Preface

This book emerged from a course given twice at Peking University (PKU), each time once a week over two months, while I was visiting the Beijing International Center for Mathematics Research.

I essentially taught character theory of finite groups, insisting during the first year on questions of rationality (hence making no assumption for the ground field to be "big enough"). During the second year, I insisted more on applications, and also on related areas, like questions concerning polynomial invariants of finite groups, or the more recent notion of Drinfeld double of a finite group. Throughout the two courses, I tried systematically—but discretely—to help students become familiar with the language of categories.

The audience consisted of second-year and third-year students. There are not that many places in the world where you could teach such a material in front of undergraduate students… PKU undergraduate students are indeed exceptional: quick, curious, open-minded, just as disbelieving as brilliant students must be, humorous, and obstinate.

Nevertheless, this book contains more than what had actually been taught and, in order to be consistent, it is organized differently from the course. My hope is that its content may be useful for various introductory courses on this fantastic—although not recent—theory. As it is now, the Texte book will usually be better suited for a graduate course (or to exceptionally bright undergraduate students). In exchange for the extra effort, reading this book will give the student a better algebraic formation than reading a book that focusses almost exclusively on groups.

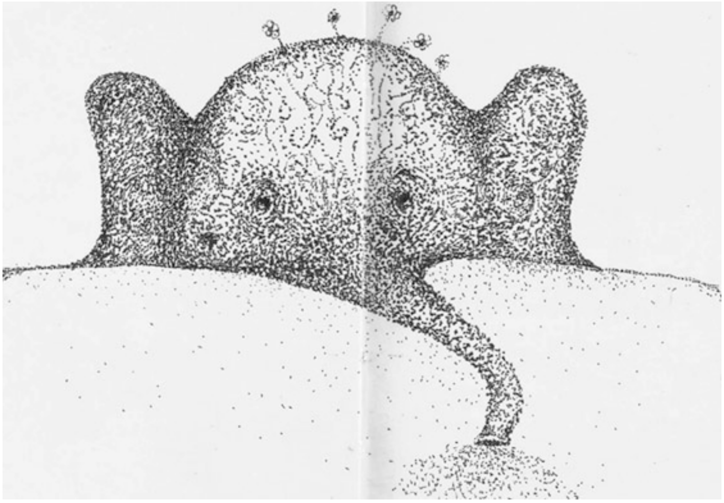
In contrast to some of the recent books on a similar topic (such as [Ste12]), I have chosen not to give classical examples such as representations of the symmetric groups; they can be found in the abundant literature on the subject. Furthermore (unlike in the rather exhaustive treatise [CR87], but also [AB95], or even [Ser12]), I chose not to make use of general "abstract" semisimple algebras: The complete structure theorems about the group algebra (including a Fourier formula over any characteristic zero field, as well as various interpretations of Schur indices) are proven without having to appeal to more general results—only using characters.

This allowed me, in a rather short book, to concentrate on less usual topics, like representations over $\mathbb{Q}$, or the graded G-modules and applications to the action of a group G on $K[X_1, \ldots, X_n]$ if G acts on K^n, or even to present a quick introduction to Hopf algebras and ribbon categories through the study of the Drinfeld double of a finite group, with a natural application to the associated representation of $\mathrm{SL}_2(\mathbb{Z})$.

A precision: The first two parts of Jean-Pierre Serre's book [Ser12] have been the main source of inspiration for our treatment of Brauer's Theorem in Chap. 5, as well as for other topics in Chaps. 2 and 3.

Michel me demande de lui faire un dessin pour son nouveau livre. Je lui demande de me raconter “ce qu'il a vu”. Je vois passer dans ses yeux l’envie de dire, et l’impossible. Il me dit: “Imagine des objets mathématiques abstraits, très complexes et très harmonieux. Pour les regarder, on les transporte dans un autre monde, un tout petit moins abstrait (qui peut quand même avoir 10 ou 196883 dimensions) et miraculeusement, on les voit beaucoup mieux, leur comportement les révèle.” Pendant qu’il parle, ses mains bougent, décrivent des courbes, des vagues, de l’harmonie. Comment dessiner ça ? Je ne vois pas. La seule image qui me vient est celle d’un mathématicien, et plus largement, d’une communauté mathématique, chanceuse de pouvoir naviguer dans tant de dimensions, chanceuse de toucher l’intouchable et d’en revenir les mains pleines. Nous autres, non mathématiciens, ne pouvons qu’admirer ces voyageurs qui prennent le large, qui voient autre chose que le haut et le bas, le concret et l’abstrait. Merci pour ces mystères récoltés dans la solitude.

Anouk Grinberg.



Abstract. The first chapter is devoted to tensor products: This basic and fundamental notion is hardly taught at undergraduate level, and I want the reader to be immediately familiar with it, once and for all.

- Chapter 2 is a general introduction to group representations (using a bit of categorical language). In particular, we treat the case of representations on sets, their classification, as well as Burnside marks, and we also introduce the reader to general linear representations, their language, basic facts such as Schur's lemma, and common examples and counterexamples.
- Chapter 3 contains the general results about characteristic zero representations, without any further assumption about the ground field (hence, the ground field may be $\mathbb{Q}$). For example, it provides a proof of the general "Fourier inversion formula" in this general context, as well as the more classical results about the Galois action on conjugacy classes in relation to the number of irreducible characters. One paragraph is devoted to what was, for Frobenius, the origin of the whole character theory: the group determinant.
- Chapter 4 "plays around with the ground ring." It contains a description, free of the theory of central simple algebras, of the integers which constitute the degree of an irreducible character: Character theory is sufficient to establish, for example, that the dimension of the occurring skewfields over their center is a square. It also contains a short introduction to reflection groups.
- Chapter 5 is devoted to induction–restriction. The first part has no assumption about the characteristic of the field, insisting on the formal equalities and isomorphisms between occurring bimodules or functors. The second part deals, more classically, with the setting of characteristic zero ground fields and class functions.
- The first part of Chap. 6 contains a proof of Brauer's characterization of characters, while its second part contains some (less classical in textbooks) applications to "subgroups controlling π-fusion" and normal π-complements. As an application, it also contains a treatment of Frobenius groups.
- In order to study representations of finite groups on graded modules, Chap. 7 starts with a short introduction to basic notions concerning graded modules and graded algebras. In the second part, we introduce the notion of graded characters with applications to generalization of Molien's Formula. As a particular case, we study the action of G on $S(V)$ (the symmetric algebra of V) if G acts on a vector space V. We conclude by an introduction to the main characterization of reflection groups.
- The last chapter is devoted to the study of the Drinfeld double of a finite group. We start by a quick introduction to Hopf algebras and their algebra representations, then we introduce the notion of universal R-matrix and show that the representation category of the Drinfeld double is a ribbon category. We conclude by the definition of the S-matrix of that ribbon category, and by explicitly computing the associated representation of $\mathrm{GL}_2(\mathbb{Z})$.

Prerequisites

This book requires familiarity with the notions of groups, rings, fields, and in particular, with the undergraduate knowledge of linear algebra. More specifically, let k be a (commutative) field. Here are some examples of what is assumed to be known.

- The results of an undergraduate course on k-linear algebra. If V is a vector space over k, we shall denote by $[V : k]$ its dimension, by $\mathrm{tr}_{V/k}(\alpha)$ the trace of an endomorphism α of V, and by V^* its dual space.
- Matrices and their determinants.

  For example, the following identity will not be proved: let M be an $n \times n$ matrix with entries in k, let ${}^t\mathrm{Com}(M)$ denote the transpose of its matrix of cofactors, let 1_n be the $n \times n$ identity matrix; then

$$ {}^t\mathrm{Com}(M) \cdot M = det(M) \cdot 1_n. $$

- Rudiments of fields extensions: multiplicativity of the degrees (if $K \subset L \subset M$ are fields, then $[M : K] = [M : L][L : K]$), algebraic elements, finite extensions.
- At some places (clearly indicated), which may be omitted by a beginner, we shall assume more familiarity with Galois theory, in particular, with Galois theory of cyclotomic extensions. All what is needed is treated in the Appendix.

We take for granted that the reader is familiar with the standard notation $\mathbb{N}$ (for "numbers")—note that by convention $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z}$ (for "Zahlen"), $\mathbb{Q}$ (for "quotients"), $\mathbb{R}$ (for "real"), $\mathbb{C}$ (for "complex"), as well as $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (for "finite"). Moreover, the letter i will denote a complex number such that $i^2 = -1$.

By convention, a *division ring* is a (not necessarily commutative) ring where all nonzero elements are invertible—a division ring is also called a skewfield. A *field* is a *commutative* division ring.

The letter k (for "Körper") will usually denote a field; K, L will usually denote *characteristic zero fields*[1], while D will be used for division rings.

The center ZD of a division ring D is a field, and if k is a subfield of ZD, we say that D is a division k-algebra.

For D a division ring and V a D-vector space, we denote by $[V : D]$ the dimension of V. In particular, if D is a division k-algebra for a field k, $[D : k]$ is the dimension of D viewed as a k-vector space. Only in the last chapter, where the field k will be fixed, will we denote the dimension of a k-vector space V by $dim\ V$.

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[1] It may happen also that L denotes a subgroup of a group G…

We shall also use the following notation.

- The *Kronecker delta function*, applied to two variables x and y, is denoted $\delta_{x,y}$—not to be confused with the characteristic functions δ_s (unbolded) used in the last chapter.
- For Ω any finite set, $|\Omega|$ will denote the number of its elements.
- A subset (subgroup, subring, submodule, ...) Ω' of a set (group, ring, module, ...) Ω is said to be *proper* if $\Omega' \neq \Omega$.
- If g is an element of a group G, we denote by g the subgroup generated by g.
- For H a subgroup of a group G, we denote by $[G/H]$ a complete set of representatives for the cosets gH of G modulo H.
- For a group G acting on a set Ω, if $\mathrm{Cl}(\Omega)$ is the set of orbits of Ω under G, we shall denote by $[\mathrm{Cl}(\Omega)]$ a complete set of representatives for the orbits.

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