

## Chapter 2

# Several Impedances from One Equation

**Abstract** The chapter is dedicated to estimate impedances and their space derivatives using the theory of the random processes. This approach is effective to estimate several unknown values from one equation. Results are prepared in the frequency domain and characterized by confidence limits. Selections of directions for interpretation are discussed. The theoretical principles considered below deal with fields induced by sources in the ionosphere or magnetosphere of the Earth only.

**Keywords** Random processes • Spectra • Transfer function

### 2.1 Introduction

The purpose of data processing of induction soundings data is the definition of impedance—the main characteristics of soundings. The induction soundings are carried out on natural fields, the intensity and position of which vary with time. Impedances, representing a ratio of the spectra of the field components, are independent of the power source variation, at least at the time of observation. Impedance depends on the frequency, on the oscillations of external fields, electrical conductivity and magnetic permeability of the medium, which allows us to perform soundings. Note that impedances are also depending on the source characteristics of the field. So, following Guglielmi, we can say: “The Earth does not have its own impedance”—it’s just a convenient feature for analysis.

Most of the magnetotelluric source fields are not stable and that is why the impedances determined at different times, i.e. for different combinations of the field sources, may be different from each other. Matching of all sources of the field is impossible, and there is a need to use the criteria for selecting the best impedance or a combination of them. Such criteria are either of negative character, as the impossibility of increasing the impedance value with the increase in the period of field variations, or based on a priori knowledge of the signal/noise ratio in individual components of the observed field.

It is most fundamental to have a theoretical criterion for measuring the link between spectra of the observed field components. This is the coherence. Usage of this criterion on the basis of the existing theory of harmonic signals is not possible: a linear relationship between the Fourier amplitudes of sine waves on the same frequency is always there! Furthermore, one field component is generally simultaneously connected not to two, but to three or more other field components. We know from school that one equation with two or more unknowns has an infinite number of solutions. However, if the registered electromagnetic fields can be simulated by a random process with some limitations, it is possible to quantify the strength of linear relationships between their components in the frequency band. As a measure of linear relationship there appears the coherence—an analog of correlation in the frequency domain.

Due to complicated multi-variate connections between the components of the field we are forced to introduce more detailed criteria, like partial and multiple coherences. In addition, the representation of the field variations by a random process allows in many cases to establish criteria for selecting the best of the impedances found by means of a priori information about the nature of the field and how to measure it. The result of this analysis is the evaluation of the impedance within the frequency bands and the amplitudes with a predetermined probability.

As in any statistics, the fulfillment of such criteria is a necessary but not sufficient condition for the existence of impedances. Therefore, the analysis of data based on the theory of random processes only warns the researchers about the false conclusions; at best, it points to the need to improve the techniques of observation and theoretical foundations of the method. Checking of the validity of the selected criteria remains to be done by a comparison of results of interpretation of sounding with the independent research.

## 2.2 Modeling of Fields by Stochastic Processes

### 2.2.1 *Random Process Definition*

At the end of the last century, a lot of papers were dedicated to the problems of spectral analysis of geophysical data (Bath 1974; Reddy and Rankin 1974; Svetov and Shimilevich 1982; and so forth). However, even in these papers the random process was used not as a strict mathematical model but as an illustration of application of stochastic approaches to data analysis. In terms of the theory of random processes it is not difficult to justify the possibility of evaluating solutions of an equation with two or three unknowns in the range of frequencies and amplitudes with a given probability. In the approach presented below, the random process is considered as a model of registered fields for induction sounding of the Earth.

A field component variation is generally written as  $X(t)$ , where  $t$  is time. We agree that the time interval of the registration changes in the infinite limits.

Processes in nature do not occur in isolation: they are accompanied by a number of factors not accounted for in the particular experiment. In other words, registered data are not reproducible and could be different, depending on unrecorded factors.

In this case, it is useful to use a set of functions, only one of which is our registered data and the remaining simulate all kinds of other outcomes of a similar experiment. That would be considered once the whole set of features is mathematically written as  $X(t, \chi)$ , where  $\chi$  is a parameter called an elementary event, fixing a specific function of the proposed set. This set is a random process that we will use as a model of the observed field.

If we fix any value of the parameter  $\chi = \chi_0$ , then the process will be only a function of time  $X = X(t, \chi_0)$  and this is the realization of a random process. A random processes is the set of all its realizations. If we fix any moment of time  $t_0$ , then it will be only a function of elementary events  $X = X(t_0, \chi)$ , which is called a random variable.

What gave such definitions? Let the components of the field entry  $X(t)$  be modeled by a random process  $X(t, \chi)$ . It means that the resulting data is the realization of a random process, and the value of the random variable is registered at each time. From these statements we know nothing about any other possible values of the field. The restrictions described below are imposed on the random process so that it could serve as a specific model of the electromagnetic field variations and to obtain reliable data in the framework of this model.

### 2.2.2 *The Distribution Function and Its Moments*

First of all, we should set the probability for a space of elementary events. To do this, it should be determined what are the acceptable values of electromagnetic field, which could be registered at time  $t_0$ ? Let us formulate the answer as follows: it is possible to have any field values, but with different probabilities. Indeed, the full-scale data show that if we exclude the regular oscillations of the field from the consideration, then we can see that significant amplitude variations occur not as often as the variation of smaller amplitude. The probability to register a variation that is great in magnitude (greater than the field of the Earth) is practically zero in the time scales used in practice. This response should be formulated now as restrictions to the random process.

As it is stated that the random variable  $X(t_0, \chi)$  may have any value, it is clear that the set of elements  $\chi$  must be large enough. Further, the probability space should be measurable, which is equivalent to the existence of a random process. In this space we define a measure, hereinafter simply referred to as probability  $P$ : to each subset of the space we assign a number that is less than one, and to the set we assign one. It should be noted that such a construction of measure approximately corresponds to the strict definition. For example, we assume  $P$  to be an area, and the area of the entire set is equal to 1. In this case, we would get an even distribution, unacceptable for us. To set the probability we can determine all finite-dimensional

distribution functions. To do this, we must assume that all implementations are continuous functions of time. Such a requirement is not contrary to the observed fields without instantaneous changes in their values.

Let us define: the finite-dimensional distribution function is a function that determines the probability of an elementary event in which the values of the random variables do not exceed the specified values. The adopted field model has an infinite number of implementations and we can safely say that it is continuous. It should also require its continuous differentiability, which will consider the moments of the distribution functions, such as expectation, variance, etc. Please note that the periodic components of the field, which is the base for soundings theory, were excluded from consideration. Otherwise, the distribution density function cannot satisfy the condition of its continuity (Bendat and Piersol 1986; Jenkins and Watts 1968).

The density distribution function can be fully characterized by all its moments, the first two of which are the expectation and variance. The expectation of a random process used below is defined as follows:

$$M[X(t, \chi_i)] = \int X(t, \chi) P(d\chi) \quad (2.1)$$

The indefinite integral of measure function  $P$  follows from the introduction of the space of elementary events. In practice, this estimation is performed using the integral of the elementary functions and, for example, in the case of a process with a finite number of realizations, takes the form:

$$M[X(t, \chi_n)] = \sum_{n=1}^N X(t, \chi_n) P(\chi_n). \quad (2.2)$$

The expectation, as well as all other aspects of the distribution function, is a non-random characteristic of a random process, and, as a result of integrating (2.1), it does not depend on parameter  $\chi$  anymore.

The following example is important for understanding the mathematical expectation of a random process. The expectation of a random process consists of one implementation, for example a sine wave, and this coincides with the realization that follows from (2.2). Hence, the expectation of a sine wave is itself a sinusoid, and an intuitive understanding of its average value of zero corresponds to the expectation of a random process consisting of a plurality of sine waves:  $X(t, \chi) = \cos[\omega t + \Psi(\chi)]$ , where  $\Psi(\chi)$  is uniformly distributed from  $-\pi$  to  $\pi$ . Obviously, in general, the expectation of the random process is time-dependent, as well as the distribution function itself. Similarly, all its moments may have this property as well. This fact greatly complicates the practical application of our model to the electromagnetic field. In the random model, it is necessary to impose additional conditions.

### 2.2.3 Stationary and Ergodic Hypothesis

Of course, considering all the finite-function distributions is almost impossible. Thus, we assume that all of the distribution functions are invariant with respect to any changes in time. Processes which satisfy this condition are called strictly stationary. In the stationary random process the expectation does not depend on time. Next, we will consider only such processes and therefore the term “stationary” can be omitted.

The electromagnetic field seems to satisfy this condition only in a first approximation. The intensity of the field sources is naturally changing over time. However, considering the variability of the field is useful for a specific purpose only. This can be, for example, a marked influence of transients on the sounding results or the study of the variability of transfer functions in time. The theory of induction sounding was built to established processes and does not cover large time scales. Thus, as long as there are sufficient grounds for the introduction of a specific type of unsteadiness in a random field model, it is better not to enter. In addition, practical conclusions can be drawn by segments’ implementations that in extreme cases can be adjusted to criteria of stationary processes within the registered segments studied. In such a way, it may be possible to study the variability of the spectra over time.

In general, a stationary random process can be characterized by a distribution function. The distribution law of a random process for modeling field variations is usually taken to be normal. The normal distribution law is simple, since it can be fully characterized by the value of the mathematical expectation and variance. The variance of a stationary process is independent of time and can be found through the autocorrelation function:

$$R_{xx}(\tau) = M[X(t, \chi)X(t + \tau, \chi)] \quad (2.3)$$

The value of autocorrelation function at  $\tau = 0$  is the variance of a random process.

For a simultaneous study of two field components, each being a realization of a random process with its own characteristics, the cross-correlation function is used:

$$R_{xx}(\tau) = M[X(t, \chi) Y(t + \tau, \chi)] \quad (2.4)$$

Equation (2.4) holds for the consideration of a two-dimensional random process.

To determine the characteristics of the non-random process, one must know all its implementations, since the integration is expected in the whole space of elementary events. Such information is not available to the researcher. In addition to the original information, a random process includes all sorts of outcomes of a similar experiment, which is impossible to observe. They can be understood as the field variations recorded in a similar experiment, e.g. registered in the same place at another time.

Then, combining a large number of records at one point in time, it would be possible to evaluate the characteristics of the non-random process. This would be natural if the characteristics determined in such a way were close to their values defined for each implementation individually. These examples of arguments illustrate to some extent the introduction of the ergodic hypothesis: a stationary random process satisfies this hypothesis, if for each function of the random process the time average is equal to the average for the set of observations.

For the expectation, for example, this condition can be written as  $P\{X(t, \chi_0)^* \equiv M[X(t, \chi_i)]\} = 1$ , where the asterisk denotes averaging over time. More rigorous definitions can be found in Wiener (1948).

In accordance with the property of ergodicity of a random process, the nonrandom characteristics defined by one of its implementations with a probability of 1 are equal to the characteristics found throughout the ensemble of realizations. An example of a stationary and ergodic process can be the function  $X(t, \chi) = \cos[\omega t + \Psi(\chi)]$ .

In general, the ergodicity of a random process allows to establish the characteristics of the field using only one of the available implementations. It should be remembered that if the original random process is a sum of two random stationary ergodic processes with normal distribution, then the original process will have the same properties.

### 2.2.4 The Spectrum of a Random Process

Let us analyze the spectra of a stochastic process for the electromagnetic field modeling. We assume that all of the above restrictions imposed on the model of the field are satisfied. There is given a realization of one model over an infinite period of time and it is required to determine its spectrum. Determination of the spectrum by the Fourier series for a periodic function is valid. However, the periodicity is not a required property of field variations. A non-periodic function can be represented by the Fourier integral:

$$X(t, \chi_0) = \int_{-\infty}^{\infty} S_x(\omega, \chi_0) \exp(i\omega t) d\omega \quad (2.5)$$

where  $S_x(\omega, \chi_0)$  is the spectrum of a random process.

For the existence of the integral (2.5) it is required to have the absolute integrability of each implementation. To perform this test, it would be necessary to assume that each implementation is damped at infinity. This would be contrary to the condition of stationarity of the process, since the dispersion processes have been changing over time. However, there is an example of undamped functions represented by Fourier integral:  $X(t) = \cos \omega t^2$ . But there is no reason to believe that the implementation of all the fields is like this. Then, it can be argued that the Fourier

integral is also not suitable for the construction of the spectrum of a random process.

Let us use a more abstract concept of the integral than in relation (2.5):

$$X(t, \chi_0) = \int_{-\infty}^{\infty} \exp(i\omega t) dZ_F(\omega, \chi_0) \quad (2.6)$$

where  $Z_F(\omega, \chi_0)$  is a function of bounded variation, i.e., the path length of the point on the  $Z$ -axis when changing  $\omega$  from  $-\infty$  to  $\infty$  is finite. If  $Z_F(\omega, \chi_0)$  is differentiable in  $\omega$ , the Stieltjes integral is equal to the Fourier integral. Yet the condition of bounded variation is tough enough: a stationary random process may include implementation presented in the form of (2.6). A way out of this difficult situation was specified by T. Kramer, who proposed to consider the  $Z_F(\omega, \chi_0)$  function as another random process with uncorrelated increments and the convergence of the integral (2.6) to understand the mean-square sense.

Then the Stieltjes integral can be rewritten as

$$X(t, \chi) = \int_{-\infty}^{\infty} \exp(i\omega t) dZ_F(\omega, \chi) \quad (2.7)$$

This integral is often called the representation of Kramer. The value  $Z_F(\omega, \chi)$  is called the spectral measure and can be regarded as an extension of the concept of Fourier coefficients on random processes.

The energy spectrum of a random process follows from the representation of Kramer. Indeed, the energy of the process  $X(t, \chi)$  in a narrow band of frequencies is the variance of the process in this band, which should be equal to the mathematical expectation of the square of its amplitude. The amplitude of the random process is an integrated value of the spectral measure.

Then the expectation of the squared module of the spectral measures is the energy spectrum of a random process. However, in practice, we use the derivative of the energy spectrum in the frequency domain, this is, the spectral density  $S_{xx}(\omega, \chi)$ , that can be calculated using the well-known techniques. Determination of the energy spectrum in a narrow band of frequencies  $\Delta\omega$  by Kramer may be written as:

$$\int_{\Delta\omega} S_{xx}(\omega, \chi) \cdot d\omega = M[dZ_x^*(\omega, \chi) \cdot dZ_x(\omega, \chi)] \quad (2.8)$$

Hereafter, the asterisk denotes complex conjugation.

The spectral density is a very handy feature of the nonrandom process, perhaps the most important in determining the impedances and its gradients.

It is known that the period range of induction soundings has a set of cycling frequencies, e.g., diurnal period and its harmonics. Therefore, a complete model of

the field should consist of part of the field that is modeled by stochastic processes, and the amount of the periodic component. Then the total spectral representation of the field should be written in the form (Volkomirskaya et al. 1979):

$$X = \int_{-\infty}^{\infty} \exp(i\omega t) dZ_{Fx}(\omega, \chi) + \sum_{k=0}^n \exp(i\omega t) X_k$$

Maybe, such a process describes most of the cases that are of interest for sounding purposes.

### 2.2.5 Properties of the Spectral Density

The spectral density of a random process can be defined by its energy spectrum. Using the mean value theorem, relation (2.8) can be rewritten as  $\int_{\Delta\omega} S'_{xx}(\omega) d\omega = \Delta\omega M[dZ_{Fx}^*(\omega, \chi) \cdot dZ_{Fx}(\omega, \chi)]$ , where the prime denotes a certain mean value of the spectral density in the frequency range  $\Delta\omega$ . Then (2.8) can be written as:

$$S'_{xx}(\omega) = (\Delta\omega)^{-1} M[dZ_{Fx}^*(\omega, \chi) \cdot dZ_{Fx}(\omega, \chi)]$$

This equation is used for determination of the spectral density. The dimension of the spectral density is equal to the square of the amplitude multiplied by the unit time.

If two components of the field, simulating two random processes,  $X(t, \chi)$  and  $Y(t, \chi)$ , are studied at the same time, then the mutual spectral density is determined by analogy:

$$S_{xy}(\omega) = (\Delta\omega)^{-1} M[dZ_{Fx}^*(\omega, \chi) \cdot dZ_{Fy}(\omega, \chi)]$$

The spectral density  $S_{xx}(\omega)$  is a real and positive value, while the mutual spectral density  $S_{xy}(\omega)$  is the complex value, and  $S_{xy}^*(\omega) = S_{yx}(\omega)$ .

Relations between the spectral density and autocorrelation functions are described by the Wiener-Khinchin expressions:

$$S_{xx}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xx}(\tau) \cdot \exp(i\omega\tau) d\tau$$

$$S_{xy}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xy}(\tau) \cdot \exp(i\omega\tau) d\tau$$



These expressions allow us to calculate the spectral density of the correlation function. Functions are called uncorrelated if  $R_{xy}(\tau) \equiv 0$ . Therefore, the mutual spectral density of uncorrelated functions is equal to zero. Another important feature of the spectral densities is: if  $X(t) = a \cdot U(t)$ , then  $S_{xx}(\omega) = a^2 \cdot S_{uu}(\omega)$ , where  $a$  is a real number.

## 2.3 Impedances as the Transfer Functions

### 2.3.1 Two-Component Analysis. Coherency

Let us consider the simplest, but important for the understanding, case which is used mainly in magnetovariation sounding methods proposed by Berdichevsky et al. (1969), Schmucker (1970) and Banks (1969). Let two orthogonal components of the magnetic and/or electric fields be observed and modeled by the corresponding random processes  $B(t, \chi)$  and  $E(t, \chi)$  with all the properties mentioned earlier. Then, these components may be expressed in terms of spectral measures (2.7) of magnetic and electric fields, respectively. The impedance definition problem reduces to the determination of the transfer function between the spectral measures, as follows from the theory of magnetotelluric soundings with scalar impedance.

In terms of random processes, this linear relation can be written as:

$$dE(\omega, \chi) = Z(\omega) \cdot dB(\omega, \chi) \quad (2.9)$$

Hereafter, we will omit the index  $F$ . It is natural to assume that the transfer function  $Z(\omega)$  is a non-random characteristic and the structure of the medium is determined only at the point of observation.

Let us express relation (2.9) in terms of the spectral density, the values of which can be evaluated with the experimental data. To do this, multiply both sides of (2.9) by the value of the complex conjugate of  $dE(\omega, \chi)$ , and then repeat this operation, multiplying both sides of the original equation to  $dB^*(\omega, \chi)$ . As a result, we get the two new equations:

$$\begin{aligned} dE(\omega, \chi) \cdot dE^*(\omega, \chi) &= Z(\omega) \cdot dB(\omega, \chi) \cdot dE^*(\omega, \chi) \\ dE(\omega, \chi) \cdot dB^*(\omega, \chi) &= Z(\omega) \cdot dB(\omega, \chi) \cdot dB^*(\omega, \chi) \end{aligned}$$

Take the expectation of both sides of each equation for  $\chi$

$$\begin{aligned} M[dE \cdot dE^*] &= Z(\omega) M[dB \cdot dE^*] \\ M[dE \cdot dB^*] &= Z(\omega) M[dB \cdot dB^*] \end{aligned} \quad (2.10)$$

Hereinafter the arguments of spectral measures are omitted for brevity. In accordance with the definitions of spectral densities,  $S'_{xx}(\omega) = (\Delta\omega)^{-1} M[dZ_{Fx}^*(\omega, \chi) \cdot dZ_{Fx}(\omega, \chi)]$  and  $S_{xy}(\omega) = (\Delta\omega)^{-1} M[dZ_{Fx}^*(\omega, \chi) \cdot dZ_{Fy}(\omega, \chi)]$ , after multiplying Eq. (2.10) by  $(\Delta\omega)^{-1}$  we obtain the relations between the spectral characteristics of the components of the field:

$$\begin{aligned} S_{EE}(\omega) &= Z(\omega) \cdot S_{EB}(\omega) \\ S_{BE}(\omega) &= Z(\omega) \cdot S_{BB}(\omega) \end{aligned} \quad (2.11)$$

Equations (2.11) determine a transfer function in excess process. The impedance can be found from any of those equations. In general, there is no evidence to suggest that the prescribed lower impedances must always be equal and therefore we denote them in different ways:

$$Z'(\omega) = S_{EE}(\omega)/S_{EB}(\omega) \text{ and } Z''(\omega) = S_{BE}(\omega)/S_{BB}(\omega)$$

The values of the transfer functions obtained by different methods are traditionally compared through the coherence  $\text{Co}(\omega)$ . It is the square root of the ratio of these functions  $[Z''(\omega)/Z'(\omega)]^{1/2}$ :

$$\text{Co}^2(\omega) = |S_{EB} \cdot S_{BE}|/(S_{BB} \cdot S_{EE}) = |S_{EB} \cdot S_{EB}^*|/(S_{BB} \cdot S_{EE}) = |S_{EB}|^2/(S_{BB} \cdot S_{EE})$$

Here the equality  $S_{xy}^*(\omega) = S_{yx}(\omega)$  was used and the argument from the spectral densities is omitted. The procedure for dividing the functions is taken so that a mutual spectral density was in the numerator and thus equal to zero in the case of uncorrelated processes. Coherence is the value of nonrandom process indeed, it is varying from zero to one similar to the correlation, but in the frequency domain. If the processes are related linearly,  $\text{Co}^2 = 1$ .

A striking example of a linear relationship is a connection between periodic processes: on one frequency the squared coherence is always equal to 1! This also holds in geophysics: the meaning of a coherence value of one is that the relationship between the two processes is linear, and that their phase difference is not random in a narrow frequency band of frequencies  $\Delta\omega$  (Bychkov et al. 1975). Consequently, the coherence value should be considered as a measure of linearity in the relation between the spectral measures of the processes.

The situation is similar if the linear system is considered by means of admittances  $B_x = Y_{xy} \cdot E_y$  and  $B_y = Y_{yx} \cdot E_x$ , where the analogue of random processes is the inverse relationship between measures of spectral components of the field:  $dB(\omega, \chi) = Y(\omega) \cdot dE(\omega, \chi)$ . Having done all the changes discussed above, we get:

$$\begin{aligned} S_{EB}(\omega) &= Y'(\omega) \cdot S_{EE}(\omega) \\ S_{BB}(\omega) &= Y''(\omega) \cdot S_{EE}(\omega) \end{aligned}$$

It is clear that the equations  $Z'(\omega) = 1/Y'(\omega)$  and  $Z''(\omega) = 1/Y''(\omega)$  will be satisfied always, regardless of noise. However, the coherence can now be rewritten, such as:  $\text{Co}^2 = Z''(\omega) \cdot Y'(\omega)$ .

Since the magnitudes of the transfer function are complex values, then the following expressions (Bendat and Piersol 1986) are used to identify their magnitude and phase separately (on an example of impedances):

$$|Z| = |S_{BE}|/S_{BB} = S_{EE}/|S_{BE}|$$

and

$$\text{Arg } Z = \arctg(\text{Im } S_{EB}/\text{Re } S_{EB}) \quad (2.12)$$

This ends the determination of impedance in two-component linear relationships, and their existence in nature is satisfied ( $\text{Co}^2 = 1$ ). However, in practice, the coherence value is never equal to unity. Let us consider the reasons for this phenomenon below.

### 2.3.2 Signals with Uncorrelated Noise. Shift Error

Until now it was assumed that a linear relation introduced by the theory is strictly implemented. But, firstly, such a connection may apply only roughly, and secondly, additional signal sources may be present and lead to uncorrelated signals in the recorded data.

We will continue considering the following situation: the registered field component is a sum of two fields: a field for which the linear relationship  $E_x = Z_{xy} \cdot B_y$  and  $E_y = -Z_{yx} \cdot B_x$  is valid, and a field for which it is not. Then the magnitude of observed fields,  $E^\circ(t, \chi)$  and  $B^\circ(t, \chi)$ , can be represented in the form of random processes expected theoretically and discussed above,  $E(t, \chi)$  and  $B(t, \chi)$ , and the noise  $e(t, \chi)$  and  $h(t, \chi)$ —two different random processes with the restrictions declared above.

Following the superposition of the fields, the spectral measures of registered signals,  $dE^\circ(\omega, \chi)$  and  $dB^\circ(\omega, \chi)$ , can be written as a sum of useful signal measures,  $dE(\omega, \chi)$  and  $dB(\omega, \chi)$ , and the corresponding noise measures referred to as  $de(\omega, \chi)$  and  $db(\omega, \chi)$ . The dependency  $dE(\omega, \chi) = Z(\omega) \cdot dB(\omega, \chi)$  between the spectral measures of useful signals through an impedance  $Z(\omega)$  can be written as:

$$dE^\circ - de = Z \cdot [dB^\circ - db] \quad (2.13)$$

Let us multiply this equality by the complex conjugate term with the measured spectral measures: the first one by  $dE^{\circ*} = dE^* + de^*$ , and then  $dB^{\circ*} = dB^* + db^*$ . As a result, we get two new equations:

$$\begin{aligned} dE^0 \cdot dE^{0*} - de \cdot (dE^* + de^*) &= Z \cdot [dB^0 \cdot dE^0 - db \cdot (dE^* + de^*)]; \\ dE^0 \cdot dB^{0*} - de \cdot (dB^* + db^*) &= Z \cdot [dB^0 \cdot dB^{0*} - db \cdot (dB^* + db^*)]. \end{aligned}$$

To move from these equations to the spectral densities, it is necessary to take the operation of the expectation of both sides of these equalities. We thus assume that the noise is not correlated either with each other or with the useful signal, i.e., the equalities

$$M[dE^* \cdot de] = M[dE^* \cdot db] = M[de^* \cdot db] = M[dH^* \cdot de] = M[dH^* \cdot db] = 0$$

In such a situation, the part of input signal will not be correlated with the part of output signal. So, taking the mathematical expectation of the operation of the two original equations and taking into account the equalities written above, we go to the spectral density:

$$\begin{aligned} S_{EE}^0 - S_{ee} &= Z(\omega) \cdot S_{EB}^0 \\ S_{BE}^0 &= Z(\omega) \cdot (S_{BB}^0 - S_{bb}) \end{aligned} \quad (2.14)$$

The impedance  $Z(\omega)$  cannot be unequivocally defined from these equations because the noise intensities,  $S_{ee}$  and  $S_{bb}$ , are not known. In this situation, the measured values are used to determine the spectral densities of the impedance via the following relations, as if there were no noise at all:

$$Z'(\omega) = S_{EE}^0 / S_{EB}^0 \quad \text{and} \quad Z''(\omega) = S_{BE}^0 / S_{BB}^0 \quad (2.15)$$

Of course, different impedance estimations are obtained, and this deviation is called shift error. It is important to remember that, in practice, this failure may significantly exceed the random error and does not decrease with increasing number of raw data. Then, the estimate of the coherence of the same measured spectral densities should be calculated:

$$\text{Co}^2 = |S_{BE}|^2 / (S_{EE} + S_{ee}) \cdot (S_{BB} + S_{bb}).$$

Here, the ratio  $S_{xy}^* = S_{yx}$  was used. It can be seen that  $\text{Co}^2$  is equal to one only when  $S_{ee} = S_{bb} = 0$ . If  $\text{Co}^2 < 1$ , the impedances have a different assessment.

It is useful to determine how a true value of the impedance  $Z(\omega)$  can be connected with its measure  $Z'(\omega)$  and  $Z''(\omega)$ . Taking into account that  $S_{BE}^0 = S_{EE}^0 / Z' = Z'' \cdot S_{BB}^0$ , we will rewrite Eq. (2.14) in the form:

$$\begin{aligned} Z'(\omega) &= Z(\omega) / [1 - S_{ee} / S_{EE}^0] \\ Z''(\omega) &= Z(\omega) \cdot [1 - S_{bb} / S_{BB}^0] \end{aligned}$$

Since the spectral density of noise is obviously less than the sums of the measured signals, the assessment of the impedance  $Z'(\omega)$  is overstated, and that of  $Z''(\omega)$ —understated in comparison with the true impedance value, so the following inequality is valid:

$$|Z'(\omega)| > |Z(\omega)| > |Z''(\omega)|$$

It should be emphasized that the presence of uncorrelated noise leads to differences in the modules of estimates only. Estimates of the argument are equal to the true impedance argument, as constructed by unbiased assessment of the mutual spectral density  $S_{BE}$  (Lejbo 1978). This fact can offer a method of determining the impedance using only unshifted estimates of cross-spectral densities (remote reference method). It is enough to multiply the original Eq. (2.13) by the spectral measures of signal from a remote observation point with uncorrelated noise  $de'$  and  $db'$ . This gives the zero value of  $S_{ee}$  and  $S_{bb}$ .

It is possible to reduce the shift of the impedance module by considering, for example, the geometric mean of the two discussed above  $Z^\times = [|Z'| \cdot |Z''|]^{1/2}$ . This estimate coincides with the true impedance of the module if the ratios of “signal/noise” are the same on both channels. Communication of such estimates can be obtained through the coherence also as the ratio of first two impedances from the relation above. Then  $|Z^\times| = |Z'| \cdot \text{Co}$  and  $Z^\times = |Z''|/\text{Co}$ , that is often used in practice. In general, estimate  $Z^\times$  has a smaller displacement than the original, but its value is unknown.

All the relations presented above can be applied to the admittances. Moreover, based on the type of impedance and admittance estimates, it can be argued that the estimate  $Z^\times$  is determined by other formulas too:  $Z^\times = [|Z''| \cdot |Y'|]^{1/2} = [|Z'| \cdot |Y''|]^{1/2}$ . Thus, the estimate  $Z^\times$  is the geometric mean of the corresponding impedance and admittance estimates. All these formulas are valid when the two processes are analyzed with uncorrelated noise.

The estimates of admittances and impedances, as described above, depend on the noise level for each of the two channels that are controlled by the coherence. Then the interpretation of any assessment can be carried out only if its value is high enough. The first two noise estimates depend on the intensity of only one of the channels. They can be used with a priori information about the noise power on each channel. For example, in the range of long periods the noise is usually concentrated in the electric field and a reliable estimate will be  $S_{EB}^0/S_{BB}^0$ . Phases of impedances and admittances are the most reliable data for interpretation.

More generally, the observed electromagnetic field is a sum of fields used for soundings and other fields which can also be linked together, for example, linearly, i.e.  $de = \eta(\omega) \cdot db$ . The sources of the fields are different and we will consider them to remain uncorrelated. Then the values of cross-spectral density of noise,  $S_{eb}$  and  $S_{be}$ , are not zero, and after the capture of the expectations of the spectral measures, instead the equality (2.14) we will obtain the equations in the form:

$$\begin{aligned} S_{EE}^o - S_{ee} &= Z(\omega) \cdot (S_{BE}^o - S_{be}) \\ S_{EB}^o - S_{eb} &= Z(\omega) \cdot (S_{BB}^o - S_{bb}) \end{aligned}$$

Since the intensity of the noise is unknown beforehand, we shall seek the transfer functions, as before, in the form of relationships, not taking into account the noise (2.15). According to  $S_{be} = S_{ee}/\eta(\omega)$  and  $S_{eb} = \eta(\omega)S_{bb}$ , we get:

$$\begin{aligned} Z'(\omega) &= Z(\omega) / [1 - S_{be}/S_{HE}^o] + S_{be} \cdot \eta(\omega) / S_{BE}^o \\ Z''(\omega) &= Z(\omega) [1 - S_{bb}/S_{BB}^o] + \eta(\omega) \cdot S_{bb} / S_{BB}^o \end{aligned}$$

The last equation shows that if  $\eta(\omega) = Z(\omega)$ , i.e. both sources are the same, then the result of sounding will not differ from the case of a single source without noise. If these two transfer functions are different, then not only the module but also the phase of the impedance value will be shifted.

So, a brief analysis of the impedance estimates in the presence of correlated noise with unknown transfer function indicates a lack of baseline information for any practical conclusions on the shifted values of estimates. So, it gives the opportunity to make the soundings. However, this approach makes it possible to perform the soundings in the presence of several field sources.

## 2.4 Data Processing of Dst Field Components

The theoretical principles of magnetovariation sounding dealt with electromagnetic fields induced in the ionosphere and/or magnetosphere of the Earth, i.e., with the external sources which are used for these electromagnetic soundings. However, in practice a lot of electromagnetic fields of other kind can be registered simultaneously (Junge 1986). In this situation, we have to treat the registered field components as a sum of the signals and noise. The main purpose of the data processing is to separate these sources and to estimate the transfer functions corresponding to the responses considered in the theory.

Let us analyze the simplest situation with two measured field components, as it usually done in the magnetovariation method. Assuming  $H$  and  $Z$  to be the signal spectra (Fourier transform) of the horizontal and vertical magnetic field components, and  $h$  and  $z$ , the noise spectra of the corresponding measured components, we can write the observed spectra in the following form:

$$\tilde{H} = H + h \quad \text{and} \quad \tilde{Z} = Z + z$$

Then, the searched response function  $W(\omega) = B_Z/B_H$  in equation  $Z = i\omega\mu/2 \cdot R \cdot \text{tg}\theta \cdot B_Z/B_H$  can be written in our case as:

$$(\tilde{Z} - z) = W \cdot (\tilde{H} - h) \quad (2.16)$$

Let us multiply both parts of Eq. (2.16) by complex conjugate values of the measured field components: first by  $\tilde{Z}^* = Z^* + z^*$  and then by  $\tilde{H}^* = H^* + h^*$ . In consequence, we will obtain two equations which can be written in the form:

$$\begin{aligned} \tilde{Z} \cdot \tilde{Z}^* - (Z^* + z^*) \cdot z &= W \cdot [\tilde{H} \cdot \tilde{Z}^* - (Z^* + z^*) \cdot h], \\ \tilde{Z} \cdot \tilde{H}^* - (H^* + h^*) \cdot z &= W \cdot [\tilde{H} \cdot \tilde{H}^* - (H^* + h^*) \cdot h]. \end{aligned} \quad (2.17)$$

To obtain the spectral densities it is necessary to average the values like  $\tilde{H} \cdot \tilde{H}^*$  through several time series or to smooth them through several frequencies. Both procedures are equal for linear analysis and yield the spectral density estimation  $S$ , which can be written in the form

$$S_{ij}(\omega) = (\Delta\omega)^{-1} \left\langle H_i(\omega, \kappa) \cdot H_j^*(\omega, \kappa) \right\rangle \quad (2.18)$$

Here  $\kappa$  is the realization number in a simplest case,  $\langle \dots \rangle$  is the averaging over  $\kappa$  or smoothing operation over  $\omega$ . More details were considered by Semenov (1985), with the explanation why this method cannot be used for regular oscillation like Sq. If  $H_i$  and  $H_j$  are not correlated, the spectral density will be equal to zero. Using Eq. (2.18), expression (2.17) can be rewritten for spectral densities as follows:

$$\tilde{S}_{ZZ} - S_{ZZ} = W \cdot \tilde{S}_{HZ} \quad \text{and} \quad \tilde{S}_{ZH} = W \cdot (\tilde{S}_{HH} - S_{hh}) \quad (2.19)$$

It was taken into consideration here that both noises are not correlated with signals and with each other, i.e., we had:

$$\langle Z^* \cdot z \rangle = \langle Z^* \cdot h \rangle = \langle z^* \cdot h \rangle = \langle H^* \cdot z \rangle = \langle H^* \cdot h \rangle = 0$$

It is obvious that transfer function  $W$  cannot be found from Eq. (2.19) because the noise densities  $S_{zz}$  and  $S_{hh}$  are unknown. In this situation we can only estimate the transfer function. For this purpose, let us assume that the noises are neglected. Then, from two Eq. (2.19) without noise spectra we can determine two estimates:

$$W_1 = \tilde{S}_{ZZ} / \tilde{S}_{HZ} \quad \text{and} \quad W_2 = \tilde{S}_{ZH} / \tilde{S}_{HH} \quad (2.20)$$

To verify the justification of our assumption, let us consider the ratio of these two estimates of  $W$ :

$$\frac{W_2}{W_1} = \frac{\tilde{S}_{ZH} \cdot \tilde{S}_{HZ}}{\tilde{S}_{HH} \cdot \tilde{S}_{ZZ}} = \frac{|\tilde{S}_{ZH}|^2}{\tilde{S}_{HH} \cdot \tilde{S}_{ZZ}} = \text{Co}^2 \quad (2.21)$$

Quantity  $\text{Co}^2$  is named a common coherence function which is a measure of the linear relationship in the frequency domain. Using expressions with noises it is easy to show that  $\text{Co} \leq 1$ . If  $\text{Co} = 0$ , relation (2.16) cannot describe the measured field, i.e. only uncorrelated noises were recorded. If  $\text{Co} = 1$ , both estimations,  $W_1$  and  $W_2$ , are equal and noises are absent.

In fact,  $1 \geq \text{Co} \geq 0$ , and it is interesting to establish how the searched response function is related to its estimation,  $W_1$  and  $W_2$ . From Eqs. (2.19) and (2.20) it is easy to show that  $W_2 = W_1 \cdot (1 - \tilde{S}_{hh}/\tilde{S}_{HH})$  and  $W_1 = W/(1 - \tilde{S}_{zz}/\tilde{S}_{ZZ})$ . Assuming the noises less than the signals:  $|\tilde{S}_{zz}/\tilde{S}_{ZZ}| < 1$  and  $|\tilde{S}_{hh}/\tilde{S}_{HH}| < 1$ , we have:

$$|W_1(\omega)| \geq |W(\omega)| \geq |W_2(\omega)| \quad (2.22)$$

i.e., the searched value lies between the two estimates. However, this fact is valid only for module values. The phases of both estimations are equal:

$$\text{Arg } W_1 = \text{arc tg } (\text{Im} \tilde{S}_{ZH} / \text{Re} \tilde{S}_{ZH}) = \text{Arg } W_2 \quad (2.23)$$

This is the reason why we can say that the phase estimations are more reliable than the module ones for the magnetovariation sounding. These phase estimations are free from the shift errors, a characteristic of which is the coherence.

## 2.5 Principal and Selected Directions in Magnetotelluric

Many methods were suggested to choose the principal directions in the magnetotelluric investigation (Yee and Paulson 1987). As a rule, these methods were based on the selection of directions in which the additional impedance has a minimum. However, the spectral analysis cannot estimate zero. From the point of view of this kind of data processing, two possibilities exist:

$$E_y = Z_m \cdot B_x + Z_a \cdot B_y, \quad \text{or} \quad E_y = Z_m \cdot B_x + e. \quad (2.24)$$

Here  $E_y$  is the electric field component in the direction of the  $B_y$  magnetic component,  $B_x$  is the magnetic field component perpendicular to both  $E_y$  and  $B_y$  components,  $Z_m$  and  $Z_a$  are main and additional impedances, respectively.

The difference between these equivalencies is obvious: the term  $Z_a \cdot B_y$  is replaced by the noises  $e$  in the second equivalence. Then, instead of attempts to find  $Z_a = 0$ , we can establish which one of the relationships of Eq. (2.24) we are dealing with. It was shown (Semenov and Kaikkonen 1986) that the second equivalence can be considered instead of the first one provided that there is the equality to zero of common coherency,  $\text{Co}_{BB} = 0$ , and partial coherency,  $\text{Co}_{EB/B} = 0$ , simultaneously. This means that under these conditions the measured component  $E_y$  correlates only



with the component  $B_x$  which is not related with component  $B_y$ . This corresponds to the zero value of the additional impedance in the deterministic theory. If those coherence functions are zero for one or two nonorthogonal directions, we have found the selected directions. If both selected directions are orthogonal, we have found the principal directions.

As it was mentioned above, almost all kinds of coherence functions are not invariant during the reference system rotation (Bath 1974). This fact allows us to use the diagrams of these coherencies.

## 2.6 Confidence Limits

Many methods exist to estimate the spectral densities (2.18), which differ as a rule in the calculations algorithms. However, the general characteristic exists for all these methods: it is the degree of freedom  $\nu$ . For example, for smoothing through several frequencies,  $\nu$  is twice the number of them, for the averaging of the Fourier transforms for several time series,  $\nu$  is twice the number of the time series used. The same value can be calculated from other methods. Some special case is the analysis in the time domain only (Wieladek and Ernst 1977; Svetov and Shimilevich 1982).

Let any estimation of the response  $Z$  be found in the narrow frequency range known from the analysis. Then the confidence limits can be estimated according to the formulae (Bendat and Piersol 1986):

$$\begin{aligned} \delta|\tilde{Z}| &= \frac{2q}{\nu - 2q} \cdot \mathbf{F} \cdot \frac{[1 - Co_{EBB}^2] \cdot \tilde{S}_{EE}}{[1 - Co_{BB}^2] \cdot \tilde{S}_{BB}} \\ \delta \text{Arg } \tilde{Z} &= \arcsin(\delta|\tilde{Z}|/|\tilde{Z}|), \end{aligned} \quad (2.25)$$

where  $q$  is the number of input signals (two in our case),  $\mathbf{F}$  is Fisher's parameter,  $Co_{BB}$  is the common coherency between inputs and  $Co_{EBB}$  is the multiple coherency.

Theoretically, expression (2.25) is valid for the case of normal distributions of the field intensities. Formula (2.25) estimates the random errors, while the coherencies estimate the possible shift error.

## 2.7 Mean and Robust Estimations

The task to average the obtained results for impedances, admittances and apparent resistivities appears very often in many situations. The apparent resistivity is connected with the impedance by formula presented in Chap. 1.

Let us recall that all transfer functions are complex ones. Then the mean complex values can be written for two apparent resistivities  $\rho$  (for one direction, for example) in the form:

$$\bar{\rho} = \sqrt{(\rho_1 \cdot \rho_2)} = \sqrt{(|\rho_1| \cdot |\rho_2|)} \cdot e^{-i(\varphi_1 + \varphi_2)/2}. \quad (2.26)$$

As long as we are going to average the modules and phases separately, the mean geometrical values of the modules will be in accordance with the arithmetical mean values of phases.

As a response function, the real and imaginary parts of  $C$  are widely used too. They are connected with the apparent resistivity by the formulae:

$$\begin{aligned} \rho &= \left[ (ImC)^2 + (ReC)^2 \right] \cdot 0.8 \cdot \pi^2 \cdot f, \\ Arg \rho &= \pm 2 \cdot \arctg \left( \frac{ImC}{ReC} \right) + 90^\circ \end{aligned} \quad (2.27)$$

Here  $\rho$  is expressed in Ohm-m,  $C$  in km,  $f$  is frequency in cycles per second,  $Arg \rho$  is in degrees and the sign in the second expression depends on the assumed oscillation term  $\exp(\pm i\omega t)$ . For example, this sign is minus for data of Schultz and Larsen (1987) and it is plus for data of Roberts (1984). The mean values of the  $C$  responses can be considered as the arithmetic mean values:  $\overline{ReC} = (ReC_1 + ReC_2)/2$ , for example.

The other way to determine the averaged value of response functions is robust estimation. This technique uses the absolute values as the error criteria, instead of the least-squares ones (Claerbout and Muir 1973). The idea is simple: a set of  $N$  real samples  $|\rho_i|$  or  $\varphi_i$  may be sorted in the order:

$$\rho_1 \leq \rho_2 \leq \dots \leq \rho_m \leq \dots \rho_{N-1} \leq \rho_N \quad (2.28)$$

The median (middle value  $\rho_m$ ) can be different from the arithmetical or geometric means due to several blunders. In this case, to construct the error bars, the following differences are considered:

$$\Delta\rho_i = |\rho_i - \rho_m|,$$

which are also sorted in the order  $\Delta\rho_1 \leq \Delta\rho_2 \leq \dots \Delta\rho_m \leq \dots \Delta\rho_{N-1} \leq \Delta\rho_N$ .

The middle values of this set is named the median absolute deviation (MAD). Then the estimation of  $\rho$  can be written as follows (with the theoretically known probability)

$$\rho_m + MAD \geq \rho \geq \rho_m - MAD$$

The question arises how to improve this set of  $\rho_i$  in order to reduce the MAD value. It is obvious that the highest errors are concentrated at edges of the row

(2.28). Then, if several samples are removed from both edges, we can reach the aim. We have a possibility to make this process iteratively. As a criterion of finishing these iterations, the approximate coincidence of the mean and median estimations for a new row can be considered. More details can be found in Egbert and Booker (1986) and Chave et al. (1987).

The apparent resistivities or impedances averaged by different methods may not coincide exactly with each other. This can be the reason why the same initial responses can produce somewhat different geoelectrical structures even if the conditions of inversion are the same.

## 2.8 Conclusions

In this chapter we have considered the principles how to estimate solutions of one equation with several unknown values by means of statistical methods.

## References

- Banks, R.J.: Geomagnetic variations and the electrical conductivity of the upper mantle. *Geophys. J. Roy. Astron. Soc.* **17**, 457–487 (1969)
- Bath, M.: *Spectral Analysis in Geophysics*. Elsevier, Amsterdam (1974)
- Bendat, J., Piersol, A.: *Random Data Analysis and Measurement Procedures*, 2nd edn. Wiley, New York (1986)
- Berdichevsky, M.N., Vanyan, L.L., Feinberg, E.B.: The frequency sounding of the Earth using spherical analysis results of geomagnetic variations. *Geomag. Aeron.* **9**, 372–374 (1969) (in Russian)
- Bychkov, V.S., Lejbo, A.B., Semenov, V.: Interpretation of coherency coefficient and phase difference function. *Geomagnetic Studies*, 16. Moscow, Nauka (1975) (in Russian)
- Chave, A.D., Thomson, D.J., Ander, M.E.: The robust estimation of power spectra, coherencies, and transfer functions. *J. Geophys. Res.* **92**(B1), 633–648 (1987)
- Claerbout, J.F., Muir, F.: Robust modelling with erratic data. *Geophysics* **38**(5), 826–844 (1973)
- Egbert, G.D., Booker, J.R.: Robust estimation of geomagnetic transfer functions. *Geophys. J. Roy. Astron. Soc.* **87**, 173–194 (1986)
- Jenkins, G.M., Watts, D.G.: *Spectral Analysis and Its Application*. Holden-Day, San-Francisco (1968)
- Junge, A.: Characterization of and correction for cultural noise. *Surv. Geophys.* **17**, 361–391 (1986)
- Lejbo, A.B.: Probabilistic model for estimating the impedance in a horizontally-homogeneous media. *Geomag. Aeron.* **20**(4), 706–710 (1978) (in Russian)
- Reddy, I.K., Rankin, D.: Coherence functions for magnetotelluric analysis. *Geophysics* **39**(3), 312–320 (1974)
- Roberts, R.G.: The long period electromagnetic response of the Earth. *Geophys. J. Roy. Astron. Soc.* **78**(2), 547–572 (1984)
- Semenov, V.: Data processing of magnetotelluric soundings/Obrabotka dannyh magnitotelluricheskogo zondirovaniya. Moscow, Nedra, 133 p. (1985) (in Russian)

- Semenov, V., Kaikkonen, P.: On magnetotelluric data analysis with example for the Kuhmo region in Eastern Finland. Report 13 of the Oulu University, Finland, 1–13 (1986)
- Svetov, B.S., Shimilevich, 1982. Determination of the linear relation between the components of magnetotelluric field. *Izv. AN USSR, Fizika Zemli*, 5, 59–67 (1982) (in Russian)
- Schmucker, U.: Anomalies of geomagnetic variations in the southwestern United States. *Bull. Scripps Inst. Ocean. Univ. Calif.* 1–165
- Schultz, A., Larsen, J.C.: On the electrical conductivity of the mid-mantle: I. Calculation of equivalent scalar magnetotelluric response function. *Geophys. J. Roy. Astron. Soc.* **88**, 733–761 (1987)
- Volkomirskaya, L.B., Lejbo, A.B., Semenov, V.: On the statistical treatment of the electromagnetic sounding data. *Geomag. Aeron.* **19**(5), 530–537 (1979) (in Russian)
- Wiener, N.: *Cybernetics: Or Control and Communication in the Animal and the Machine*. The M.I. T Press, Cambridge, Massachusetts (1948)
- Wieladek, R., Ernst, T.: Application of the method of least squares to determining impulse responses and transfer functions. *Publ. Inst. Geophys. Pol. Acad. Sci. G-1*, **110**, 3–12 (1977)
- Yee, E., Paulson, K.V.: The canonical decomposition and its relationship in other forms of magnetotelluric impedance tensor analysis. *J. Geophys.* **63**, 173–189 (1987)

Induction Soundings of the Earth's Mantle

Semenov, V.; Petrishchev, M.

2018, XVIII, 100 p. 39 illus., 12 illus. in color., Hardcover

ISBN: 978-3-319-53794-8