

Chapter 2

Basics of Fuzzy Control

2.1 Introduction

Since the introduction of fuzzy set theory by Zadeh [1], the fuzzy logic theory has been developed in a variety of directions. Now, the fuzzy logic theory has found widespread applications in the control engineering, signal and information processing, pattern recognition, expert system, decision-making, and so on. As one of the most successful applications of fuzzy sets and fuzzy logic, the research on fuzzy control problems has undergone great progress in the past decades [2]. Since the fuzzy logic-based control was first used in the controller design for the steam engine [3–5], it has been extensively applied to system modeling, intelligence control, system identification, pattern recognition and classification, neural network, etc. Compared with conventional crisp control, fuzzy logic-based control can model humans experience more accurately in a linguistic manner, thereby providing an efficient way to realize the intelligent control in industrial application.

In general, the fuzzy control system can be classified into two types: the model-free and the model-based fuzzy control system. The model-free fuzzy control scheme has witnessed great developments in recent years because, under certain circumstances, it outperforms other conventional model-free approaches such as the nonlinear adaptive or PID control. A large number of important results have appeared on this issue, see [25, 26]. The model-based fuzzy control systems include three types: type-1, type-2, and type-3 [27]. Presentation of these three different types of fuzzy control systems can be found in [6], from which one can know that type-2 fuzzy control system is actually a special case of type-1. There are many applications of type-1 fuzzy control systems, see [21–24].

In recent years, it has been recognized that fuzzy system models are qualified to represent a certain class of nonlinear dynamic systems following the T-S fuzzy model [8]. Since then there have been several approaches for the study of stability analysis and robust controller synthesis using the so-called parallel distributed compensation (PDC) method for uncertain nonlinear systems [9].

2.2 Classical Sets and Fuzzy Sets

In the classical set, its characteristic function assigns a value of either 1 or 0 to each individual in the universal set, thereby discriminating between members and nonmembers of the crisp set under consideration. The values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set. Larger values denote higher degrees of set membership such a function is called a membership function and the set is defined by it is a fuzzy set.

The concept of the fuzzy set is only an extension of the concept of a classical or crisp set. The fuzzy set is actually a fundamentally broader set compared with the classical or crisp set. *A fuzzy set is thus a set containing elements that have varying degrees of membership in the set.* This idea is therefore in contrast with classical or crisp, set because members of a crisp set would not be members unless their membership was full or complete.

2.2.1 Illustrative Example 2.1

To clarify ideas, if a temperature is defined as a crisp high, its range must be between 80 °F and higher and it has nothing to do with 70 °F or even 60 °F. But the *fuzzy* set will take care of a much broader range for this high temperature. In other words, the fuzzy set will consider a much larger temperature range such as from 0 °F to higher degrees as a high temperature. The exact degree to which the 0 °F can contribute to that high temperature depends on the membership function. This means that the *fuzzy* set uses a universe of discourse as its base and it considers an infinite number of degrees of membership in a set. In this way, the ‘classical’ or crisp set can be considered as a subset of the *fuzzy* set.

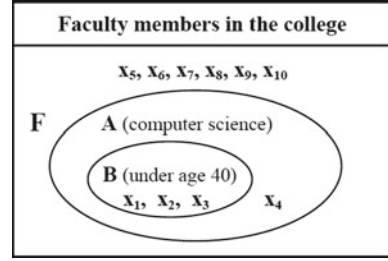
2.2.2 Set Operations and Properties

Simply stated, *a classical set is a collection of objects in a given range with a sharp boundary.* An object can either belong to the set or not belong to the set. For example, we assume to create a faculty set or a faculty collection **F** with 10-faculty members $x_1, x_2, \dots, x_{10}$ in a college:

$$\mathbf{F} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}. \quad (2.1)$$

The entire object of discussion **F** is called a *universe of discourse*, and each member x_i is called an element. Assuming that elements $x_1, \dots, x_4$ belong to the department of computer science, which can be considered as another **set A**. The

Fig. 2.1 The ‘classical’ set of faculty members in a college



elements $x_1, \dots, x_3$ are under age 40, which can be considered as a **set B**. Therefore the following relations exist:

$$\begin{aligned} X &\in F \\ \text{set } \mathbf{A} &= \{x_1, x_2, x_3, x_4\} \in F \\ \text{set } \mathbf{B} &= \{x_1, x_2, x_3\} \subset \text{set } \mathbf{A}. \end{aligned} \quad (2.2)$$

It can be seen that all elements in **set B** belong to **set A**, or the **set A** contains **set B**. In this case, **set B** can be considered as a subset of **set A** and can be expressed as **set B** $\subset$ **set A**.

The relationship among different sets we discussed above can be described in Fig. 2.1. The basic ‘classical’ set operations include complement, intersection, and union, which are represented as

$$\begin{aligned} \text{Complement of } \mathbf{A} (\mathbf{A}^c) & \quad \mathbf{A}^c(x) = 1 - \mathbf{A}(x) \\ \text{Intersect of } \mathbf{A} \text{ and } \mathbf{B} (\mathbf{A} \cap \mathbf{B}) & \quad \mathbf{A} \cap \mathbf{B} = \mathbf{A}(x) \cap \mathbf{B}(x) \\ \text{Union of } \mathbf{A} \text{ and } \mathbf{B} (\mathbf{A} \cup \mathbf{B}) & \quad \mathbf{A} \cup \mathbf{B} = \mathbf{A}(x) \cup \mathbf{B}(x). \end{aligned} \quad (2.3)$$

The representations of those classical set operations are shown in Fig. 2.2.

It is quite clear that an element either belongs to a set or does not belong to that set in the ‘classical’ set and its operation. There is a sharp boundary between different elements for different sets and these cannot be mixed with each other. However, for the fuzzy set, it has different laws. Further details can be found in the texts [28–30].

2.2.3 Fuzzy Sets and Operations

As we learned from the previous section, the ‘classical’ set has a sharp boundary, which means that a member either belongs to that set or does not. Also, this ‘classical’ set can be mapped to a function with two elements, 0 or 1. For example, in the previous section, we defined the faculty member in the department of computer science as set A. A faculty either fully belongs to this set ($\mu_A(x) = 1$) if one is a faculty of the computer science department or has nothing to do with **set A** ($\mu_A(x) = 0$) if

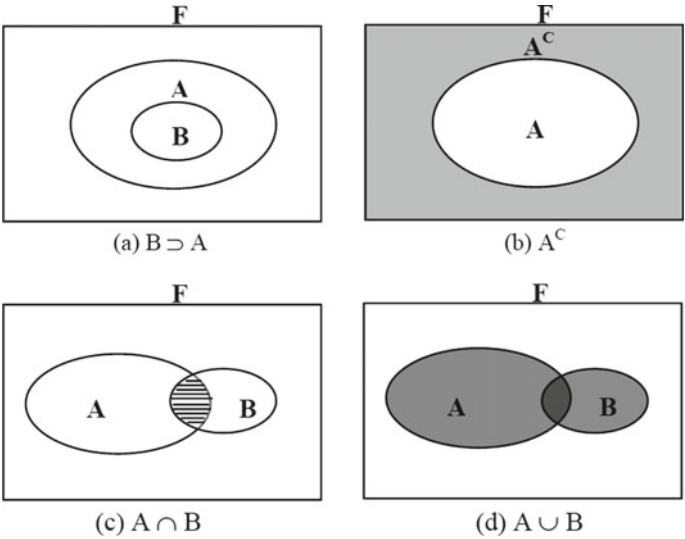


Fig. 2.2 The ‘classical’ set and operations

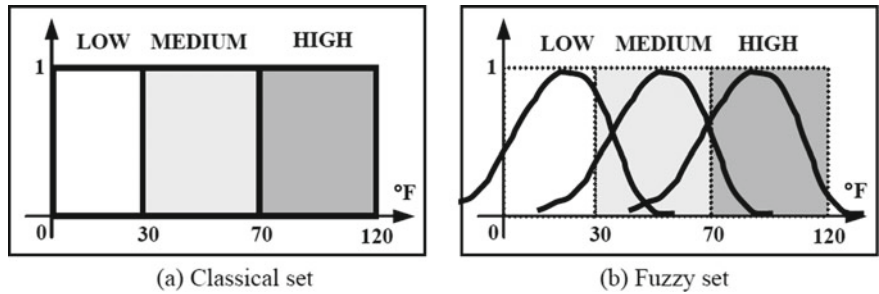


Fig. 2.3 Representations of classical and fuzzy sets

he is not a faculty in that department. This mapping is straightforward with sharp boundary without any ambiguity. In other words, this ‘fully belonging to’ can be mapped as a member of **set A** with degree of 1, and ‘not belong to’ can be mapped as a member of **set A** with degree of 0. This mapping is similar to a black-and-white binary categorization.

Compared with a ‘classical’ set, a fuzzy set allows members to have a smooth boundary. In other words, a fuzzy set allows a member to belong to a set to some partial degree. For instance, still using the temperature as an example, the temperature can be divided into three categories: LOW (70 °F through 120°F), MEDIUM (0 °F through 30 °F), and HIGH (30 °F through 70 °F) from the point of view of the classical set, which is shown in Fig. 2.3a.

Recall in the ‘classical’ set that any temperature can only be categorized into one subset, either LOW, MEDIUM, or HIGH, and the boundary is quite clear. However in the fuzzy set, such as shown in Fig. 2.3b, these boundaries become vague or smooth. One temperature can be categorized into two or maybe even three subsets simultaneously. For example, the temperature 40 °F can be considered to belong to LOW to a certain degree, say 0.5°, but at the same time it can belong to MEDIUM to about 0.7°. Another interesting thing is the temperature 50 °F, which can be considered to belong to LOW and HIGH to around 0.2° and belong to MEDIUM to almost 1°. The dash-line in Fig. 2.3b represents the ‘classical’ set boundary.

It is clear that a *fuzzy set* contains elements which have varying degrees of membership in the set, and this is contrasted with the ‘classical’ or crisp sets because members of a classical set cannot be members unless their membership is full or complete in that set. A *fuzzy set* allows a member to have a partial degree of membership and this partial degree membership can be mapped into a function or a universe of membership values. Assume that we have a *fuzzy set* A, and if an element x is a member of this *fuzzy set* A, this mapping can be denoted as

$$\mu_A(x) \in [0, 1] \quad (A = (x, \mu_A(x)) | x \in \mathbf{X}). \quad (2.4)$$

A *fuzzy set* A with an element x has a membership function of $\mu_A(x)$. When the universe of discourse $\mathbf{X}$ is discrete and finite, this mapping can be expressed as

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_1)}{x_1} + \dots = \sum_j \frac{\mu_A(x_j)}{x_j}. \quad (2.5)$$

It is simple to see that when the universe $\mathbf{X}$ is continuous and infinite, the *fuzzy set* A can be represented as

$$A = \int \frac{\mu_A(x)}{x} \quad (2.6)$$

By similarity to the ‘classical’ set operations, the basic *fuzzy set* operations also include intersection, union, and complement, and those operations are defined as

$$\begin{aligned} \text{Union :} \quad & \mathbf{A} \cup \mathbf{B} = \mu_A(x) \cup \mu_B(x) = \mathbf{max}(\mu_A(x), \mu_B(x)) \\ \text{Intersect :} \quad & \mathbf{A} \cap \mathbf{B} = \mu_A(x) \cap \mu_B(x) = \mathbf{min}(\mu_A(x), \mu_B(x)) \\ \text{Complement :} \quad & \mathbf{A}^c = \mathbf{F} \ominus \mathbf{A} \end{aligned} \quad (2.7)$$

where A and B are two fuzzy sets and x is an element in the universe of discourse, $\mathbf{X}$. The graphical illustrations of those operations are shown in Fig. 2.4. According to the definition of fuzzy set operations, a fuzzy set union operation is exactly equivalent to select the maximum member from those members in the sets, and the intersection operation is selecting the minimum member from the sets.

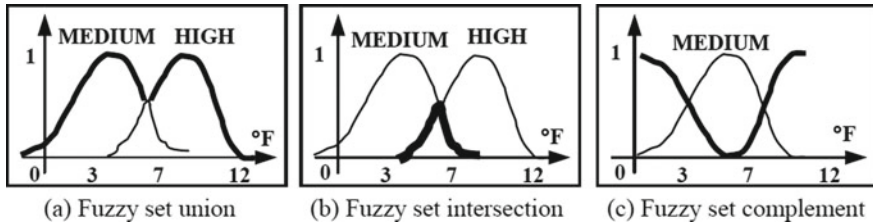


Fig. 2.4 Fuzzy set operations

2.2.4 Classification of Fuzzy Sets

The *fuzzy sets* can be classified based on the membership functions. They are:

- A. *Normal fuzzy set*. If the membership function has at least one element in the universe whose value is equal to 1, then that set is called as normal fuzzy set.
- B. *Subnormal fuzzy set*. If the membership function has the membership values less than 1, then that set is called as subnormal fuzzy set.
- C. *Convex fuzzy set*. If the membership function has membership values those are monotonically increasing, or, monotonically decreasing, or they are monotonically increasing and decreasing with the increasing values for elements in the universe, those *fuzzy set A* is called convex fuzzy set.
- D. *Nonconvex fuzzy set*. If the membership function has membership values which are not strictly monotonically increasing or monotonically decreasing or both monotonically increasing and decreasing with increasing values for elements in the universe, then this is called as nonconvex fuzzy set.

Note that when intersection is performed on two convex fuzzy sets, the intersected portion is also a convex fuzzy set.

2.2.5 Membership Functions

A fuzzy set is characterized by an MF. Among the different constituents of a fuzzy system, the MFs are the central as they need to be representative of the input output space of the system. Therefore, the MFs directly affect the modeling accuracy and the system performance. In fuzzy system, the input–output behavior of the system is represented by the rule-base. Consequently, the interpretability of the fuzzy rule-base primarily relies upon the MFs. Therefore, determining appropriate MFs is a decisive task in the realization of a well-behaved fuzzy system. By appropriate MFs, it is meant particularly the shape of MFs within the universe of discourse. Eventually, the shape of the MFs plays a critical role in fuzzy system and fuzzy control. Therefore, defining appropriate MFs are important toward the development of any application involving fuzzy logic. MFs can have any form of regular or irregular shapes as long

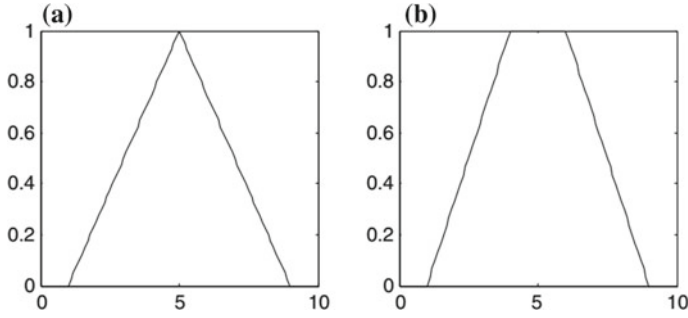


Fig. 2.5 Linear MFs **a** Triangular MF = [1 5 9]; **b** Trapezoidal MF = [1 4 6 9]

as they are convenient to be described mathematically. A convenient and concise way to construct an MF is to parametrize it and then express the MF mathematically in terms of parameters. MFs that are highly irregular shaped cannot be parameterized so easily, even if it can, computation would be excessive. A common practice is that the designer adopts regular shaped or known parameterized MFs such as triangular, trapezoidal, sigmoidal, Gaussian and bell-shaped MFs, see Fig. 2.5. Some common and widely used MFs are discussed in the following:

1. **Piecewise Linear MF:** Piecewise linear functions are the simplest form of MFs. Triangular and trapezoidal MFs are the most widely used among them. These MFs can be either symmetric or asymmetric shaped.

- *Triangular MF*

Triangular MFs are very common in fuzzy system computation because of their simplicity and ease of computation. A triangular MF is specified by three parameters $\{a, b, c\}$ and defined by:

$$\mu(x) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right). \quad (2.8)$$

The parameters $\{a, b, c\}$ with $\{a < b < c\}$ determine the x coordinates of the three corners of the underlying triangular MF. The parameters a and c locate the feet of the triangle and the parameter b locates the peak. A symmetric triangular MF is shown in Fig. 2.5a.

- *Trapezoidal MF*

A trapezoidal MF has the shape of a truncated triangle. A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$ and defined by

$$\mu(x) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right). \quad (2.9)$$

The parameters $\{a, b, c\}$ with $\{a < b < c\}$ determine the x coordinates of the four corners of the underlying trapezoidal MF. The parameters a and d locate the feet and b and c locate the shoulder of the trapezoid. A trapezoidal MF can have either narrow shoulder or wide shoulder. A symmetric trapezoidal MF is shown in Fig. 2.5b.

- Due to simple formulae and computational efficiency, linear MFs, that is, both triangular and trapezoidal MFs, are extensively used, especially in real-time applications. Although trapezoidal type MF has often been used in fuzzy control literature, triangular MFs are most commonly used almost intuitively for all the variables.

2. Nonlinear Smooth MF:

Piecewise linear MFs may not always be suitable for all applications. Therefore, nonlinear smooth functions are also in widespread use in fuzzy systems and control. Gaussian, bell-shaped, z-shaped, s-shaped, and sigmoidal functions are very common among them.

- *Gaussian MF*

Gaussian MFs are also popular among fuzzy systems researchers due to the fact that it can be specified by only two parameters m , σ and defined by:

$$\mu(x) = \exp\left[\frac{-1}{2}\left(\frac{x-m}{\sigma}\right)^2\right] \quad (2.10)$$

The parameters m and σ represent the center and width of the Gaussian MF, respectively. Gaussian MF is smooth, symmetric, and nonzero at all points as illustrated in Fig. 2.6a.

- *Two-Sided Gaussian MF*

In two-sided Gaussian MF, there are two Gaussian functions. The first function $f_1(x)$, described by the parameters (m_1, σ_1) , determines the shape of the left-side curve. The second function $f_2(x)$, described by the parameters (m_2, σ_2) , determines the shape of the right-side curve.

$$f_1(x) = \begin{cases} \exp\left[\frac{-1}{2}\left(\frac{x-m_1}{\sigma_1}\right)^2\right], & x \leq m_1 \\ 1, & \text{Otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} 1, & x \geq m_2 \\ \exp\left[\frac{-1}{2}\left(\frac{x-m_2}{\sigma_2}\right)^2\right], & \text{Otherwise} \end{cases} \quad (2.11)$$

Then the two-sided Gaussian membership function is defined by

$$\mu(x) = f_1(x) \times f_2(x) \quad (2.12)$$

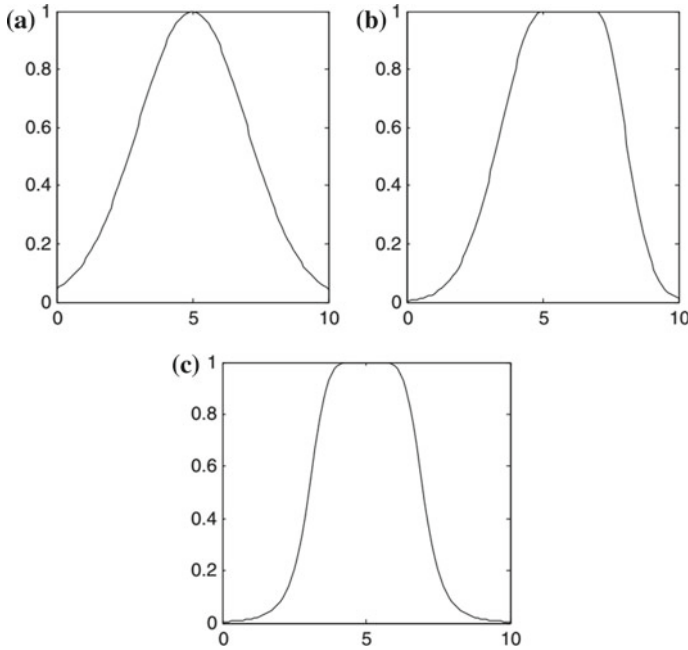


Fig. 2.6 Nonlinear MFs: **a** Gaussian MF = [2 5]; **b** Two-sided Gaussian MF = [1.5 5 1 7]; **c** Bell-shaped MF = [2 3 5]

The argument x must be a real number or a nonempty vector of strictly increasing real numbers, and m_1, σ_1 and m_2, σ_2 must be real numbers. Two-sided Gaussian membership function always returns a continuously differentiable curve with values within the range of $[0, 1]$. Usually when $m_1 < m_2$, the two-sided Gaussian function is a normal membership function and has a maximum value of 1, with the rising curve identical to that of $f_1(x)$ and a falling curve identical to that of $f_2(x)$ defined above. If $m_1 \geq m_2$, the two-sided Gaussian membership function is a subnormal membership function and has a maximum value less than 1. The two-sided Gaussian MF is asymmetric and it is illustrated in Fig. 2.6b.

- *Bell-Shaped MF*

A bell-shaped MF is specified by three parameters $\{m, \sigma, a\}$ shown in Fig. 2.6c and defined by

$$\mu(x) = \frac{1}{1 + \left| \frac{x-m}{\sigma} \right|^{2a}}. \quad (2.13)$$

The parameters m and σ represent the center and width of the bell-shaped MF, respectively. Parameter a , usually positive, controls the slope of the MF

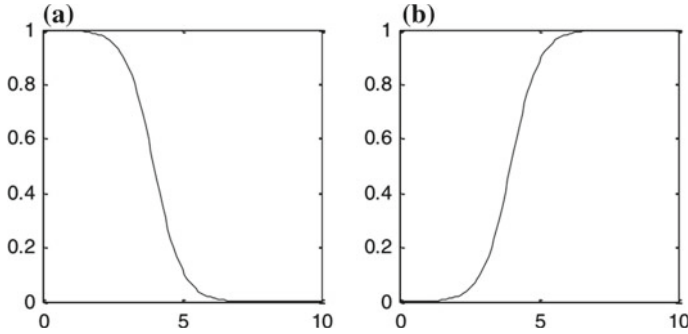


Fig. 2.7 Sigmoidal MF: **a** left open MF = $[-2 \ 4]$; **b** right open MF = $[2 \ 4]$

at crossover point. Bell-shaped MF is symmetric and has the features of being smooth and nonzero at all points.

3. **Sigmoidal MF:** A sigmoidal MF, which is either open left or right, are useful in many applications. A sigmoidal MF is specified by two parameters $\{a, c\}$ shown in Fig. 2.7a, b and defined as

$$\mu(x) = \frac{1}{1 + \exp^{-a(x-c)}} \quad (2.14)$$

The parameter c is the center of the sigmoidal function. The sign of the parameter a determines the spread of the sigmoidal membership function, that is, inherently open to the right or to the left. Thus the parameter a is appropriate for representing concepts of linguistic hedges such as *very large* or *more or less small*. If a is a positive number, the MF will open to the right. If a is a negative number, the MF will open to the left. This simple property of the sigmoidal MF helps to represent the fuzzy concepts such as *very large positive* or *very large negative* in linguistic terms.

4. **Closed Symmetrical or Asymmetrical Sigmoidal MFs** (that is, not open to the right or left) can be constructed by using either the difference or product of the two sigmoidal MFs described above. The MF formed by the difference between the two sigmoidal MFs is defined as difference-sigmoid, and the MF formed by the product of these is defined as product-sigmoid. Both the MFs in this family are smooth and nonzero at all points.

5. **Difference Sigmoidal MF:**

The difference sigmoidal function depends on four parameters, $a_1, c_1, a_2,$ and c_2 , and is the difference between two sigmoidal functions defined by

$$f_1(x, a_1, c_1) - f_2(x, a_2, c_2) = \frac{1}{1 + \exp^{-a_1(x-c_1)}} - \frac{1}{1 + \exp^{-a_2(x-c_2)}}. \quad (2.15)$$

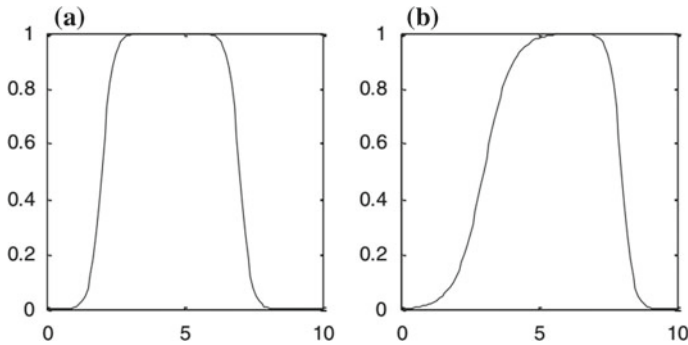


Fig. 2.8 Sigmoidal MF: **a** Difference sigmoidal MF = [5 2 5 7]; **b** Product sigmoidal MF = [2 3 -5 8]

The difference sigmoidal function is shown in Fig. 2.8a. The parameters of difference sigmoidal are listed as $\{a_1, c_1, a_2, c_2\}$.

6. Product Sigmoidal MF:

The product sigmoidal function is simply the product of two sigmoidal curves defined by

$$f_1(x, a_1, c_1) \times f_2(x, a_2, c_2) = \frac{1}{1 + \exp^{-a_1(x-c_1)}} \times \frac{1}{1 + \exp^{-a_2(x-c_2)}} \quad (2.16)$$

The product sigmoidal function is shown in Fig. 2.8b. The parameters of product sigmoidal are listed as $\{a_1, c_1, a_2, c_2\}$.

7. *The Γ -function*: an increasing membership function with straight lines.
8. *The L -function*: a decreasing membership function with straight lines.
9. *The Δ -function*: a triangular function with straight lines.
10. *The Singleton-function*: a membership function with membership function value 1 for only one value and the rest is zero.

The last four types are shown in Fig. 2.9.

Figure 2.10 provides a description of the various features of membership functions of fuzzy sets.

2.2.6 A Comparison

In the foregoing sections, three sets are defined:

- A college faculty **set F**,
- A faculty set for the computer science department, **set A**, and
- A set of faculty members who are under age 40, or **set B**.

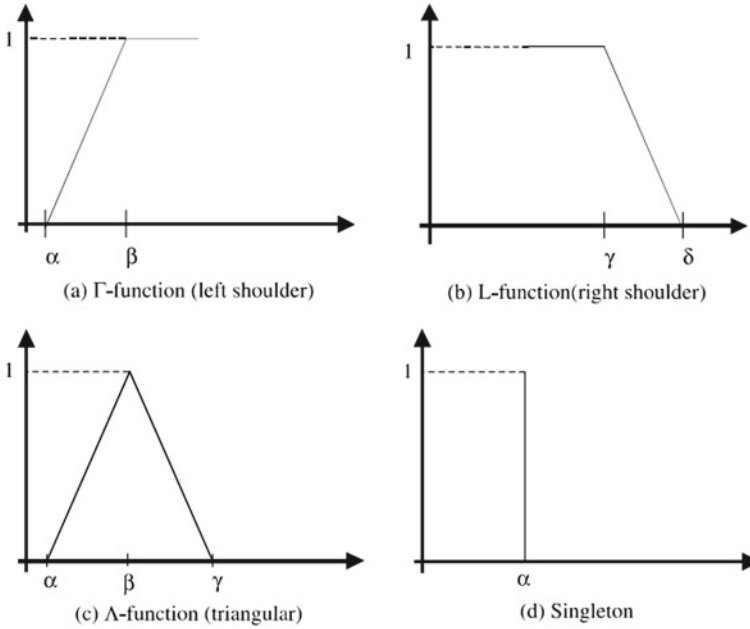
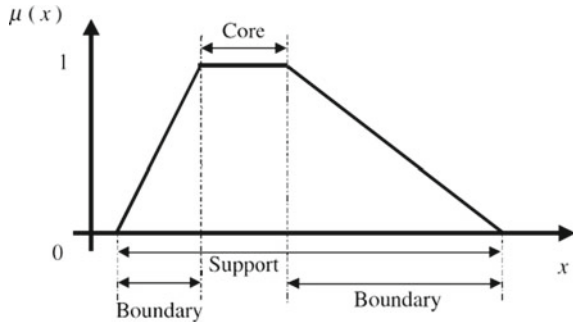


Fig. 2.9 Additional examples of membership functions

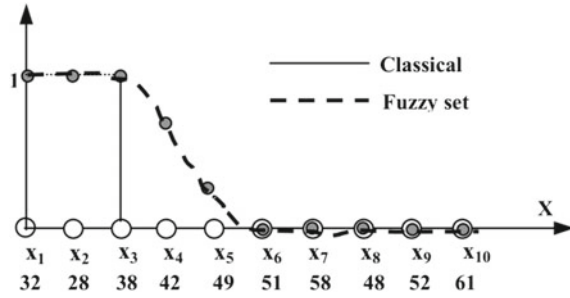
Fig. 2.10 Description of fuzzy membership functions



In what follows, we make a clear comparison for the ‘classical’ set and *fuzzy* set using this example.

According to ‘classical’ or crisp set theory, **set B** includes only three members who are under age 40 (x_1, x_2, x_3), so the boundary between the faculty who are under age 40 and the faculty who are above age 40 is obviously clear (see solid-line in Fig. 2.11). However, for the fuzzy set theory, **set B** contains not only those three members, x_1, x_2, x_3 , but also some other members to some varying degree. The actual age distribution of all 10 faculty members is displayed in the x-axis in Fig. 2.11.

Fig. 2.11 A comparison between classical and fuzzy sets



It can be found from Fig. 2.11 that the member x_4 and x_5 do not belong to the set **B** in the view of the crisp set, but they can be considered as partial members in a partial membership of 0.7 and 0.3°, respectively. Mapping this to a membership function, it can be expressed as

$$\mu_B(x) = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{0.7}{x_4} + \frac{0.3}{x_5} \quad (2.17)$$

where the division operator is called a separator, and ‘+’ is the or operator. All other members whose membership is 0 are omitted from this function.

2.3 Fuzzification

The fuzzy set is a powerful tool and allows us to represent objects or members in a vague or ambiguous way. The fuzzy set also provides a way that is similar to a human beings concepts and thought process. However, just the fuzzy set itself cannot lead to any useful and practical products until the fuzzy inference process is applied. To implement fuzzy inference to a real product or to solve an actual problem, as we discussed before, three consecutive steps are needed, which are:

- *Fuzzification*—convert classical data or crisp data into fuzzy data or Membership Functions (MFs),
- *Fuzzy Inference Process*—combine membership functions with the control rules to derive the fuzzy output and
- *Defuzzification*—use different methods to calculate each associated output and put them into a table: the lookup table. Pick up the output from the lookup table based on the current input during an application

In *Fuzzification* most variables existing in the real world are crisp or classical variables. One needs to convert those crisp variables (both input and output) to fuzzy variables, and then apply fuzzy inference to process those data to obtain the desired

output. Finally, in most cases, those fuzzy outputs need to be converted back to crisp variables to complete the desired control objectives.

2.3.1 Illustrative Example 2.2

In fuzzy control applications, the observed data are usually crisp. Since the data manipulation in an FLC is based on fuzzy sets, fuzzification is necessary. Fuzzification performs a scale transformation of physical values of the current state variables into normalized universe of discourse. It means that it maps the normalized value of the control output variable onto physical domain. Therefore, fuzzification can be looked at as a mapping from an observed input space to fuzzy sets in certain input universe of discourse.

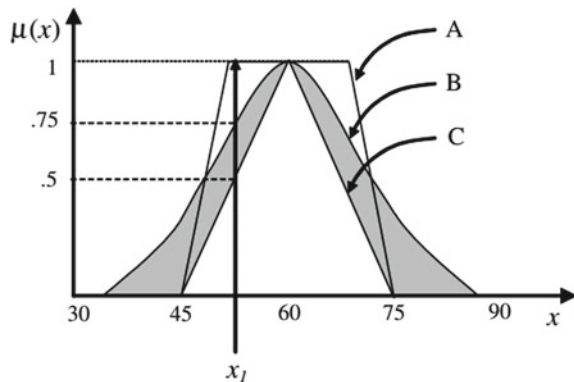
Considering this view, the fuzzification of three different types of MFs is shown in Fig. 2.12. It is to be noted that due to the shape of the MFs the membership values are different for each MF, that is, $\mu_A(x_1) = 1$, $\mu_B(x_1) = 0.75$ and $\mu_C(x_1) = 0.5$ are different. This is again evident that the choice of MF plays an important role in any fuzzy system.

Generally, *fuzzification* involves two processes:

- Derive the membership functions for input and output variables and
- Represent them with linguistic variables.

This process is equivalent to converting or mapping classical set to fuzzy set to varying degrees. As we discussed before, membership functions can have multiple different types. The exact type depends on the actual applications. For those systems that need significant dynamic variation in a short period of time, a triangular or trapezoidal waveform should be utilized. For those system that need very high control accuracy, a Gaussian or S-curve waveform should be selected.

Fig. 2.12 Fuzzification of different membership functions



2.3.2 Illustrative Example 2.3

We proceed to illustrate the process of fuzzification by recalling the air conditioner example developed in the previous section. Assume that we have an air conditioner control system that is under the control of only a heater. Naturally if the temperature is high, the heater control motor should be turned off, and if the temperature is low, that heater motor should be turned on, which are common sense. Regularly, the normal temperature range is from 20 to 90 °F. This range can be further divided into three sub-range or subset, which are

- Low temperature:* 20 to 40 °F 30 °F is center
- Medium temperature:* 30 to 80 °F 55 °F is center
- High temperature:* 60 to 90 °F 75 °F is center.

Next, those three ranges need to be converted to linguistic variables: **LOW**, **MEDIUM**, and **HIGH**, which correspond to the three temperature ranges listed above.

The membership function of these temperatures is shown in Fig. 2.13. To make thing simple, a trapezoidal waveform is utilized for this type of membership function. A crisp low temperature can be considered as a medium temperature to some degree

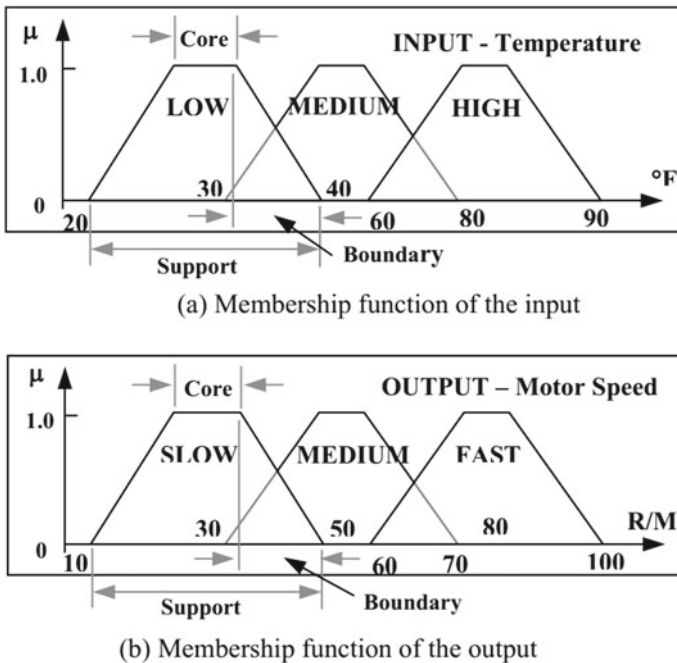


Fig. 2.13 Membership function of input and output

in this fuzzy membership function representation. For instance, 35 °F will belong to **LOW** and **MEDIUM** to 0.5°. Some terminologies used for the membership function are also shown in Fig. 2.13.

The support of a *fuzzy set*, says **LOW**, is the set of elements whose degree of membership in **LOW** is greater than 0. This support can be expressed in a function form as

$$\text{Support}(\text{LOW}) = \{x \in \mathbf{T} | \mu_{\text{LOW}}(x) > 0\}. \quad (2.18)$$

It can be shown that the support of a *fuzzy set* is a ‘classical’ set. The core of a *fuzzy set* is the set of elements whose degree of membership in that set is equal to 1, which is equivalent to a crisp set. The boundary of a *fuzzy set* indicates the range in which all elements whose degree of membership in that open set (0 1). After the membership functions are defined for both input and output, the next step is to define the fuzzy control rule.

2.4 Fuzzy Rule-Based System

Fuzzy control rule can be considered as the knowledge of an expert in any related field of application. The fuzzy rule is represented by a sequence of the form **IF-THEN**, leading to algorithms describing what action or output should be taken in terms of the currently observed information, which includes both input and feedback if a closed-loop control system is applied. The law to design or build a set of fuzzy rules is based on a human being’s knowledge or experience, which is dependent on each different actual application.

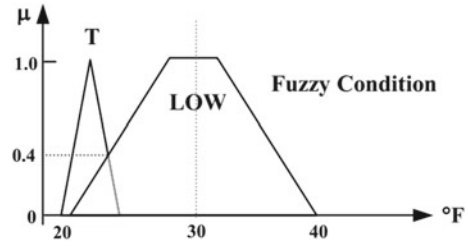
A fuzzy **IF-THEN** rule associates a condition described using linguistic variables and fuzzy sets to an output or a conclusion. The **IF** part is mainly used to capture knowledge by using the elastic conditions, and the **THEN** part can be utilized to give the conclusion or output in linguistic variable form. This **IF-THEN** rule is widely used by the fuzzy inference system to compute the degree to which the input data matches the condition of a rule. Figure 2.14 illustrates a way to calculate the degree between a fuzzy input **T**(temperature) and a fuzzy condition **LOW**, see the air conditioner system.

$$M(\mathbf{T}, \text{LOW}) = \text{Support } \min(\mu_{\mathbf{T}}(x), \mu_{\text{LOW}}(x)) \quad (2.19)$$

2.4.1 Fuzzy If-Then Rules

Knowledge is the main source of intelligence. Therefore, an efficient knowledge representation is vital in designing an intelligent system so that an easy knowledge

Fig. 2.14 Matching a fuzzy input with a fuzzy condition



maneuvering is possible and can perform by far computationally. This is essential for any intelligent system that helps system reason and eventually learns from experience and infers new knowledge. There have been many methods developed by researchers to represent knowledge [30]. One of the simplest forms of knowledge representation is the If-then rules. In other words If-then rules offer a convenient format for expressing knowledge. A fuzzy system is characterized by a set of linguistic statements based on expert knowledge. The core of such fuzzy systems is the rule-base that consists of if-then rules and conform the main knowledge-base of the fuzzy system in the sense that all other components are used to implement these rules in a reasonable and efficient manner. The expert knowledge is usually in the form of if-then rules, which are easily implemented by fuzzy conditional statements in fuzzy logic. The fuzzy conditional statement is expressed as

$$\text{If } \langle \text{fuzzy proposition} \rangle, \text{ Then } \langle \text{fuzzy proposition} \rangle \quad (2.20)$$

The collection of fuzzy conditional statements forms the rule-base or the rule set of a fuzzy system.

2.4.2 Fuzzy Proposition

A *fuzzy proposition* is a statement which can have a value within the interval of [0 1]. There are two types of fuzzy propositions: *atomic proposition*, and *compound proposition*. An *atomic proposition* is a single statement whereas a *compound proposition* is a composition of more than one atomic propositions using the connectives ‘and,’ ‘or,’ and ‘not’ which represent fuzzy intersection, fuzzy union, and fuzzy complement, respectively. For example, if x represents the speed of the car, then the following is fuzzy propositions (Fig. 2.15):

where x is a linguistic variable, and A , B , and C are linguistic values of x (that is, A , B , and C are fuzzy sets defined within the physical domain of x) and denote the fuzzy sets ‘slow,’ ‘medium,’ and ‘fast,’ respectively. Finally, the fuzzy rule-base comprises of the fuzzy propositions of the form:

$$\text{If } x_1 \text{ is } A_i \text{ and } x_2 \text{ is } B_j \text{ and } x_3 \text{ is } C_k; \text{ Then } y \text{ is } Z_\ell \quad (2.21)$$

Fig. 2.15 Examples of propositions

$$\begin{aligned}
 &x \text{ is } A \\
 &x \text{ is } B \text{ or } x \text{ is not } C \\
 &x \text{ is not } B \text{ and } x \text{ is not } C \\
 &(x \text{ is } B \text{ and } x \text{ is not } C) \text{ or } x \text{ is } C
 \end{aligned}$$

where A_i , B_j , C_k and Z_ℓ are fuzzy sets of the linguistic variables x_1, x_2, x_3 , and y in the respective universe of discourses. Let $\mathbf{M}$ be the number of rules in the fuzzy rule-base, that is, $r = 1; 2; 3; \dots; \mathbf{M}$ in (2.21). The rules in the form of (2.21) are called ‘canonical fuzzy If-Then rules.’

2.4.3 Methods for Construction of Rule-Base

In general, the fuzzy ‘if-then’ rules are derived by human experts by applying rules of physical laws and experience. Experts can map the inputs and outputs to generate rules for fuzzy system with few inputs and outputs. The rule-base is the mapping between the input and the output spaces defined as

$$\Phi : X_1 \times X_2 \rightarrow Y \quad (2.22)$$

where X_1 and X_2 are inputs and Y is the output. The mapping Φ can be visualized pictorially as shown in Fig. 2.22. The mapping is sometimes a kind of intuition, which requires a trial and error process to refine it.

As the number of inputs and outputs grows, the rule-base grows drastically. In such cases, it is nearly impossible for a human expert to map the inputs and outputs of the system and leads to difficulty in defining the rules for the system. Therefore, a systematic approach for constructing rule-base seems demanding (Fig. 2.16).

2.4.4 Illustrative Example 2.4

For most actual applications, the input variables are commonly more than one dimension. For example, in our air conditioner system, the inputs include both current temperature and the change rate of the temperature. The fuzzy control rules should also be extended to allow multiple inputs to be considered to derive the output. In Fig. 2.17, a table is given as an example of fuzzy control rules applied in our air conditioner system.

The rows and columns represent two inputs, the temperature input and the change rate of the temperature input, and those inputs are related to IF parts in IF-THEN

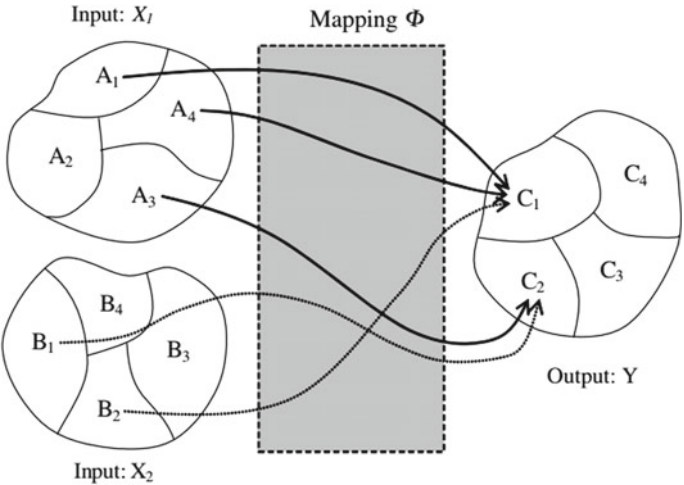


Fig. 2.16 Rule-base as a mapping

Fig. 2.17 An example of fuzzy rules

$\dot{T} \backslash T$	LOW	MEDIUM	HIGH
LOW	FAST	MEDIUM	MEDIUM
MEDIUM	FAST	SLOW	SLOW
HIGH	MEDIUM	SLOW	SLOW

rules. The conclusion or control output can be considered as a third dimensional variable that is located at the cross point of each row (temperature) and each column (change rate of the temperature), and that conclusion is associated with the THEN part in IF-THEN rules. For example, when the current temperature is LOW, and the current change rate of the temperature is also LOW, the heater motors speed should be FAST to increase the temperature as soon as possible.

This can be represented by the IF-THEN rule as
IF the temperature is LOW, and the change rate of the temperature is LOW, THEN the conclusion or output (heater motor speed) should be FAST.

2.5 Defuzzification

One should observe that the conclusion or control output derived from the combination of input, output membership functions and fuzzy rules so far is a vague or fuzzy element, and this process is called *fuzzy inference*. To make that conclusion or fuzzy output available to real applications, a defuzzification process is needed.

The *defuzzification* process is meant to convert the fuzzy output back to the crisp or classical output to the control objective. Remember, the fuzzy conclusion or output

is still a linguistic variable, and this linguistic variable needs to be converted to the crisp variable via the defuzzification process. Three defuzzification techniques are commonly used, which are: Mean of Maximum method, Center of Gravity method and the Height method.

2.5.1 Mean of Maximum (MOM) Method

The Mean of maximum (MOM) defuzzification method computes the average of those fuzzy conclusions or outputs that have the highest degrees. For example, the fuzzy conclusion is: *the heater motor x is rotated FAST*. By using the MOM method, this defuzzification can be expressed as

$$MOM(FAST) = \frac{\sum_{x' \in T} x'}{T}, \quad T = \{x' | \mu_{FAST}(x') = \text{Support}_{\mu_{FAST}(x)}\} \quad (2.23)$$

where T is the set of output x that has the highest degrees in the set $FAST$. A graphic representation of the MOM method is shown in Fig. 2.18.

A shortcoming of the MOM method is that it does not consider the entire shape of the output membership function, and it only takes care of the points that have the highest degrees in that function. For those membership functions that have different shapes but the same highest degrees, this method will produce the same result.

2.5.2 Center of Gravity (COG) Method

The Center of Gravity method (COG) is the most popular defuzzification technique and is widely utilized in actual applications. This method is similar to the formula for calculating the center of gravity in physics. The weighted average of the membership function or the center of the gravity of the area bounded by the membership function curve is computed to be the most crisp value of the fuzzy quantity. For example, for the conclusion: *the heater motor x is rotated FAST*. The COG output can be represented as

Fig. 2.18 Graphic representation of MOM method

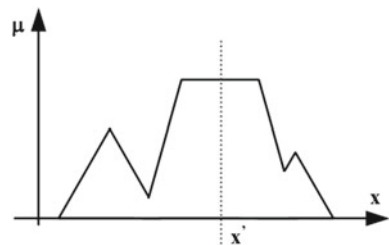
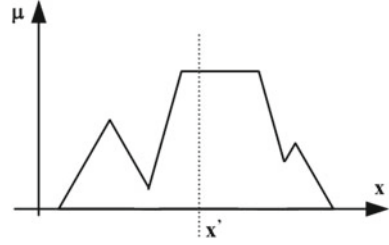


Fig. 2.19 Graphic representation of COG method



$$COG(FAST) = \frac{\sum_x \mu_{FAST}(x) \times x}{\sum_x \mu_{FAST}(x)} \quad (2.24)$$

If x is a continuous variable, this defuzzification result becomes

$$COG(FAST) = \frac{\int \mu_{FAST}(x) x dx}{\int \mu_{FAST}(x) dx} \quad (2.25)$$

A graphic representation of the COG method is shown in Fig. 2.19.

2.5.3 The Height Method (HM)

This defuzzification method is valid only for the case where the output membership function is an aggregated union result of symmetrical functions. This method can be divided into two steps. First, the consequent membership function F_j can be converted into a crisp consequent $x = f_j$ where f_j is the center of gravity of F_j . Then the COG method is applied to the rules with crisp consequents, which can be expressed as

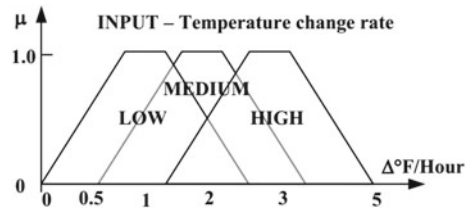
$$x = \frac{\sum_{j=1}^M w_j f_j}{\sum_{j=1}^M w_j} \quad (2.26)$$

where w_j is the degree to which the j th rule matches the input data. The advantage of this method is its simplicity. Therefore many neuro-fuzzy models use this defuzzification method to reduce the complex of calculations.

2.5.4 The Lookup Table

The terminal product of defuzzification is the lookup table. Defuzzification needs to be performed for each subset of a membership function, both inputs and outputs. For instance, in the air conditioner system, one needs to perform defuzzification for

Fig. 2.20 The membership function of the change rate of temperature



each subset of temperature input such as **LOW**, **MEDIUM**, and **HIGH** based on the associated fuzzy rules. The defuzzification result for each subset needs to be stored in the associated location in the lookup table according to the current temperature and temperature change rate.

2.5.5 Illustrative Example 2.5

For the purpose of illustration, we use the air conditioner system as an example to illustrate the defuzzification process and the creation of the lookup table. Seeking simplicity in exposition, we make two assumptions:

- I. Assume that the membership function of the change rate of the temperature can be described as in Fig. 2.20;
- II. only four rules are applied to this air conditioner system

The four rules are as follows:

- (1) IF the temperature is **LOW**, and the change rate of the temperature is **LOW**, THEN the heater motor speed should be **FAST**,
- (2) IF the temperature is **MEDIUM**, and the change rate of the temperature is **MEDIUM**, THEN the heater motor speed should be **SLOW**,
- (3) IF the temperature is **LOW**, and the change rate of the temperature is **MEDIUM**, THEN the heater motor speed should be **FAST**,
- (4) IF the temperature is **MEDIUM**, and the change rate of the temperature is **LOW**, THEN the heater motor speed should be **MEDIUM**

Based on the assumption made for the membership function and fuzzy rules, we can illustrate this defuzzification process using a graph. Four fuzzy rules can be interpreted as functional diagrams, as shown in Fig. 2.21.

As an illustration, consider the current input temperature is 35°F and the change rate of the temperature is 1°F per hour. From Fig. 2.21, it can be found that the points of intersection between the temperature values of 35°F and the graph in the first column (temperature input T) have the membership functions of 0.6, 0.8, 0.5 and 0.8. Likewise, the second column (temperature change rate ΔT) shows that a temperature change rate of 1°F per hour has the membership functions of 1.0, 0.4, 0.4

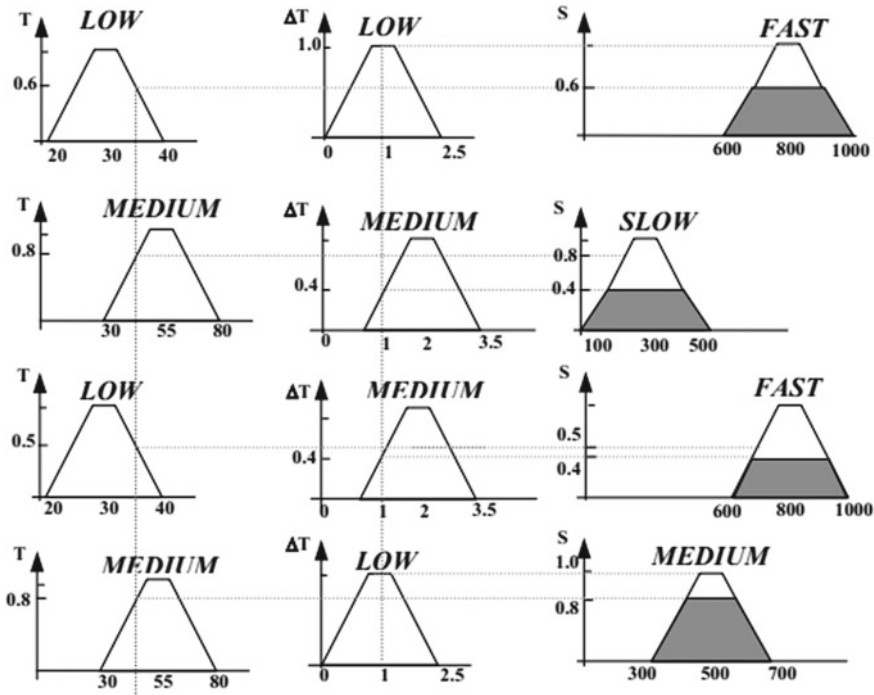


Fig. 2.21 An illustration of fuzzy output calculation

and 1.0. The fuzzy output for the four rules is the intersection of the paired values obtained from the graph, or the AND result between the temperature input and the temperature change rate input. According to fuzzy set properties (2.7), this operation result should be: $\min(0.6, 1.0)$, $\min(0.8, 0.4)$, $\min(0.5, 0.4)$, and $\min(0.8, 1.0)$, which produces to 0.6, 0.4, 0.4, and 0.8, respectively.

2.6 Fuzzy Inference System

Fuzzy inference systems (FISs) are also known as fuzzy rule-based systems. This is a major unit of a fuzzy logic system where the decision-making is an important part in the entire system. A functional block is described in Fig. 2.22. *Fuzzy Inference* is the process of formulating a nonlinear mapping from a given input space to output space. The mapping then provides a basis from which decisions can be made. The process of fuzzy inference involves all input/output membership functions, fuzzy logic operators, and if-then rules [31, 32]. There are three basic types of fuzzy inference, which have been widely employed in various control applications. The differences between these

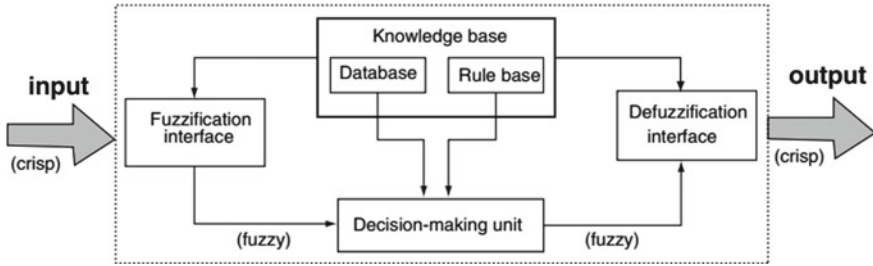


Fig. 2.22 Fuzzy inference system

three fuzzy inferences, also called fuzzy models, lie in the consequents of their fuzzy rules, aggregations, and defuzzification procedures. These fuzzy models are:

1. Mamdani fuzzy inference,
2. Tsukamoto fuzzy inference,
3. Sugeno fuzzy inference.

2.6.1 Mamdani Fuzzy Inference

The *Mamdani-type* fuzzy modeling was first proposed as the first attempt to control a steam engine and boiler by a set of linguistic control rules obtained from experienced human operator [5]. Figure 2.23 is an illustration of a two-input single-output Mamdani-type fuzzy model. The choice of T-conorm (e.g., Max) and T-norm (e.g., min or product) operators can be max-min and max-product. Max-min is the most common rule of composition. In max-min rule of composition, the inferred output of each rule is a fuzzy set chosen from the minimum firing strength. In max-product rule of composition the inferred output of each rule is a fuzzy set scaled down by its firing strength via algebraic product. A typical rule in *Mamdani-type* fuzzy model with two-input single-output has the form

$$\text{If } x \text{ is } \mathbf{A} \text{ and } y \text{ is } \mathbf{B} \text{ then } z \text{ is } \mathbf{C} \quad (2.27)$$

where x and y are the input variables and z is the output variable. $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are the MFs of the corresponding input and output variables.

2.6.2 Tsukamoto Fuzzy Inference

In the *Tsukamoto* fuzzy model, the consequent of each fuzzy Ifthen rule is represented by a fuzzy set with a monotonic MF [33]. As a result, the inferred output of each rule

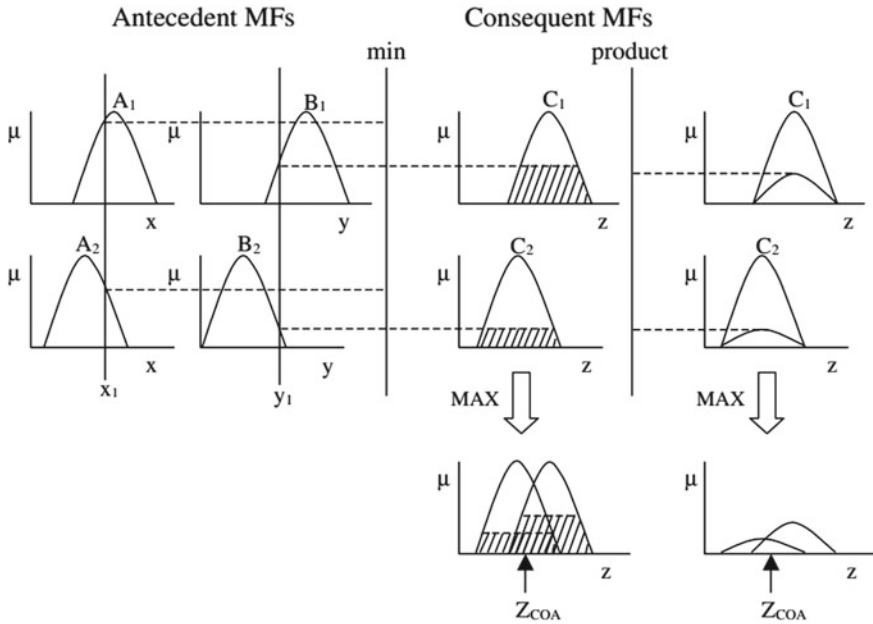


Fig. 2.23 Mamdani inference model

is defined as crisp value included by the rule's firing strength. The overall output is taken as the weighted average of each rule's output. Since each rule infers a crisp output, the Tsukamoto fuzzy model aggregates each rules output by the method of weighted average and thus avoids the time-consuming process of defuzzification. Figure 2.24 illustrates a two-input single-output Tsukamoto fuzzy model. A typical rule in Tsukamoto-type fuzzy model with two-input single-output has the form same as (2.27).

2.6.3 Sugeno Fuzzy Method

The Sugeno fuzzy model, also known as the TSK fuzzy model, was proposed by [34] in an effort to develop a systematic approach to generate fuzzy rules from a given input/output data set. Figure 4.18 illustrates a two-input single-output Sugeno fuzzy model. A typical fuzzy rule in Sugeno fuzzy model has the form

$$\text{If } x \text{ is } \mathbf{A} \text{ and } y \text{ is } \mathbf{B} \text{ then } z = \mathbf{f}(x, y) \quad (2.28)$$

where $\mathbf{A}$ and $\mathbf{B}$ are MFs in the antecedent part, while $z = \mathbf{f}(x, y)$; y is a linear function in the consequent part. Usually $\mathbf{f}(x, y)$ is polynomial in the input variables

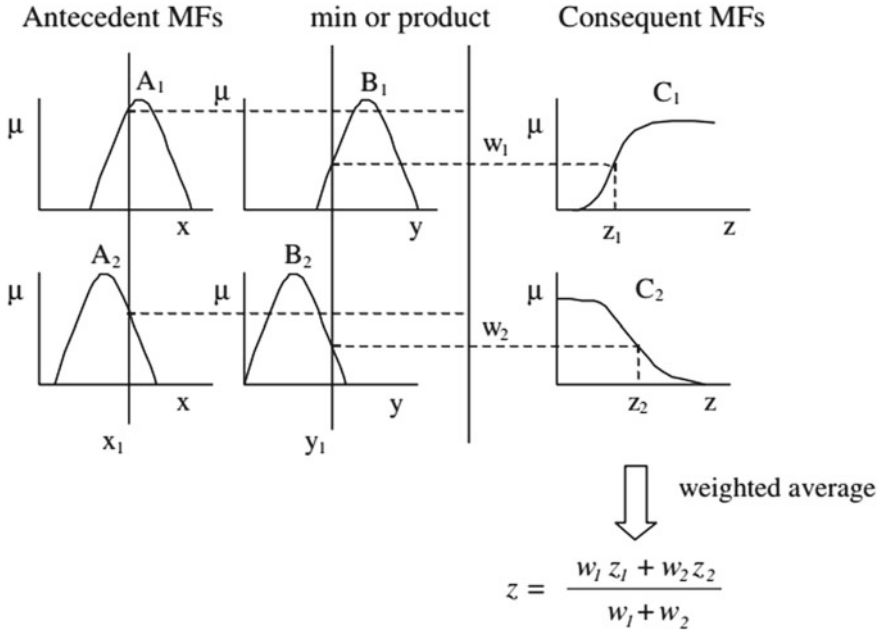


Fig. 2.24 Tsukamoto inference model

x and y but it can be any function as long as it can appropriately describe the output of the model within the fuzzy region specified by the antecedent of the rule. When $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is a first-order polynomial, the resulting fuzzy inference system is called a first-order Sugeno fuzzy model which was proposed in [34, 35]. When $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is a constant, it is zero-order Sugeno fuzzy model, which can be considered as a special case of *Mamdani* fuzzy model, in which the consequent of each rule is specified by a fuzzy singleton or by a pre-defuzzified consequent or a special case of Tsukamoto fuzzy model in which the consequent of each rule is specified by an MF of a step function.

Moreover, the output of zero-order Sugeno model is a smooth function of its input variables as long as neighboring MFs in the antecedent have enough overlap. In other words, the overlap of MFs in the consequent of a *Mamdani* model does not have a decisive effect on the smoothness; it is the overlap of the antecedent MFs that determines the smoothness of the resulting input/output behavior. The overall output of a *Sugeno* fuzzy model is obtained via weighted average of the crisp output, thus avoiding the time-consuming process of defuzzification required by *Mamdani* model (Fig. 2.25).

In practice, the weighted average operator is sometimes replaced with the weighted sum operator to reduce computation further, especially in the training of a fuzzy inference system. However, this simplification could lead to the loss of MF linguistic meaning unless the sum of firing strengths is close to unity.

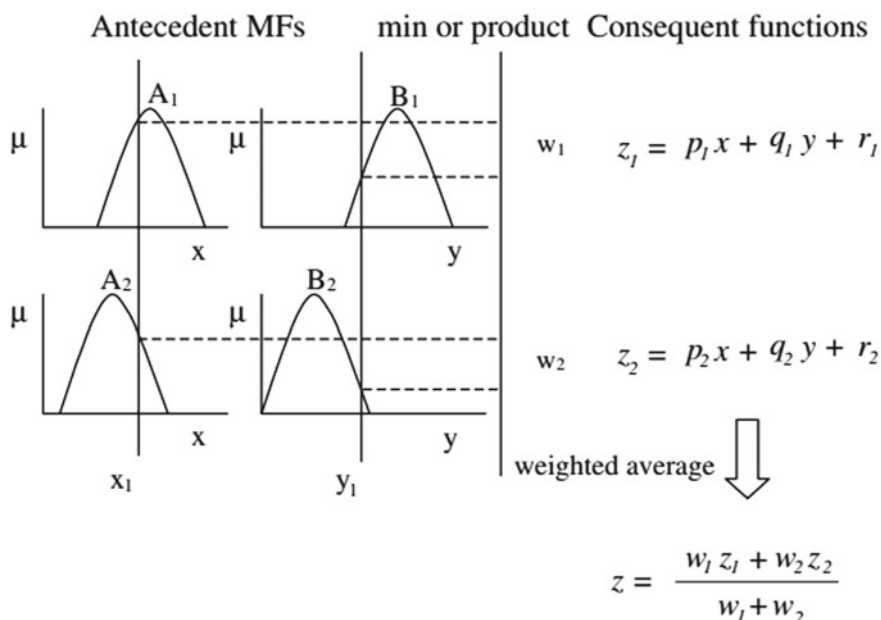


Fig. 2.25 Sugeno inference model

2.7 Fuzzy Control Basics

A block diagram of a fuzzy control system is shown in Fig. 2.26. The fuzzy controller sometimes is called a ‘fuzzy logic controller’ (FLC) or even a ‘fuzzy linguistic controller’ since, as we will see, it uses fuzzy logic in the quantification of linguistic descriptions. In this book, we will avoid these phrases and simply use ‘fuzzy controller.’ It is basically composed of the following four elements:

1. A *rule-base* (a set of If-Then rules), which contains a fuzzy logic quantification of the expert’s linguistic description of how to achieve good control.

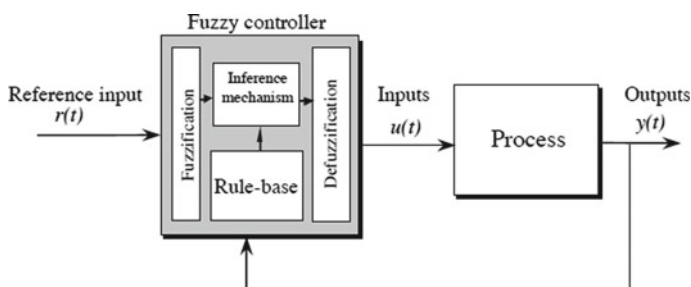
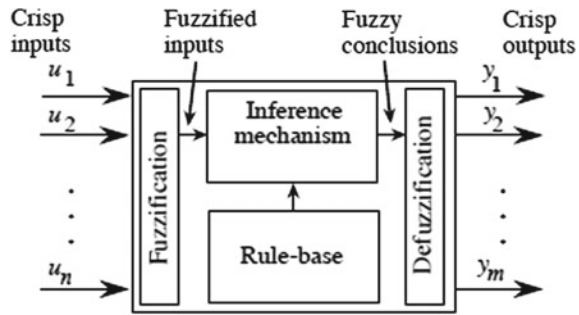


Fig. 2.26 Schematic of fuzzy controller

Fig. 2.27 Schematic of fuzzy controller



2. An *inference mechanism* (also called an ‘inference engine’ or ‘fuzzy inference’ module), which emulates the expert’s decision-making in interpreting and applying knowledge about how best to control the plant.
3. A *fuzzification interface*, which converts controller inputs into information that the inference mechanism can easily use to activate and apply rules.
4. A *defuzzification interface*, which converts the conclusions of the inference mechanism into actual inputs for the process.

2.7.1 Linguistic Variables, Values, and Rules

A fuzzy system is a static nonlinear mapping between its inputs and outputs (that is, it is not a dynamic system). Sometimes, the preprocessing of the inputs to the fuzzy system (that is, differentiators or integrators) is included in the definition of the fuzzy system and thereby obtain a ‘fuzzy system’ that is dynamic. In this book, we adopt the convention that such preprocessing is not part of the fuzzy system, and hence the fuzzy system will always be a memoryless nonlinear map.

From now onwards, it is assumed that the fuzzy system has inputs $u_j \in \mathcal{U}_j$, $j = 1, \dots, n$ and outputs $y_j \in \mathcal{Y}_j$, $j = 1, \dots, m$, as depicted in Fig. 2.27.

The inputs and outputs are *crisp* that is, they are real numbers, not fuzzy sets. The fuzzification block converts the crisp inputs to fuzzy sets, the inference mechanism uses the fuzzy rules in the rule-base to produce fuzzy conclusions (that is, the implied fuzzy sets), and the defuzzification block converts these fuzzy conclusions into the crisp outputs.

2.7.2 Universes of Discourse

The ordinary (‘crisp’) sets $\mathcal{U}_j$, $\mathcal{Y}_j$ are called the ‘universes of discourse’ for u_j , y_j , respectively (in other words, they are their domains). In practical applications, most often the universes of discourse are simply the set of real numbers or some interval or subset of real numbers.

2.7.3 Linguistic Variables

To specify rules for the rule-base, the expert will use a ‘linguistic description’; hence, linguistic expressions are needed for the inputs and outputs and the characteristics of the inputs and outputs. We will use ‘linguistic variables’ (*constant symbolic descriptions of what are in general time-varying quantities*) to describe fuzzy system inputs and outputs. For our fuzzy system, linguistic variables denoted by $\tilde{u}_j$ are used to describe the inputs u_j . Similarly, linguistic variables denoted by $\tilde{y}_j$ are used to describe outputs y_j . For instance, an input to the fuzzy system may be described as $\tilde{u}_1 = \text{‘position error’}$ or $\tilde{u}_2 = \text{‘velocity error’}$ and an output from the fuzzy system may be $\tilde{y}_1 = \text{‘voltagein’}$.

2.7.4 Linguistic Values

Just as u_j and y_j take on values over each universe of discourse $\mathcal{U}_j$ and $\mathcal{Y}_j$, respectively, linguistic variables $\tilde{u}_j$ and $\tilde{y}_j$ take on ‘linguistic values’ that are used to describe characteristics of the variables. Let $\tilde{A}_j^k$ denote the k th linguistic value of the linguistic variable $\tilde{u}_j$ defined over the universe of discourse $\mathcal{U}_j$. If we assume that there exist many linguistic values defined over $\mathcal{U}_j$, then the linguistic variable $\tilde{u}_j$ takes on the elements from the set of linguistic values denoted by

$$\tilde{A}_j = \{\tilde{A}_j^k, k = 1, \dots, N_j\}$$

Sometimes for convenience we will let the j indices take on negative integer values.

In the same way, let $\tilde{B}_j^p$ denote the p th linguistic value of the linguistic variable $\tilde{y}_j$ defined over the universe of discourse $\mathcal{Y}_j$. The linguistic variable $\tilde{y}_j$ takes on elements from the set of linguistic values denoted by

$$\tilde{B}_j = \{\tilde{B}_j^p, p = 1, \dots, M_j\}$$

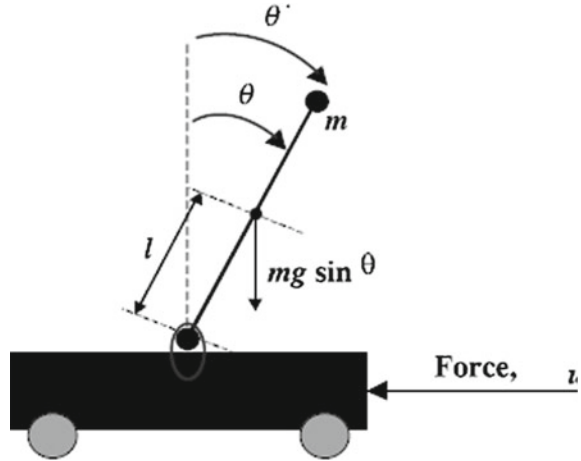
Sometimes for convenience we will let the j indices take on negative integer values.

Linguistic values are generally descriptive terms or adjectives. For example, if we assume that $\tilde{u}_1$ denotes the linguistic variable ‘speed,’ then we may assign $\tilde{A}_1^1 = \text{‘slow,’}$ $\tilde{A}_1^2 = \text{‘medium’}$ and $\tilde{A}_1^3 = \text{‘fast’}$ so that $\tilde{u}_1$ has a value from $\tilde{A}_1 = \{\tilde{A}_1^1, \tilde{A}_1^2, \tilde{A}_1^3, \}$

2.8 Notes

In this chapter, we have provided the reader with a ‘tutorial introduction’ to direct fuzzy control. In our tutorial introduction, we presented a step-by-step overview of the operations of the fuzzy controller. We have addressed several basic issues in the

Fig. 2.28 The inverted pendulum problem



design of fuzzy controllers. This introduction is designed to provide the reader with an intuitive understanding of the mechanics of the operation of the fuzzy controller. Our mathematical characterization served to show how the fuzzy controller can handle more inputs and outputs, the range of possibilities for the definition of universes of discourse, the membership functions, the rules, the inference mechanism, and defuzzification methods.

Problem

2.1 Consider the linearized model of inverted pendulum in Fig. 2.28, described by:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 15.79 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1.46 \end{bmatrix} u \quad (2.29)$$

with $\ell = 0.5$ m, $m = 100$ g, and initial conditions $x'(0) = \dot{\theta}[1 \ 0]^t$. It is desired to stabilize the system using fuzzy rules.

Fuzzy Control, Estimation and Diagnosis

Single and Interconnected Systems

Mahmoud, M.S.

2018, XXVI, 689 p. 341 illus., 186 illus. in color. With

online files/update., Hardcover

ISBN: 978-3-319-54953-8