

Chapter 2

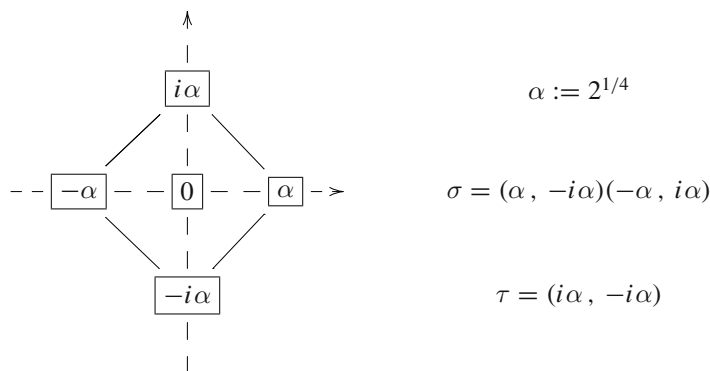
On Representations

2.1 Generalities on Representations

2.1.1 Introduction

When groups were discovered, during the XIXth century, there were no “abstract groups” (like “*a set endowed with a law which is associative, etc.*”). Groups were given as *permutation groups*, and most of the time as acting on roots of polynomials by preserving the algebraic relations between them.

For example, the group which is now known as the dihedral group of order 8, denoted by D_8 , and which is defined by generators σ and τ subject to relations $\sigma^2 = \tau^2 = (\sigma\tau)^4 = 1$, was known as the group which permutes the four complex roots of $X^4 - 2$ by preserving their rational relations.



One sees that this group may also be seen as the group of isometries of \mathbb{R}^2 which preserve a square centered in 0: it is “*represented*” as a subgroup of $\text{GL}_2(\mathbb{R})$. Choosing $(\alpha, i\alpha)$ as a basis of \mathbb{C} (identified with \mathbb{R}^2), we see that

$$\sigma \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad , \quad \tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

A representation of a group, roughly, consists in viewing that group as acting on a mathematical object (the set of roots of a polynomial, or a vector space, or a topological space, etc.) preserving the underlying structure.

2.1.2 General Representations

First definitions.

Let \mathcal{C} be a category:

- \mathcal{C} may be the category **Top** of topological spaces, in which case its objects are topological spaces and its morphisms are the continuous maps between the objects,
- \mathcal{C} may be the category **Vect** $_k$ of vector spaces over the field k , in which case its objects are k -vector spaces and its morphisms are the linear maps between the objects, or the subcategory **vect** $_k$ whose objects are the finite dimensional k -vector spaces and morphisms are again the linear maps between the objects (such a subcategory, where the objects run over a subclass, is called a full subcategory),
- \mathcal{C} may be the category **Set** of sets, in which case its objects are sets and its morphisms are the maps between the objects, or the full subcategory **set** $_k$ of finite sets,
- etc.

For X and Y objects of \mathcal{C} , the set of morphisms from X to Y will be denoted by $\text{Mor}_{\mathcal{C}}(X, Y)$ or simply $\text{Mor}(X, Y)$.

In the case where $\mathcal{C} = \mathbf{Vect}_k$, $\text{Mor}(X, Y)$ is the space $\text{Hom}(X, Y)$ (sometimes denoted, when necessary, $\text{Hom}_k(X, Y)$) of k -linear maps from X to Y .

If X is an object of \mathcal{C} , the group $\text{Aut}(X)$ of automorphisms of X consists of all morphisms from X to X which admit an inverse morphism.

- If $\mathcal{C} = \mathbf{Top}$, $\text{Aut}(X)$ is the group of all homeomorphisms of X into itself,
- if $\mathcal{C} = \mathbf{Vect}_k$, $\text{Aut}(X) = \text{GL}(X)$, the *general linear group* comprising all bijective linear endomorphisms of X ,
- if $\mathcal{C} = \mathbf{Set}$, $\text{Aut}(X) = \mathfrak{S}(X)$, the *symmetric group* of X , group of all bijections of X into itself.

Let now G be a group.

Definitions 2.1.1 • A *representation* of G on \mathcal{C} (or a \mathcal{C} -*representation* of G) is a pair (X, ρ) where

- * X is an object of \mathcal{C} ,
- * ρ is a group morphism from G into the group $\text{Aut}(X)$ of automorphisms of X .

- A representation (X, ρ) is said to be *faithful* if the morphism ρ is injective.
- A *morphism* between representations (X, ρ) and (X', ρ') of G on the same category \mathcal{C} is a morphism $f : X \rightarrow X'$ in \mathcal{C} such that, for all $g \in G$, the following diagram is commutative :

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \rho(g) \downarrow & & \downarrow \rho'(g) \\ X & \xrightarrow{f} & X' \end{array}$$

If (X, ρ) is a representation, we say that G acts on X .

The reader is invited to define the category $\mathbf{Rep}(G, \mathcal{C})$ of representations of G on \mathcal{C} , and in particular to check the following lemma.

Lemma 2.1.2 *A morphism $f : (X, \rho) \rightarrow (X', \rho')$ in $\mathbf{Rep}(G, \mathcal{C})$ is an isomorphism if and only if the morphism $f : X \rightarrow X'$ is an isomorphism in \mathcal{C} .*

In particular, two representations (X, ρ) and (X, ρ') on the same object X are isomorphic (one says then that they are *equivalent*) if there exists an automorphism f of X such that, for all $g \in G$,

$$\rho'(g) = f \cdot \rho(g) \cdot f^{-1},$$

i.e, if ρ and ρ' are “uniformly conjugate”.

Notation 2.1.3 If (X, ρ) and (X', ρ') are isomorphic, we write

$$(X, \rho) \simeq (X', \rho').$$

We end this paragraph with a remark on morphisms of representations.

Let (X, ρ) and (Y, σ) be \mathcal{C} -representations of G .

The group $\text{Aut}(X) \times \text{Aut}(Y)$ acts on the set $\text{Mor}(X, Y)$ of morphisms from X to Y , by the formula

$$\text{for } \alpha \in \text{Aut}(X), \beta \in \text{Aut}(Y), f \in \text{Mor}(X, Y), (\alpha, \beta) \cdot f := \beta f \alpha^{-1}.$$

Composed with the diagonal morphism

$$G \rightarrow \text{Aut}(X) \times \text{Aut}(Y), g \mapsto (\rho(g), \sigma(g)),$$

that operation defines an action of G on $\text{Mor}(X, Y)$:

$$\text{for } g \in G, f \in \text{Mor}(X, Y), g \cdot f := \sigma(g) f \rho(g)^{-1},$$

that is, the following diagram is commutative:

$$\begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 \rho(g) \downarrow & & \downarrow \sigma(g) \\
 X & \xrightarrow{g \cdot f} & Y
 \end{array}$$

The following lemma is then clear.

Lemma 2.1.4 *The set of morphisms $(X, \rho) \rightarrow (Y, \sigma)$ is the set of fixed points of G in its action on $\text{Mor}(X, Y)$.*

Case where objects have elements.

Remark 2.1.5 Although it is indeed the case for the categories **Top**, **Vect_k**, **Set**, the objects of a category need not be “sets containing elements” and morphisms need not “act on elements”: for example, the objects of the category **Rep**(G, \mathcal{C}) are not “sets containing elements”.

Nevertheless, if this is the case, and when there is no ambiguity, if x is an element of X , we shall write

$$gx := \rho(g)(x).$$

In that case, a morphism $(X, \rho) \rightarrow (X', \rho')$ is a morphism $f : X \rightarrow X'$ in \mathcal{C} such that, for all $g \in G$ and $x \in X$,

$$f(gx) = gf(x).$$

We add a few technical remarks for the case where X contains elements.

If (X, ρ) is a representation of G on \mathcal{C} and if $x \in X$, the fixator of x in G is the subgroup defined by

$$G_x := \{g \in G \mid gx = x\}.$$

The proof of the following lemma is left as an exercise to the reader.

Lemma 2.1.6 *Let (X, ρ) be a representation of G .*

(1) *For $x \in X$ and $g \in G$, we have*

$$G_{gx} = gG_xg^{-1}.$$

(2) *If $f : (X, \rho) \xrightarrow{\sim} (Y, \sigma)$ is an isomorphism of representations, then for $x \in X$ we have $G_{f(x)} = G_x$.*

Action of $\text{Aut}(G)$ on representations.

Let us denote by $\text{Aut}(G)$ the group of automorphisms of G . Any element $g \in G$ defines an element $\text{Inn}(g)$ acting on the left by

$$\text{Inn}(g) : G \xrightarrow{\sim} G, \quad h \mapsto ghg^{-1},$$

and we denote by $\text{Inn}(G)$ the group of all $\text{Inn}(g)$ for $g \in G$.

The group $\text{Inn}(G)$ is *normal* in $\text{Aut}(G)$, since for all $a \in \text{Aut}(G)$ and $g \in G$,

$$a\text{Inn}(g)a^{-1} = \text{Inn}(a(g)).$$

The group $\text{Aut}(G)$ acts on \mathcal{C} -representations, as we shall see now.

Let $a \in \text{Aut}(G)$.

- Whenever (X, ρ) is a \mathcal{C} -representation of G , we set ${}^a(X, \rho) := (X, {}^a\rho)$ where ${}^a\rho : G \rightarrow \text{Aut}(X)$ is the group morphism defined by ${}^a\rho := \rho \cdot a^{-1}$.

It is immediate to check that ${}^{aa'}(X, \rho) = {}^a({}^{a'}(X, \rho))$.

- If $f : X \rightarrow X'$ induces a morphism of representations

$$(X, \rho) \rightarrow (X', \rho'),$$

then it also induces a morphism of representations

$${}^a(X, \rho) \rightarrow {}^a(X', \rho').$$

Thus in particular $\text{Aut}(G)$ acts on the set of isomorphism classes of \mathcal{C} -representations.

- For $g \in G$, $\rho(g)$ is an isomorphism of representations

$$\rho(g) : {}^{\text{Inn}(g)}(X, \rho) \xrightarrow{\sim} (X, \rho).$$

Thus in particular $\text{Inn}(G)$ acts trivially on the set of isomorphism classes of \mathcal{C} -representations.

The group $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$ of *outer automorphisms* then acts on the set of isomorphism classes of \mathcal{C} -representations.

2.2 Set-Representations

2.2.1 Union and Product

Given sets X and Y , we know how to construct

- their (disjoint) union $X \sqcup Y$,
- their product $X \times Y$,

and if Z is a set, we have

$$Z \times (X \sqcup Y) = (Z \times X) \sqcup (Z \times Y).$$

Now given a group G and representations (on **Set**) (X, ρ) and (Y, σ) , we define their union and their product by

$$\begin{aligned} (\rho \sqcup \sigma)(g)(z) &:= \begin{cases} \rho(g)(z) & \text{if } z \in X, \\ \sigma(g)(z) & \text{if } z \in Y, \end{cases} \\ (\rho \times \sigma)(g)(x, y) &:= (\rho(g)(x), \sigma(g)(y)) \quad \text{for } x \in X, y \in Y. \end{aligned}$$

It is clear that distributivity of the product on the union remains valid for representations.

Let (X, ρ) be a **Set**-representation of G .

If X_1 is a subset of X which is stable by all the bijections $\rho(g)$ for $g \in G$, it gives rise to a *subrepresentation* (X_1, ρ_1) of (X, ρ) (where $\rho_1(g)$ is the restriction of $\rho(g)$ to X_1).

Example 2.2.1 Let (X, ρ) be a representation of G , and let $x \in X$. Then the subset of X defined by $X_1 := \{\rho(g)(x) \mid g \in G\}$ is stable by G , hence defines a subrepresentation.

Let X'_1 be the complement of X_1 in X . Then X'_1 is also stable by all the $\rho(g)$ for $g \in G$, and we have an isomorphism of representations (with obvious notations)

$$(X, \rho) \simeq (X_1, \rho_1) \sqcup (X'_1, \rho'_1).$$

2.2.2 Transitive Representations

The proof of the following proposition is immediate, and left to the reader as an exercise.

Proposition – Definition 2.2.2 *Let (X, ρ) be a **Set**-representation of G , $X \neq \emptyset$. The following assertions are equivalent.*

- (i) (X, ρ) is “simple” or “irreducible”, that is, if (X_1, ρ_1) is a subrepresentation of (X, ρ) , then either $X_1 = X$ or $X_1 = \emptyset$.
- (ii) (X, ρ) is “indecomposable”, that is, if $(X, \rho) \simeq (X_1, \rho_1) \sqcup (X'_1, \rho'_1)$, then either $X_1 = \emptyset$ or $X'_1 = \emptyset$.
- (iii) (X, ρ) is “transitive”, that is, for all $x, y \in X$, there exists $g \in G$ such that $y = \rho(g)(x)$.

Example 2.2.3 Let H be a subgroup of G . We denote by ρ_H the representation (by left multiplication) of G on the set G/H of left cosets of G modulo H . Thus, for $g, x \in G$, we have

$$\rho_H(g)(xH) := gxH .$$

- (1) The representation $(G/H, \rho_H)$ is transitive. Indeed, given xH and $yH \in G/H$, we have $yH = \rho_H(yx^{-1})(xH)$.
- (2) The fixator of xH is xHx^{-1} . Indeed, it can be proved either by a direct computation or by applying Lemma 2.1.6, (1).

We shall see (Theorem 2.2.5 below) that the above example is “universal”.

Let us first check that any representation is isomorphic to a disjoint union of transitive representations.

Let (X, ρ) be a **Set**-representation of G .

We define the equivalence relation \sim_G on X as follows:

$$(x \sim_G y) \iff (\exists g \in G)(y = \rho(g)(x)) .$$

We denote by $G \backslash X$ the set of equivalence classes of the above relation.

The proof of the following proposition is obvious.

Proposition 2.2.4 (1) Any equivalence class Ω of \sim_G defines a transitive subrepresentation (Ω, ρ_Ω) of (X, ρ) .

(2) Thus

$$(X, \rho) \simeq \bigsqcup_{\Omega \in G \backslash X} (\Omega, \rho_\Omega) .$$

2.2.3 Classification of Transitive Representations

Let us now classify all transitive representations. The next theorem shows that the isomorphism classes of transitive representations of G are naturally parametrized by conjugacy classes of subgroups of G .

Theorem 2.2.5 (1) Let (X, ρ) be a transitive representation of G . For $x \in X$, let G_x denote the fixator of x in G . Then the map

$$G/G_x \rightarrow X , \quad gG_x \mapsto \rho(g)(x)$$

defines an isomorphism $(G/G_x, \rho_{G_x}) \xrightarrow{\sim} (X, \rho)$:

$$\begin{array}{ccc} G/G_x & \xrightarrow{\sim} & X \\ \downarrow g & & \downarrow \rho(g) \\ G/G_x & \xrightarrow{\sim} & X \end{array}$$

- (2) For subgroups H and H' of G , the representations $(G/H, \rho_H)$ and $(G/H', \rho_{H'})$ are isomorphic if and only if H and H' are conjugate.

Proof The proof of (1) is easy and left to the reader.

(2) Assume first that H and H' are conjugate. Since $(G/H, \rho_H)$ is transitive and since H is the fixator of $H \in G/H$ (yes!), it follows from Lemma 2.1.6, (1), that H' is the fixator of an element of G/H . Thus by (1) it follows that $(G/H, \rho_H)$ and $(G/H', \rho_{H'})$ are isomorphic.

Assume now that $(G/H, \rho_H)$ and $(G/H', \rho_{H'})$ are isomorphic. It follows from Lemma 2.1.6, (2), that both H and H' are fixators of elements of G/H , hence are conjugate by Lemma 2.1.6, (1). \square

If (X, ρ) is a **Set**-representation, the cardinality of X is called the *degree* of the representation.

Corollary 2.2.6 *If G is a finite group, the degree of any transitive representation of G divides the order of G .*

Notation 2.2.7 From now on, we also call G -set a set X endowed with an action of G , that is, a group morphism $\rho : G \rightarrow \mathfrak{S}(X)$.

The category $\mathbf{Rep}(G, \mathbf{Set})$ is also denoted ${}_G\mathbf{Set}$.

Notation 2.2.8 When a finite group G acts on an object X “with elements” (e.g. a set, a vector space, or a set of morphisms $\text{Mor}(Y, Z)$), we denote by $\text{Fix}^G(X)$ the subset of fixed points of X under G .

2.2.4 Burnside’s Marks

From now on, all our G -sets are assumed to be finite. In other words, we consider the category ${}_G\mathbf{set}$.

Definition and properties of marks.

Definition 2.2.9 Let G be a finite group and let X be a (finite) G -set. The *mark* of X is the function m_X defined on the set of all subgroups of G , such that, for any subgroup H of G ,

$$m_X(H) := |\text{Fix}^H(X)|.$$

Let us denote by $\mathcal{S}(G)$ the set of *conjugacy classes* of subgroups of G .

Define the following partial order on $\mathcal{S}(G)$:

$$(C \leq C') :\iff (\exists H \in C) (\exists H' \in C') (H \subseteq H').$$

Notice that:

- the value $m_X(H)$ depends only on the conjugacy class of the group H ,
- by Theorem 2.2.5, the set of isomorphism classes of transitive G -sets is in natural bijection with the set $\mathcal{S}(G)$ of conjugacy classes of subgroups of G .

Thus the function mark induces a function on $\mathcal{S}(G) \times \mathcal{S}(G)$:

$$m(C, C') := m_{G/H'}(H) \quad \text{for } H \in C \text{ and } H' \in C'.$$

For H a subgroup of G , we denote by

$$N_G(H) := \{g \in G \mid gHg^{-1} = H\}$$

the *normalizer* of H in G .

Lemma 2.2.10 (1) $m(C, C') \neq 0$ if and only if $C \leq C'$.

(2) $m(C, C) = |N_G(H)/H|$ for $H \in C$.

Proof (1) A group H fixes an element in G/H' if and only if it is contained in the fixator of an element of G/H' . But the fixators of elements in G/H' are the conjugates of H' . Hence $m_{G/H'}(H) \neq 0$ if and only if H is contained in H' up to conjugation.

(2) We have

$$\begin{aligned} \text{Fix}^H(G/H) &= \{gH \in G/H \mid HgH = gH\} \\ &= \{gH \in G/H \mid g^{-1}HgH = H\} \\ &= \{gH \in G/H \mid g \in N_G(H)\} = N_G(H)/H. \end{aligned}$$

□

Marks characterize representations.

Proposition 2.2.11 Let X and Y be finite G -sets. The following assertions are equivalent.

- (i) X and Y are isomorphic as G -sets.
- (ii) $m_X = m_Y$.

The proof will use the notion of *Möbius function of a poset* — the reader may refer to [Aig79, Chap. IV] for details and examples. It may be viewed as an easy computation of the inverse of a non singular “triangular matrix on a partially ordered set”.

Given the function $m : \mathcal{S}(G) \times \mathcal{S}(G) \rightarrow \mathbb{Q}$, let us define the function $\mu : \mathcal{S}(G) \times \mathcal{S}(G) \rightarrow \mathbb{Q}$ by the following conditions:

- (1) for all $C, D \in \mathcal{S}(G)$, $\mu(C, D) \neq 0$ implies $C \leq D$,
- (2) for all $C \in \mathcal{S}(G)$, $\mu(C, C)m(C, C) = 1$,

(3) if $C < E$, then

$$\sum_{C \leq D \leq E} \mu(C, D)m(D, E) = 0.$$

Note that condition (3) allows to compute $\mu(C, E)$ knowing $\mu(C, D)$ for $C \leq D < E$, hence defines $\mu(C, E)$ for all $C \leq E$ by induction since $m(E, E) \neq 0$. Note also that μ takes indeed its values in \mathbb{Q} .

The following lemma is an inversion formula.

Lemma 2.2.12 *Let $f : S(G) \rightarrow \mathbb{Q}$ be a function. Define $\hat{f} : S(G) \rightarrow \mathbb{Q}$ by*

$$\hat{f}(C) := \sum_{D \geq C} m(C, D)f(D).$$

Then

$$f(C) = \sum_{D \geq C} \mu(C, D)\hat{f}(D).$$

Proof It is an immediate computation:

$$\begin{aligned} \sum_{D \geq C} \mu(C, D)\hat{f}(D) &= \sum_{D \geq C} \mu(C, D) \left(\sum_{E \geq D} m(D, E)f(E) \right) \\ &= \sum_{E \geq C} f(E) \left(\sum_{C \leq D \leq E} \mu(C, D)m(D, E) \right) \\ &= f(C). \end{aligned}$$

□

Proof of Proposition 2.2.11 For $D \in S(G)$ (a conjugacy class of subgroups of G), let us denote by X_D the transitive G -set associated with some $H \in D$.

Let X be a G -set, which is then isomorphic to a disjoint union of some X_D for $D \in S(G)$. We denote by $f_X(D)$ the multiplicity of G -sets isomorphic to X_D in such a decomposition, which we write

$$X \simeq \bigsqcup_{D \in S(G)} f_X(D) \cdot X_D. \quad (*)$$

To prove Proposition 2.2.11, it suffices to prove that the function m_X determines the function f_X .

By equality (*) above, since by definition $m(C, D) = m_{X_D}(C)$ and since $m(C, D) \neq 0 \Rightarrow C \leq D$, we have

$$m_X(C) = \sum_{D \geq C} f_X(D)m(C, D).$$

Lemma 2.2.12 shows then that

$$f_X(C) = \sum_{D \geq C} \mu(C, D) m_X(D),$$

which proves that m_X determines f_X . □

2.3 Linear Representations

Throughout this section, k is a field. We shall only consider representations of finite groups on finite dimensional vector spaces – in other words, **vect** $_k$ -representations.

2.3.1 Generalities

A (finite dimensional) k -linear representation of G is a pair (M, ρ) where M is a finite dimensional k -vector space and $\rho : G \rightarrow \text{GL}(M)$ is a group morphism. We shall often abbreviate (if there is no ambiguity) (M, ρ) by M .

The set of morphisms from (M, ρ) to (N, σ) is the set of fixed points of G acting on the space $\text{Hom}_k(M, N)$ of linear maps from M to N (see Lemma 2.1.4). This set of morphisms will be denoted by $\text{Hom}_{kG}(M, N)$.

Thus the category **Rep** (G, vect_k) may be seen as follows:

- its objects are the pairs (M, ρ) , where M is a finite dimensional k -vector space and $\rho : G \rightarrow \text{GL}(M)$ a group morphism – or, simpler, the objects are *finite dimensional k -vector spaces endowed with a k -linear action of G* ,
- the set of morphisms between objects M and N is

$$\text{Hom}_{kG}(M, N) = \{ \alpha \in \text{Hom}_k(M, N) \mid (\forall g \in G)(\alpha g = g \alpha) \}.$$

The dimension of M is called the *degree* of the representation.

The *trivial representation* is (k, ρ_{triv}) where $\rho_{\text{triv}} : G \rightarrow \text{GL}(k) = k^\times$ is the trivial morphism (i.e. $\rho_{\text{triv}}(g) = 1$ for all $g \in G$).

Matrix representations.

Assume M has dimension r . The choice of a basis of M defines a group isomorphism $\text{GL}(M) \xrightarrow{\sim} \text{GL}_r(k)$ (where $\text{GL}_r(k)$ denotes the group of all invertible $r \times r$ matrices with entries in k).

Thus a representation of G may be seen as a pair (r, ρ) where r is a natural integer and $\rho : G \rightarrow \text{GL}_r(k)$ is a group morphism.

In this context, two representations (r, ρ) and (s, σ) are isomorphic (equivalent) if and only if $r = s$ and there is an element $f \in \text{GL}_r(k)$ such that

$$\forall g \in G, \sigma(g) = f \cdot \rho(g) \cdot f^{-1},$$

i.e. if the matrices $\rho(g)$ and $\sigma(g)$ (for $g \in G$) are *uniformly similar*.

Extension of scalars.

Assume that k is a subfield of a field k' .

• Given a k -representation (M, ρ) , its extension of scalars is the k' -representation $(k' \otimes_k M, k' \otimes_k \rho)$, where we denote by $k' \otimes_k \rho$ the morphism $G \rightarrow \text{GL}(k' \otimes_k M)$ defined by

$$g \mapsto \text{Id}_{k'} \otimes \rho(g).$$

A basis of M over k is identified with a basis of $k' \otimes_k M$ over k' . Thus, from the point of view of a matrix representation $\rho : G \rightarrow \text{GL}_r(k)$, extending the scalars consists in viewing the matrix entries of $\rho(g)$ as elements of k' .

• Conversely, a k' -representation (M', ρ') is said to be *rational over k* if there exists a k -representation (M, ρ) such that

$$(M', \rho') \simeq (k' \otimes_k M, k' \otimes_k \rho).$$

From the point of view of matrix representations, a representation $G \rightarrow \text{GL}_r(k')$ is rational over k if there exists an element $U \in \text{GL}_r(k')$ such that, for all $g \in G$, all the entries of $U\rho(g)U^{-1}$ belong to k .

Construction of representations.

If (M, ρ) and (N, σ) are representations of G , one may build

- their direct sum $(M \oplus N, \rho \oplus \sigma)$ (the construction is left to the reader), the analog of the disjoint union of **Set**-representations,
- their tensor product $(M \otimes_k N, \rho \otimes_k \sigma)$ (the construction is left to the reader), the analog of the product of **Set**-representations.

If M_1 is a subspace of M which is stable by the action of all $\rho(g)$ for $g \in G$, we get a *subrepresentation* (M_1, ρ_1) of (M, ρ) (where $\rho_1(g)$ is the restriction of $\rho(g)$ to M_1).

In this case, we also have a *quotient representation* $(M/M_1, \bar{\rho})$ (where $\bar{\rho}(g)$ is the automorphism of M/M_1 induced by $\rho(g)$).

① Contrary to what happens for **Set**-representations, given a representation (M, ρ) and a G -stable subspace M_1 , such a subspace need not have a G -stable complement (see the following exercise).

Exercise 2.3.1 Assume that G is the cyclic (additive) group $\mathbb{Z}/p\mathbb{Z}$ where p is a prime number, and let k be a field of characteristic p .

Consider the morphism

$$G \rightarrow \mathrm{GL}_2(k), \quad \lambda \mapsto \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

Prove that the line generated by the first standard basis vector of k^2 is G -stable but has no G -stable complement.

Indecomposable and irreducible representations.

As for **Set**-representations, we say that a representation (M, ρ) is

- *simple* (or *irreducible*) if $M \neq \{0\}$ and the only G -stable subspaces are M and $\{0\}$,
- *indecomposable* if $M \neq \{0\}$ and $(M, \rho) \simeq (M_1 \oplus M'_1, \rho_1 \oplus \rho'_1)$ imply either $M_1 = \{0\}$, or $M'_1 = \{0\}$.

⌚ Contrary to what happens for **Set**-representations, a representation may be indecomposable without being irreducible (see below Exercise 2.3.2).

⌚ Nevertheless, we shall see below (3.1.5) that over *characteristic zero* fields, again the indecomposable and the irreducible representations coincide (for a *finite* group).

Exercise 2.3.2 Prove that the representation defined in Exercise 2.3.1 is not irreducible but is indecomposable.

The first assertion of the following theorem is a consequence of an easy induction on the degree of the representation. The second assertion is a much deeper result, known as *Krull–Schmidt Theorem*, which won't be proved here.

Theorem 2.3.3 *Let (M, ρ) be a representation of finite degree.*

- (1) *It is a direct sum of indecomposable representations.*
- (2) *If G is finite, and if $(M, \rho) \simeq \bigoplus_{i \in I} (M_i, \rho_i) \simeq \bigoplus_{j \in J} (M'_j, \rho'_j)$ where the (M_i, ρ_i) and the (M'_j, ρ'_j) are indecomposable, there is a bijection $\alpha : I \xrightarrow{\sim} J$ such that, for all $i \in I$, $(M_i, \rho_i) \simeq (M'_{\alpha(i)}, \rho'_{\alpha(i)})$.*

From set-representations to vect_k-representations.

Let (Ω, ρ) be a **set**-representation of G . One denotes by $k\Omega$ the k -vector space with basis Ω . Then it is clear that ρ induces a group morphism

$$k\rho : G \rightarrow \mathrm{GL}(k\Omega).$$

The reader may check that this construction transforms

- the disjoint union of **set**-representations into the direct sum of the corresponding **vect_k**-representations,
- the product of **set**-representations into the tensor product of the corresponding **vect_k**-representations.

Remark 2.3.4 Using the notion of *functor* between two categories (see Appendix E), one can check easily that the above construction induces a functor

$$\mathbf{Rep}(G, \mathbf{set}) \rightarrow \mathbf{Rep}(G, \mathbf{vect}_k).$$

⚠ **Attention** ⚠ Even if (Ω, ρ) is transitive with degree at least 2, $(k\Omega, k\rho)$ is *not* irreducible.

Indeed, the element $S\Omega := \sum_{\omega \in \Omega} \omega \in k\Omega$ spans a line which is stable under G .

Definition 2.3.5 The group G acts on itself by left multiplication. The corresponding k -linear representation is called the *regular representation* of G .

The preceding example is used in the following proposition.

Proposition 2.3.6 Assume that k has nonzero characteristic p which divides the order $|G|$ of G . Then the line kSG has no G -stable complement in the regular representation kG .

Proof Let us choose a complement of kSG in kG , that is, let us choose a projector $\pi : kG \rightarrow kSG$. Hence $\pi(SG) = SG$.

Now if the corresponding complement were G -stable, we would have $g\pi(x) = \pi(gx)$ for all $x \in kG$, which would imply $\pi(SG) = |G|\pi(1) = 0$ since the characteristic of k divides $|G|$. This is a contradiction. \square

⚠ Nevertheless, we shall see below (3.1.3) that over characteristic zero fields such a situation cannot happen, since then for every subrepresentation there is a G -stable complement.

Contragredient, homomorphisms and tensor products.

Let (M, ρ) and (N, σ) be \mathbf{vect}_k -representations of G , which we denote (by abuse of notation) by their “vector spaces parts” M and N .

- There is a natural representation associated with $\mathrm{Hom}_k(M, N)$ (the space of linear maps from M to N), defined by the following formula:

$$\text{for } \alpha \in \mathrm{Hom}_k(M, N) \text{ and } g \in G, \quad g \cdot \alpha := \sigma(g)\alpha\rho(g)^{-1}.$$

- In particular, there is a natural representation associated with the dual $M^* := \mathrm{Hom}_k(M, k)$ of linear maps from M to k , defined by the following formula:

$$\text{for } \alpha \in \mathrm{Hom}_k(M, k) \text{ and } g \in G, \quad g \cdot \alpha := \alpha\rho(g)^{-1}.$$

This representation is called the *contragredient representation* of (M, ρ) .

Exercise 2.3.7 Compare the matrices associated with a representation and the matrices associated with the contragredient representation with respect to a pair of dual bases.

- There is a natural representation associated with the tensor space $M \otimes_k N$, defined by the following formula:

$$\text{for } m \in M, n \in N \text{ and } g \in G, g \cdot (m \otimes_k n) := \rho(g)(m) \otimes_k \sigma(g)(n).$$

Proposition 2.3.8 (1) *The morphism*

$$M^* \otimes_k N \rightarrow \text{Hom}_k(M, N), \varphi \otimes_k n \mapsto (m \mapsto \varphi(m)n),$$

defines a morphism of representations.

(2) *If M and N are finite dimensional, this is an isomorphism.*

Proof (1) Applying $g \in G$ to

- the elementary tensor $\varphi \otimes_k n$ gives $\varphi \rho(g)^{-1} \otimes_k \sigma(g)(n)$,
- the map $m \mapsto \varphi(m)n$ gives the map $m \mapsto \varphi(\rho(g)^{-1}(m))\sigma(g)(n)$, and this establishes (1).

(2) follows from the fact that the above morphism is an isomorphism if M and N are finite dimensional (see Proposition 1.2.14). \square



Issai Schur (1875–1941)

Irreducible representations, Schur's Lemma.

We recall that a representation (S, ρ) (S a k -vector space, $\rho : G \rightarrow \text{GL}(S)$) is said to be *irreducible* if $S \neq \{0\}$ and the only G -stable subspaces of S are $\{0\}$ and S .

Proposition 2.3.9 (Schur's Lemma) *Let (S, ρ) and (S', ρ') be irreducible representations of G .*

- (1) *If (S, ρ) and (S', ρ') are not isomorphic, then $\text{Hom}_{kG}(S, S') = \{0\}$.*
- (2) *$\text{End}_{kG}(S) := \text{Hom}_{kG}(S, S)$ is a division k -algebra, a finite extension of k .*
- (3) *If k is algebraically closed, $\text{End}_{kG}(S) = k\text{Id}_S$.*

Proof Let $f \in \text{Hom}_{kG}(S, S')$. Assume $f \neq 0$. Then

- $\ker(f)$ is a proper G -stable subspace of S , hence $\ker(f) = 0$;
- $\text{im}(f)$ is a nonzero subspace of S' , hence $\text{im}(f) = S'$.

This proves that a nonzero kG -morphism from S to S' is an isomorphism, whence (1) and (2).

To prove (3), we may for example notice that any $f \in \text{End}_{kG}(S)$ has an eigenvalue λ . Then $\ker(f - \lambda \text{Id}_S)$ is a G -stable nonzero subspace of S , hence $\ker(f - \lambda \text{Id}_S) = S$ and $f = \lambda \text{Id}_S$. \square

Representations of degree 1.

A representation of degree 1 is obviously irreducible. It corresponds to a morphism

$$G \rightarrow \text{GL}_1(k) = k^\times.$$

Since k^\times is abelian, morphisms $G \rightarrow k^\times$ correspond naturally to morphisms

$$G/[G, G] \rightarrow k^\times,$$

where $[G, G]$ denotes the *derived subgroup* of G , which is the subgroup generated by the *commutators* $[g, h] := ghg^{-1}h^{-1}$ for $g, h \in G$.

Example 2.3.10 Consider the dihedral group $G = \langle \sigma, \tau \rangle$ of order 8 mentioned in the introduction of this chapter.

Its derived subgroup is equal to its center, the subgroup of order 2 generated by $(\sigma\tau)^2$. It is not difficult to check that $G/[G, G]$ is isomorphic to the direct product of two groups of order 2.

Hence $G/[G, G]$ has four degree 1 representations over a field of characteristic different from 2, namely

$$\begin{aligned} \rho_1 : \sigma &\mapsto 1, \tau \mapsto 1; & \rho_\sigma : \sigma &\mapsto -1, \tau \mapsto 1; \\ \rho_\tau : \sigma &\mapsto 1, \tau \mapsto -1; & \rho_{\sigma\tau} : \sigma &\mapsto -1, \tau \mapsto -1. \end{aligned}$$

If (M, ρ) is any representation and if (k, σ) is a degree one representation, the tensor product $(M \otimes_k k, \rho \otimes_k \sigma)$ is naturally equivalent to the representation $(M, \rho\sigma)$ where

$$\rho\sigma : G \rightarrow \text{GL}(M), \quad g \mapsto \sigma(g)\rho(g).$$

The following lemma is immediate.

Lemma 2.3.11 *Let (S, ρ) be an irreducible representation of G and let (k, σ) be a degree one representation. Then the tensor product $(S, \sigma\rho)$ is still irreducible.*

ⓘ **Attention** ⓘ Tensoring an irreducible representation by a nontrivial representation of degree one need not change the equivalence class of the representation.

As an exercise, the reader may check that for the dihedral group of order 8, multiplying its degree 2 real representation (as defined in the introduction of this chapter) by any degree 1 representation provides equivalent representations — a fact which will become obvious with characters.

2.3.2 Finite Groups: The Group Algebra

The group algebra.

Let G be a finite group. The *group algebra* kG is the k -vector space kG endowed with multiplication defined by the multiplication of elements of G .

Thus, for $x := \sum_{s \in G} x_s s$ and $y := \sum_{t \in G} y_t t$ in kG , we have

$$xy = \sum_{s, t \in G} x_s y_t st = \sum_{g \in G} \left(\sum_{s \in G} x_s y_{s^{-1}g} \right) g.$$

Remark 2.3.12 One sees that the group algebra kG may be seen as the convolution algebra of k -valued functions on G .

Let $\tau : kG \rightarrow k$ be the linear form on kG defined by

$$\tau(g) = 0 \text{ if } g \neq 1 \text{ and } \tau(1) = 1.$$

The next lemma shows that τ is a *symmetrizing form* on kG (thus giving kG the structure of a *symmetric algebra*).

Lemma 2.3.13 (1) τ is a “class function”: for all $x, y \in kG$, $\tau(xy) = \tau(yx)$.

(2) The map

$$\widehat{\tau} : \begin{cases} kG \rightarrow (kG)^* = \text{Hom}_k(kG, k) \\ x \mapsto (y \mapsto \tau(xy)) \end{cases}$$

is a k -linear isomorphism.

Proof (1) It is clear that, for all $g, h \in G$, $\tau(gh) = \tau(hg)$. This implies (1) by linearity.

(2) The basis $(\widehat{\tau}(g^{-1}))_{g \in G}$ is the dual basis of the basis $(g)_{g \in G}$ of kG . Thus the image of $\widehat{\tau}$ contains a basis of $\text{Hom}_k(kG, k)$, which shows that $\widehat{\tau}$ is an isomorphism. \square

We identify functions $G \rightarrow k$ with linear forms on kG , and we denote by $F(G, k)$ the k -vector space of functions on G .

The following property is easy to check.

2.3.14. For $f \in F(G, k)$, the inverse image of f under the isomorphism given in assertion (2) of Lemma 2.3.13 is

$$f^\circ := \sum_{g \in G} f(g^{-1})g.$$

Convention 2.3.15 Let $f : G \rightarrow k$ be a class function on G , that is, such that $f(gh) = f(hg)$ for all $g, h \in G$. We identify f (without changing notation) with the corresponding linear form $f : kG \rightarrow k$, which is also a class function, i.e. $f(xy) = f(yx)$ for all $x, y \in kG$. Let $\text{CF}(G, k)$ denote the space of class functions on G (or kG).

The following lemma will be useful later (see Theorem 3.3.5).

Lemma 2.3.16 Let $a \in kG$ and let $f \in \text{F}(G, k)$. Let us denote by $a \cdot f$ the element of $\text{F}(G, k)$ defined by

$$(a \cdot f)(x) := f(xa) \text{ for all } x \in kG.$$

Then

$$(a \cdot f)^0 = af^0.$$

Proof By definition of f^0 , we have $f(x) = \tau(f^0x)$, hence $(a \cdot f)(x) = f(xa) = \tau(f^0xa) = \tau(af^0x)$, which establishes the lemma. \square

Let $\text{Cl}(G)$ denote the set of conjugacy classes of G , and for $C \in \text{Cl}(G)$ let us set

- * γ_C the characteristic function of C , such that $\gamma_C(g) = 0$ if $g \notin C$ and $\gamma_C(g) = 1$ if $g \in C$,
- * $\mathcal{S}C := \sum_{g \in C} g \in kG$.

For f a class function, we have

$$f = \sum_{C \in \text{Cl}(G)} f(g_C) \gamma_C \quad \text{and} \quad f^\circ = \sum_{C \in \text{Cl}(G)} f(g_C^{-1}) \mathcal{S}C,$$

where g_C denotes an element of C .

Lemma 2.3.17 Let $\text{Zk}G$ denote the center of the group algebra kG .

- (1) The family $(\gamma_C)_{C \in \text{Cl}(G)}$ is a basis of $\text{CF}(G, k)$, and the family $(\mathcal{S}C)_{C \in \text{Cl}(G)}$ is a basis of $\text{Zk}G$. Hence

$$[\text{CF}(G, k) : k] = |\text{Cl}(G)|.$$

- (2) The isomorphism of Lemma 2.3.13, (2) restricts to a k -linear isomorphism $\text{CF}(G, k) \xrightarrow{\sim} \text{Zk}G$. Hence

$$[\text{Zk}G : k] = |\text{Cl}(G)|.$$

kG-modules and representations of G.

• Let (M, ρ) be a representation of G . Then the morphism $\rho : G \rightarrow \text{GL}(M)$ extends by linearity to an *algebra morphism* $\rho : kG \rightarrow \text{End}_k(M)$.

Reciprocally, given an algebra morphism $\rho : kG \rightarrow \text{End}_k(M)$ (by which we mean in particular that it sends 1 to 1), it restricts to a group morphism $\rho : G \rightarrow \text{GL}(M)$.

• Now, given an algebra morphism $\rho : kG \rightarrow \text{End}_k(M)$ amounts to defining a multiplication

$$kG \times M \rightarrow M, (x, m) \mapsto xm$$

(defined by $xm := \rho(x)(m)$) satisfying the conditions (for $x, y \in kG, m, n \in M, \lambda \in k$)

$$\begin{cases} (x + y)m = xm + ym, & x(ym) = (xy)m \text{ and } 1m = m, \\ x(m + n) = xm + xn, & \lambda(xm) = (\lambda x)m = x(\lambda m), \end{cases} \quad (\text{mod})$$

“as if M were a kG -vector space” (except that kG is not a field...). We then say that M is a *kG-module*.

Reciprocally, given a multiplication $kG \times M \rightarrow M$ satisfying (mod) defines an algebra morphism $kG \rightarrow \text{End}_k(M)$, hence a representation of G on M .

Notation 2.3.18 From now on, we shall speak interchangeably, either of representations, or of kG -modules.

We also call *kG-module* a k -vector space M endowed with an action of G , that is, a group morphism $\rho : G \rightarrow \text{GL}(M)$.

The category $\mathbf{Rep}(G, \mathbf{vect}_k)$ is also denoted $kG\mathbf{mod}$.

Notice that a morphism of representations

$$f : (M, \rho) \rightarrow (N, \sigma)$$

corresponds to a morphism of kG -modules, i.e. a k -linear map

$$f : M \rightarrow N \quad \text{such that } f(xm) = xf(m)$$

for all $x \in kG$ and $m \in M$.

Notice that the regular representation of G , viewed in terms of kG -modules, consists just of the group algebra acted on by itself by left multiplication.

Lemma 2.3.19 Let $\rho_{kG} : kG \rightarrow \text{End}_k(kG)$ denote the morphism associated with the regular representation. Then ρ_{kG} is injective, i.e. the regular representation of kG is faithful.

Proof Indeed, notice that for all $x \in kG, x = \rho_{kG}(x)(1)$. Hence $\rho_{kG}(x) = 0$ implies $x = 0$. \square

More Exercises on Chap. 2

Exercise 2.3.20 Let \mathfrak{S}_n denote the n -th symmetric group, the group of all permutations of $\{1, \dots, n\}$, of order $n!$.

- (1) Check that the set $D_4 := \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is a normal subgroup of \mathfrak{S}_4 .

Deduce that this defines a group morphism $\mathfrak{S}_4 \rightarrow \mathfrak{S}_3$.

- (2) Prove that this morphism is surjective.

Exercise 2.3.21 (1) Let G be a finite group acting on a finite set Ω . Let $\text{Fix}^G(\Omega)$ denote the set of fixed points of Ω under G , and let $\Omega^\#$ denote the set of orbits of G on $\Omega \setminus \text{Fix}^G(\Omega)$. Prove that

$$|\Omega| = |\text{Fix}^G(\Omega)| + \sum_{\omega \in \Omega^\#} |\omega|.$$

From now on, we assume that p is a prime number and that G is a nontrivial p -group.

- (2) Assume now that $|\Omega|$ is a power of p different from 1. Prove that $|\text{Fix}^G(\Omega)|$ is divisible by p .
- (3) Let k be a (commutative) field of characteristic p . Let M be a kG -module. Prove that $\text{Fix}^G(M) \neq 0$.

HINT.

- (a) Let $(e_i)_{i=1, \dots, r}$ be a k -basis of M . Prove that the abelian group generated by $(ge_i)_{(i=1, \dots, r)(g \in G)}$ is an $\mathbb{F}_p G$ -module.
- (b) Apply question (2) above to conclude.
- (4) Prove that, up to isomorphism, the trivial representation of G is the unique irreducible kG -module.

Exercise 2.3.22 Let G be the dihedral group of order 8, generated by elements σ and τ subject to relations $\sigma^2 = \tau^2 = (\sigma\tau)^4 = 1$.

- (1) Check that the map

$$\rho : \begin{cases} G & \rightarrow \text{GL}_2(\mathbb{C}) \\ \sigma & \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{cases}, \quad \tau \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

defines a complex representation of G .

- (2) Prove that this representation is rational over \mathbb{Q} .

Exercise 2.3.23 Let G be a finite group and let (M, ρ) be a faithful \mathbb{C} -linear representation of G .

Assume (M, ρ) irreducible. Prove that the center of G is cyclic.

Exercise 2.3.24 Assume that the (commutative) field k has characteristic $p > 1$. Let $G = \langle g \rangle$ be a cyclic group of order n . We set $n = p^a m$ with $p^a := n_p$ (the largest power of p which divides n) and $m := n_{p'}$ (the largest divisor of n which is prime to p). We assume the field k contains ζ , a primitive m -th root of unity.

- (1) For M a kG -module, let $\rho_M : kG \rightarrow \text{End}_k(M)$ denote the “structural morphism”.
 - (a) What are the possible eigenvalues of $\rho_M(g)$?
 - (b) Prove that there exists a direct sum decomposition

$$M = M_1 \oplus \cdots \oplus M_m,$$

such that the endomorphism $\rho_{M_i}(g) - \zeta^i \text{Id}_{M_i}$ is nilpotent (for all $i = 1, \dots, m$).

- (2) For all pairs of integers (i, j) such that $0 \leq i < m$ and $1 \leq j \leq p^a$, we denote by $M_{i,j}$ the kG -module whose underlying k -vector space is k^j , and where (on the canonical basis of k^j)

$$\rho_{M_{i,j}}(g) := \begin{pmatrix} \zeta^i & 1 & 0 & \cdots & 0 \\ 0 & \zeta^i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \zeta^i \end{pmatrix}$$

Prove that, for all (i, j) , the kG -module $M_{i,j}$ is indecomposable.

- (3) Prove that the family $(M_{i,j})_{0 \leq i < m, 1 \leq j \leq p^a}$ is a complete set of representatives of indecomposable kG -modules.

On Characters of Finite Groups

Broué, M.

2017, XVI, 246 p. 9 illus., 5 illus. in color., Hardcover

ISBN: 978-981-10-6877-5