x, y) + d(y, z) \geq d(x, z)$.

We shall denote a metric space by (X, d) , or simply by X when d is clear in the context. The function d is called the *distance function*. To each point x

in a metric space (X, d) and every $\epsilon > 0$ we have the set

$$B_\epsilon(x) := \{y \in X \mid d(x, y) < \epsilon\}$$

called the *open ball* of radius ϵ , centered at x . Given $y \in B_\epsilon(x)$ with $r = d(x, y) < \epsilon$, we observe that $B_{\epsilon-r}(y) \subseteq B_\epsilon(x)$. When $\epsilon = 0$ then $B_\epsilon(x) = \emptyset$.

Example 2.2.2. We present several well-known examples.

- (1) Let d be the Euclidean distance on $X = \mathbb{R}^n$. The open balls in this metric are actually the open balls in the usual sense.
- (2) Let $X = \mathbb{R}^n$ with

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) := \max_i \{|x_i - y_i|\}.$$

It can be seen that the open balls (of radius ϵ) are open cubes with side length 2ϵ .

- (3) Let X be any set and let $d(x, y) = 1$ if $x \neq y$. It follows that open balls are either the point or the whole space.

- (4) Given a metric space (X, d) we may consider a new function

$$d(x, y) := \frac{d(x, y)}{1 + d(x, y)}.$$

This defines a distance function and (X, d) becomes a metric space where the distance between any two points is uniformly bounded by 1. Note that in the new metric $B_1(x) = X$ for any $x \in X$.

Let τ_d consist of all subsets of X which are arbitrary unions of finite intersections of open balls. By construction τ_d contains \emptyset and X . Moreover, it is closed under arbitrary unions. Given $U, V \in \tau_d$ one verifies that $U \cap V$ is also in τ_d . This implies that τ_d is closed under finite intersections. This collection τ_d is called the *topology induced by the metric* on X .

Remark 2.2.3. It is well-known that the topology induced by the metrics in (1) and (2) are the same, i.e., they both have the same collection of open sets. In the case of (3) we see that $\tau_d = \mathcal{P}(X)$, the power set of X . For (4) one can check that $\tau_d = \tau_d$.

Before we explain why this passage from d to τ_d is important, we recall the notion of convergence of sequences and continuous maps (between metric spaces).

Definition 2.2.4. Let (X, d) be a metric space. A sequence $\{x_n\}_{n \geq 1} \subseteq X$ is said to *converge* to $x \in X$ if given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$.

A map $f : (X, d) \rightarrow (Y, d)$ is called *continuous* if it maps convergent sequences to convergent sequences, i.e., if $\{x_n\}$ is a sequence converging to x then $\{f(x_n)\}$ converges to $f(x)$.

Note that a sequence (in a metric space) can converge to at most one point. An alternate but equivalent definition of continuity states that a map is *continuous at x* if for any $\epsilon > 0$ there exists $\delta > 0$, depending on ϵ and x , such that if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$. A function is called *continuous* if it is continuous at each point. If a common δ , independent of x , exists for a given ϵ then we say that f is *uniformly continuous*.

The significance of τ_d stems from the following key observation.

Proposition 2.2.5. *Let $f : (X, d) \rightarrow (Y, d)$ be a map between metric spaces. Then the following are equivalent:*

- (1) *the map f is continuous;*
- (2) *if $U \in \tau_d$ then $f^{-1}(U) \in \tau_d$.*

Proof. Let $\{x_n\}$ be a sequence converging to x . Assuming (2) and setting $U = B_\epsilon(f(x))$ we gather that

$$f^{-1}(U) = \{w \in X \mid d(f(w), f(x)) < \epsilon\} \in \tau_d.$$

Thus, $f^{-1}(U) = \cup_{i \in I} U_i$, where U_i is a finite intersections of open balls in X . Choose an U_i such that $x \in U_i$. It follows that there exists $\delta > 0$ such that $B_\delta(x) \subseteq U_i$. Moreover, there exists N such that $x_n \in B_\delta(x)$ for all $n \geq N$. In particular, $f(x_n) \in U$ for all such n .

To prove the converse it suffices to show that $f^{-1}(B_\epsilon(y)) \in \tau_d$ for any $y \in Y$. Let $x \in f^{-1}(B_\epsilon(y))$, i.e., $d(f(x), y) = r < \epsilon$. By continuity of f at x there exists δ (possibly depending on x) such that if $w \in B_\delta(x)$ then $d(f(w), f(x)) < \epsilon - r$. By the triangle inequality, $d(f(w), y) < \epsilon$. Consequently, $B_\delta(x) \subseteq f^{-1}(B_\epsilon(y))$ and the latter is a union of $B_\delta(x)$ as x varies. \square

2.3 Topological spaces: Definition and examples

Having extracted the abstract essence of continuity, we now define topological spaces.

Definition 2.3.1. A *topological space* (X, τ) is a set X equipped with a collection τ of subsets of X satisfying the following:

- (i) the empty set and X are in τ ;
- (ii) the collection τ is closed under finite intersections;
- (iii) the collection τ is closed under arbitrary unions.

An element $U \in \tau$ is called an *open set* while its complement is called a *closed set*.

Definition 2.3.2. A map $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ between topological spaces is called *continuous* if $f^{-1}(\tau_Y) \subseteq \tau_X$. If f maps open (respectively closed) sets to open (respectively closed) sets then f is called an *open* (respectively *closed*) map.

Given two topologies τ, τ' on X , we say they are the same if $\tau = \tau'$.

Remark 2.3.3. The condition for continuity of a function can be restated in terms of closed sets by saying that inverse images of closed sets should be closed sets.

Observe that metric spaces are topological spaces and different metrics can induce the same topology (cf. Remark 2.2.3). A topological space (X, τ) is called *metrizable* if there is a metric which induces the topology τ . If a space X is metrizable then any two distinct points *can be separated by open sets*, i.e., given $x \neq y$ there exists $U, V \in \tau$ such that $U \cap V = \emptyset$ and $x \in U, y \in V$. This follows by setting $r = d(x, y)$, $U = B_{r/2}(x)$ and $V = B_{r/2}(y)$. Topological spaces where points can be separated by open sets are called *Hausdorff*.

Example 2.3.4. Let us consider several standard examples of topology on a given set X .

- (1) The *trivial topology* on X is $\tau_{\text{tr}} = \{\emptyset, X\}$. Any map $f : (Y, \tau) \rightarrow (X, \tau_{\text{tr}})$ is continuous while f is open if $f(U) = X$ for every non-empty open set U .
- (2) The *discrete topology* on X is $\tau_{\text{dis}} = \mathcal{P}(X)$. This corresponds to Example 2.2.2 (3). Any map $f : (X, \tau_{\text{dis}}) \rightarrow (Y, \tau)$ is continuous.
- (3) The *cofinite topology* on X is the collection consisting of complements of finite sets along with the empty set. If X is finite then this is exactly the discrete topology. However, if X is infinite then any two non-empty open sets intersect. In this case the topology is not Hausdorff and cannot be induced by a metric. For instance, a map $f : (\mathbb{R}, \tau_{\text{cofin}}) \rightarrow (\mathbb{R}, \tau_d)$ with d as in Example 2.2.2 (2) is continuous if and only if it is constant. The cofinite topology on \mathbb{R} is often called the *Zariski topology*.

The usual topology on the real line has open sets which are finite or countable disjoint unions of sets of the form $(-\infty, b)$, (a, b) , (a, ∞) . Topological spaces can be weird in the sense that some of its properties may deviate sharply from what we are used to in Euclidean spaces and its open subsets. We present a few examples below.

Example 2.3.5. (1) The *profinite topology* on \mathbb{Z} consists of open sets which are defined to be arbitrary unions of (non-constant) arithmetic progressions that extend in both directions. This topology is neither discrete nor trivial.

(2) Given a space (X, τ) and a subset $A \subseteq X$ we define the *induced topology* on A by declaring $\tau_A := \{U \cap A \mid U \in \tau\}$ to be a topology on A . For instance, the induced topology on \mathbb{Q} from the standard real line is different than $(\mathbb{Q}, \tau_{\text{dis}})$.

(3) The real line admits another, quite different, topology. Consider $X = \mathbb{R}$ and $\tau = \{(-r, r) \mid r > 0\} \cup \{\emptyset, \mathbb{R}\}$. This topology is not Hausdorff. Moreover, the identity map $i : (\mathbb{R}, \tau) \rightarrow \mathbb{R}$, where the latter is the standard topology on the real line, is not continuous. If we switch the topology then the identity map is continuous.

(4) Finite sets as topological spaces are rather interesting objects. There are exactly four topologies on $X = \{a, b\}$ given by $\tau_{\text{tr}}, \tau_{\text{cofin}}, \tau_1 = \{\emptyset, \{a\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, X\}$. A well-known example is the *pseudo-circle*, a non-Hausdorff

topological space $X = \{a, b, c, d\}$ with

$$\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}.$$

This space is weakly homotopy equivalent to the circle.

The notion of convergent sequences make sense for topological spaces as well.

Definition 2.3.6. A sequence $\{x_n\}$ in a topological space (X, τ) is said to be *convergent* if there exists $x \in X$ such that for any open set U containing x there exists $N \in \mathbb{N}$ such that $\{x_n\}_{n \geq N} \subseteq U$. We say that the sequence *converges* to x and that x is a *limit* of the sequence.

If we consider the sequence $\{1, -1, 1, -1, \dots\}$ in Example 2.3.5 (3) then this converges to both -1 as well as 1 . Although a convergent sequence may have several limits, in a Hausdorff space this limit is unique. This is one of the main reasons to work primarily with Hausdorff spaces in analytical geometry and topology.

For a subset A of a space X , the *interior* of A is defined to be the largest open set contained in A , i.e.,

$$\text{Int } A = \bigcup_{U \in \tau, U \subseteq A} U.$$

The *closure* of A is defined to be the smallest closed set containing A , i.e.,

$$\overline{A} = \bigcap_{F^c \in \tau, A \subseteq F} F.$$

The *boundary* of A is the set $\overline{A} - A$. The set A is called *dense* if $\overline{A} = X$.

2.4 Topological spaces: Some key properties

In order to distinguish between topological spaces we need an appropriate notion of equivalence.

Definition 2.4.1. A map $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is called a *homeomorphism* if f is a bijection, and both f and f^{-1} are continuous. We say two topological spaces are *equivalent* (or *homeomorphic*) if there exists a homeomorphism between the two spaces.

It can be shown that any open ball in \mathbb{R}^n is homeomorphic to \mathbb{R}^n (in the standard topology). The requirement that both f and f^{-1} be continuous is necessary as a continuous bijection need not have a continuous inverse. To see this, consider the unit circle S^1 with the topology inherited from \mathbb{R}^2 . The map

$$f : [0, 2\pi) \rightarrow S^1, \quad t \mapsto e^{it}$$

is continuous and bijective but f is not open. Using stereographic projection we may prove that S^1 minus a point is homeomorphic to \mathbb{R} . This generalizes to higher dimensions for the unit sphere S^n in \mathbb{R}^{n+1} . Let us now look at a slightly more interesting construction.

Example 2.4.2. Consider $\widehat{\mathbb{R}^n} := \mathbb{R}^n \sqcup \{\infty\}$ equipped with the topology

$$\tau = \tau_d \cup \{(\mathbb{R}^n - K) \cup \{\infty\} \mid K \text{ is a closed and bounded subset of } \mathbb{R}^n\}.$$

It can be shown that $\widehat{\mathbb{R}^n}$ is homeomorphic to S^n .

Several key properties which are preserved under continuity are often used to distinguish between topological spaces.

Definition 2.4.3. A topological space (X, τ) is called *connected* if X cannot be written as the disjoint union of two non-empty open sets. It is called *path-connected* if for any two points $x, y \in X$ there exists a *path* joining x to y , i.e., continuous map $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x, \gamma(1) = y$.

It is clear that these properties are preserved under homeomorphisms. In fact, the image of a connected (respectively path-connected) space under a continuous map is connected (respectively path-connected). Euclidean spaces with the usual metric/topology are connected and path-connected. For instance, $[0, 1]$ is connected for if $[0, 1] = U \sqcup V$ is a disconnection with $U = [0, a_0) \sqcup (a_1, a_2) \sqcup \dots$ then $a_0 \in V$ which is open. Therefore, $(a_0 - \epsilon, a_0 + \epsilon) \subseteq V$ for some $\epsilon > 0$. This violates the assumption that $U \cap V = \emptyset$. Finally note that any connected subset of \mathbb{R} is of the form

$$\emptyset, (a, b), [a, b], [a, b), (a, b], (a, \infty), [a, \infty), (-\infty, b), (-\infty, b], \mathbb{R},$$

where $a, b \in \mathbb{R}$ and $a \leq b$.

Remark 2.4.4. If (X, τ) is path-connected then let $X = U \sqcup V$ be a disconnection of X , if possible. For $x \in U$ and $y \in V$ choose a path γ joining x to y . Now consider the equality $[0, 1] = \gamma^{-1}(U) \sqcup \gamma^{-1}(V)$. This is impossible as $[0, 1]$ is connected. Thus, path-connectivity implies connectivity. However, there are examples of connected spaces which are not path-connected.

The space $\mathbb{R}^2 - \mathbb{Q}^2$ is path-connected. Euclidean spaces minus the origin is path-connected if the dimension is at least 2. However, there is a difference between punctured Euclidean spaces in terms of *higher connectivity*.

Yet another key notion is that of compactness. Recall that an open cover of a space X is a way of expressing X as the union of a collection of open sets. A finite subcover (of a given cover) simply means a finite subcollection whose union is X . We shall work with the following.

Definition 2.4.5. A topological space (X, τ) is called *sequentially compact* if every sequence $\{x_n\}$ has a convergent subsequence. A topological space (X, τ) is called *compact* if every open cover has a finite subcover.

It is a known fact from elementary real analysis that for subsets of \mathbb{R}^n the two notions of compactness agree. The proof generalizes to metric spaces. Before we present some examples we note that closed subsets of compact sets are compact.

Example 2.4.6. (1) The set $(0, 1]$ is non-compact; just consider the cover given by $\{(1/n, 1]\}_{n \geq 1}$.

(2) The compact subsets of \mathbb{R}^n are precisely the closed and bounded sets. This is known as the Heine-Borel Theorem.

(3) The construction outlined in Example 2.4.2 is an example of what is known as the *one-point compactification* of a locally compact, non-compact Hausdorff space.

Compactness is preserved under continuous maps, i.e., the image of a compact set is compact. A very useful observation (involving all of the notions we have encountered so far) is the following result.

Proposition 2.4.7. *A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.*

Proof. We need only show that the map f is closed. Now any closed subset of a compact set is compact and f takes compact sets to compact sets. Finally note that a compact subset of a Hausdorff space is closed. \square

2.5 Quotient topology

Suppose we have an equivalence relation \sim on X . If X is equipped with a topology τ then consider the set X/\sim of equivalence classes. It is natural to want the projection map $q : X \rightarrow X/\sim$ to be continuous, i.e., to find a topology τ_q on X/\sim such that $q^{-1}(\tau_q) \subseteq \tau$.

Definition 2.5.1. Given an equivalence relation \sim on (X, τ) , the collection of sets $U \subseteq X/\sim$ such that $q^{-1}(U) \in \tau$ defines a topology called the *quotient topology*.

Observe that if τ' is a topology on X/\sim such that q is continuous then τ' is a subset of the quotient topology. We say that the quotient topology is the finest topology with respect to which q is continuous.

Example 2.5.2. Let us illustrate with a few basic examples.

(1) Consider $X = [0, 1]$ and the equivalence relation \sim that identifies 0 with 1, i.e., every $x \neq 0, 1$ is its own equivalence classes while $[0] = [1]$. The quotient space is Hausdorff. We define a bijective map

$$f : S^1 \rightarrow [0, 1]/\sim, \quad e^{2\pi it} \mapsto [t].$$

We can check that this is continuous. It follows from Proposition 2.4.7 that f is a homeomorphism.

The example above generalizes to higher dimensions. If \mathbb{D}^n denotes the

closed unit disk in \mathbb{R}^n then the equivalence relation \sim that identifies all its boundary points together induces a homeomorphism between S^n and \mathbb{D}^n / \sim (cf. Example 2.1.2 (6)).

(2) The previous example is a special case of the following identification. Given a subset A of (X, τ) we define the equivalence relation \sim_A that identifies all points of A together. We shall denote the quotient space by X/A . In general, this space may fail to be Hausdorff even if X is. For instance, let $X = \mathbb{R}$ with $A = \mathbb{R} - \{0\}$. Then $X/A = \{[0], [1]\}$ has $[0]$ as closed and $[1]$ as open. Perhaps a stranger example arises out of $X = \mathbb{R}$ and $A = \mathbb{Q}$. The quotient space is an uncountable set with a distinguished point $*$ where every point other than $*$ is closed. Moreover, any open set containing $\alpha \neq *$ must be the whole set.

(3) Consider $X = [0, 1] \times [0, 1]$ with the equivalence relation that identifies $(0, t)$ with $(1, 1 - t)$. The quotient space can be realized as a subspace inside \mathbb{R}^3 and is called the *Möbius strip* (cf. Figure 2.2).

We note that if (X, τ) is connected (respectively path-connected) then X/\sim is connected (respectively path-connected). Compactness is also preserved under taking quotients. As observed in Example 2.5.2 a quotient of a Hausdorff space need not be Hausdorff. Groups acting on spaces provide a plethora of examples of quotient topology.

To discuss actions of groups let us recall a few notions.

Definition 2.5.3. A *topological group* G is a topological space (G, τ) such that the underlying set G is a group, and the inverse and multiplication maps are continuous.

Any group is a topological group with the discrete topology. The first examples of (non-discrete) topological groups are the Euclidean spaces with vector addition as the group operation. The set of positive real numbers, under multiplication, is also a (topological) group. It can be seen that S^1 and \mathbb{C}^\times are both topological groups. In fact, \mathbb{C}^\times is isomorphic (as topological groups) to $S^1 \times (0, \infty)$.

Example 2.5.4. Consider the set $GL_n(\mathbb{R})$ of invertible real $n \times n$ matrices. This is a (discrete) group with several distinguished subgroups including $SL_n(\mathbb{R})$ and $O_n(\mathbb{R})$. We may equip $GL_n(\mathbb{R})$ with the subspace topology induced from $M_n(\mathbb{R})$ which can be identified with \mathbb{R}^{n^2} . The determinant, being a polynomial in entries, is continuous. Thus, the set $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} - 0)$ is open in the space of all $n \times n$ matrices. The multiplication of matrices is a polynomial map on \mathbb{R}^{n^2} . This implies that $GL_n(\mathbb{R})$ is actually a topological group.

Since the topology on $M_n(\mathbb{R})$ is induced by the Euclidean metric on \mathbb{R}^{n^2} , the group $GL_n(\mathbb{R})$ is metrizable. In fact, $GL_n(\mathbb{R})$ is open and dense in $M_n(\mathbb{R})$. It is known that $SL_n(\mathbb{R})$ is non-compact and is evident from the sequence of diagonal matrices A_k with $a_{11} = k, a_{22} = k^{-1}$ and $a_{ii} = 1$. However, $O_n(\mathbb{R})$ can be described as the common zeroes of $n(n+1)/2$ polynomials of degree 2.

Thus, $O_n(\mathbb{R})$ is a closed subset and is contained in

$$\{(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{n1}, \dots, x_{nn}) \in \mathbb{R}^{n^2} \mid \sum_i x_{ji}^2 = 1 \text{ for any } j\}.$$

The space above is the n -fold product $S^{n-1} \times \dots \times S^{n-1}$. Thus, $O_n(\mathbb{R})$ is compact.

Definition 2.5.5. We say a (topological) group G acts on a topological space X (from the left) if there exists a continuous map $\varphi : G \times X \rightarrow X$ satisfying $\varphi(e, \cdot) = \text{id}_X$ and $\varphi(gh, x) = \varphi(g, \varphi(h, x))$ for any $g, h \in G$ and $x \in X$.

Note that when G is discrete then a group action is equivalent to a group homomorphism from G to $\text{Homeo}(X)$, the group of all homeomorphisms of X . In general, a group action induces a natural equivalence relation on X ; the quotient space is denoted by X/G .

Example 2.5.6. (1) Let \mathbb{Z} act on the real line by translations. Then \mathbb{R}/\mathbb{Z} is homeomorphic to S^1 . A word of caution: we have earlier used the notation X/A in Example 2.5.2 (2). If we think of \mathbb{Z} as a subset of \mathbb{R} then \mathbb{R}/\mathbb{Z} , in our old notation, is not S^1 . In a similar way, we can define the action of $\mathbb{Z} \times \mathbb{Z}$ on \mathbb{R}^2 and the quotient is the torus $S^1 \times S^1$ (cf. Figure 2.1).

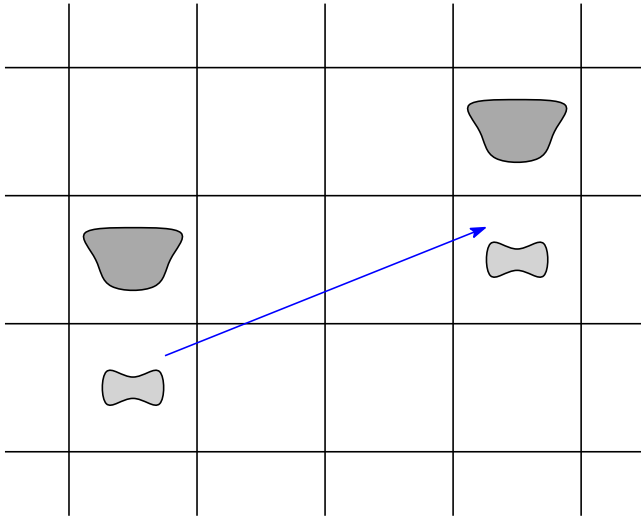


Figure 2.1: The action of the element $(3, 1) \in \mathbb{Z}^2$ on the plane.

(2) Consider the $\mathbb{Z}/2\mathbb{Z}$ -action on $X \times X$ given by $(x, y) \mapsto (y, x)$. The diagonal $\Delta := \{(x, x) \in X \times X \mid x \in X\}$ is fixed pointwise by this action. When $X = \mathbb{R}$ we observe that the quotient space is homeomorphic to the upper half plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. When $X = S^1$ we can check that the quotient space is the Möbius strip (cf. Figure 2.2).

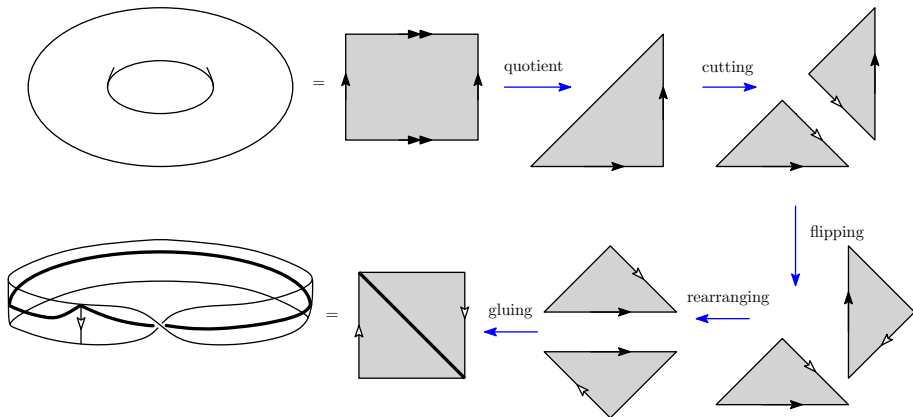


Figure 2.2: The quotient of the torus under the reflection map.

(3) The group $\mathbb{Z}/n\mathbb{Z}$ acts on S^1 via the map $([k], z) \mapsto e^{2\pi i k/n} z$. It is clear that $S^1/(\mathbb{Z}/n\mathbb{Z})$ is again the circle.

(4) The group $\mathbb{Z}/2\mathbb{Z}$ acts on S^n via identifying a point with its antipode. The quotient space is denoted by $\mathbb{R}P^n$ and called the *real projective space* of dimension n .

(5) Consider the action of $SO_2(\mathbb{R})$ on \mathbb{R}^2 . The origin is fixed by the group while all points at a distance d away from the origin are identified with each other via elements of $SO_2(\mathbb{R})$. The quotient space is $[0, \infty)$.

(6) Consider the group $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm I$ acting on the upper half-plane

$$\mathbb{H} := \{x + iy \in \mathbb{C} \mid y > 0\}$$

by $z \mapsto \frac{az+b}{cz+d}$. Here an element $A \in SL_2(\mathbb{Z})$ has entries a, b, c, d . This group and its action is important in many branches of mathematics. This group is called the *modular group* and is generated by $T(z) = z + 1$ and $S(z) = -z^{-1}$. The quotient space can be shown to be a 2-sphere minus one point.

2.6 Topological manifolds

We are ready to discuss a class of topological spaces which form the most prevalent family in algebraic and differential topology. We shall need a few preliminaries.

Definition 2.6.1. A *base* for a topology τ on X is a subcollection β such that any $U \in \tau$ can be written as a union of elements from β . A space with a countable base is called *second countable*.

Euclidean spaces are second countable. In fact, most spaces we encounter in analysis and geometry are second countable.

Definition 2.6.2. A *topological manifold* M is a topological space which is Hausdorff, second countable and for each point there exists an open set which is homeomorphic to an open ball in \mathbb{R}^n for some n . A pair (U, φ) consisting of such an open set and a homeomorphism $\varphi : U \rightarrow B \subset \mathbb{R}^n$ is called a *chart*.

Remark 2.6.3. What we have defined is a manifold without boundary. We often discuss spaces which are locally like \mathbb{R}^n or the upper half-plane in \mathbb{R}^n . The points where a local chart looks like the latter is called a *boundary point*. Closed disks and closed upper half-planes are examples of such spaces, also called *manifolds with boundary*. A manifold without boundary may be thought of as a manifold with empty boundary.

Proposition 2.6.4. A connected topological manifold is also path-connected.

Proof. Fix $x \in M$ and consider the set S of all points that can be joined by a path to x . As $x \in S$, it is non-empty. If $y \in S$ then a small open path-connected set containing y is also in S , whence S is open. If $S^c \neq \emptyset$ then a similar argument proves that S^c is open, contradicting the connectedness of M . \square

If M is connected then n is constant and called the *dimension* of the manifold. This follows from proving *invariance of dimension*, i.e., that \mathbb{R}^m and \mathbb{R}^n cannot be homeomorphic if $m \neq n$. It is one of the basic and non-trivial facts of (algebraic) topology.

The subset of \mathbb{R}^2 given by the union of the x and y -axis is not a manifold as the origin does not admit a neighbourhood which looks like \mathbb{R} . However, there are examples of spaces which are locally Euclidean but fail to be non-Hausdorff.

Example 2.6.5. We may consider $X = \mathbb{R} \sqcup \{*\}$ equipped with the topology

$$\tau = \tau_d \cup \{(U - 0) \cup \{*\} \mid U \text{ is an open set in } \mathbb{R} \text{ containing } 0\}.$$

This space is called the *line with two origins*. Every point in $\mathbb{R} \subset X$ has an open set, homeomorphic to $(-1, 1)$, containing it. As for $*$, the set $((-1, 1) - 0) \cup \{*\}$ is homeomorphic to $(-1, 1)$. However, 0 and $*$ cannot be separated by open sets.

Most of what we have seen in earlier sections are examples of manifolds - spheres, real projective spaces, $GL_n(\mathbb{R})$, $O_n(\mathbb{R})$. Configuration spaces are examples of manifolds which are useful in applications as well as in theory. For example, the configuration space of a rod rotating in \mathbb{R}^3 about a fixed hinge at the midpoint of the rod is $\mathbb{R}P^2$. In practice, one finds the following result quite useful.

Proposition 2.6.6. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function. If c is a non-critical value of f then $f^{-1}(c)$ is a manifold of dimension $n - 1$.

This can be proven using the Implicit Function Theorem from several variable calculus. Note that $f^{-1}(c)$ can be empty. The empty set is also considered

a manifold of any given dimension. The above result implies, for instance, that $SL_n(\mathbb{R})$ is a manifold.

Example 2.6.7. The Grassmannians are manifolds which appear in several branches of mathematics and are of considerable importance. For a fixed n and $1 \leq k \leq n$ the *Grassmann manifold* $G_{k,n}$ (over \mathbb{R}) is defined to be the set of linear k -planes in \mathbb{R}^n . The topology is induced by the metric

$$d(L, L') = \|P_L - P_{L'}\|$$

where P_L denotes the orthogonal projection of \mathbb{R}^n to the subspace L and $\|A\|$, for a linear map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$, is defined to be

$$\|A\| = \max_{v \in S^{n-1}} \|Av\|.$$

In other words, $\|A\|$ is the maximum possible size of the image of any unit vector under A . In fact, it is known that $G_{k,n}$ is homeomorphic to $O_n(\mathbb{R})/(O_k(\mathbb{R}) \times O_{n-k}(\mathbb{R}))$. Observe that when $k = 1$ the Grassmannian is the real projective space $\mathbb{R}P^{n-1}$ (cf. Example 2.5.6 (4)). In fact we had encountered this even earlier in Example 2.1.2 (4) as $\mathbb{P}(\mathbb{R}^n)$.

We finally come back a full circle to metric spaces. The question of when a topological space is metrizable, i.e., it admits a metric with the induced topology being the given topology, has been answered in various forms. For us, the following is of relevance.

Theorem 2.6.8 (Urysohn's metrization theorem). *Every Hausdorff, second-countable, regular space is metrizable.*

Recall that a space (X, τ) is called *regular* if points and closed sets can be separated by open sets. More precisely, given x and a closed set F not containing x there exists $U, V \in \tau$ such that $x \in U, F \subseteq V$ and $U \cap V = \emptyset$. Since a manifold can be shown to be regular, it is metrizable.

II A Tutorial on Equivalence Relations and Quotient Sets

Equivalence relation and quotient sets

In English, we say that two objects are *related* if there is some rule connecting them together. For example, two persons are said to be related by ‘blood’ if they belong to the same family, by ‘friendship’ if they are friends of each other, by ‘nationality’ if they come from the same country. Thus one way to describe a *relation* would be to collect all possible pairs of objects connected by the same relation. Mathematically a relation on a set X can be viewed as a way of identification between elements of X and can be described by a collection of ordered pairs of elements of X . The ordering is important as two pairs (a, b) and (b, a) do not necessarily represent the same relation; e.g., if a is the mother of b , then the relation between (a, b) and (b, a) are different.

Definition 2.6.1. Binary relations: Let X be a non-empty set. A *binary relation* R on X is defined as a subset of the cartesian product $X \times X$ or equivalently a collection of ordered pair of elements of X . Two elements $x, y \in X$, are said to be related by R if $(x, y) \in R$. In that case, one writes $x R y$.

Example 2.6.2. (a) On the set of real numbers \mathbb{R} , the following well known (binary) relations, e.g., *greater than* ($>$), *greater than or equal to* (\geq), *less than* ($<$), *less than or equal to* (\leq), *is equal to* ($=$) and *divides*, can be represented by the respective subsets $R_>$, R_\geq , $R_<$, R_\leq , $R_ =$ and $R_ /$ of $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as described below.

$$\begin{aligned} R_< &= \{(x, y) \in \mathbb{R}^2 : x > y\}, \\ R_\geq &= \{(x, y) \in \mathbb{R}^2 : x \geq y\}, \\ R_< &= \{(x, y) \in \mathbb{R}^2 : x < y\}, \\ R_\leq &= \{(x, y) \in \mathbb{R}^2 : x \leq y\}, \\ R_ = &= \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\} \text{ and} \\ R_ / &= \{(x, y) \in \mathbb{R}^2 : x \text{ divides } y\}. \end{aligned}$$

(b) Let $P(X)$ denote the power set of a set X (i.e., $P(X)$ is the collection of all subsets of X). On the set $P(X)$, one can define the binary relation $R_\subset (\subset P(X) \times P(X))$ by

$$R_\subset = \{(A, B) \in P(X) \times P(X) : A \subset B\}.$$

(c) For a given positive integer n , one can define the binary relation called *congruence modulo n* on the set of integers \mathbb{Z} , as follows. Two integers a and b are said to be *congruent modulo n* , often denoted by $a \equiv b \pmod{n}$, if $a - b$ is divisible by n . For example, 23 and 73 are congruent modulo 10. Thus one can describe the relation as the subset $R_{\equiv(n)}$ of $\mathbb{Z} \times \mathbb{Z}$ defined by

$$R_{\equiv(n)} = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a \equiv b \pmod{n}\}.$$

(d) Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Given two vectors $v, w \in V$, v is said to be *orthogonal to* w , denoted by $v \perp w$, if $\langle v, w \rangle = 0$. This defines the binary relation *orthogonal to* (on V ,) described by the subset R_\perp of $V \times V$ defined by

$$R_\perp = \{(v, w) \in V \times V : v \perp w\}.$$

Definition 2.6.3. Equivalence relations: A binary relation R on a set X is said to be an *equivalence relation* if the following conditions hold:

- R is *reflexive*, i.e., $(x, x) \in R$ for each $x \in X$.
- R is *symmetric*, i.e., if $(x, y) \in R$, then $(y, x) \in R$.
- R is *transitive*, i.e., if $(x, y), (y, z) \in R$, then $(x, z) \in R$.

Example 2.6.4. (a) $R_=$ and $R_{\equiv(n)}$ described in Example 2.6.2, are both examples of equivalence relations whereas none of the relations $R_>$, R_\geq , $R_<$, R_\leq , $R_!$, R_\subset or R_\perp in Example 2.6.2 is. Clearly, $R_>$, $R_<$ are neither reflexive nor symmetric; R_\perp is neither reflexive nor transitive and R_\geq , R_\leq , $R_!$, R_\subset are not symmetric.

(b) Consider a (real or complex) vector space V and let $X = V \setminus \{0\}$. Define the binary relation R_p by

$$R_p = \{(v, w) \in X \times X : w = \lambda v \text{ for some } \lambda \in \mathbb{K}\}$$

where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} if V is a real or complex vector space respectively. Naturally, if $(v, w) \in R_p$, then $w = \lambda v$ and $\lambda(\in \mathbb{K}) \neq 0$ as both v and w are non-zero vectors. Then one can re-write R_p as

$$R_p = \{(v, \lambda v) : v \in X \text{ and } \lambda \in \mathbb{K} \setminus \{0\}\}$$

and it is easy to see that R_p defines an equivalence relation on X .

(c) Let V be a real vector space and W be any linear subspace of V . Then one can define the binary relation R_W on V by

$$R_W = \{(v_1, v_2) \in V \times V : v_1 - v_2 \in W\}.$$

Using the fact that W is also a vector space, it follows that R_W is an equivalence relation.

(d) We say that a group (G, \cdot) *acts* on a non-empty set X (or X is a *G-set*) if there is a map $\rho : G \times X \rightarrow X$ satisfying

$$\begin{aligned} \rho(g_1 \cdot g_2, x) &= \rho(g_1, \rho(g_2, x)) \quad \forall x \in X \text{ and } g_1, g_2 \in G; \\ \rho(e, x) &= x \quad \forall x \in X. \end{aligned} \tag{2.1}$$

e is the identity element of G . Then ρ is called an action of G on X . Define the binary relation R_G on X by

$$R_G = \{(x, \rho(g, x)) \in X \times X : x \in X \text{ and } g \in G\}.$$

It can be shown (using the properties of ρ) that R_G is an equivalence relation.

For example, the group of integers $(\mathbb{Z}, +)$ acts on the set of real numbers \mathbb{R} by translation. More generally, for all $n \in \mathbb{N}$, the product group \mathbb{Z}^n acts on the Euclidean n -space \mathbb{R}^n by translation i.e., there is a natural map $\rho : \mathbb{Z}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $\rho(a, x) = a + x$ for all $a \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$; satisfying the conditions (2.1). Using this action of \mathbb{Z}^n on \mathbb{R}^n , one obtains the equivalence relation $R_{\mathbb{Z}^n}$ given by

$$R_{\mathbb{Z}^n} = \{(x, a + x) : x \in \mathbb{R}^n \text{ and } a \in \mathbb{Z}^n\}.$$

Definition 2.6.5. Equivalence classes: Let (X, R) denote a set X together with an equivalence relation R (on X). Given any element x_0 , the *equivalence class* of x_0 denoted by $[x_0]$ is a subset of X defined by

$$[x_0] = \{x \in X : (x, x_0) \in R\}$$

i.e., $x \in [x_0]$ if and only if x is related to x_0 under the relation R .

From the definition of equivalence class, it follows that given a set X together with an equivalence relation R and any two elements $x, y \in X$, $[x] = [y]$ if and only if x is related to y by R . In such a case, the elements x and y are said to be *equivalent* under the relation R .

Example 2.6.6. (a) For the equivalence relation $R_=_$ as in Example 2.6.4(a), the equivalence class $[x_0]$ for any $x_0 \in \mathbb{R}$, is the singleton subset $\{x_0\}$ of X .

For the equivalence relation $R_{\equiv(n)}$, check that there are only n distinct equivalence classes, namely, $[0], [1], \dots, [n-1]$.

(b) Let V and X be as in Example 2.6.4(b). Given any vector $v \in X$, its equivalence class corresponding to the relation R_p is the subset

$$[v] = \{\lambda v : \lambda \in \mathbb{R}, \lambda \neq 0\}.$$

Thus if the vector space V is \mathbb{R}^n or \mathbb{C}^n , the equivalence class $[v]$ is the line (real or complex) in \mathbb{R}^n or \mathbb{C}^n respectively, passing through the origin and the vector v with the point 0 removed. Moreover, it is easy to see that in this case, two vectors $v, w \in X$ will be equivalent under R_p if and only if they both lie on the same straight line (real or complex) passing through the origin in \mathbb{R}^n or \mathbb{C}^n respectively.

(c) Consider the equivalence relation R_W on a vector space V corresponding to a given subspace W of V described in Example 2.6.4(c). It is easy to see that with respect to the relation R_W , the equivalence class $[0]$ of the null vector $0 \in V$, is the subspace W , as using the definitions of R_W and the equivalence class $[0]$, it follows that $v \in [0]$ if and only if $(v - 0) = v \in W$. Also each vector

in W is equivalent to the null vector 0 under R_W and hence $[w] = [0] = W$ for all $w \in W$. For a vector $v \in V$ which lies outside W , the equivalence class $[v]$ is different from W (as v is not equivalent to any vector in W). In this case, one can check that the equivalence class is $[v] = v + W = \{v + w : w \in W\}$ which is an affine subspace of V .

(d) As discussed in Example 2.6.4(d), consider a G -set X for some group G and let ρ be the corresponding group action. Then for any x_0 , its equivalence class $[x_0]$ with respect to the relation R_G , is the following subset of X described by

$$[x_0] = \{\rho(g, x_0) : g \in G\}.$$

Thus when $X = \mathbb{R}^n$, $G = \mathbb{Z}^n$ and the group action ρ is just the usual translation, then for any x_0 , its equivalence class is

$$[x_0] = \{a + x_0 : a \in \mathbb{Z}^n\}$$

i.e., $[x_0]$ consists of all translates of x_0 by the elements of \mathbb{Z}^n . Also in this case, any two elements $x_0, y_0 \in \mathbb{R}^n$, will be equivalent or $[x_0] = [y_0]$ if and only if $(x_0 - y_0) \in \mathbb{Z}^n$.

Definition 2.6.7. Quotient sets: Let (X, R) denote a set X together with an equivalence relation R (on X). The set of all equivalence classes of X with respect to the relation R , denoted by X/R and defined by $X/R := \{[x] \mid x \in X\}$, is said to be the *quotient set* of X by the relation R .

If X has some additional (algebraic/topological/geometric) properties, there are standard techniques of transferring these properties to the quotient set X/R (details of which we avoid in this note). For example, if X is a topological space, one can transform X/R also into a topological space in a canonical manner.

Example 2.6.8. (a) For the equivalence relation $R_=\$ (cf. Example 2.6.4(a)), the quotient set X/R (here $X = \mathbb{R}$) is the same as the set of real numbers (one can easily see that in this case, the map which sends each real number to its equivalence class under $R_=\$, defines a bijection of sets).

The quotient set for the relation $R_{\equiv(n)}$, commonly denoted by $\mathbb{Z}/n\mathbb{Z}$ or \mathbb{Z}/n is the finite set $\mathbb{Z}/n\mathbb{Z} = \{[0], [1], \dots, [n-1]\}$. This quotient set $\mathbb{Z}/n\mathbb{Z}$ has several important mathematical properties that are related to many branches of Mathematics.

(b) In Example 2.6.4(b), the quotient set for the equivalence relation R_p is

$$X/R = \{[v] : v \in X\}$$

which consists of all one dimensional subspaces of V minus the null vector. Thus if V is \mathbb{R}^n (respectively \mathbb{C}^n), the corresponding quotient set is the set of all real (complex) lines minus the origin in \mathbb{R}^n (\mathbb{C}^n) passing through the origin. The quotient space has very rich topological and geometric structures and are known as the real (complex) projective space.

(c) For the equivalence relation R_W on a vector space V , (cf. Example 2.6.4(c)), it is easy to see that the quotient set V/R_W (commonly denoted by V/W), is a vector space of dimension $\dim V - \dim W$ with addition and scalar multiplication of vectors in V/W given by

$$\begin{aligned}[v_1] + [v_2] &= [v_1 + v_2] \quad \forall v_1, v_2 \in V, \text{ and} \\ c[v_1] &= [cv_1] \quad \forall v_1 \in V, c \in \mathbb{K}\end{aligned}$$

($\mathbb{K} = \mathbb{R}$ or \mathbb{C} if V is a real or complex vector space respectively).

If V is equipped with an inner product $\langle \cdot, \cdot \rangle$, then there is a nice description for the quotient space V/W as follows. Recall that in this case, for any subspace W of V , $(V, \langle \cdot, \cdot \rangle)$ can be decomposed as the *orthogonal direct sum* $V = W \oplus W^\perp$ (i.e., each $v \in V$ has a unique expression $v = v_0 + v_1$ such that $v_0 \in W$ and $v_1 \in W^\perp$) where $W^\perp = \{v \in V : \langle v, w \rangle = 0 \quad \forall w \in W\}$ is the subspace of V called the *orthogonal complement* of W in V and $\dim W^\perp = \dim V - \dim W$. Also recall the *orthogonal projection* map $P : V \rightarrow W^\perp$ which is a surjective linear map sending any vector $v = v_0 + v_1 \in V$ (as described above), to v_1 . It can be checked that there is a canonical isomorphism between the vector spaces V/W and W^\perp . As both the vector spaces are of the same dimension, it suffices to construct an injective linear map between them. Define $L : V/W \rightarrow W^\perp$ by $L([v]) = P(v)$, for $v \in V$.

Then L is well defined (i.e., for any two vectors $u, v \in V$ with $[u] = [v]$ (or $(u - v) \in W$), one has $L([u]) = L([v])$): In fact, $(u - v) = (u_1 - v_1) + (u_0 - v_0) \in W$ where $u = u_0 + u_1$ and $v = v_0 + v_1$ with $u_0, v_0 \in W$ and $u_1, v_1 \in W^\perp$ respectively, implies that $u_1 = v_1$ and thus $P(u) = P(v)$.

It is easy to see that L is a linear map. Injectivity of L follows from the fact that $L([v]) = P(v) = 0$ if and only if $v \in W$ (property of an orthogonal projection) or equivalently, $[v] = 0$.

(d) Consider a G -set X for some group G (cf. Example 2.6.4(d)) and let ρ be the corresponding group action. The quotient space consisting of all equivalence classes with respect to the relation R_G , is commonly denoted by X/G .

In particular, when $X = \mathbb{R}$, $G = \mathbb{Z}$ and the group action ρ is given by $\rho(a, x) = a + x$, $x \in \mathbb{R}$ and $a \in \mathbb{Z}$; then the quotient space \mathbb{R}/\mathbb{Z} is the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ in the complex plane via the bijection $f : \mathbb{R}/\mathbb{Z} \rightarrow S^1$ defined by $f([x]) = e^{2\pi i x}$, for $x \in \mathbb{R}$. Clearly, f is well defined i.e., for any two equivalent elements $x, y \in \mathbb{R}$ (i.e., $[x] = [y]$ and $(x - y) \in \mathbb{Z}$), one has $f([x]) = f([y])$.

f is surjective by construction. Finally f is injective as $f([x]) = f([y])$ implies that $e^{2\pi i(x-y)} = 1$ i.e., $(x - y) \in \mathbb{Z}$ and hence $[x] = [y]$. Similarly, for $X = \mathbb{R}^n$, $G = \mathbb{Z}^n$ ($n \geq 1$) and the group action ρ given by $\rho(a, x) = a + x$, $x \in \mathbb{R}^n$ and $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$; the quotient space $\mathbb{R}^n/\mathbb{Z}^n$ is the n -torus $T^n = S^1 \times S^1 \times \dots \times S^1$ (n -times) via the bijection $F : \mathbb{R}^n/\mathbb{Z}^n \rightarrow T^n$ defined by $f([x]) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. It turns out that these quotient sets have very special structures from algebraic, topological as well as geometric points of view.

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