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Model-Reference Adaptive Control

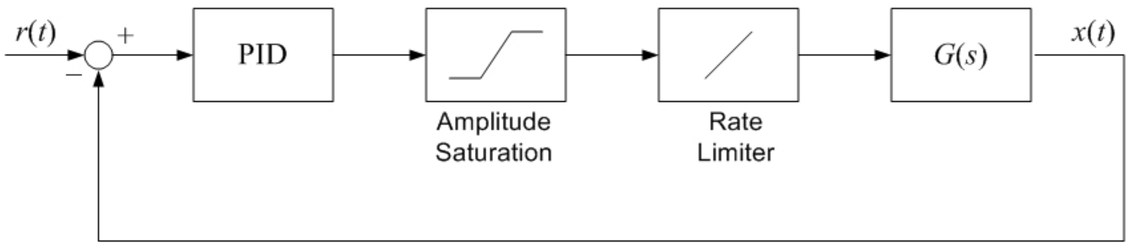
A Primer

Problem Solutions to Exercises and Suggested Exam Questions

Springer

Chapter 2 Exercises

1. Consider the following PID control system with the actuator amplitude saturation and rate limiting:



$$G(s) = \frac{5}{s^2 + 5s + 6}$$

with $k_p = 15$, $k_i = 8$, and $k_d = 5$ as the control gains. The actuator has both amplitude and rate limits between -1 and 1.

a. Compute the characteristic roots of the ideal closed-loop system without consideration for actuator amplitude saturation and rate limiting.
b. Construct a Simulink model for a sinusoidal input $r(t) = \sin t$. Plot the input, the ideal output without the actuator amplitude saturation and rate limiting, and the actual output for a simulation time $t = 10$ sec. Also plot the actuator command signal $u_c(t)$ and the control input signal to the plant $u(t)$.
c. Comment on the effect of rate limiting.

Solution:

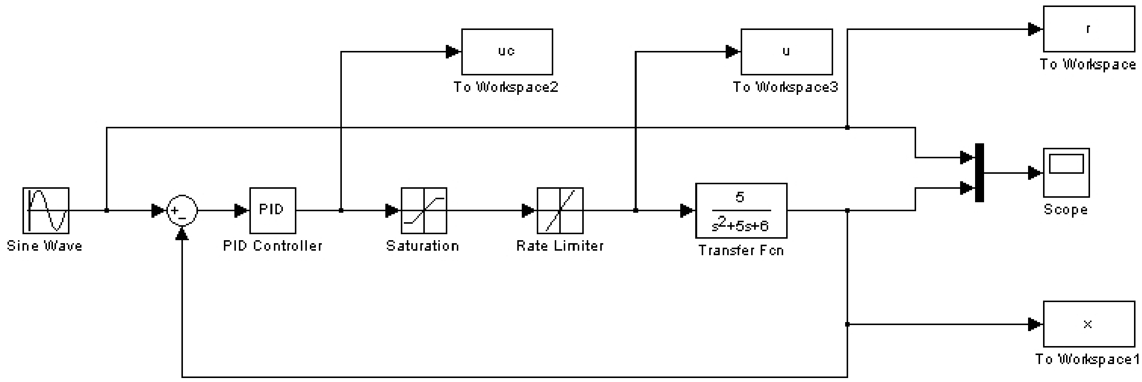
a. The characteristic equation is

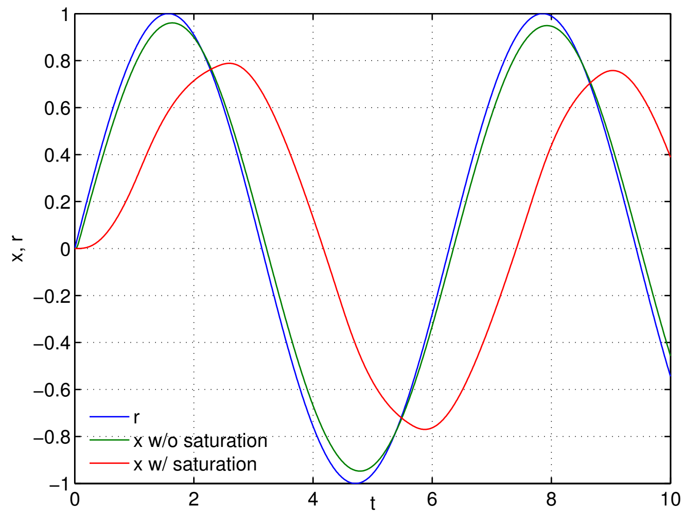
$$s^3 + (2\zeta\omega_n + bk_d)\, s^2 + \left(\omega_n^2 + bk_p\right) s + bk_i = 0$$

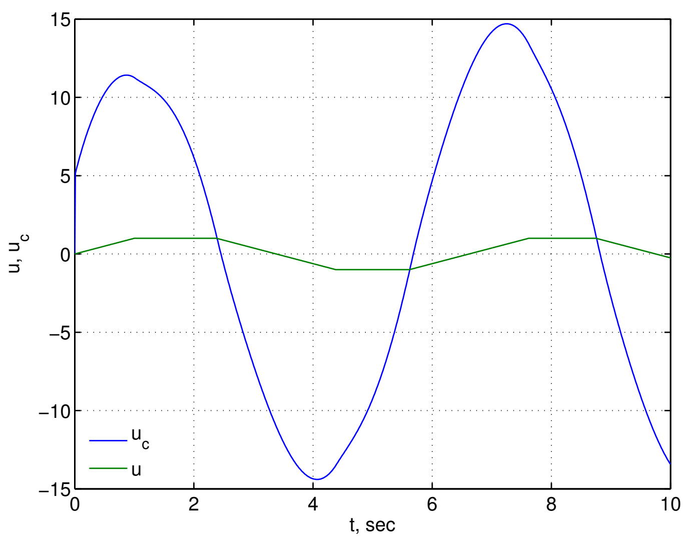
$$s^3 + 30s^2 + 81s + 40 = 0$$

The roots are $s_{1,2,3} = -0.6443,\ -2.2943,\ -27.0614$.

b. The Simulink model is as shown.







c. Amplitude saturation causes an amplitude reduction and signal distortion, whereas rate limiting causes a phase delay in the signal. This phase delay can potentially cause instability if the ideal closed-loop system does not have a sufficient phase margin.

2. Given

$$\ddot{\theta} + c\dot{\theta} + 2\sin\theta - 1 = 0$$

a. Find all the equilibrium points of the system for $-\pi \leq \theta(t) \leq \pi$.
b. Linearize the system and compute the eigenvalues about all the equilibrium points.
c. Classify the types of the equilibrium points on a phase plane and plot the phase portraits of the nonlinear system.

Solution:

a. Let $x_1(t) = \theta(t)$ and $x_2(t) = \dot{\theta}(t)$, then the state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -cx_2 - 2\sin x_1 + 1 \end{bmatrix}$$

The equilibrium points are

$$x_1^* = \frac{\pi}{6}, \frac{5\pi}{6}$$
$$x_2^* = 0$$

b. The Jacobian matrix is

$$J(x) = \begin{bmatrix} 0 & 1 \\ -2\cos x_1 & -c \end{bmatrix}$$

For the equilibrium point $\left(\frac{\pi}{6}, 0\right)$

$$J\left(\frac{\pi}{6}, 0\right) = \begin{bmatrix} 0 & 1 \\ -\sqrt{3} & -c \end{bmatrix}$$

the linearized equation is

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sqrt{3} & -c \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

and the eigenvalues are

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4\sqrt{3}}}{2}$$

For the equilibrium point $\left(\frac{5\pi}{6}, 0\right)$

$$J\left(\frac{\pi}{6}, 0\right) = \begin{bmatrix} 0 & 1 \\ \sqrt{3} & -c \end{bmatrix}$$

the linearized equation is

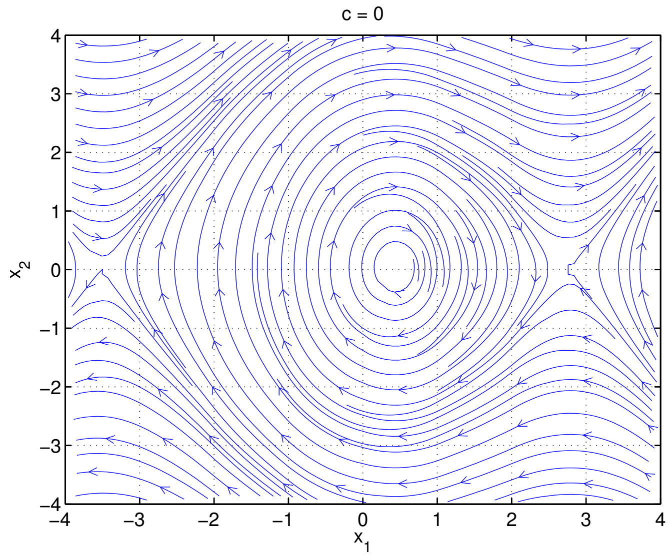
$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sqrt{3} & -c \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

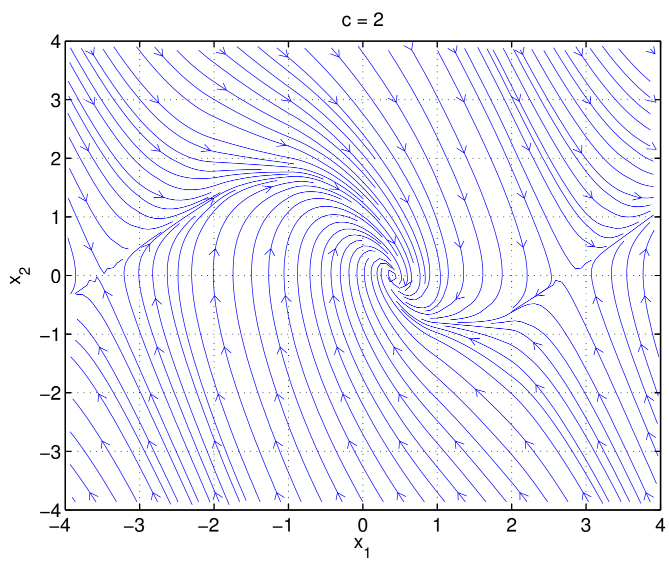
and the eigenvalues are

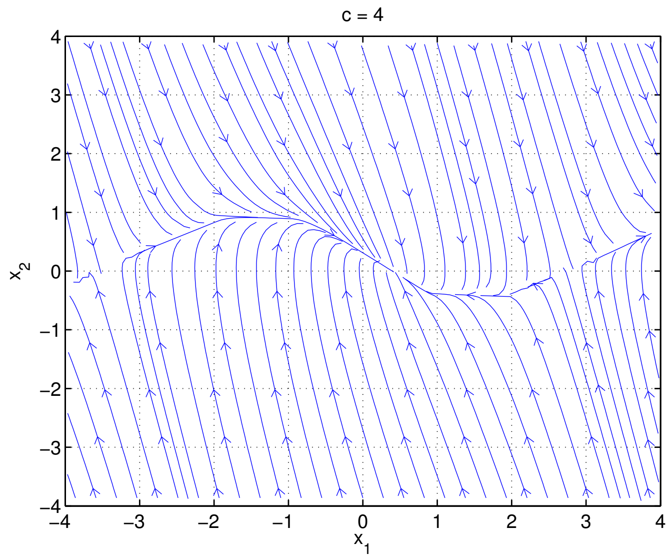
$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 + 4\sqrt{3}}}{2}$$

c. For the equilibrium point $\left(\frac{\pi}{6}, 0\right)$, consider the following cases:
   i. $c = 0$: the eigenvalues are purely imaginary. So the equilibrium point is a center.
   ii. $0 < c < 2\sqrt{\sqrt{3}}$: the eigenvalues are a complex conjugate pair with negative real part. So the equilibrium point is a stable focus.
   iii. $-2\sqrt{\sqrt{3}} < c < 0$: the eigenvalues are a complex conjugate pair with positive real part. So the equilibrium point is an unstable focus.
   iv. $c \geq 2\sqrt{\sqrt{3}}$: the eigenvalues are real and negative. So the equilibrium point is a stable node.
   v. $c = -2\sqrt{\sqrt{3}}$: the eigenvalues are real and positive. So the equilibrium point is an unstable node.

For the equilibrium point $\left(\frac{5\pi}{6}, 0\right)$, the eigenvalues are real and have opposite signs for all values of c. So the equilibrium point is a saddle.







3. Repeat Exercise 2.2 for

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$$

Solution:

a. The equilibrium points are $(0,0)$ and $(1,1)$.

b. The Jacobian matrix is

$$J(x) = \begin{bmatrix} -1 + x_2 & x_1 \\ -x_2 & 1 - x_1 \end{bmatrix}$$

For the equilibrium point $(0,0)$

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

the linearized equation is

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

and the eigenvalues are

$$\lambda_{1,2} = \pm 1$$

For the equilibrium point $(1,1)$

$$J(1,1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

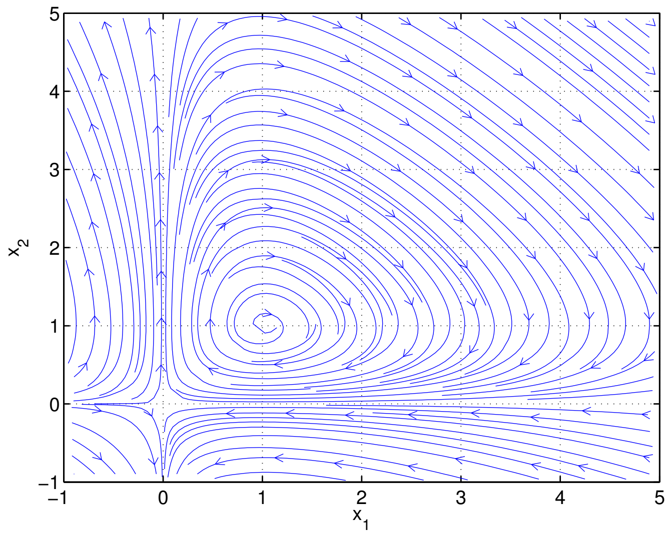
the linearized equation is

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

and the eigenvalues are

$$\lambda_{1,2} = \pm i$$

c. The equilibrium point $(0,0)$ is a saddle point. The equilibrium point $(1,1)$ is a center.



4. Analytically determine the solution of the following nonlinear system:

$$\dot{x} = |x|\, x^2$$

with a general initial condition $x(0) = x_0$.

a. Let $x_0 = 1$. Does the solution have a finite escape time? If so, determine it.
b. Repeat part (a) with $x_0 = -1$.
c. Comment on the effect of initial condition on the stability of the system.

Solution:

a. The equation can be expressed as

$$\dot{x} = |x|\, x^2 = \begin{cases} -x^3 & x < 0 \\ 0 & x = 0 \\ x^3 & x > 0 \end{cases}$$

which has a general solution

$$x(t) = \begin{cases} \frac{x_0}{\sqrt{1+2x_0^2 t}} & x < 0 \\ 0 & x = 0 \\ \frac{x_0}{\sqrt{1-2x_0^2 t}} & x > 0 \end{cases}$$

The solution for $x_0 = 1$ has a finite escape time at $t = \frac{1}{2x_0^2}$.

b. The solution for $x_0 = -1$ does not have a finite escape time.
c. This problem illustrates that the stability of a nonlinear system is highly dependent upon the initial condition.

Chapter 3 Exercises

1. Verify that the 1-norm of $x \in \mathbb{R}^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

satisfies the norm conditions.

Solution:
To verify $\|x\|_1$ is a norm, note that $\|x\|_1$ can be expressed as

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = |x_1| + |x_2| + \cdots + |x_n|$$

It is obvious that $\|x\|_1 \geq 0$ and $\|x\|_1 = 0$ if and only if $x_i = 0 \ \forall i = 1, 2, \ldots, n$. Thus $\|x\|_1$ satisfies the positivity and positive-definiteness conditions.Since

$$\|\alpha x\|_1 = |\alpha x_1| + |\alpha x_2| + \cdots + |\alpha x_n| = |\alpha| \left(|x_1| + |x_2| + \cdots + |x_n| \right)$$

then $\|x\|_1$ satisfies the homogeneity condition.
Let $y \in \mathbb{R}^n$. Then,

$$\|x + y\|_1 = |x_1 + y_1| + |x_2 + y_2| + \cdots + |x_n + y_n|$$

But

$$|x_1 + y_1| \leq |x_1| + |y_1|$$

So

$$\|x + y\|_1 \leq |x_1| + |y_1| + |x_2| + |y_2| + \cdots + |x_n| + |y_n| = \|x\|_1 + \|y\|_1$$

Thus, $\|x\|_1$ satisfies the triangle inequality.

2. Compute analytically the 1-, 2-, infinity, and Frobenius norms of

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 4 & 0 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

and verify the answers with Matlab using the function “norm”.
Note: Matlab may be used to compute the eigenvalues.

Solution:
The 1-, 2-, infinity, and Frobenius norms of

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 4 & 0 & 2 \\ -1 & 3 & 2 \end{bmatrix}$$

are computed as follows

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}| = \max(1+4+1, 3, 2+2+2) = 6$$

$$A^* A = A^\top A = \begin{bmatrix} 18 & -3 & 4 \\ -3 & 9 & 6 \\ 4 & 6 & 12 \end{bmatrix}$$

$$\lambda_{max}(A^* A) = 20$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^* A)} = \sqrt{20}$$

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| = \max(1+2, 4+2, 1+3+2) = 6$$

$$\|A\|_F = \sqrt{\operatorname{trace}(A^* A)} = \sqrt{18+9+12} = \sqrt{39}$$

The results agree with Matlab answers using "norm" function.

3. Decompose A into its symmetric part P and anti-symmetric part Q. Write the quadratic function $V(x) = x^\top P x$. Is $V(x)$ positive (semi-)definite, negative (semi-)definite, or neither?

Solution:
A can be decomposed into a symmetric part and anti-symmetric part as

$$A = P + Q$$

where the symmetric part is

$$P = \frac{1}{2}\left(A + A^\top\right) = \begin{bmatrix} \frac{1}{2} & 2 & -\frac{3}{2} \\ 2 & 0 & \frac{5}{2} \\ -\frac{3}{2} & \frac{5}{2} & 2 \end{bmatrix}$$

and the anti-symmetric part is

$$Q = \frac{1}{2}\left(A - A^\top\right) = \begin{bmatrix} 0 & -2 & -\frac{1}{2} \\ 2 & 0 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$V(x) = x^\top P x = x_1^2 + 4x_1x_2 - 3x_1x_3 + 5x_2x_3 + 2x_3^2$$

The eigenvalues of P are

$$\lambda(P) = -3.1837,\ 2.4817,\ 3.7020$$

Therefore, P is neither positive or negative definite, and so is $V(x)$.

4. Given a set $\mathcal{C} \subset \mathbb{R}^2$

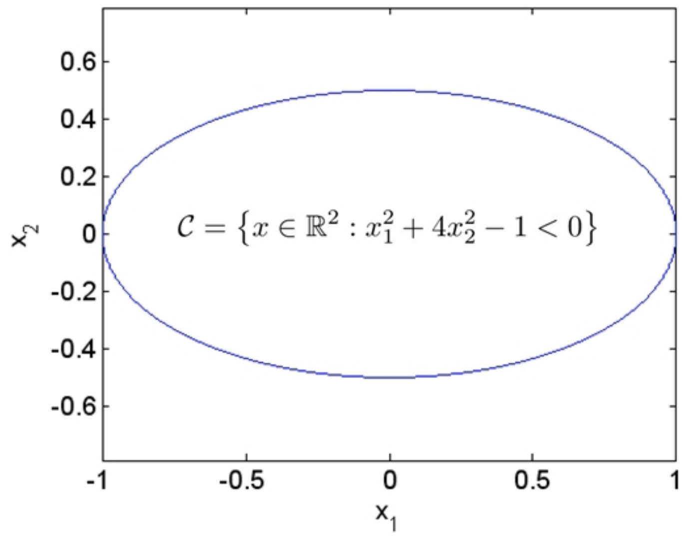
$$\mathcal{C} = \left\{x \in \mathbb{R}^2 : x_1^2 + 4x_2^2 - 1 < 0\right\}$$

Is $\mathcal{C}$ a compact set? Write the set notation for the complementary set $\mathcal{C}^c$. Plot and illustrate the region in $\mathbb{R}^2$ that represents $\mathcal{C}$.

Solution:
$\mathcal{C}$ is not a closed set so it is not a compact set. Its complementary set is

$$\mathcal{C}^c = \left\{x \in \mathbb{R}^2 : x_1^2 + 4x_2^2 - 1 \ge 0\right\}$$

Geometrically, $\mathcal{C}$ is a region enclosed by an ellipse not including the boundary.



5. For each of the following equations, determine if $f(x)$ is locally Lipschitz at $x = x_0$ or globally Lipschitz:

a. $\dot{x} = \sqrt{x^2+1}$, $x_0 = 0$.
b. $\dot{x} = -x^3$, $x_0 = 1$.
c. $\dot{x} = \sqrt{x^3+1}$, $x_0 = 0$.

Solution:

a. $\dot{x} = \sqrt{x^2+1}$, $x_0 = 0$.

$$f(x) = \sqrt{x^2+1}$$
$$f'(x) = \frac{x}{\sqrt{x^2+1}}$$

$f'(x)$ is bounded for all $x(t) \in \mathbb{R}$ since

$$\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}} = \pm 1$$

So $f(x)$ is globally Lipschitz.

b. $\dot{x} = -x^3$, $x_0 = 1$.

$$f(x) = -x^3$$
$$f'(x) = -3x^2$$

$f'(x)$ is unbounded as $x(t) \to \pm\infty$. So $f(x)$ is not globally Lipschitz. Since $f'(x_0 = 1) = -3$ is well defined, therefore $f(x)$ is locally Lipschitz in the neighborhood of $x_0 = 1$.

c. $\dot{x} = \sqrt{x^3+1}$, $x_0 = 0$.

$$f(x) = \sqrt{x^3+1}$$
$$f'(x) = \frac{3x^2}{2\sqrt{x^3+1}}$$

$f'(x)$ is unbounded as $x(t) \to \pm\infty$ and for $x = -1$. So $f(x)$ is not globally Lipschitz. Since $f'(x_0 = 0) = 0$ is well defined, therefore $f(x)$ is locally Lipschitz in the neighborhood of $x_0 = 1$ sufficiently away from $x = -1$.

Chapter 4 Exercises

1. Given

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1\left(x_1^2 + x_2^2 - 1\right) - x_2 \\ x_1 + x_2\left(x_1^2 + x_2^2 - 1\right) \end{bmatrix}$$

   a. Determine all the equilibrium points of the system and linearize the system about the equilibrium points to classify the types of equilibrium points.
   b. Use the Lyapunov candidate function

$$V(x) = x_1^2 + x_2^2$$

   to determine the types of Lyapunov stability of the equilibrium points and their corresponding regions of attraction, if any.

Solution:

a. The equilibrium point is determined from

$$x_1\left(x_1^2 + x_2^2 - 1\right) - x_2 = 0$$

$$x_1 + x_2\left(x_1^2 + x_2^2 - 1\right) = 0$$

Multiplying the first equation by $-x_2(t)$ and the second equation by $x_1(t)$ and adding them together yield

$$x_1^2 + x_2^2 = 0$$

Thus, the equilibrium point is at $x_1^* = 0$ and $x_2^* = 0$. There is only one equilibrium point.
The Jacobian is computed as

$$J(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 + 2x_1^2 & 2x_1x_2 - 1 \\ 1 + 2x_1x_2 & x_1^2 + x_2^2 - 1 + 2x_2^2 \end{bmatrix}$$

$$J(x^*) = \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues are

$$\lambda_{1,2} = -1 \pm i$$

The equilibrium point is a stable focus.

b. Choose the Lyapunov candidate function

$$V(x) = x_1^2 + x_2^2$$

Then,

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1\left[x_1\left(x_1^2+x_2^2-1\right)-x_2\right] + 2x_2\left[x_1 + x_2\left(x_1^2+x_2^2-1\right)\right]$$
$$= 2\left(x_1^2+x_2^2\right)\left(x_1^2+x_2^2-1\right)$$

$\dot{V}(x) < 0$ for all $x(t) \in \mathcal{R}_A$ where $\mathcal{R}_A$ is the region of attraction

$$\mathcal{R}_A = \left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 1 < 0\right\}$$

Thus, the equilibrium point is asymptotically stable for all $x(t) \in \mathcal{R}_A$.

2. Given

$$\dot{x} = x\left(-1 + \frac{1}{2}\sin x\right)$$

subject to $x(0) = 1$

a. Determine the upper and lower bound solutions.
b. Use the Lyapunov candidate function

$$V(x) = x^2$$

to determine the type of Lyapunov stability and the upper bound of $V(x)$ as an explicit function of time.

Solution:

a. Since $-1 \le \sin x \le 1$, therefore

$$-\frac{3}{2}x \le x\left(-1 + \frac{1}{2}\sin x\right) \le -\frac{1}{2}x$$

Thus

$$-\frac{3}{2}x \le \dot{x} \le -\frac{1}{2}x$$

The bounded solutions are with $x(0) = 1$

$$e^{-\frac{3}{2}t} \le x(t) \le e^{-\frac{1}{2}t}$$

b. Choose the Lyapunov candidate function

$$V(x) = x^2$$

Then,

$$\dot{V}(x) = 2x\dot{x} = 2x^2\left(-1 + \frac{1}{2}\sin x\right) \le -x^2 = -V(x) < 0$$

Since $\dot{V}(x) \le -V(x)$, the equilibrium point is exponentially stable. The upper bound solution of $V(x)$ is

$$V(t) \le V(0)\,e^{-t}$$

where $V(0) = x^2(0) = 1$, so

$$V(t) \le e^{-t}$$

3. Use the Lyapunov candidate function

$$V(x) = x_1^2 + x_2^2$$

to study stability of the origin of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)\left(x_1^2 + x_2^2\right) \\ (x_1 + x_2)\left(x_1^2 + x_2^2\right) \end{bmatrix}$$

Solution:
$\dot{V}(x)$ is evaluated as

$$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(x_2 - x_1)\left(x_1^2 + x_2^2\right) + 2x_2(x_1 + x_2)\left(x_1^2 + x_2^2\right) = -2\left(x_1^2 - 2x_1x_2 - x_2^2\right)\left(x_1^2 + x_2^2\right)$$

$\dot{V}(x) < 0$ if

$$x_1^2 - 2x_1x_2 - x_2^2 > 0$$

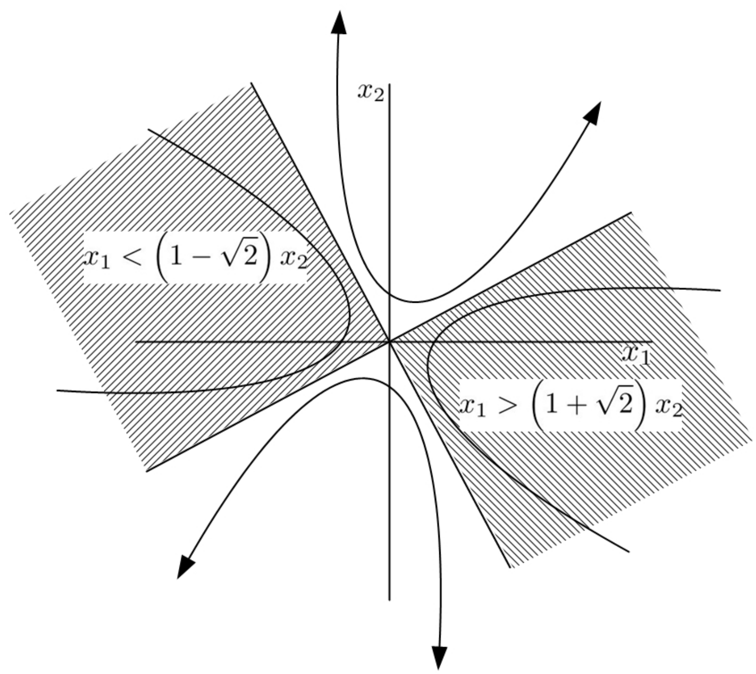
This inequality yields two solutions

$$x_1 > \left(1 + \sqrt{2}\right)x_2$$

or

$$x_1 < \left(1 - \sqrt{2}\right)x_2$$

The equilibrium is a saddle point type as illustrated. So it is unstable.



4. Given

$$\dot{x} = Ax$$

a. Calculate analytically P that solves

$$A^\top P + PA = -2I$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$$

and verify the result using the Matlab function "lyap".

b. Determine if P is positive or negative (semi-)definite. What can be said about stability of the origin of this system.

Solution:

a. P is computed from the Lyapunov equation as

$$A^\top P + PA = -2I$$

$$\begin{bmatrix} 0 & -4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -8p_{12} & p_{11} - 4p_{12} - 4p_{22} \\ p_{11} - 4p_{12} - 4p_{22} & 2p_{12} - 8p_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$p_{12} = \frac{1}{4}$$

$$p_{22} = \frac{2p_{12} + 2}{8} = \frac{5}{16}$$

$$p_{11} = 4p_{12} + 4p_{22} = \frac{9}{4}$$

$$P = \begin{bmatrix} \frac{9}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{5}{16} \end{bmatrix}$$

The Matlab command "lyap(A',-2*eye(2))" yields the same result.

b. P is computed as

$$\begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -8p_{12} & p_{11} + 4p_{12} - 4p_{22} \\ p_{11} + 4p_{12} - 4p_{22} & 2p_{12} + 8p_{22} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$p_{12} = \frac{1}{4}$$

$$p_{22} = -\frac{2p_{12} + 2}{8} = -\frac{5}{16}$$

$$p_{11} = -4p_{12} + 4p_{22} = -\frac{9}{4}$$

$$P = \begin{bmatrix} -\frac{9}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{5}{16} \end{bmatrix}$$

The eigenvalues of P are

$$\lambda_{1,2}(P) = -2.2817, \ -0.2808$$

Since $\lambda_{1,2}(P) < 0$, then P is negative definite. Choose a Lyapunov function

$$V(x) = -x^\top P x > 0$$

Evaluating $\dot{V}(x)$ yields

$$\dot{V}(x) = -\dot{x}^\top P x - x^\top P \dot{x} = -x^\top A^\top P x - x^\top P A x = -x^\top \left(A^\top P + PA\right) x = 2x^\top x > 0$$

Since $\dot{V}(x) > 0$, the equilibrium at the origin is unstable.

5. Given

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1\left(1 - x_1^2 - x_2^2\right) + x_2 \\ -x_1 + x_2\left(1 - x_1^2 - x_2^2\right) \end{bmatrix}$$

a. Use the Lyapunov candidate function

$$V(x) = x_1^2 + x_2^2$$

to determine the type of Lyapunov stability of the origin.

b. Find an invariant set.
c. Solve for $V(t)$ as an explicit function of time and plot the trajectories of $V(t)$ for $V(0) = 0.01,\ 0.5,\ 1,\ 1.5,\ 2$.

Solution:

a. Choose a Lyapunov function

$$V(x) = x_1^2 + x_2^2$$

Then,

$$\dot{V}(x) = 2x_1\left[x_1\left(1 - x_1^2 - x_2^2\right) + x_2\right] + 2x_2\left[-x_1 + x_2\left(1 - x_1^2 - x_2^2\right)\right] = 2\left(x_1^2 + x_2^2\right)\left(1 - x_1^2 - x_2^2\right)$$

$\dot{V}(x) < 0$ for all $x(t) \in \mathcal{S}$ where

$$\mathcal{S} = \left\{x(t) \in \mathbb{R}^2 : \dot{V}(x) < 0 \Rightarrow x_1^2 + x_2^2 > 1\right\}$$

Since $\mathcal{S}$ does not include the origin, therefore the equilibrium is unstable in the sense of Lyapunov.

b. Let $\mathcal{R}$ be

$$\mathcal{R} = \left\{x(t) \in \mathbb{R}^2 : \dot{V}(x) = 0 \Rightarrow g(x) = x_1^2 + x_2^2 - 1 = V(x) - 1 = 0\right\}$$

Then,

$$\dot{g}(x) = \dot{V}(x) = -2V(x)\left[V(x) - 1\right] = 0$$

Therefore, $\mathcal{R}$ is an invariant set.

c. Since

$$\dot{V}(x) = -2V(x)\left[V(x) - 1\right]$$

Then,

$$\frac{dV}{V(V-1)} = -2dt$$

Using partial fraction, this can be expressed as

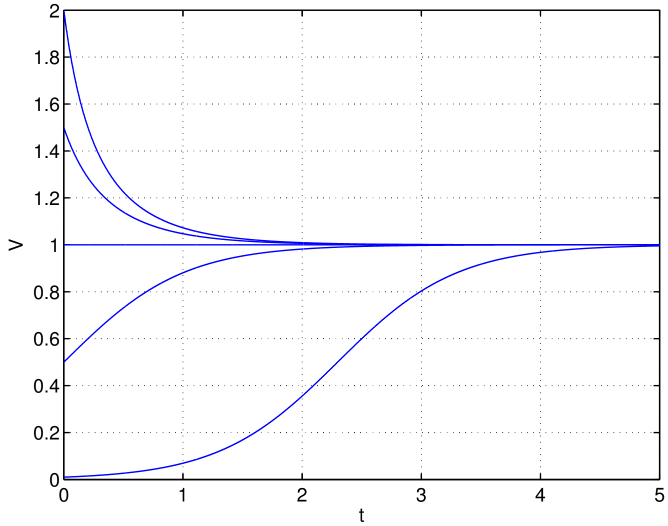
$$\left(\frac{1}{V-1} - \frac{1}{V}\right)dV = -2dt$$

which yields the following general solution

$$V(t) = \frac{V_0}{V_0 - (V_0 - 1)e^{-2t}}$$

As $t \to \infty$, $V(t)$ tends to a constant solution

$$\lim_{t\to\infty} V(t) = V(x \in \mathcal{R}) = 1$$



6. Given

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

Determine whether or not A is Hurwitz. If so, compute P using the Euler method to integrate the differential Lyapunov equation

$$\frac{dP}{d\tau} = PA + A^\top P + I$$

subject to $P(0) = 0$, where τ is time-to-go. Plot all 6 elements of P on the same plot and verify the result at the final time-to-go with the Matlab function "lyap".

Solution:
The eigenvalues of A are

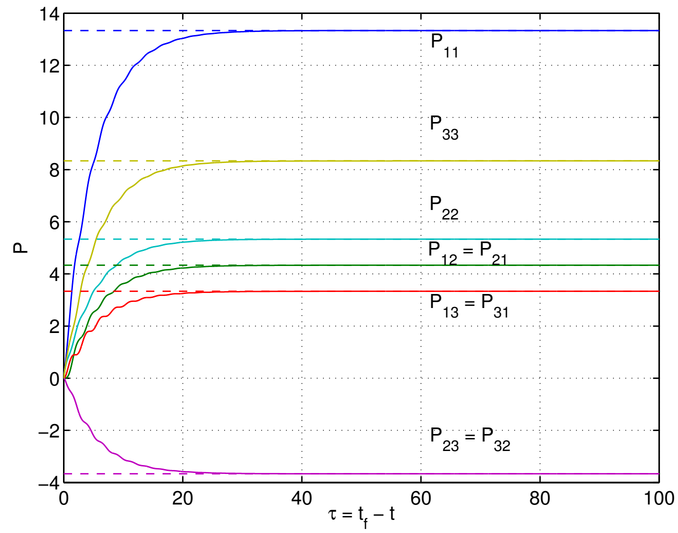
$$\lambda_{1,2,3}(A) = -1.8105,\ -0.0947 \pm 1.2837i$$

Thus A is Hurwitz. The numerical solution of P that solves

$$A^\top P + PA = -2I$$

is as shown where the solid lines are the numerical results by integrating the differential Lyapunov equation backward in time and the dash lines are the results from the Matlab function "lyap".
The numerical results check with the exact solution of P

$$P = \frac{1}{3}\begin{bmatrix} 40 & 13 & 10 \\ 13 & 16 & -11 \\ 10 & -11 & 25 \end{bmatrix}$$



7. Use the Lyapunov's direct method to determine an ultimate bound of the solution $x(t)$ for the following equation:

$$\dot{x} = -x + \cos t \sin t$$

subject to $x(0) = 1$. Plot the solution $x(t)$ for $0 \le t \le 20$.

Solution:
Choose a Lyapunov candidate function

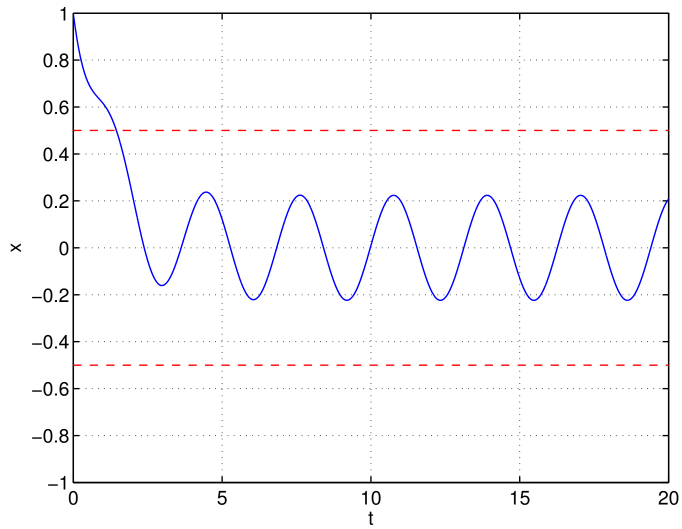
$$V(x) = x^2$$

Then,

$$\dot{V}(x) = 2x(-x + \cos t \sin t) = -2x^2 + x \sin 2t \le -2|x|\left(|x| - \frac{1}{2}\right)$$

$$\dot{V}(x) = 2x\dot{x} = -2x^2 + x \sin 2t \le -2|x|\left(|x| - \frac{1}{2}\right)$$

$\dot{V}(x) \le 0$ if $|x| > \frac{1}{2}$. So the ultimate bound is $\frac{1}{2}$.



8. Given a non-autonomous system

$$\dot{x} = (-2 + \sin t)\, x - \cos t$$

a. Show that the system is uniformly ultimately bounded by the Lyapunov theorem for non-autonomous systems. Also determine the ultimate bound of $\|x\|$.
b. Plot the solution by numerically integrating the differential equation and show that it satisfies the ultimate bound.

Solution:

a. Choose a Lyapunov candidate function

$$V(x) = x^2$$

Then,

$$\dot{V}(x) = 2x\dot{x} = 2x\left[(-2 + \sin t)\, x - \cos t\right]$$

Note that

$$-2 + \sin t \leq -1$$

$$-2x \cos t \leq 2\|x\|$$

Therefore,

$$\dot{V}(x) \leq -2\|x\|^2 + 2\|x\|$$

We see that

$$\dot{V}(x) \leq -2V(x) + 2\sqrt{V(x)}$$

Let $W = \sqrt{V} = \|x\|$. Then,

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}} = -\sqrt{V} + 1 \leq -W + 1$$

The solution of $W(t)$ is

$$W(t) \leq (\|x_0\| - 1)\, e^{-t} + 1$$

Thus

$$\lim_{t\to\infty} \|x\| = \lim_{t\to\infty} W(t) \leq 1 = R$$

Choose

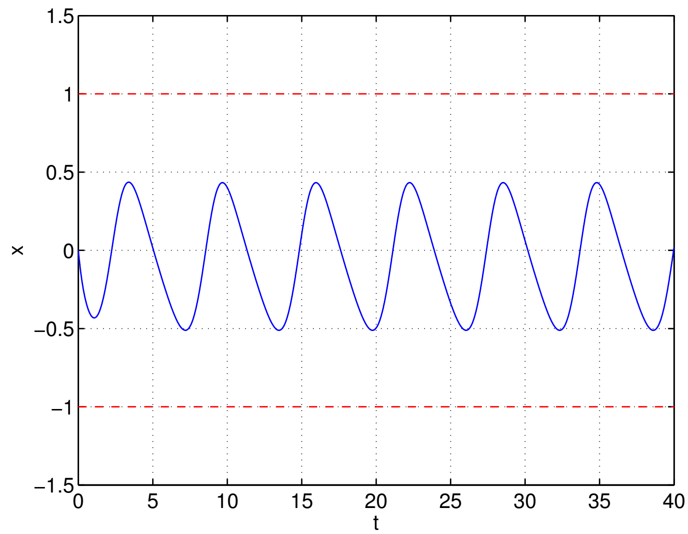
$$\varphi_3(\|x\|) = 2\|x\|^2 - 2\|x\|$$

Then, it follows that

$$\dot{V}(x) \le -\varphi_3(\|x\|)$$

$\dot{V}(x) \le 0$ for $\|x\| \ge 1$. Therefore, the solution $x(t)$ is uniformly ultimately bounded with a Lyapunov ultimate bound of 1.

b. The solution of $x(t)$ is shown in the following plot. The largest value of $\|x\|$ is 0.5112 which is less than the Lyapunov ultimate bound.



9. Given

$$\dot{x} = -\left(1 + \sin^2 t\right) x + \cos t$$

a. Use the Lyapunov candidate function

$$V(x) = x^2$$

to determine the upper bound of $\dot{V}(x)$ as a function of $V(x)$.

b. Let $W = \sqrt{V}$. Solve for the inequality $W(t)$ as an explicit function of time and determine the ultimate bound of the system.

c. Show that the system is uniformly ultimately bounded.

Solution:

a. Given the Lyapunov candidate function

$$V(x) = x^2 = \|x\|^2$$

$\dot{V}(x)$ is evaluated as

$$\dot{V}(x) = 2x\left[-\left(1 + \sin^2 t\right)x + \cos t\right] = -2x^2\left(1 + \sin^2 t\right) + 2x\cos t \le -2\|x\|^2 + 2\|x\| = -2V(x) + 2\sqrt{V(x)}$$

Then, $\dot{V}(x) \le 0$ if $\|x\| > 1$.

b. Let $W = \sqrt{V} = \|x\|$. Then,

$$\dot{W} = \frac{\dot{V}}{2\sqrt{V}} \le \frac{-2V + 2\sqrt{V}}{2\sqrt{V}} = -W + 1$$

The explicit solution is

$$W(t) \le [W(0) - 1]\, e^{-t} + 1$$

The ultimate bound is determined by

$$\lim_{t\to\infty} W(t) = \|x\| \leq 1 = R$$

Therefore, the ultimate bound is 1.

c. We show that the Lyapunov theorem for uniform ultimate boundedness for non-autonomous systems is satisfied. Let $\varphi_1(\|x\|) = a\|x\|^2 \in \mathcal{KR}$ and $\varphi_2(\|x\|) = b\|x\|^2 \mathcal{KR}$ where $a < 1$ and $b > 1$. Then, $\varphi_1(\|x\|) \leq V(x) \leq \varphi_2(\|x\|)$ and $\dot{V}(x) \leq 0$ for $\|x\| > 1$. Furthermore, let $\varphi_3(\|x\|) = 2\|x\|^2 - 2\|x\| \in \mathcal{KR}$. Then, $\dot{V}(x) \leq -\varphi_3(\|x\|)$ for $\|x\| > 1$. Therefore, the solution is uniformly ultimately bounded. Alternatively, the solution is uniformly ultimately bounded since $\dot{V}(x) \leq 0$ outside the compact set $\|x\| \leq 1$.

10. For the following functions:

a. $f(t) = \sin\left(e^{-t^2}\right)$

b. $f(t) = e^{-\sin^2 t}$

Plot $f(t)$ for $t \in [0,5]$. Determine whether or not the limit of $f(t)$ exists as $t \to \infty$ and $\dot{f}(t)$ is uniformly continuous. If so, use the Barbalat's lemma to show that $\dot{f}(t) \to 0$ as $t \to \infty$ and verify by taking the limit of $\dot{f}(t)$ as $t \to \infty$.

Solution:

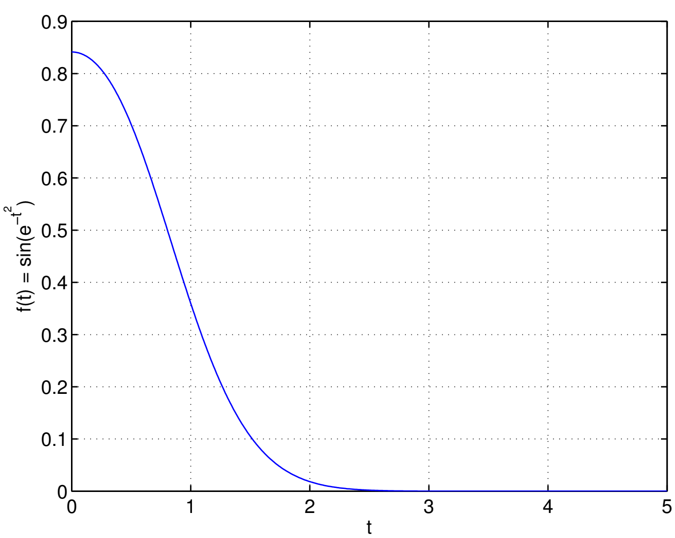
a. $f(t) = \sin\left(e^{-t^2}\right)$. The limit of $f(t)$ exists as $t \to \infty$ because $\lim_{t\to\infty} \sin\left(e^{-t^2}\right) = 0$. Taking the derivative

$$\dot{f}(t) = -2te^{-t^2} \cos\left(e^{-t^2}\right)$$

To show that $\dot{f}(t)$ is uniformly continuous, we need to determine if the limit of $\dot{f}(t)$ exists as $t \to \infty$ and that its derivative $\ddot{f}(t)$ is bounded. We see that the limit of $\dot{f}(t)$ exists as $t \to \infty$. Evaluating $\ddot{f}(t)$ yields

$$\ddot{f}(t) = -2e^{-t^2} \cos\left(e^{-t^2}\right) + 4t^2 e^{-t^2} \cos\left(e^{-t^2}\right) + 4t^2 e^{-2t^2} \sin\left(e^{-t^2}\right)$$

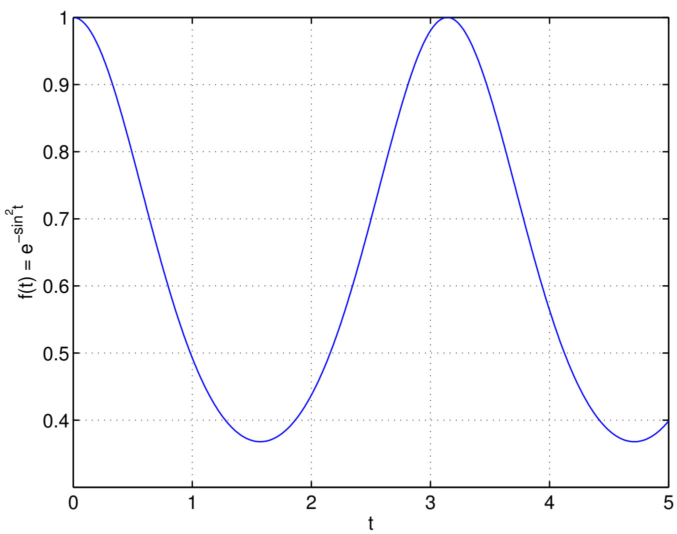
We see that $\ddot{f}(t)$ is bounded as $t \to \infty$. Therefore, $\dot{f}(t)$ is uniformly continuous. Then, according to the Barbalat's lemma, $\dot{f}(t) \to 0$ as $t \to \infty$. This can easily be verified by taking the limit of $\dot{f}(t)$ as $t \to \infty$ which gives $\lim_{t\to\infty} \dot{f}(t) = 0$.



b. $f(t) = e^{-\sin^2 t}$. The limit of $f(t)$ does not exist. Taking the derivative

$$\dot{f}(t) = -2\sin t \cos t e^{-\sin^2 t}$$

The limit of $\dot{f}(t)$ does not exists. So $\dot{f}(t)$ is not uniformly continuous.



11. Consider the following adaptive control system:

$$\dot{e} = -e + \theta x$$

$$\dot{\theta} = -xe$$

where $e(t) = x_m(t) - x(t)$ is defined as the tracking error between a given explicit reference time signal $x_m(t)$ which is assumed to be bounded; i.e., $x_m(t) \in \mathcal{L}_\infty$, and the state variable $x(t)$. Show that the adaptive system is stable and that $e(t) \to 0$ as $t \to \infty$.

Solution:
Choose a Lyapunov candidate function

$$V(e,\theta) = e^2 + \theta^2$$

Then,

$$\dot{V}(e,\theta) = 2e(-e+\theta x) + 2\theta(-xe) = -2e^2 \le 0$$

Since $\dot{V}(e,\theta)$ is negative semi-definite, $e(t) \in \mathcal{L}_\infty$ and $\theta(t) \in \mathcal{L}_\infty$, i.e., they are bounded. Since $\dot{V}(e,\theta) \le 0$, then

$$V(e(t\to\infty), \theta(t\to\infty)) - V(e(t_0), \theta(t_0)) = \int_{t_0}^{\infty} \dot{V}(e,\theta)\,dt = -2\int_{t_0}^{\infty} e^2(t)\,dt = -2\|e\|_2^2$$

$$V(e(t\to\infty), \theta(t\to\infty)) = V(e(t_0), \theta(t_0)) - 2\|e\|_2^2 = e^2(t_0) + \theta^2(t_0) - 2\|e\|_2^2 < \infty$$

So, $V(e,\theta)$ has a finite limit as $t \to \infty$. Since $\|e\|_2$ exists, therefore $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.Also

$$\ddot{V}(e,\theta) = -4e(-e+\theta x)$$

Since $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\theta(t) \in \mathcal{L}_\infty$ by the virtue that $\dot{V}(e,\theta) \le 0$, and $x(t) \in \mathcal{L}_\infty$ since $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $x_m(t) \in \mathcal{L}_\infty$ by assumption, then $\ddot{V}(e,\theta) \in \mathcal{L}_\infty$. Therefore, $\dot{V}(e,\theta)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}(e,\theta) \to 0$ and hence $e(t) \to 0$ as $t \to \infty$.

Chapter 5 Exercises

1. Consider a first-order nonlinear SISO system with a matched uncertainty

$$\dot{x} = ax + b\left[u + \theta^* \phi\left(x\right)\right]$$

where a is unknown but b is known, θ^* is unknown, and $\phi\left(x\right) = x^2$.
A reference model is specified by

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m < 0$ and b_m are known, and $r\left(t\right)$ is a bounded command signal.

a. Design and implement in Simulink a direct adaptive controller that enables the plant output $x\left(t\right)$ to track the reference model signal $x_m\left(t\right)$, given $b = 2$, $a_m = -1$, $b_m = 1$, and $r\left(t\right) = \sin t$. For adaptation rates, use $\gamma_x = 1$ and $\gamma = 1$. For simulation purposes, assume $a = 1$ and $\theta^* = 0.2$ for the unknown parameters. Plot $e\left(t\right)$, $x\left(t\right)$, $x_m\left(t\right)$, $u\left(t\right)$, and $\theta\left(t\right)$ for $t \in [0, 50]$.
b. Show by the Lyapunov stability analysis that the tracking error is asymptotically stable; i.e., $e\left(t\right) \to 0$ as $t \to \infty$.
c. Repeat part (a) for $r\left(t\right) = 1\left(t\right)$ where $1\left(t\right)$ is the unit-step function. Plot the same sets of data as in part (a). Comment on the convergence of $k_x\left(t\right)$ and $\theta\left(t\right)$ to the ideal values k_x^* and θ^*.

Solution:

a. Define the ideal gain k_x^* that satisfies one of the model matching conditions

$$a + bk_x^* = a_m$$

and the known gain k_r that satisfies the other model matching condition

$$bk_r = b_m$$

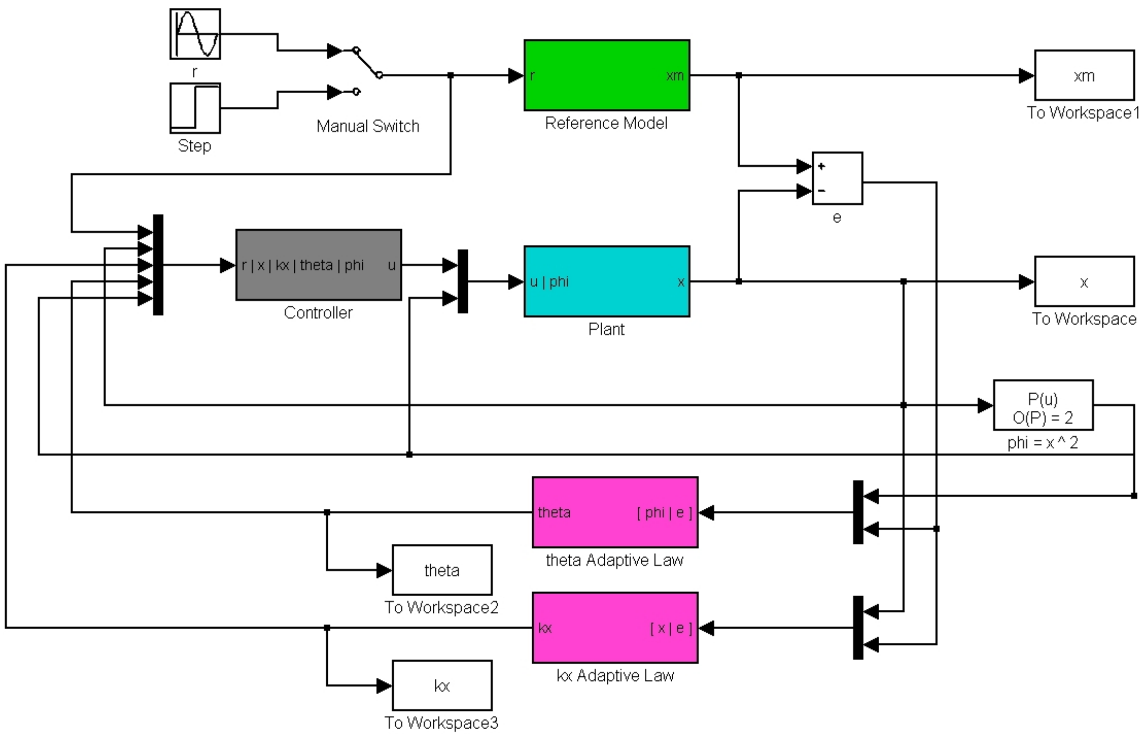
since b is known. Numerically, $k_x^* = -1$ and $k_r = 0.5$.
The adaptive controller is then given by

$$u = k_x\left(t\right) x + k_r r - \theta\left(t\right) x^2$$

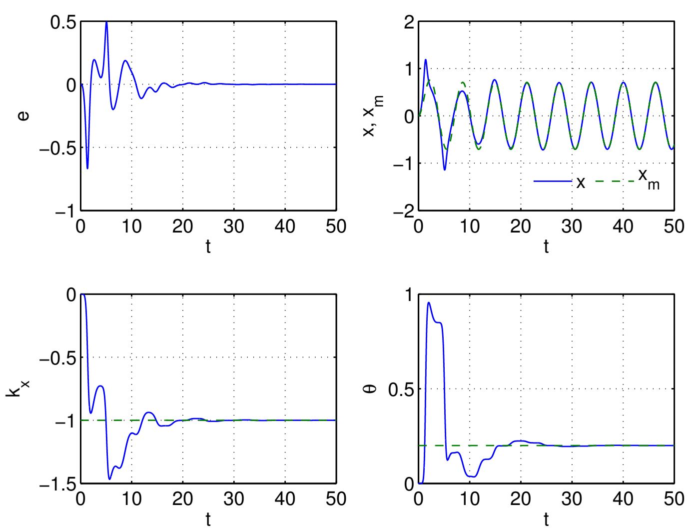
with the following adaptive laws:

$$\dot{k}_x = \gamma_x x e b$$

$$\dot{\theta} = -\gamma x_\theta^2 e b$$



The Simulink model of the adaptive controller is as shown.
The response of the adaptive controller with $r(t) = \sin t$ is shown in the following plot. Note that the parameters $k_x(t)$ and $\theta(t)$ converge to their ideal values k_x^* and θ^*.



b. Let $\tilde{k}_x(t) = k_x(t) - k_x^*$ and $\tilde{\theta}(t) = \theta(t) - \theta^*$ be the estimation errors. Then, the closed-loop plant model is

$$\dot{x} = \left(\underbrace{ax + bk_x^*}_{a_m} + b\tilde{k} \right) x + \underbrace{bk_r}_{b_m} r - b\tilde{\theta} x^2$$

The closed-loop tracking error equation is obtained as

$$\dot{e} = \dot{x}_m - \dot{x} = a_m e - b\tilde{k}_x x + b\tilde{\theta}x^2$$

Choose a Lyapunov candidate function

$$V\left(e, \tilde{k}_x, \tilde{\theta}\right) = e^2 + \frac{\tilde{k}_x^2}{\gamma_x} + \frac{\tilde{\theta}^2}{\gamma}$$

Then,

$$\dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right) = 2e\left(a_m e - b\tilde{k}_x x + b\tilde{\theta}x^2\right) + \frac{2\tilde{k}_x\dot{\tilde{k}}_x}{\gamma_x} + \frac{2\tilde{\theta}\dot{\tilde{\theta}}}{\gamma} = 2a_m e^2 - 2\tilde{k}_x\left(xeb - \frac{\dot{\tilde{k}}_x}{\gamma_x}\right) + 2\tilde{\theta}\left(x^2 eb + \frac{\dot{\tilde{\theta}}}{\gamma}\right)$$

Substituting in the adaptive laws $\dot{k}_x(t) = \dot{\tilde{k}}_x(t)$ and $\dot{\theta}(t) = \dot{\tilde{\theta}}(t)$ yields

$$\dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right) = 2a_m e^2 \leq 0$$

Since $\dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right)$ is negative semi-definite, $e(t) \in \mathcal{L}_\infty$, $k_x(t) \in \mathcal{L}_\infty$, and $\theta(t) \in \mathcal{L}_\infty$, i.e., all signals are bounded. Also

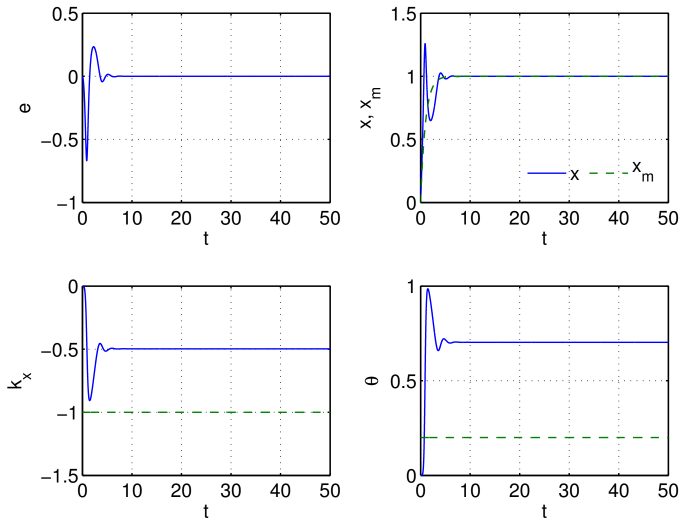
$$V(t \to \infty) - V(t_0) = \int_{t_0}^{\infty} \dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right) dt = 2a_m \int_{t_0}^{\infty} e^2(t)\, dt = 2a_m \|e\|_2^2$$

So, $V\left(e, \tilde{k}_x, \tilde{\theta}\right)$ has a finite limit as $t \to \infty$. Since $\|e\|_2$ exists, therefore $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Differentiating $\dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right)$ yields

$$\ddot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right) = 4a_m e\left(a_m e - b\tilde{k}_x x + b\tilde{\theta}x^2\right)$$

Since $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, $k_x(t) \in \mathcal{L}_\infty$, and $\theta(t) \in \mathcal{L}_\infty$ by the virtue that $\dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right) \leq 0$, and $x(t) \in \mathcal{L}_\infty$ since $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $x_m(t) \in \mathcal{L}_\infty$ because $r(t) \in \mathcal{L}_\infty$ by assumption, therefore $\ddot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right) \in \mathcal{L}_\infty$. Therefore, $\dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}\left(e, \tilde{k}_x, \tilde{\theta}\right) \to 0$ which implies $e(t) \to 0$ as $t \to \infty$. Therefore, the tracking error is asymptotically stable.

c. The response of the adaptive controller with $r(t) = 1(t)$ is as shown in the following plot. The parameters $k_x(t)$ and $\theta(t)$ do not converge to their corresponding ideal values in this case. This is due to the fact the the Lyapunov stability proof only shows that $e(t) \to 0$ as $t \to \infty$, but not $\tilde{k}_x(t)$ or $\tilde{\theta}(t)$. Therefore, MRAC does not guarantee parameter convergence.



2. Consider the following first-order plant

$$\dot{x} = ax + b\left[u + \theta^{*}\phi\left(x\right)\right]$$

where a , $b > 0$, θ^{*} is unknown, and $\phi\left(x\right) = x^{2}$. Design an indirect adaptive controller in Simulink by estimating a, b, and θ^{*} so that the plant follows a reference model

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r\left(t\right) = \sin t$. For simulation purposes, use $a = 1$, $b = 1$, $\theta^{*} = 0.1$, $x\left(0\right) = x_m\left(0\right) = 1$, $\hat{a}\left(0\right) = 0$, $\hat{b}\left(0\right) = 1.5$, $\gamma_a = \gamma_b = \gamma_\theta = 1$. Also assume that a lower bound of b is $b_0 = 0.5$. Plot the time histories of $e\left(t\right)$, $x\left(t\right)$ vs. $x_m\left(t\right)$, $\hat{a}\left(t\right)$, $\hat{b}\left(t\right)$, and $\hat{\theta}\left(t\right)$ for $t \in [0, 50]$

Solution:
The adaptive laws are

$$\dot{\hat{a}} = -\gamma_a x e$$

$$\dot{\hat{b}} = \begin{cases} -\gamma_b \bar{u} e & \text{if } \left|\hat{b}\right| > b_0, \text{ or if } \left|\hat{b}\right| = b_0 \text{ and } \left|\dot{\hat{b}}\right| \geq 0 \\ \epsilon \operatorname{sgn}\left(b\right) & \text{otherwise, } \epsilon \approx 0 > 0 \end{cases}$$

$$\dot{\theta} = -\gamma_\theta \phi\left(x\right) e \operatorname{sgn}\left(b\right)$$

The adaptive controller is given by

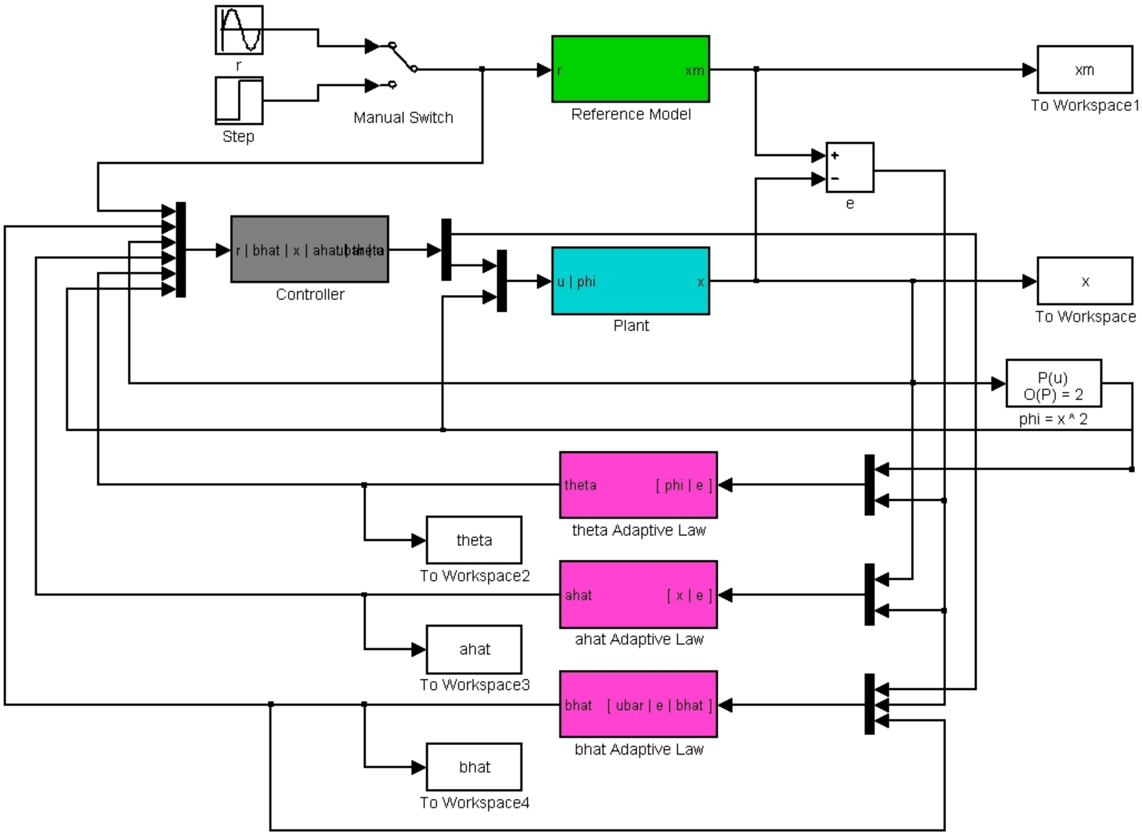
$$u = k_x\left(t\right)x + k_r\left(t\right)r - \theta\left(t\right)\phi\left(x\right)$$

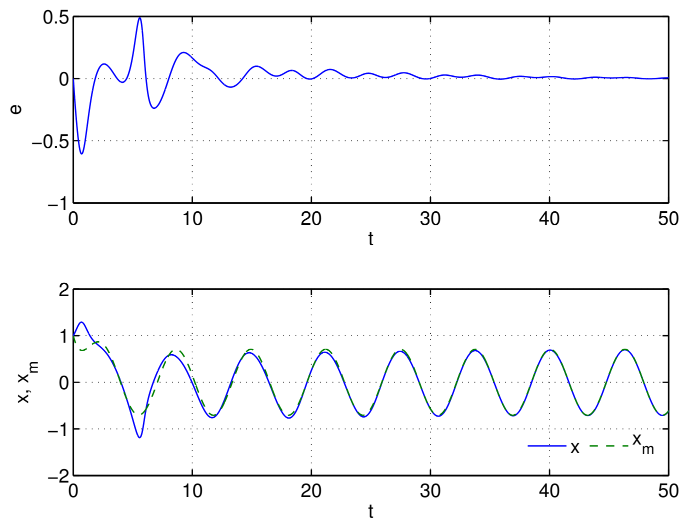
where

$$k_x = \frac{a_m - \hat{a}}{\hat{b}}$$

$$k_r = \frac{b_m}{\hat{b}}$$

The Simulink model and simulation results are as shown.


r
Step
Manual Switch
r
xm
Reference Model
xm
To Workspace1
e
r | bhat | x | ahat | theta
Controller
u | phi
x
Plant
x
To Workspace
P(u)
O(P) = 2
phi = x ^ 2
theta
[phi | e]
theta Adaptive Law
theta
To Workspace2
ahat
[x | e]
ahat Adaptive Law
ahat
To Workspace3
bhat
[ubar | e | bhat]
bhat Adaptive Law
bhat
To Workspace4


e
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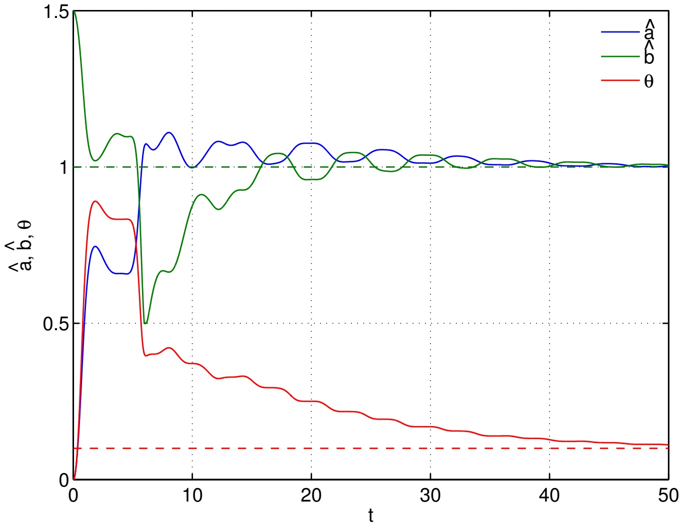


3. Derive direct MRAC laws for a second-order SISO system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = b\left[u + \Theta^{*\top}\Phi(y)\right]$$

where ζ and ω_n are unknown, but b is known. Show by applying the Barbalat's lemma that the tracking error is asymptotically stable.
Design a direct adaptive controller for a second-order system using the following information: $b = 1$, $\zeta_m = 0.5$, $\omega_m = 2$, $b_m = 4$, $r(t) = \sin 2t$, and

$$\Phi(y) = \begin{bmatrix} 1 \\ y^2 \end{bmatrix}$$

For simulation purposes, the unknown parameters may be assumed to be $\zeta = -0.5$, $\omega_n = 1$, and $\Theta^{*\top} = \begin{bmatrix} 0.5 & -0.1 \end{bmatrix}$, and all initial conditions are assumed to be zero. Use $\Gamma_x = \Gamma_\Theta = 100I$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $K_x(t)$, and $\Theta(t)$ for $t \in [0, 100]$.

Solution:
Assuming there exist ideal gains K_x^* and k_r^* that satisfy the model matching conditions

$$A + BK_x^* = A_m$$

$$Bk_r^* = B_m$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ b \end{bmatrix}, \; A_m = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m\omega_m \end{bmatrix}, \; B_m = \begin{bmatrix} 0 \\ b_m \end{bmatrix}$$

Since A_m and B_m have the same structures as A and B, K_x^* and k_r^* actually exist. In fact, k_r^* can be used as the command feedforward gain k_r where

$$k_r = k_r^* = \left(B^\top B\right)^{-1} B^\top B_m = \frac{b_m}{b}$$

Define an adaptive controller as

$$u = K_x(t)\,x + k_r r - \Theta^\top(t)\,\Phi(x)$$

Let $\tilde{K}_x(t) = K_x(t) - K_x^*$ and $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ be the estimation errors, then the closed-loop plant model is

$$\dot{x} = \left(\underbrace{A + BK_x^*}_{A_m} + B\tilde{K}_x \right) x + \underbrace{Bk_r}_{B_m} r - B\tilde{\Theta}^\top \Phi(x)$$

Then, the closed-loop tracking error equation is obtained as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\tilde{K}_x x + B\tilde{\Theta}^\top \Phi(x)$$

Choose a Lyapunov candidate function

$$V\left(e, \tilde{K}_x, \tilde{\Theta}\right) = e^\top P e + \tilde{K}_x \Gamma_x^{-1} \tilde{K}_x^\top + \tilde{\Theta}^\top \Gamma_\Theta^{-1} \tilde{\Theta}$$

where $P = P^\top > 0$ solves the Lyapunov equation

$$PA + A^\top P = -Q$$

with $Q = Q^\top > 0$.
Then,

$$\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) = -e^\top Q e + 2e^\top P\left[-B\tilde{K}_x x + B\tilde{\Theta}^\top \Phi(x)\right] + 2\tilde{K}_x \Gamma_x^{-1} \dot{\tilde{K}}_x^\top + 2\tilde{\Theta}^\top \Gamma_\Theta^{-1} \dot{\tilde{\Theta}}$$

Since $e^\top PB$ is a scalar quantity, then

$$2e^\top PB\tilde{K}_x x = 2\tilde{K}_x x e^\top PB$$

$$2e^\top PB\tilde{\Theta}^\top \Phi(x) = 2\tilde{\Theta}^\top \Phi(x) e^\top PB$$

Thus,

$$\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) = -e^\top Q e + 2\tilde{K}_x\left(-x e^\top PB + \Gamma_x^{-1} \dot{\tilde{K}}_x^\top\right) + 2\tilde{\Theta}^\top \left[\Phi(x) e^\top PB + \Gamma_\Theta^{-1} \dot{\tilde{\Theta}}\right]$$

Setting the trace terms to zero yields the adaptive laws for $K_x(t)$ and $\Theta(t)$

$$\dot{K}_x^\top = \Gamma_x x e^\top PB$$

$$\dot{\Theta} = -\Gamma_\Theta \Phi(x) e^\top PB$$

Therefore,

$$\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) = -e^\top Q e \leq -\lambda_{min}(Q) \|e\|^2 \leq 0$$

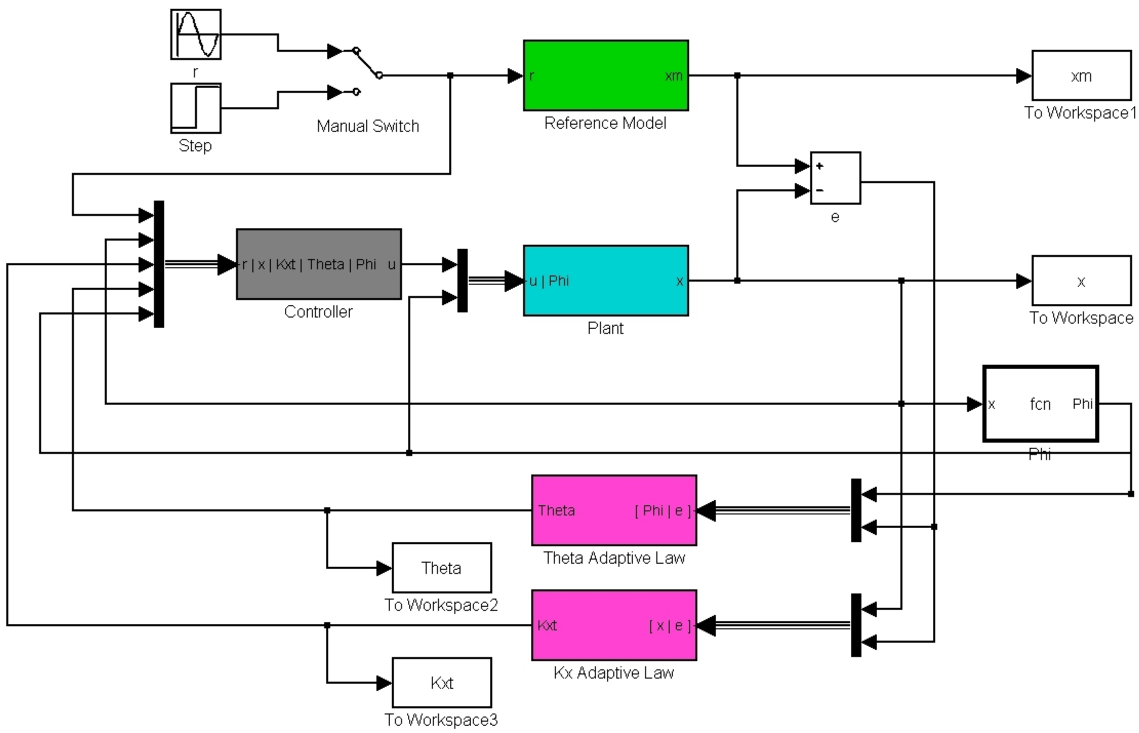
Since $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) \leq 0$, therefore $e(t)$, $K_x(t)$, and $\Theta(t)$ are bounded. Then,

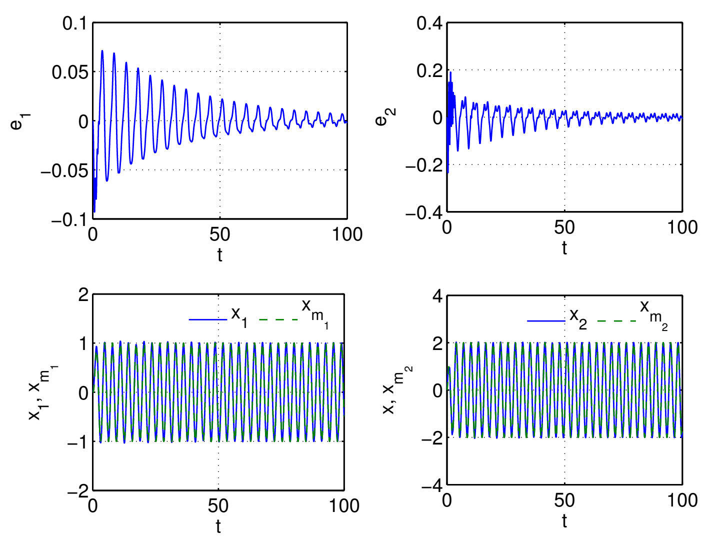
$$V(t \to \infty) = V(t_0) - \int_{t_0}^{\infty} e^\top Q e dt \leq V(t_0) - \lambda_{min}(Q) \|e\|^2$$

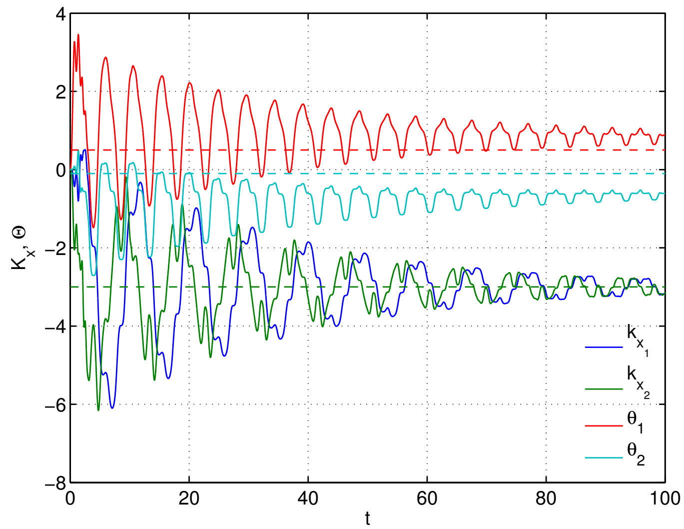
So, $V\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ has a finite limit as $t \to \infty$. Since $\|e\|$ exists, therefore $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, but $\|\dot{e}\| \in \mathcal{L}_\infty$. $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ can be shown to be uniformly continuous by examining its derivative to see if it is bounded, where

$$\ddot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) = -\dot{e}^\top Q e - e^\top Q \dot{e} = -e^\top \left(QA + A^\top Q\right) e - 2e^\top Q\left[A_m e - B\tilde{K}_x x + B\tilde{\Theta}^\top \Phi(x)\right]$$

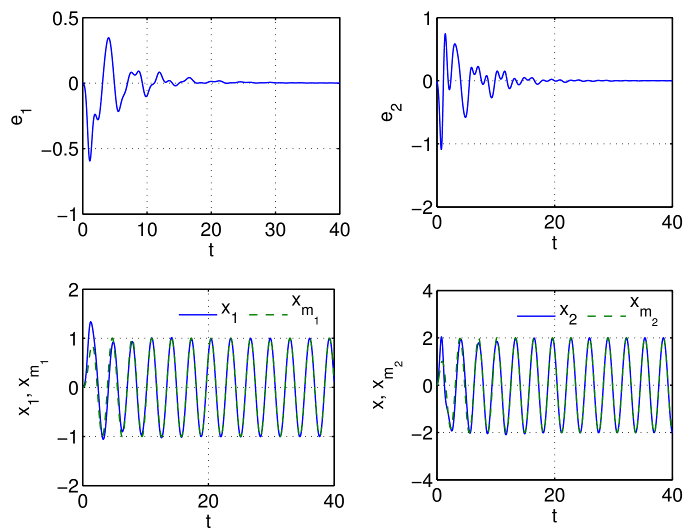
Since $e(t)$, $K_x(t)$, and $\Theta(t)$ are bounded by the virtue that $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) \leq 0$, $x(t)$ is bounded because $e(t)$ and $x_m(t)$ bounded, $r(t)$ is bounded by the problem statement, and $\Phi(x)$ is bounded because $x(t)$ is bounded, therefore $\ddot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ is bounded. Thus, $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) \to 0$ and hence $e\left(t\right) \to 0$ as $t \to \infty$. Therefore, the tracking error is asymptotically stable.The Simulink model and simulation results are as shown.

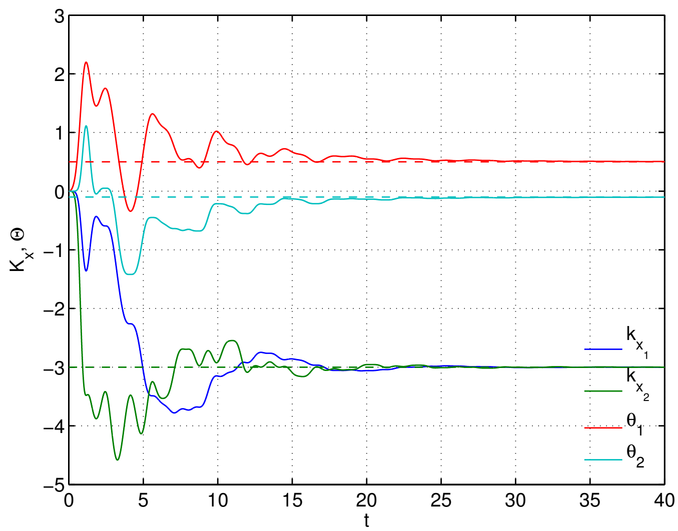






Notice that with $\mathit{\Gamma}_x = \mathit{\Gamma}_\Theta = 100I$, the signals are highly oscillatory and the parameter convergence for $\Theta(t)$ is not obtained. Re-running the simulation with $\mathit{\Gamma}_x = \mathit{\Gamma}_\Theta = 10I$, the simulation results are much more improved as shown in the following plots.





4. For Exercise 5.3, suppose b is unknown but $b > 0$ is known. Design an indirect adaptive controller in Simulink. For simulation purposes, all initial conditions are assumed to be zero, except for $\hat{\omega}_n(0) = 0.8$ and $\hat{b}(0) = 0.6$. For simplicity, use the unmodified adaptive laws for $\hat{\omega}_n(t)$ and $\hat{b}(t)$. Use $\gamma_\omega = \gamma_\zeta = \gamma_b = 10$ and $\Gamma_\Theta = 10I$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $\hat{\omega}_n(t)$, $\hat{\zeta}(t)$, $\hat{b}(t)$, and $\Theta(t)$ for $t \in [0, 100]$.

Solution:
The indirect adaptive laws are

$$\dot{\hat{b}} = -\gamma_b \bar{u} e^\top \bar{P}$$

$$\dot{\hat{\omega}}_n = \frac{\gamma_\omega x_1 e^\top \bar{P}}{2\hat{\omega}_n}$$

$$\dot{\hat{\zeta}} = \frac{\left(\gamma_\zeta \dot{x}_1 \hat{\omega}_n - \gamma_\omega x_1 \hat{\zeta}\right) e^\top \bar{P}}{2\hat{\omega}_n^2}$$

$$\dot{\Theta} = -\Gamma_\Theta \Phi(x) e^\top \bar{P} \operatorname{sgn} b$$

The adaptive controller is given by

$$u = K_x(t) x + k_r(t) r - \Theta^\top(t) \Phi(x)$$

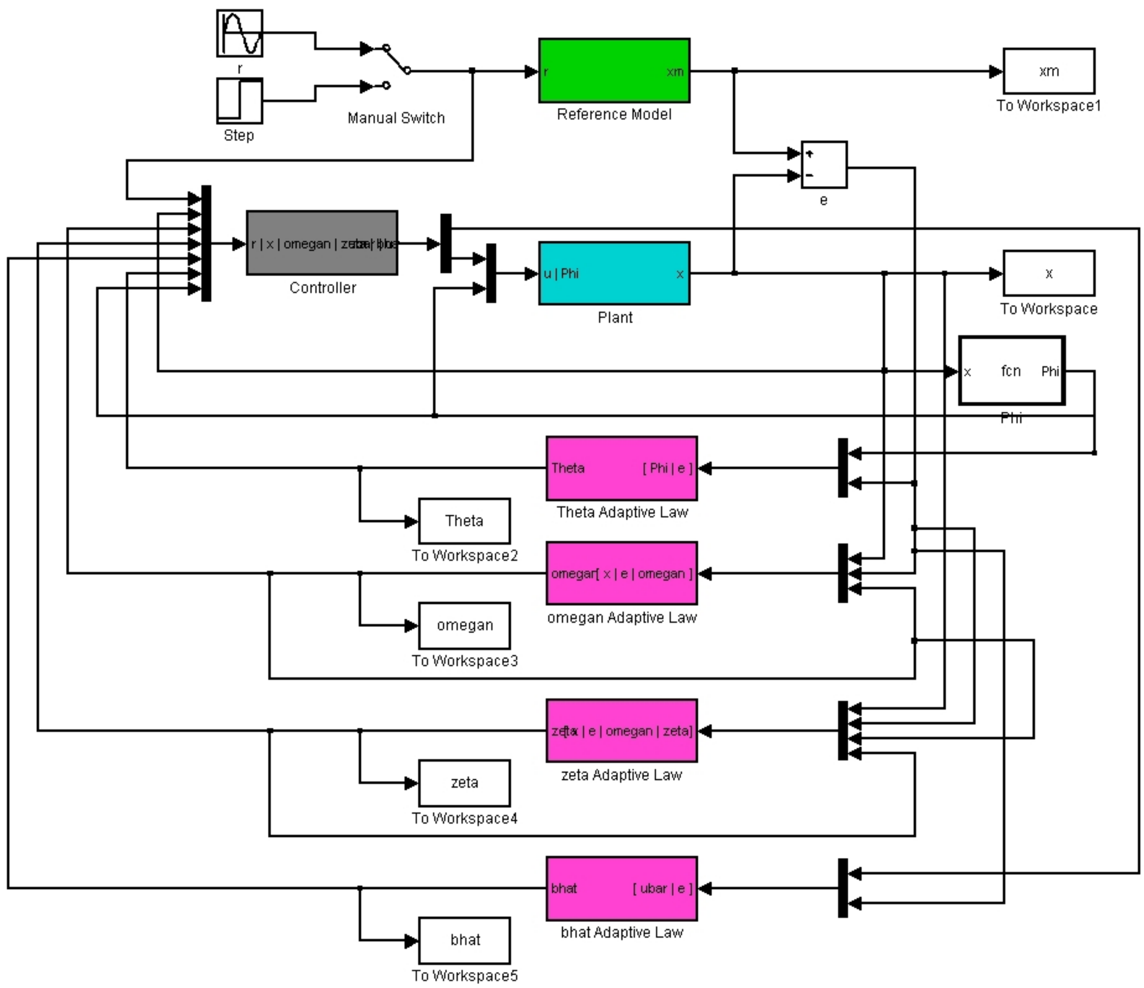
where

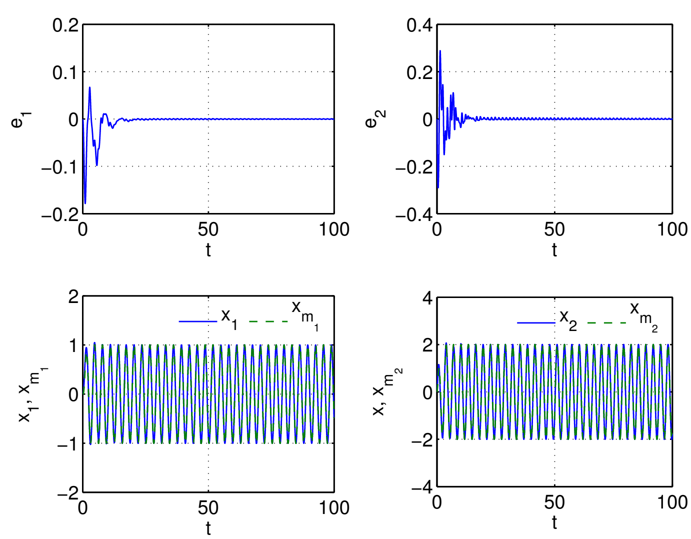
$$\hat{A} = \begin{bmatrix} 0 & 1 \\ -\hat{\omega}_n^2 & -2\hat{\zeta}\hat{\omega}_n \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ \hat{b} \end{bmatrix}, \ A_m = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m \omega_m \end{bmatrix}, \ B_m = \begin{bmatrix} 0 \\ b_m \end{bmatrix}$$

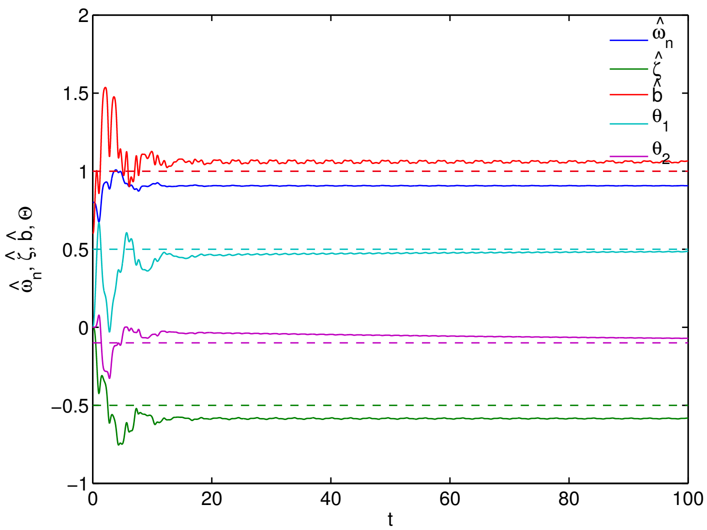
$$K_x = \left(\hat{B}^\top \hat{B}\right)^{-1} \hat{B}^\top \left(A_m - \hat{A}\right)$$

$$k_r = \left(\hat{B}^\top \hat{B}\right)^{-1} \hat{B}^\top B_m$$

The Simulink model and simulation results are as shown.


r
Step
Manual Switch
Reference Model
xm
To Workspace1
e
Controller
Plant
x
To Workspace
fcn
Phi
Theta Adaptive Law
Theta
To Workspace2
omegan Adaptive Law
omegan
To Workspace3
zeta Adaptive Law
zeta
To Workspace4
bhat Adaptive Law
bhat
To Workspace5


e1
e2
t
x1
xm1
x2
xm2



5. Thus far, we have considered adaptive control with a matched uncertainty as a function of x. In physical systems, an external disturbance is generally a function of t. Adaptive control can be used for disturbance rejection if the disturbance structure is known. Suppose the matched uncertainty is a function of t, then all the adaptive laws can still be used by just replacing $\Phi(x)$ by $\Phi(t)$, assuming $\Phi(t)$ is known and bounded. Consider the following first-order plant:

$$\dot{x} = ax + b\left[u + \theta^* \phi(t)\right]$$

where a , b, and θ^* are unknown, but $b > 0$ is known, and $\phi(t) = \sin 2t - \cos 4t$. Design an indirect adaptive controller in Simulink by estimating a, b, and θ^* so that the plant follows a reference model

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$. For simulation purposes, use $a = 1$, $b = 1$, $\theta^* = 0.1$, $x(0) = x_m(0) = 1$, $\hat{a}(0) = 0$, $\hat{b}(0) = 1.5$, $\gamma_a = \gamma_b = \gamma_\theta = 1$. Also assume that a lower bound of b is $b_0 = 0.5$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $\hat{a}(t)$, $\hat{b}(t)$, and $\hat{\theta}(t)$ for $t \in [0, 50]$.

Solution:
The indirect adaptive laws are

$$\dot{\hat{a}} = -\gamma_a x e$$

$$\dot{\hat{b}} = \begin{cases} -\gamma_b \bar{u} e & \text{if } \left|\hat{b}\right| > b_0, \text{ or if } \left|\hat{b}\right| = b_0 \text{ and } \frac{d\left|\hat{b}\right|}{dt} \geq 0 \\ \epsilon \operatorname{sgn}(b) & \text{otherwise, } \epsilon \approx 0 > 0 \end{cases}$$

$$\dot{\theta} = -\gamma_\theta \phi(t)\, e \operatorname{sgn}(b)$$

The adaptive controller is given by

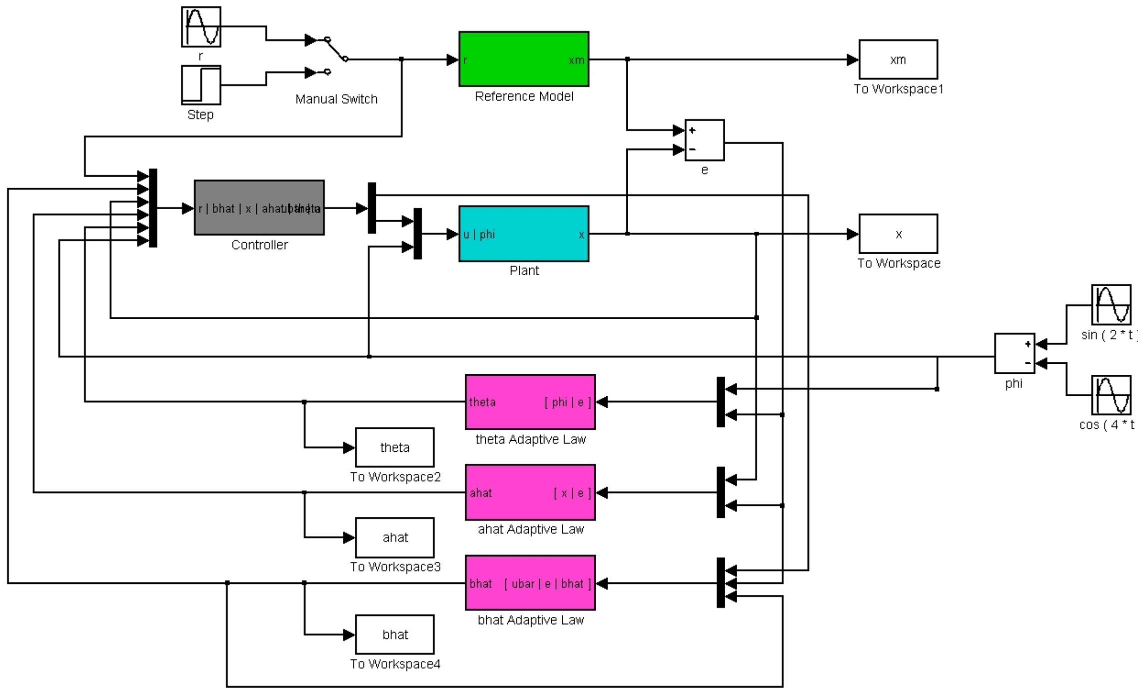
$$u = k_x(t)\, x + k_r(t)\, r - \theta(t)\, \phi(t)$$

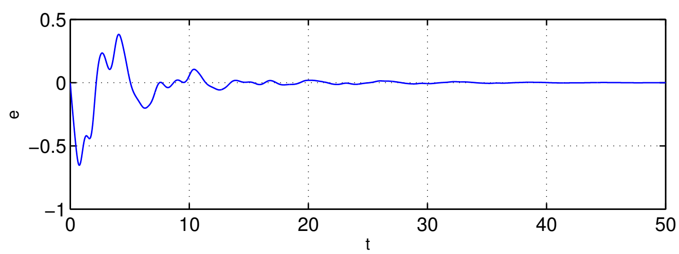
where

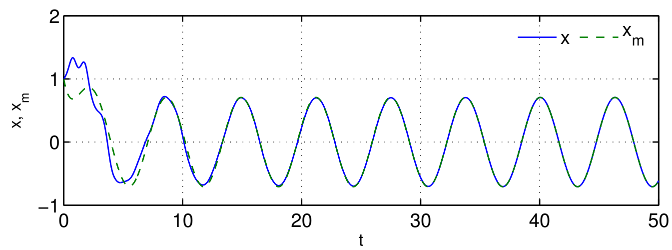
$$k_x = \frac{a_m - \hat{a}}{\hat{b}}$$

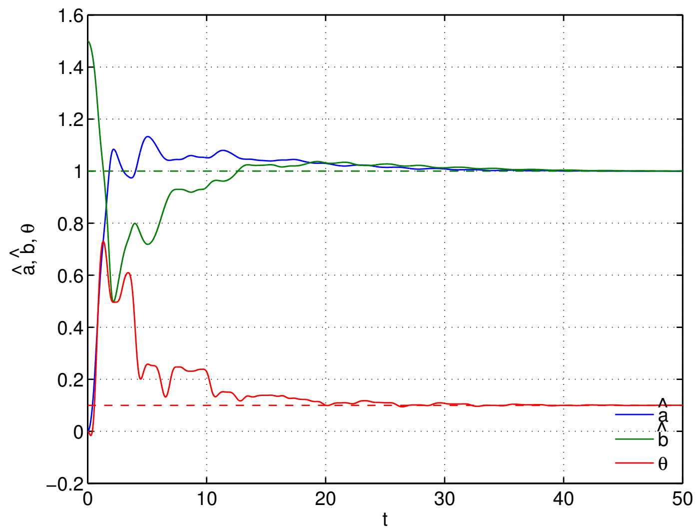
$$k_r = \frac{b_m}{\hat{b}}$$

The Simulink model and simulation results are as shown.









6. Derive direct MRAC laws for a MIMO system

$$\dot{x} = Ax + B\left[u + \Theta^{*\top}\Phi(x)\right]$$

where A is unknown, but B is known. Show by applying the Barbalat's lemma that the tracking error is asymptotically stable.
Given $x(t) = \left[x_1(t)\ x_2(t)\right]^\top$, $u(t) = \left[u_1(t)\ u_2(t)\right]^\top$, $\Phi(x) = \left[x_1^2\ x_2^2\right]^\top$, and

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

design a direct adaptive controller in Simulink for the MIMO system to follow a second-order SISO system specified by

$$\dot{x}_m = A_m x + B_m r$$

where $r(t) = \sin 2t$ and

$$A_m = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}, \ B_m = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

For simulation purposes, the unknown parameters may be assumed to be

$$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \ \Theta^* = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.1 \end{bmatrix}$$

and all initial conditions are assumed to be zero. Use $\Gamma_x = \Gamma_\Theta = 10I$. Plot the time histories of $e(t)$, $x(t)$ vs. $x_m(t)$, $K_x(t)$, and $\Theta(t)$ for $t \in [0, 100]$.

Solution:
Assuming that there exist K_x^* and K_r that satisfy the model matching conditions

$$A + BK_x^* = A_m$$

$$BK_r = B_m$$

The adaptive controller is given by

$$u = K_x(t)\, x + K_r r - \Theta^\top(t)\, \Phi(x)$$

Let $\tilde{K}_x(t) = K_x(t) - K_x^*$ and $\tilde{\Theta}(t) = \Theta(t) - \Theta^*$ be the estimation errors, then the closed-loop plant model is

$$\dot{x} = \left(\underbrace{A + BK_x^*}_{A_m} + \tilde{K}_x \right) x + \underbrace{BK_r}_{B_m} r - B\tilde{\Theta}^\top \Phi(x)$$

The closed-loop tracking error equation is obtained as

$$\dot{e} = \dot{x}_m - \dot{x} = A_m e - B\tilde{K}_x x + B\tilde{\Theta}^\top \Phi(x)$$

Choose a Lyapunov candidate function

$$V\left(e, \tilde{K}_x, \tilde{\Theta}\right) = e^\top P e + \text{trace}\left(\tilde{K}_x \Gamma_x^{-1} \tilde{K}_x^\top\right) + \text{trace}\left(\tilde{\Theta}^\top \Gamma_\Theta^{-1} \tilde{\Theta}\right)$$

Then,

$$\begin{aligned}\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) &= -e^\top Q e + 2e^\top PB\left[-\tilde{K}_x x + \tilde{\Theta}^\top \Phi(x)\right] + 2\text{trace}\left(\tilde{K}_x \Gamma_x^{-1} \dot{\tilde{K}}_x^\top\right) + 2\text{trace}\left(\tilde{\Theta}^\top \Gamma \dot{\tilde{\Theta}}\right) \\ &= -e^\top Q e + 2\text{trace}\left(\tilde{K}_x\left[-xe^\top PB + \Gamma_x^{-1}\dot{\tilde{K}}_x^\top\right]\right) + 2\text{trace}\left(\tilde{\Theta}^\top\left[\Phi(x)\, e^\top PB + \Gamma\dot{\tilde{\Theta}}\right]\right)\end{aligned}$$

The adaptive laws are

$$\begin{aligned}\dot{K}_x^\top &= \Gamma_x x e^\top PB \\ \dot{\Theta} &= -\Gamma_\Theta \Phi(x)\, e^\top PB\end{aligned}$$

Thus,

$$\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) = -e^\top Q e \le -\lambda_{min}(Q)\,\|e\|^2 \le 0$$

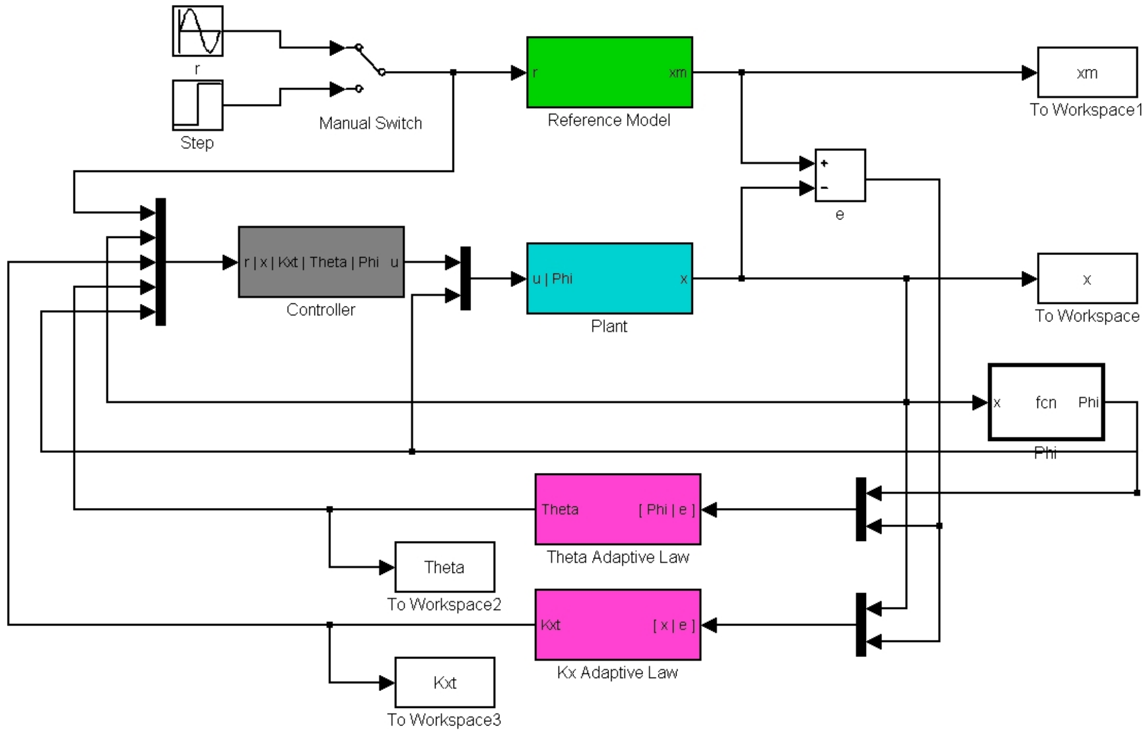
Since $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) \le 0$, therefore $e(t)$, $K_x(t)$, and $\Theta(t)$ are bounded. Then,

$$V(t \to \infty) = V(t_0) - \int_{t_0}^{\infty} e^\top Q e\, dt \le V(t_0) - \lambda_{min}(Q)\,\|e\|^2$$

So, $V\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ has a finite limit as $t \to \infty$. Since $\|e\|$ exists, therefore $e(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, but $\|\dot{e}\| \in \mathcal{L}_\infty$. $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ can be shown to be uniformly continuous by examining its derivative to see if it is bounded, where

$$\ddot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) = -\dot{e}^\top Q e - e^\top Q \dot{e} = -e^\top\left(QA + A^\top Q\right) e - 2e^\top Q\left[A_m e - B\tilde{K}_x x + B\tilde{\Theta}^\top \Phi(x)\right]$$

Since $e(t)$, $K_x(t)$, and $\Theta(t)$ are bounded by the virtue that $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) \le 0$, $x(t)$ is bounded because $e(t)$ and $x_m(t)$ bounded, $r(t)$ is bounded by the problem statement, and $\Phi(x)$ is bounded because $x(t)$ is bounded, therefore $\ddot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ is bounded. Thus, $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}\left(e, \tilde{K}_x, \tilde{\Theta}\right) \to 0$ and hence $e(t) \to 0$ as $t \to \infty$. Therefore, the tracking error is asymptotically stable.
The Simulink model and simulation results are as shown.


r
Step
Manual Switch
r
xm
Reference Model
xm
To Workspace1
+
-
e
r | x | Kxt | Theta | Phi
u
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u | Phi
x
Plant
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To Workspace
x
fcn
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Theta Adaptive Law
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To Workspace2
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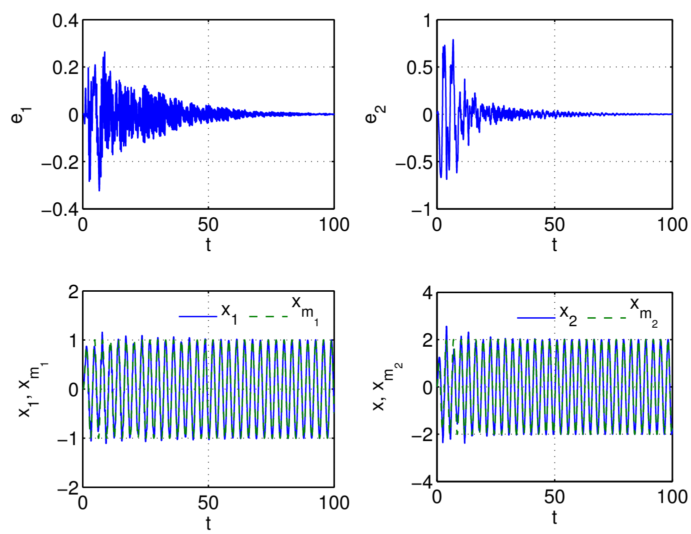
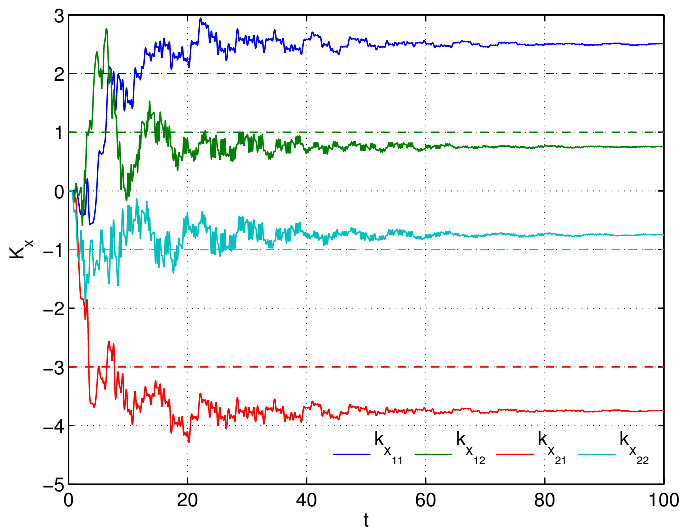
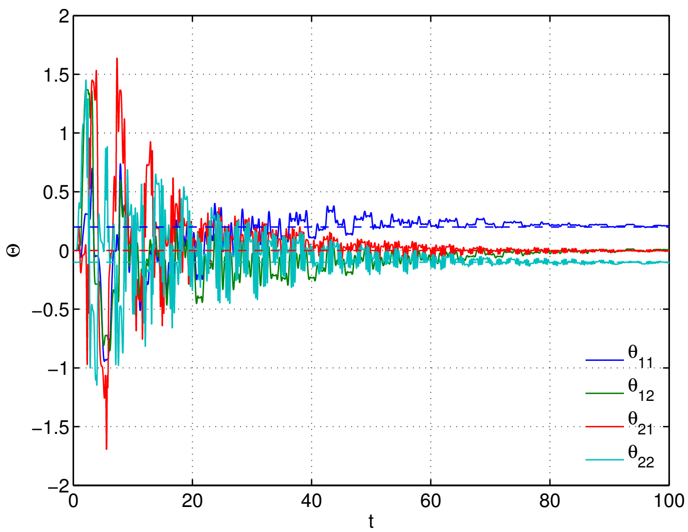
Kx Adaptive Law
Kxt
To Workspace3


e1
e2
t
x1, xm1
x, xm2
x1
xm1
x2
xm2



K_x
t
$k_{x_{11}}$
$k_{x_{12}}$
$k_{x_{21}}$
$k_{x_{22}}$



Θ
t
θ_{11}
θ_{12}
θ_{21}
θ_{22}

Chapter 6 Exercises

1. A process is represented by a set of data (t, x, y) given in the Matlab file "Process_Data.mat" where the output $y(t)$ can be approximated by a $4-th$ degree polynomial in terms of $x(t)$ with end point conditions $y = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$. Determine numerically the matrix A and vector B and solve for the coefficients θ_i, $i = 2, 3, 4$. Compare the result with the Matlab function "polyfit".

   Solution:
   The output $y(x)$ is approximated as follows:

   $$\hat{y} = \theta_0 + \theta_1 x + \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4$$

   Applying the end point conditions $y(0) = 0$ and $y^{'}(0) = 0$ results in $\theta_0 = 0$ and $\theta_1 = 0$. Therefore,

   $$\hat{y} = \theta_2 x^2 + \theta_3 x^3 + \theta_4 x^4 = \Theta^\top \Phi(x)$$

   where $\Theta = \begin{bmatrix} \theta_2 \; \theta_3 \; \theta_4 \end{bmatrix}^\top$, $\Phi(x) = \begin{bmatrix} x^2 \; x^3 \; x^4 \end{bmatrix}^\top$.
   Θ is solved by the batch least-squares method as

   $$\Theta = A^{-1} B$$

   where

   $$A = \sum_{i=1}^{N} \Phi(x_i) \Phi^\top(x_i) = \begin{bmatrix} \sum_{i=1}^N x_i^4 & \sum_{i=1}^N x_i^5 & \sum_{i=1}^N x_i^6 \\ \sum_{i=1}^N x_i^5 & \sum_{i=1}^N x_i^6 & \sum_{i=1}^N x_i^7 \\ \sum_{i=1}^N x_i^6 & \sum_{i=1}^N x_i^7 & \sum_{i=1}^N x_i^8 \end{bmatrix}$$

   $$B = \sum_{i=1}^{N} \Phi(x_i) y_i^\top = \begin{bmatrix} \sum_{i=1}^N x_i^2 y_i \\ \sum_{i=1}^N x_i^3 y_i \\ \sum_{i=1}^N x_i^4 y_i \end{bmatrix}$$

   The solution is $\theta_2 = -0.5$, $\theta_3 = 0.1$, and $\theta_4 = 0.3$. Matlab function "polyfit(x,y,4)" yields the same answer.

2. Write Matlab code to solve Exercise 6.1 using the least-squares gradient method with $\Theta(0) = 0$ and $\Gamma = 10$. Plot $\theta_i(t)$ versus t. Compare the result with that in Exercise 6.1.
   Note that the Euler method for the least-squares gradient method is expressed as

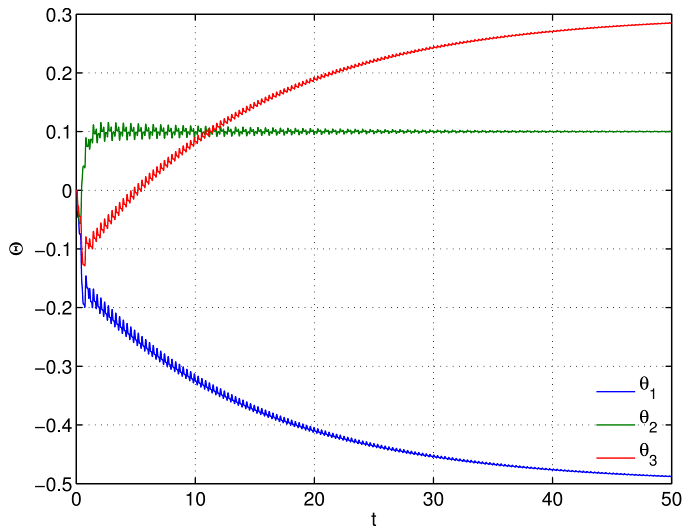
   $$\Theta_{i+1} = \Theta_i - \Delta t \Gamma \Phi(x_i) \left[\Phi^\top(x_i) \Theta_i - y_i \right]$$

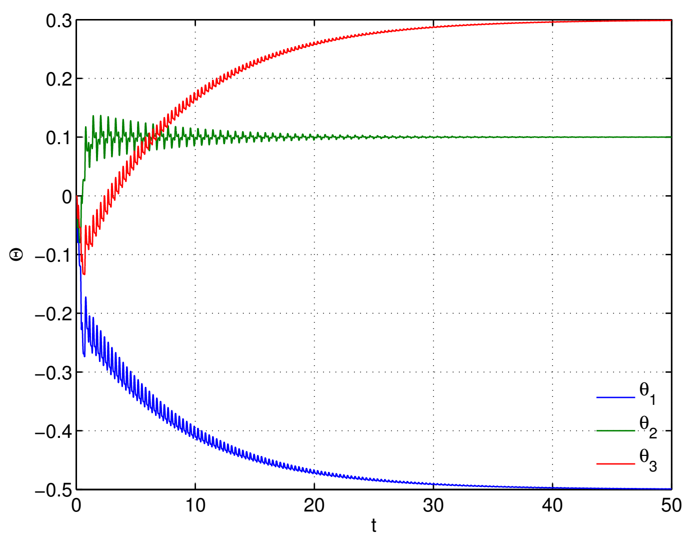
   Solution:
   The least-squares estimation by the least-squares gradient method is expressed as

$$\Phi(x) = \begin{bmatrix} x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

$$\dot{\Theta} = -\Gamma \Phi(x) \left[\Phi^{\top}(x) \Theta - y \right]$$

For $\Gamma = 10I$, $\Theta(t)$ does not quite converge to the correct values. Changing $\Gamma = 30I$ causes $\Theta(t)$ to converge to the same values as those in Exercise 6.1.





3. Determine if the following functions are persistently exciting (PE), and if so, determine T and α.

   a. $\phi(t) = e^{-t}$. (Hint: find limit of $\tilde{\theta}(t)$ as $t \to \infty$)

   b. $\Phi(t) = \begin{bmatrix} \cos \pi t \\ \sin \pi t \end{bmatrix}$

Solution:

a. $\phi(t) = e^{-t}$. Consider a scalar estimation error equation

$$\dot{\tilde{\theta}} = -\gamma\phi^2(t)\,\tilde{\theta}$$

Then,

$$\tilde{\theta}(t) = \exp\left[-\gamma\int_0^t e^{-2\tau}d\tau\right] = \exp\left[\frac{\gamma}{2}\left(e^{-2t}-1\right)\right]$$

As $t \to \infty$, $\tilde{\theta}(t) \to e^{-\frac{\gamma}{2}} \neq 0$. Thus, $\phi(t)$ is not PE and does not guarantee a parameter convergence.

b. $\Phi(t) = \begin{bmatrix}\cos\pi t\\ \sin\pi t\end{bmatrix}$. Then,

$$\Phi(t)\,\Phi^\top(t) = \begin{bmatrix}\cos^2\pi t & \sin\pi t\cos\pi t\\ \sin\pi t\cos\pi t & \sin^2\pi t\end{bmatrix}$$

The PE condition is evaluated as

$$\frac{1}{T}\int_t^{t+T}\Phi(\tau)\,\Phi^\top(\tau)\,d\tau = \frac{1}{T}\begin{bmatrix}\frac{T}{2}+\frac{\sin 2\pi(t+T)-\sin 2\pi t}{4\pi} & -\frac{\cos 2\pi(t+T)-\cos 2\pi t}{4\pi}\\ -\frac{\cos 2\pi(t+T)-\cos 2\pi t}{4\pi} & \frac{T}{2}-\frac{\sin 2\pi(t+T)-\sin 2\pi t}{4\pi}\end{bmatrix}$$

Let $T = 1$. Then, $\Phi(x)$ is PE since

$$\frac{1}{T}\int_t^{t+T}\Phi(\tau)\,\Phi^\top(\tau)\,d\tau = \frac{1}{T}\begin{bmatrix}\frac{T}{2} & 0\\ 0 & \frac{T}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Thus, $\alpha = \frac{1}{2}$. The estimation error is exponentially stable and parameter convergence is guaranteed.

4. Consider a first-order system with a matched uncertainty

$$\dot{x} = ax + b\left[u + \theta^*\phi(t)\right]$$

where a and θ^* are unknown, but $b = 2$, and $\phi(t) = \sin t$. For simulation purposes, use $a = 1$ and $\theta^* = 0.2$. The reference model is given by

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$.
Implement in Simulink an indirect adaptive control using the recursive least-squares method with normalization. All initial conditions are zero. Use $R(0) = 10$. Plot $e(t)$, $x(t)$ versus $x_m(t)$, $\hat{a}(t)$, and $\theta(t)$, for $t \in [0, 40]$.

Solution:
Let $\Omega(t) = \begin{bmatrix}\hat{a}(t) & b\theta(t)\end{bmatrix}^\top$ and $\Psi(x,t) = \begin{bmatrix}x & \sin t\end{bmatrix}^\top$. Then, the RLS adaptive laws with normalization are given by

$$\dot{\Omega} = -R\Psi(x,t)\,\epsilon$$

$$\dot{R} = -\frac{R\Psi(x,t)\,\Psi^\top(x,t)\,R}{1+\Psi^\top(x,t)\,R\Psi(x,t)}$$

where

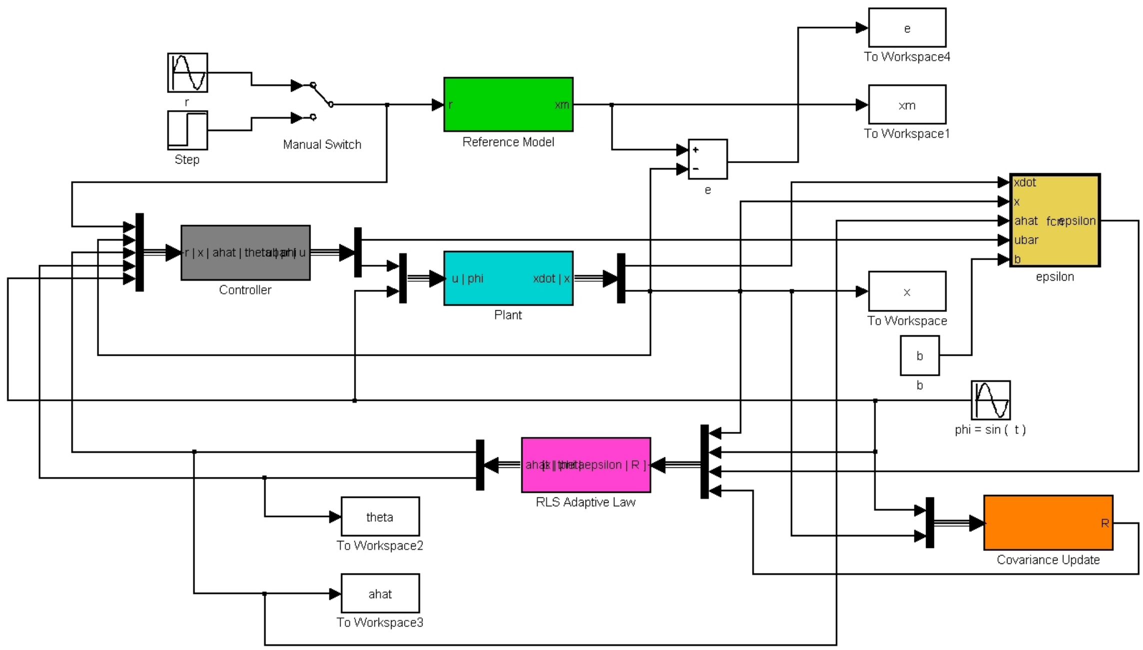
$$\epsilon = a_m x + b_m r - \dot{x}$$

$$\bar{u} = \frac{a_m - \hat{a}}{b}x + \frac{b_m}{b}r$$

The adaptive controller is given by

$$u = \bar{u} - \theta(t)\,\phi(t)$$

The Simulink model and simulation results are as shown.


e
To Workspace4
xm
To Workspace1
r
Step
Manual Switch
r
xm
Reference Model
e
xdot
x
ahat
ubar
b
fcn
epsilon
epsilon
Controller
u | phi
xdot | x
Plant
x
To Workspace
b
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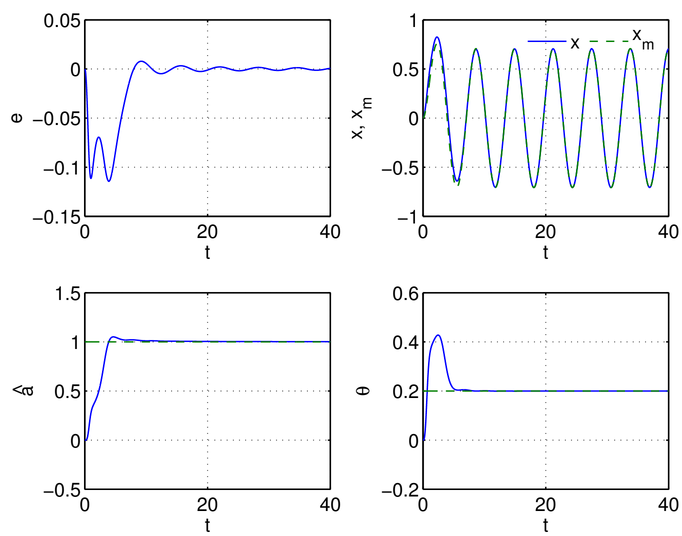
phi = sin (t)
RLS Adaptive Law
theta
To Workspace2
ahat
To Workspace3
R
Covariance Update


e
t
x, x_m
x
x_m
t
$\hat{a}$
t
θ
t

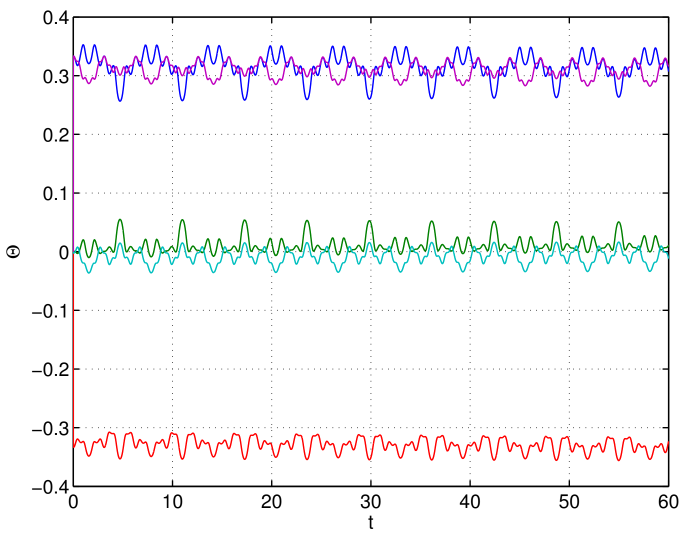
Chapter 7 Exercises

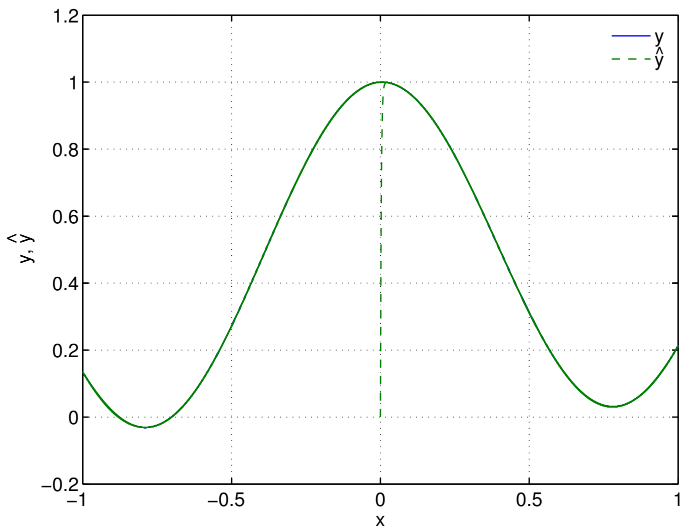
1. Approximate

$$y = 0.1 \sin 0.4x + \cos^2 2x$$

where $x(t) = \sin t$ for $t \in [0, 60]$, by a 4^{th}-degree Chebyshev polynomial using the least-squares gradient method with $\Gamma = 100I$ and $\Delta t = 0.001$. Initialize $\Theta(t)$ with zero. Plot $\Theta(t)$ versus t. Plot $y(t)$ and $\hat{y}(t)$ versus $x(t)$ on the same plot. Compute the root mean square error between $y(t)$ and $\hat{y}(t)$.

Solution:
$\Theta(t)$ and $\hat{y}$ are shown in the following plots. The root mean square error is 0.0058.





2. Implement a sigmoidal neural network

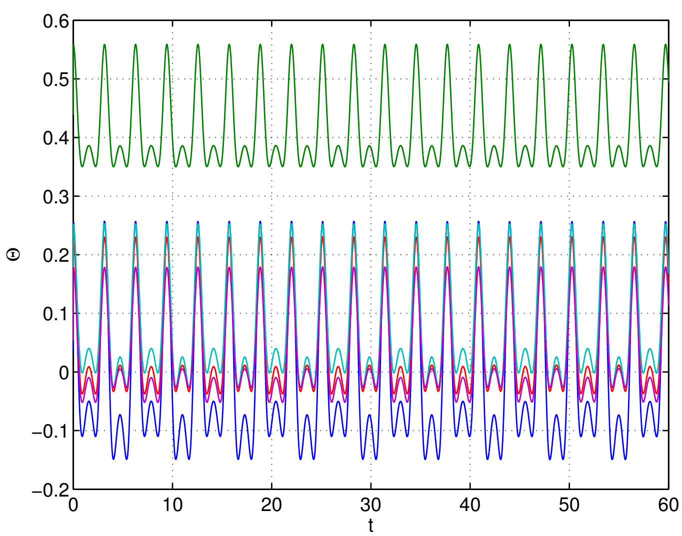
$$\hat{y} = \hat{f}(x) = V^\top \sigma\left(W_x^\top x + W_0\right) + V_0 = \Theta^\top \Phi\left(W^\top \bar{x}\right)$$

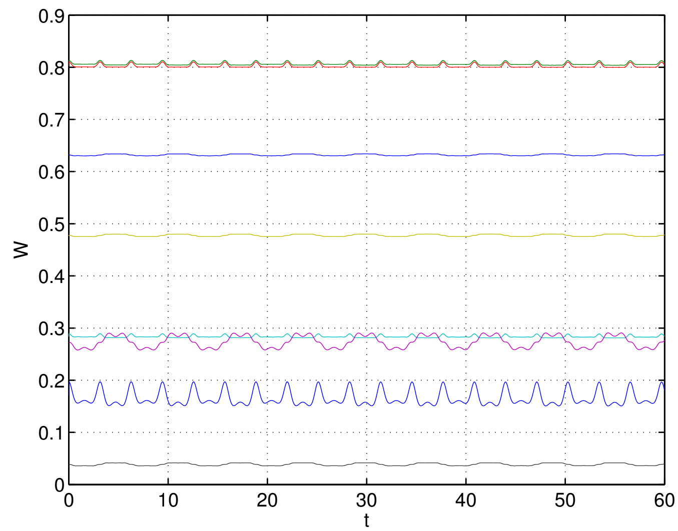
where

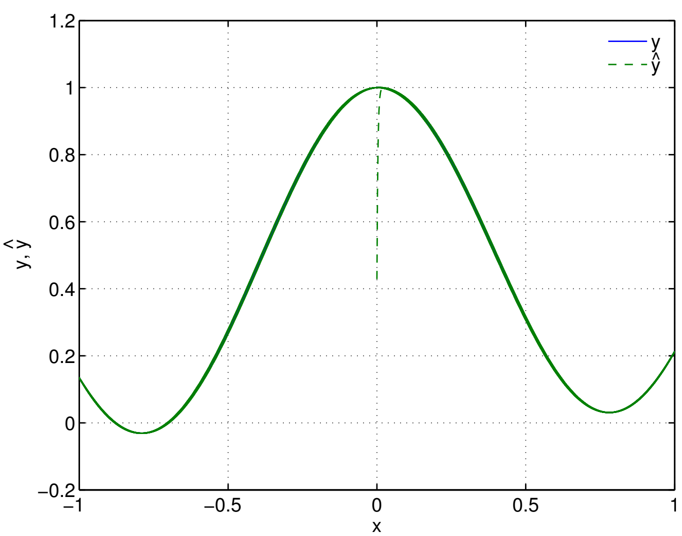
$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

to approximate $y(t)$ in Exercise 7.1 with $\Theta(t) \in \mathbb{R}^5$, $W(t) \in \mathbb{R}^2 \times \mathbb{R}^4$ and $\Gamma_\Theta = \Gamma_W = 100I$ and $\Delta t = 0.001$. The initial conditions $\Theta(0)$ and $W(0)$ are to be generated by a random number generator. Plot $\Theta(t)$ and $W(t)$ versus t. Plot $y(t)$ and $\hat{y}(t)$ versus $x(t)$ on the same plot. Compute the root mean square error between $y(t)$ and $\hat{y}(t)$.

Solution:
$\Theta(t)$ and $\hat{y}$ are shown in the following plots. The root mean square error is 0.0051.







3. Consider a first-order system with a matched unstructured uncertainty

$$\dot{x} = ax + b\left[u + f\left(x\right)\right]$$

where a and $f\left(x\right)$ are unknown, but $b = 2$. For simulation purposes, $a = 1$ and $f\left(x\right) = 0.1 \sin 0.4x + \cos^2 2x$. The reference model is given by

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r\left(t\right) = \sin t$.
Implement in Simulink a direct adaptive control using a least-squares gradient method to approximate $f\left(x\right)$ by a 4^{th}-degree Chebyshev polynomial. All initial conditions are zero. Use $\Gamma = 0.2I$. Plot $e\left(t\right)$, $x\left(t\right)$ versus $x_m\left(t\right)$, $k_x\left(t\right)$, and $\Theta\left(t\right)$ for $t \in [0, 60]$.

Solution:
Let $\Omega^\top\left(t\right) = \left[\Omega_1\left(t\right)\ \Omega_2^\top\left(t\right)\right]^\top = \left[bk_x\left(t\right)\ b\Theta^\top\left(t\right)\right]$ and $\Psi\left(x,t\right) = \left[-x\ \Phi^\top\left(x\right)\right]^\top$ where $\Phi\left(x\right) = \left[1\ x\ 2x^2-1\ 4x^3-3x\ 8x^4-8x^2+1\right]^\top$. Then, the least-squares gradient adaptive laws are

$$\dot{\Omega} = -\Gamma \Psi\left(x, t\right) \epsilon$$

$$k_x = \frac{\Omega_1}{b}$$

$$\Theta = \frac{\Omega_2}{b}$$

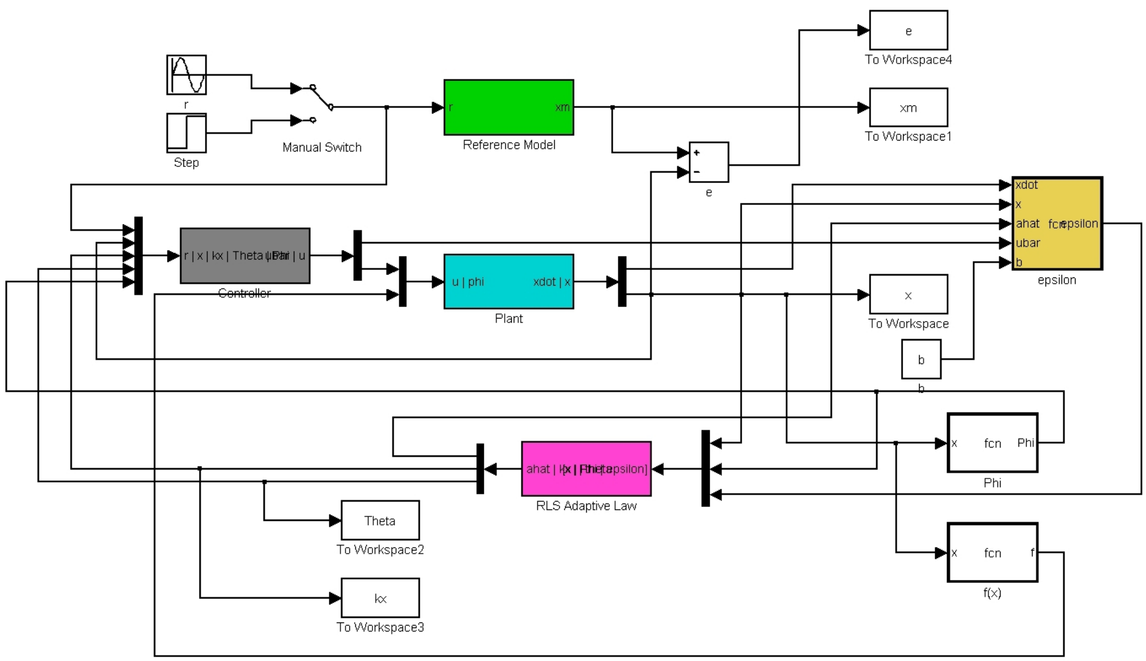
where

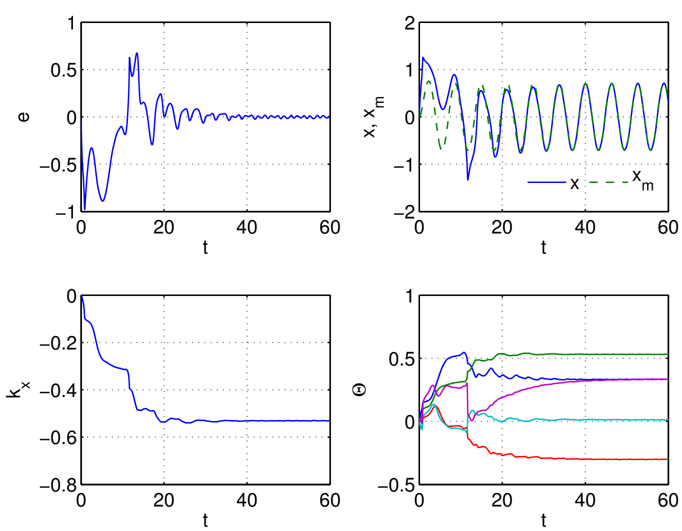
$$\epsilon = a_m x + b_m r - \dot{x}$$

The adaptive controller is then given by

$$u = k_x\left(t\right) x + k_r r - \Theta^\top\left(t\right) \Phi\left(x\right)$$

The Simulink model and simulation results are as shown.





Chapter 8 Exercises

1. Consider a first-order SISO system

$$\dot{x} = ax + bu + w$$

where $w(t)$ is a bounded disturbance and $u(t)$ is an adaptive controller defined as

$$u = k_x x$$

$$\dot{k}_x = -\gamma_x x^2 b$$

Suppose the solution of $x(t)$ is given by

$$x = t(1+t)^p$$

a. Analyze parameter drift behaviors of the closed-loop system by finding all values of p that result in unbounded feedback gain $k_x(t)$ and all values of p that result in a completely bounded system.
b. Implement the adaptive controller in Simulink using the following information: $a = 1$, $b = 1$, $\gamma_x = 1$, $x(0) = 0$, and $k_x(0) = 0$ with a time step $\Delta t = 0.001$ sec for two different values of p: one for unbounded $k_x(t)$ and the other for all bounded signals. Plot the time histories of $x(t)$, $u(t)$, $w(t)$, and $k_x(t)$ for each of the values of p for $t \in [0, 20]$ sec.

Solution:

a. $x(t)$ is bounded if $p \le -1$. $k_x(t)$ is evaluated as

$$k_x - k_x(0) = -\gamma_x b \int_0^t \tau^2 (1+\tau)^{2p} d\tau$$

Let $u = 1 + \tau$, then

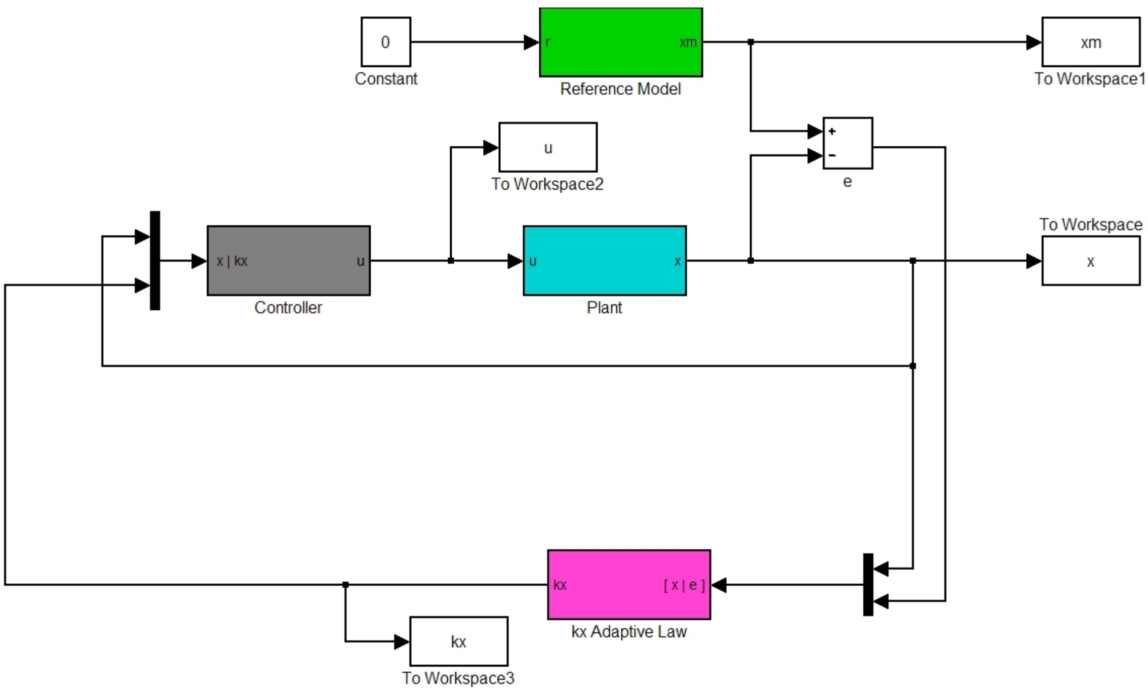
$$k_x - k_x(0) = -\gamma_x b \int_0^t (u-1)^2 u^{2p} du = -\gamma_x b \left[\frac{(1+t)^{2p+3} - 1}{2p+3} - \frac{2(1+t)^{2p+2} - 2}{2p+2} + \frac{(1+t)^{2p+1} - 1}{2p+1} \right]$$

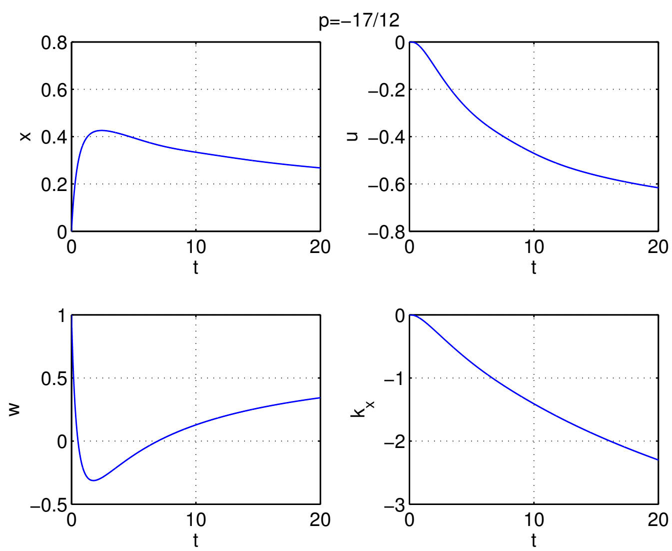
Thus, $k_x(t)$ is bounded if $2p + 3 < 0$ or $p < -\frac{3}{2}$.
$w(t)$ is obtained as

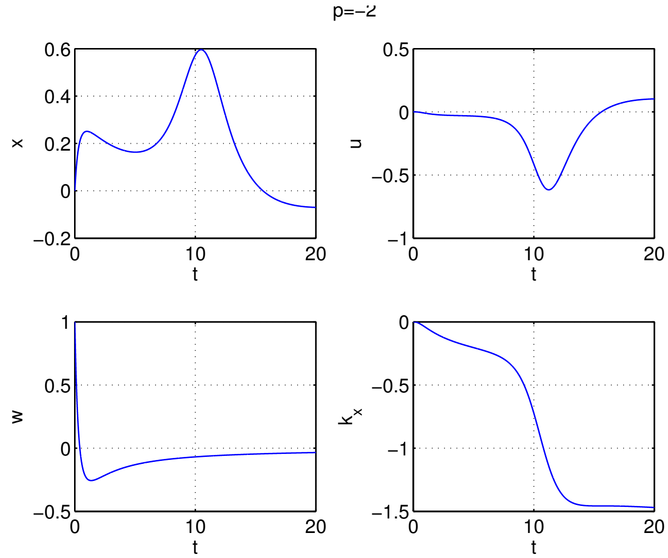
$$\begin{aligned} w = \dot{x} - ax - bk_x x &= pt(1+t)^{p-1} + (1+t)^p - at(1+t)^p \\ &- b\left\{ -\gamma_x b \left[\frac{(1+t)^{2p+3} - 1}{2p+3} - \frac{2(1+t)^{2p+2} - 2}{2p+2} + \frac{(1+t)^{2p+1} - 1}{2p+1} \right] + k_x(0) \right\} t(1+t)^p \end{aligned}$$

$w(t)$ is bounded if $3p + 3 \le -1$ or $p \le -\frac{4}{3}$ and $p \ne -\frac{3}{2}$, $p \ne -1$, $p \ne -\frac{1}{2}$.
Thus, $x(t)$ and $w(t)$ are bounded but $k_x(t)$ is unbounded if $-\frac{3}{2} < p \le -\frac{4}{3}$. The system is completely bounded if $p < -\frac{3}{2}$.

b. The Simulink model and the simulation results are as shown.
Choose $p = -\frac{17}{12}$ for unbounded $k_x(t)$. Note that $k_x(t)$ is unbounded, but $x(t)$, $u(t)$, and $w(t)$ are bounded. Choose $p = -2$ for bounded closed-loop signals. The closed-loop system is completely bounded.







2. Consider a time delay second-order SISO system

$$\ddot{y} - \dot{y} + y = u\,(t - t_d)$$

where t_d is an unknown time delay.
The unstable open-loop plant is stabilized with a linear derivative controller

$$u = k_d^* \dot{y}$$

where $k_d^* = -7$.

a. Calculate analytically the cross-over frequency ω and the time delay margin t_d that corresponds to neutral stability of the closed-loop system.
b. Now, suppose an adaptive controller is designed to follow the delay-free closed-loop system with the linear derivative controller as the reference model

$$\ddot{y}_m + 6\dot{y}_m + y_m = 0$$

Let $x\,(t) = \begin{bmatrix} y\,(t) & \dot{y}\,(t) \end{bmatrix}^\top \in \mathbb{R}^2$, then the open-loop plant is designed with an adaptive derivative controller

$$u = K_x x$$
$$\dot{K}_x^\top = -\varGamma_x x x^\top P B$$

where $K_x\,(t) = \begin{bmatrix} 0 & k_d\,(t) \end{bmatrix}$ and $\varGamma_x = \mathrm{diag}(0, \gamma_x)$ and γ_x is an adaptation rate.
Implement the adaptive controller in Simulink using the following information: $Q = I$, $y\,(0) = 1$, $\dot{y}\,(0) = 0$, and $K_x\,(0) = 0$ with a time step $\varDelta t = 0.001$ sec. Determine $\gamma_{x_{max}}$ that causes the closed-loop system to be on the verge of instability by trial-and-error to within 0.1 accuracy. Calculate $k_{d_{min}}$ that corresponds to $\gamma_{x_{max}}$. Plot the time histories of $x\,(t)$, $u\,(t)$, and $k_d\,(t)$ for $t \in [0, 10]$ sec.

Solution:

a. The closed-loop plant is

$$y = \frac{(s-1)\,y\,(0) + \dot{y}\,(0)}{s^2 - s + 1 + 7se^{-t_d s}}$$

Substituting $s = j\omega$ into the characteristic equation in the denominator yields

$$-\omega^2 - j\omega + 1 + 7j\omega\left(\cos\omega t_d - j\sin\omega t_d\right) = 0$$

Then, separating the real and imaginary parts yields

$$-\omega^2 + 1 + 7\omega\sin\omega t_d = 0$$

$$-\omega + 7\omega\cos\omega t_d = 0 \Rightarrow \cos\omega t_d = \frac{1}{7}$$

The cross-over frequency equation is obtained as

$$\omega^4 - 50\omega^2 + 1 = 0$$

The cross-over frequency and time delay are computed to be

$$\omega = \sqrt{\frac{50}{2} + \sqrt{\frac{50^2 - 4}{4}}} = \sqrt{25 + 4\sqrt{39}} = 7.0697\,\mathrm{rad/sec}$$

$$t_d = \frac{1}{\omega}\cos^{-1}\frac{1}{7} = \frac{1}{\sqrt{25 + 4\sqrt{39}}}\cos^{-1}\frac{1}{7} = 0.2019\,\mathrm{sec}$$

b. The open-loop plan is expressed as

$$\begin{bmatrix}\dot{y}\\ \ddot{y}\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}}_{A}\begin{bmatrix}y\\ \dot{y}\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B} u\left(t - t_d\right)$$

The reference model is expressed as

$$\begin{bmatrix}\dot{y}_m\\ \ddot{y}_m\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -1 & -6\end{bmatrix}}_{A_m}\begin{bmatrix}y_m\\ \dot{y}_m\end{bmatrix}$$

Let $Q = I$, then

$$P = \begin{bmatrix}\frac{19}{6} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{6}\end{bmatrix}$$

The adaptive law then becomes

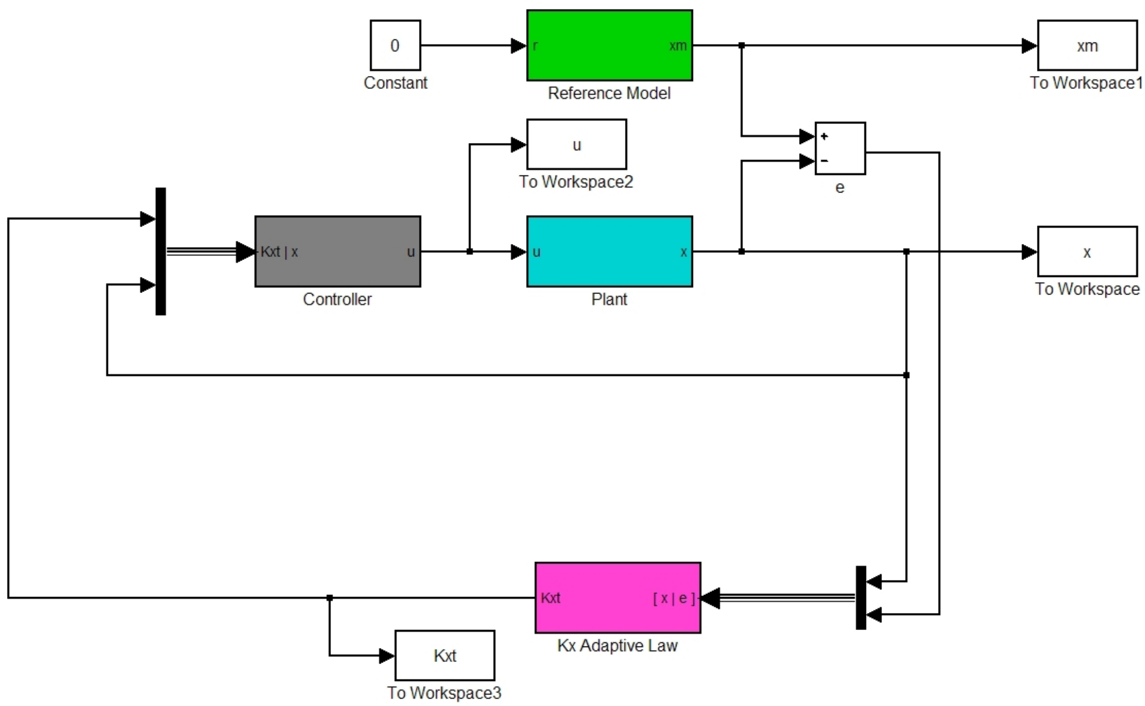
$$\dot{K}_x^\top = \begin{bmatrix}0\\ \dot{k}_d\end{bmatrix} = -\begin{bmatrix}0 & 0\\ 0 & \gamma_x\end{bmatrix}\begin{bmatrix}y\\ \dot{y}\end{bmatrix}\begin{bmatrix}y & \dot{y}\end{bmatrix}\begin{bmatrix}\frac{19}{6} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{6}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ -\gamma_x\dot{y}\left(\frac{1}{2}y + \frac{1}{6}\dot{y}\right)\end{bmatrix}$$

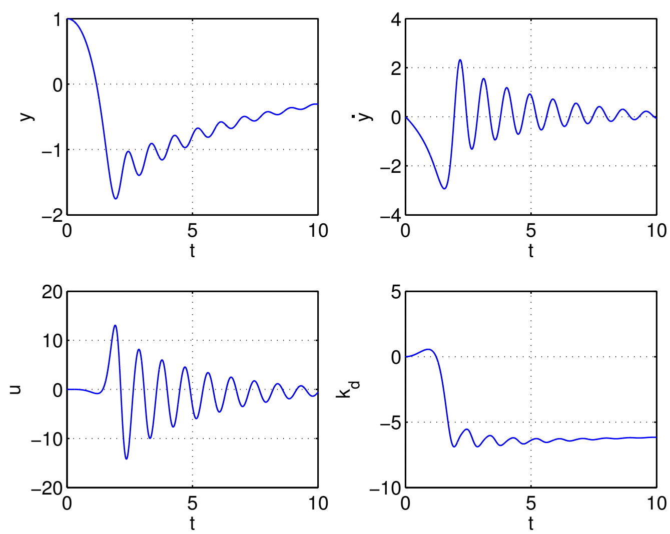
So the controller is simplified as

$$u = k_d\left(t\right)\dot{y}$$

$$\dot{k}_d = -\gamma_x\dot{y}\left(\frac{1}{2}y + \frac{1}{6}\dot{y}\right)$$

The Simulink model and simulation results are as shown with the time delay $t_d = 0.2019$ sec injected at the input.
By trial and error, $\gamma_{x_{max}}$ is determined to be 4.7, which corresponds to $k_{d_{min}} = -6.8853$. Note that the linear system is unstable at $k_d = -7$. The adaptive control result is in agreement with the result of the linear system. The plots of $x\left(t\right)$, $u\left(t\right)$, and $k_d\left(t\right)$ are as shown.





3. For the Rohrs' counterexample, stability of the closed-loop system is affected by the frequency of the reference command signal $r(t)$. Write the closed-loop transfer function from $r(t)$ to $y(t)$. Then, compute the cross-over frequency ω for the reference command signal

$$r = 0.3 + 1.85 \sin \omega t$$

to give a 60° phase margin. Also compute the ideal feedback gain k_y^* corresponding to this phase margin. Implement in Simulink the Rohrs' counterexample using the same initial conditions $k_y(0)$ and $k_r(0)$ with $\gamma_y = \gamma_r = 1$ and $\Delta t = 0.001$ sec. Plot the time histories of $y(t)$, $u(t)$, $k_y(t)$, and $k_r(t)$ for $t \in [0, 60]$ sec.

Solution:
The open-loop plant is

$$y = \frac{2}{s+1}\frac{229u}{s^2+30s+229} = \frac{458u}{s^3+31s^2+259s+229}$$

The initial value of the controller is

$$u = k_y(0)\,y + k_r(0)\,r = -0.65y + 1.14r$$

Then,

$$y = \frac{458\left(-0.65y + 1.14r\right)}{s^3+31s^2+259s+229}$$

The closed-loop transfer function is obtained as

$$\frac{y}{r} = \frac{522.12}{s^3+31s^2+259s+526.7}$$

The phase margin can be determined from

$$\tan\phi = \frac{\omega^3 - \left(\omega_n^2 - 2a\zeta\omega_n\right)\omega}{-\left(2\zeta\omega_n - a\right)\omega^2 - a\omega_n^2}$$

which results in

$$\omega^3 + \sqrt{3}\left(2\zeta\omega_n - a\right)\omega^2 - \left(\omega_n^2 - 2a\zeta\omega_n\right)\omega + \sqrt{3}a\omega_n^2 = 0$$

Substituting in $\phi = \frac{\pi}{3}$, $a = -1$, $2\zeta\omega_n = 30$, and $\omega_n^2 = 229$, we get

$$\omega^3 + 31\sqrt{3}\omega^2 - 259\omega - 229\sqrt{3} = 0$$

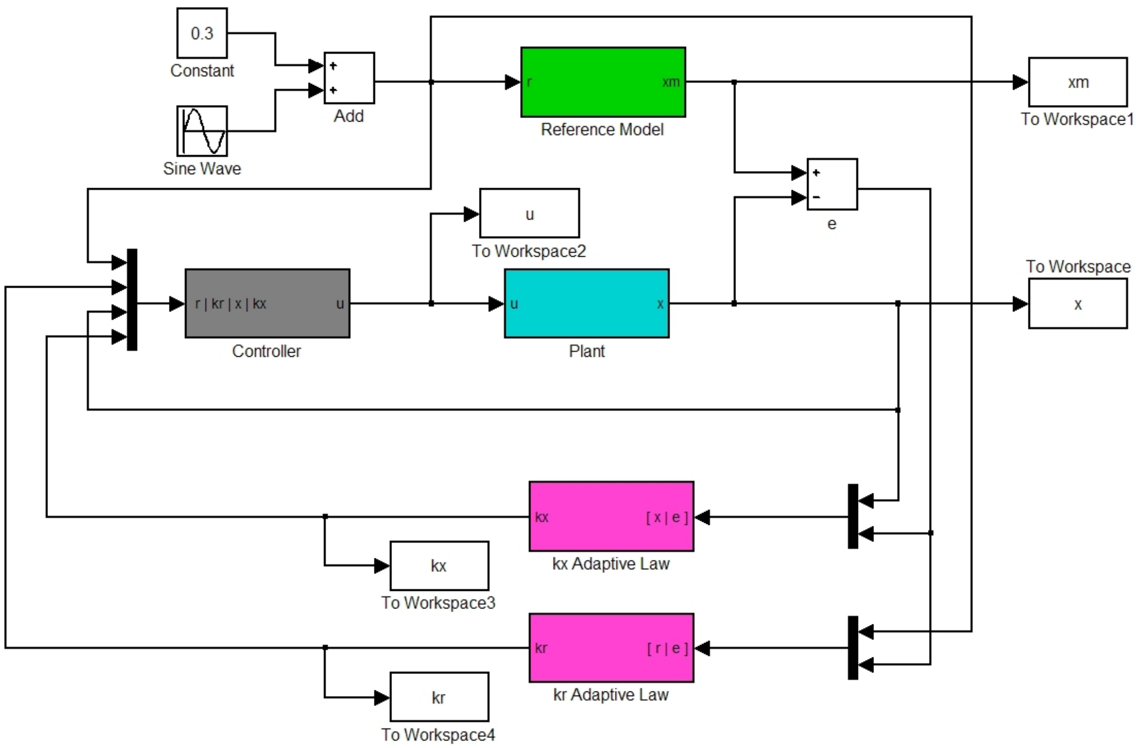
The solution is $\omega = 5.5714$ rad/sec. The feedback gain corresponding to this frequency is computed from

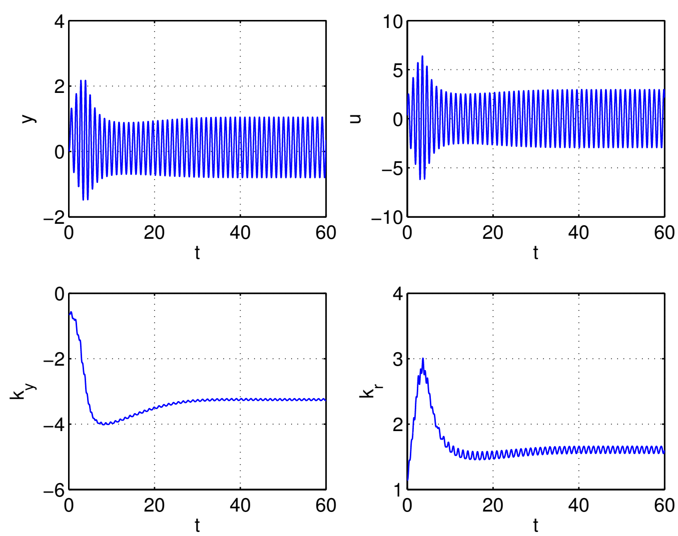
$$\phi = \omega t_d = \sin^{-1}\left[\frac{\omega^3 - \left(\omega_n^2 - 2a\zeta\omega_n\right)\omega}{b\omega_n^2 k_y}\right]$$

This yields

$$k_y = \frac{\omega^3 - \left(\omega_n^2 - 2a\zeta\omega_n\right)\omega}{b\omega_n^2\sin\phi} = -3.2021$$

The Simulink model and simulation results are as shown.





By changing the frequency of the reference command signal that corresponds to a 60° phase margin, the closed-loop system is now stable, although the response is highly oscillatory. Note that $k_y(t)$ tends to a value of -3.2783 at 60 sec which agrees reasonably well with the theoretical value.

Chapter 9 Exercises

1. Consider a time-delay second-order SISO system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = bu\left(t - t_d\right)$$

where $b = 1$, $t_d = \frac{1}{3}$ sec, and ζ and ω_n are unknown but their actual values are -0.5 and 1 rad/sec, respectively.
The system is designed to track a second-order reference model

$$\ddot{y}_m + 2\zeta_m\omega_m\dot{y}_m + \omega_m^2 y_m = b_m r\left(t\right)$$

where $\zeta_m = 0.5$, $\omega_m = 2$ rad/sec, $b_m = 4$, and $r\left(t\right) = 1$, with an adaptive controller

$$u = K_x\left(t\right)x + k_r r$$

where $x\left(t\right) = \left[\, y\left(t\right)\ \dot{y}\left(t\right) \,\right]^\top$ and $K_x\left(t\right) = \left[\, k_p\left(t\right)\ k_d\left(t\right) \,\right]$.

a. Calculate the fixed-gain values of $k_{p_{min}}$ and $k_{d_{min}}$ to achieve a phase margin of 60^o and a time delay margin of 1/3 sec.
b. Define a convex set described by an ellipse that contains $k_p\left(t\right)$ and $k_d\left(t\right)$

$$g\left(k_p, k_d\right) = \left(\frac{k_p}{a}\right)^2 + \left(\frac{k_d}{b}\right)^2 - 1 \le 0$$

where a and b are to be determined from $k_{p_{min}}$ and $k_{d_{min}}$. Design a projection method for the adaptive controller to ensure robustness in the presence of time delay. Write down the adaptive law. Implement the adaptive controller in Simulink using the following information: $y\left(0\right) = 0$, $\dot{y}\left(0\right) = 0$, $K_x\left(0\right) = 0$, and $\varGamma_x = 0.2I$ with a time step $\varDelta t = 0.001$ sec. Plot the time histories of $y\left(t\right)$, $u\left(t\right)$, $k_p\left(t\right)$ and $k_d\left(t\right)$ for $t \in [0, 600]$ sec. What happens when the projection method is removed from the adaptive law?

Solution:

a. The closed-loop transfer function is

$$\frac{y}{r} = \frac{k_r e^{-t_d s}}{s^2 + 2\zeta\omega_n s + \omega_n^2 - k_p e^{-t_d s} - k_d s e^{-t_d s}}$$

The characteristic equation with $s = j\omega$ yields the following:

$$-\omega^2 + \omega_n^2 - k_p \cos\omega t_d - k_d\omega\sin\omega t_d = 0$$

$$2\zeta\omega_n\omega + k_p\sin\omega t_d - k_d\omega\cos\omega t_d = 0$$

The cross-over frequency is computed to be

$$\omega = \frac{\phi}{t_d} = \pi$$

Then,

$$-\pi^2 + 1 - \frac{1}{2}k_p - \frac{\pi\sqrt{3}}{2}k_d = 0$$

$$-\pi + \frac{\sqrt{3}}{2}k_p - \frac{\pi}{2}k_d = 0$$

Solving for k_p and k_d gives

$$k_{p_{min}} = k_p = \frac{-\pi^2 + \pi\sqrt{3} + 1}{2} = -1.7141$$

$$k_{d_{min}} = k_d = \frac{-\pi^2\sqrt{3} - \pi + \sqrt{3}}{2\pi} = -2.9450$$

b. The constraint function is

$$g\left(k_p, k_d\right) = \left(\frac{k_p}{k_{p_{min}}}\right)^2 + \left(\frac{k_d}{k_{d_{min}}}\right)^2 - 1 \le 0$$

Then,

$$\nabla g_{K_x}\left(K_x\right) \nabla^\top g_\Theta\left(\Theta\right) = \begin{bmatrix} \frac{2k_p}{k_{p_{min}}^2} \\ \frac{2k_d}{k_{d_{min}}^2} \end{bmatrix} \begin{bmatrix} \frac{2k_p}{k_{p_{min}}^2} & \frac{2k_d}{k_{d_{min}}^2} \end{bmatrix} = \begin{bmatrix} \frac{4k_p^2}{k_{p_{min}}^4} & \frac{4k_pk_d}{k_{p_{min}}^2 k_{d_{min}}^2} \\ \frac{4k_pk_d}{k_{p_{min}}^2 k_{d_{min}}^2} & \frac{4k_d^2}{k_{d_{min}}^4} \end{bmatrix}$$

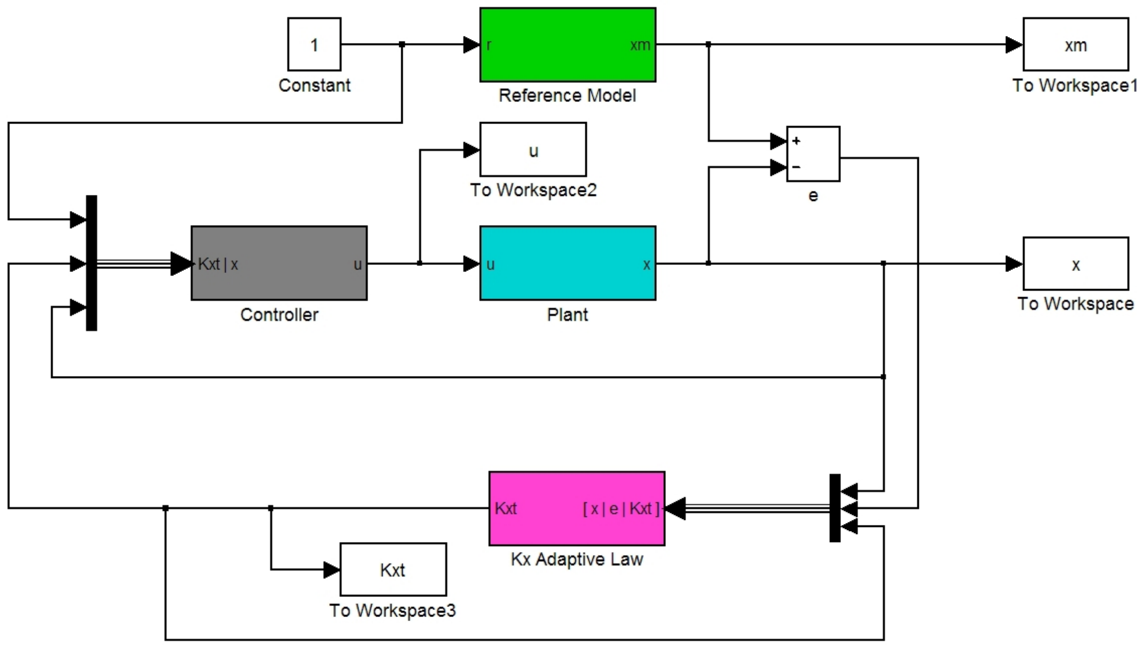
$$\nabla^\top g_\Theta\left(\Theta\right) \nabla g_\Theta\left(\Theta\right) = \frac{4k_p^2}{k_{p_{min}}^4} + \frac{8k_pk_d}{k_{p_{min}}^2 k_{d_{min}}^2} + \frac{4k_d^2}{k_{d_{min}}^4} = 4\left(\frac{k_p}{k_{p_{min}}^2} + \frac{k_d}{k_{d_{min}}^2}\right)^2$$

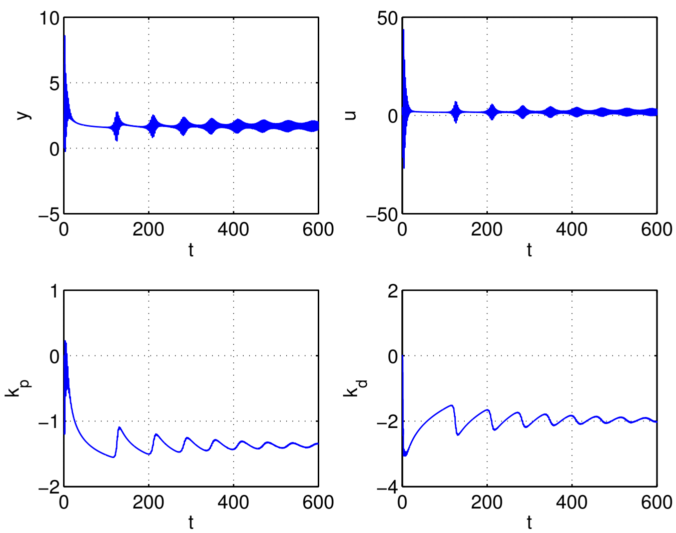
The projection method for the adaptive law is

$$\dot{K}_x = \begin{cases} \Gamma x e^\top PB & \text{if } g\left(K_x\right) < 0 \text{ or if } g\left(K_x\right) = 0 \text{ and } -\left(xe^\top PB\right)^\top \nabla g_{K_x}\left(K_x\right) \le 0 \\ \Gamma\left[I - \frac{\nabla g_\Theta(\Theta)\nabla^\top g_\Theta(\Theta)}{\nabla^\top g_\Theta(\Theta)\nabla g_\Theta(\Theta)}\right] xe^\top PB & \text{otherwise} \end{cases}$$

The Simulink model and simulation results are as shown.

The response with the projection method exhibits high frequency chattering as $k_p\left(t\right)$ and $k_d\left(t\right)$ are forced back into the compact set whenever the constraint is violated. This results in the switching behavior in the $k_p\left(t\right)$ and $k_d\left(t\right)$ signals. When the projection method is removed from the adaptive law, the closed-loop system becomes unstable as $k_p\left(t\right)$ and $k_d\left(t\right)$ exceed $k_{p_{min}}$ and $k_{d_{min}}$.





2. Implement in Simulink the σ modification and e modification for the Rohrs' counterexample with the reference command

$$r = 0.3 + 1.85 \sin 16.1t$$

using the same initial conditions $k_y(0)$ and $k_r(0)$ with $\gamma_x = \gamma_r = 1$, $\sigma = 0.2$, and $\mu = 0.2$ and $\Delta t = 0.001$ sec. Plot the time histories of $y(t)$, $u(t)$, $k_y(t)$and $k_r(t)$ for $t \in [0, 100]$ sec. Experiment with different values of σ and μ and determine by trial-and-error the values of σ and μ at which the system begins to stabilize.

Solution:
For the Rohrs' counterexample, different robust modification schemes can stabilize the closed-loop plant. The σ modification adaptive laws are given by

$$\dot{k}_y = \gamma_x (ye - \sigma k_y)$$

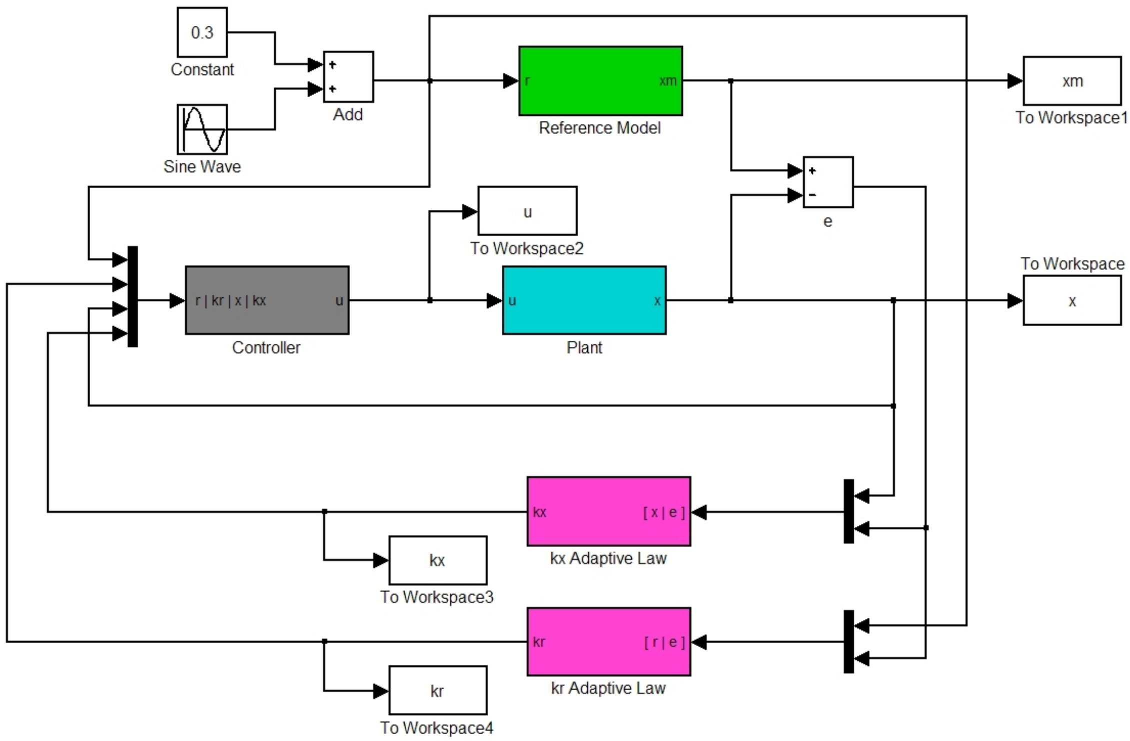
$$\dot{k}_r = \gamma_r \left(re - \sigma k_r \right)$$

The e modification adaptive laws are expressed as

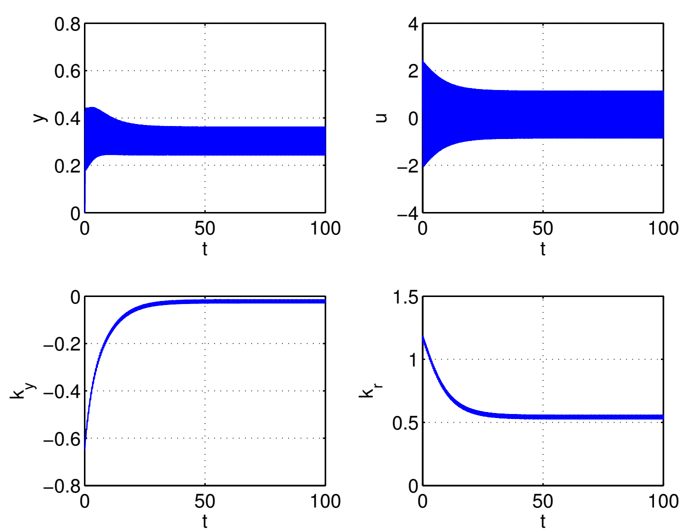
$$\dot{k}_y = \gamma_x \left(ye - \mu \left| e \right| k_y \right)$$

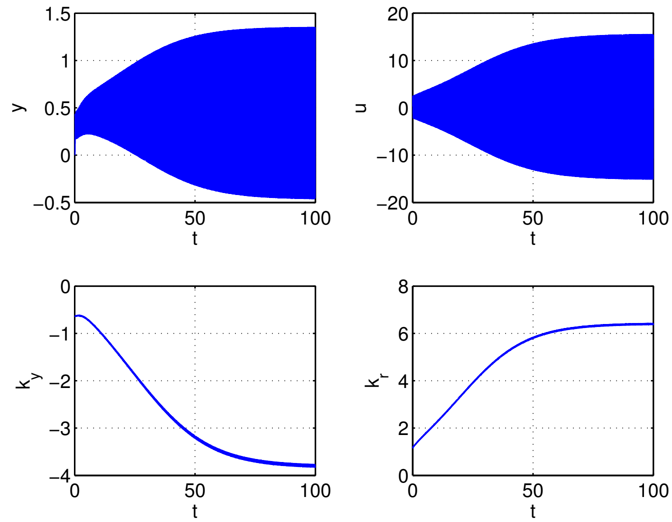
$$\dot{k}_r = \gamma_r \left(re - \mu \left| e \right| k_r \right)$$

The Simulink model and simulation results are as shown.



The closed-loop system is stable with the σ modification ($\sigma = 0.2$) as shown.





The system is also stable with the e modification ($\mu = 0.2$) as shown.
The modification parameters at which the closed-loop system begins to become stable are $\sigma = 0.12$ and $\mu = 0.17$ for $\gamma_x = \gamma_r = 1$.

3. Consider a first-order SISO system

$$\dot{x} = ax + bu + w$$

where a is unknown, b is known, and w is an unknown disturbance.
To prevent the parameter drift, the σ modification is used in an adaptive regulator design

$$u = k_x(x)\,x$$

$$\dot{k}_x = -\gamma_x\left(x^2 b + \sigma k_x\right)$$

Suppose $x(t)$ is a sinusoidal response where $x(t) = \sin t$.

a. Derive the general time-varying disturbance $w(t)$ that produces the given response $x(t)$ in terms of a, b, γ_x, σ, and $k_x(0)$. Let $a = 1$, $b = 1$, $\gamma_x = 10$, $\sigma = 0.1$, $x(0) = 0$, and $k_x(0) = 0$. Express $w(t)$.
b. Implement in Simulink the control system with a time step $\Delta t = 0.001$ sec. Plot the time histories of $x(t)$, $u(t)$, $w(t)$ and $k_x(t)$ for $t \in [0, 20]$ sec.
c. Repeat part (b) with the standard MRAC by setting $\sigma = 0$. Does the system exhibit the parameter drift?

Solution:

a. $k_x(t)$ is evaluated as

$$\frac{d}{dt}\left(e^{\gamma_x \sigma t} k_x\right) = -\gamma_x b e^{\gamma_x \sigma t} x^2 = -\gamma_x b e^{\gamma_x \sigma t} \sin^2 t$$

Using the following trigonometric identity

$$\cos 2t = 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1 - \cos 2t}{2}$$

The σ modification adaptive law is integrated as

$$e^{\gamma_x \sigma t} k_x - k_x(0) = -\frac{\gamma_x b}{2} \int_0^t e^{\gamma_x \sigma \tau} (1 - \cos 2\tau)\, d\tau$$

$$= -\frac{b}{2\sigma} \left(e^{\gamma_x \sigma t} - 1\right) + \frac{\gamma_x b}{2} \frac{e^{\gamma_x \sigma t} (\gamma_x \sigma \cos 2t + 2 \sin 2t) - \gamma_x \sigma}{\gamma_x^2 \sigma^2 + 4}$$

This results in

$$k_x = \left[k_x(0) + \frac{b}{2\sigma} - \frac{\gamma_x^2 \sigma b}{2(\gamma_x^2 \sigma^2 + 4)}\right] e^{-\gamma_x \sigma t} - \frac{b}{2\sigma} + \frac{\gamma_x b (\gamma_x \sigma \cos 2t + 2 \sin 2t)}{2(\gamma_x^2 \sigma^2 + 4)}$$

The disturbance that generates $x = \sin t$ is then obtained as

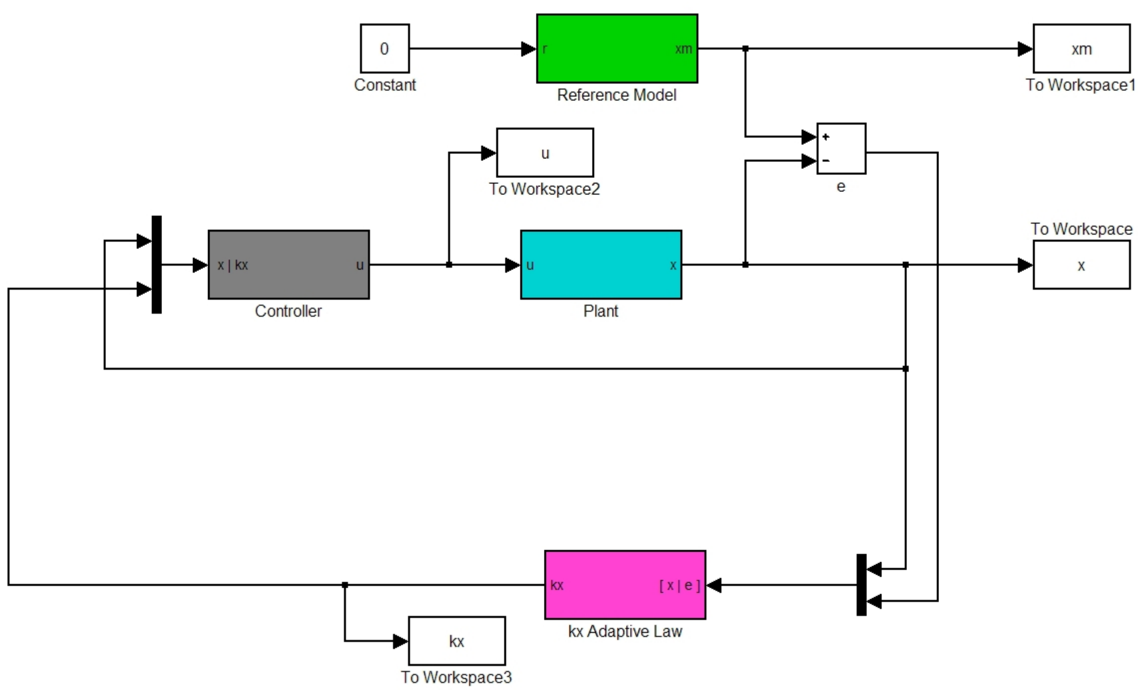
$$w = \dot{x} - ax - bk_x x = \cos t - a \sin t - b \left[k_x(0) + \frac{b}{2\sigma} - \frac{\gamma_x^2 \sigma b}{2(\gamma_x^2 \sigma^2 + 4)}\right] e^{-\gamma_x \sigma t} \sin t + \frac{b^2}{2\sigma} \sin t$$

$$- \frac{\gamma_x b^2 (\gamma_x \sigma \cos 2t + 2 \sin 2t) \sin t}{2(\gamma_x^2 \sigma^2 + 4)}$$

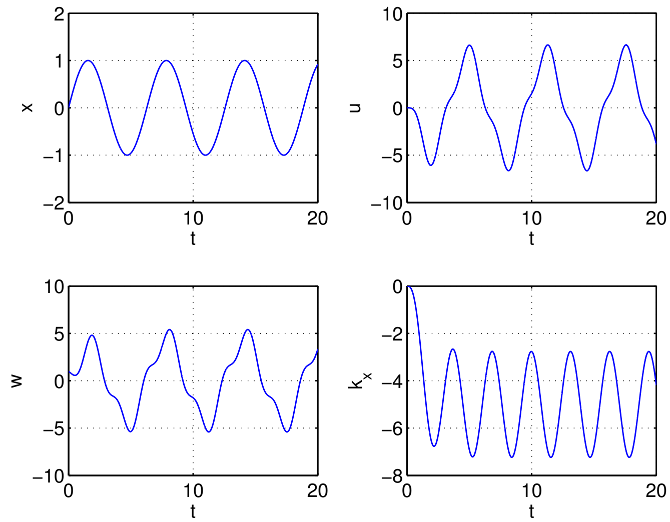
Let $a = 1$, $b = 1$, $\gamma_x = 10$, $\sigma = 0.1$, and $k_x(0) = 0$. Then, the disturbance is expressed as

$$w = \cos t + 4 \sin t - 4e^{-t} \sin t - (\cos 2t + 2 \sin 2t) \sin t$$

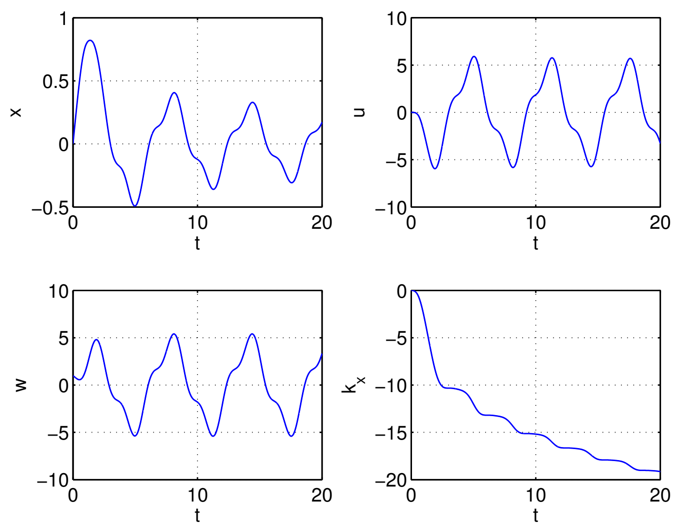
b. The Simulink model is as shown.



The closed-loop system is stable with the σ modification as shown. The response of $x(t)$ follows exactly the signal $\sin t$.



c. The system exhibits parameter drift of $k_x(t)$ when the σ modification is removed.



4. Consider a linear system

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Design a reference model for tracking the output $y(t)$ with a reference command $r(t)$ using the optimal control approach and the following cost function:

$$J = \lim_{t_f \to \infty} \frac{1}{2} \int_0^{t_f} \left[(Cx - r)^\top Q (Cx - r) + u^\top R u \right] dt$$

Derive the expressions for the optimal control gain matrices K_x and K_r for the closed-loop system

$$\dot{x} = (A + BK_x) x + BK_r r$$

Given

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$r = \sin t - 2\cos 4t - 2e^{-t}\sin^2 4t$$

Implement in Simulink the control system. Let $Q = q$ and $R = \frac{1}{q}$. Determine a suitable value of q, K_x, and K_r such that $\sqrt{\frac{1}{t_f}\int_0^{t_f}(y-r)^2\,dt} \le 0.05$ for $t \in [0, 10]$ sec. Initialize with $x(0) = \begin{bmatrix} -2 & 1 \end{bmatrix}^\top$. Plot the time histories of $y(t)$ and $r(t)$ on the same plot, and $e(t) = y(t) - r(t)$.

Solution:
The Hamiltonian function is defined as

$$H(x, u) = \frac{1}{2}(Cx - r)^\top Q (Cx - r) + \frac{1}{2}u^\top R u + \lambda^\top (Ax + Bu)$$

The adjoint equation is obtained as

$$\dot{\lambda} = -\nabla H_x^\top = -C^\top Q(Cx - r) - A^\top \lambda$$

subject to transversality condition $\lambda(t_f) = 0$.
The necessary condition of optimality is established by

$$\nabla H_u^\top = Ru + B^\top \lambda \Rightarrow u = -R^{-1}B^\top \lambda$$

We assume an adjoint solution of the form

$$\lambda = Wx + V$$

Then,

$$\dot{\lambda} = \dot{W}x + W\left[Ax - BR^{-1}B^\top (Wx + V)\right] + \dot{V} = -C^\top Q(Cx - r) - A^\top (Wx + V)$$

This yields the following equations for infinite-time horizon optimal control

$$WA + A^\top W - WBR^{-1}B^\top W + C^\top QC = 0$$

$$V = \left(A^\top - WBR^{-1}B^\top\right)^{-1} C^\top Q r$$

Therefore,

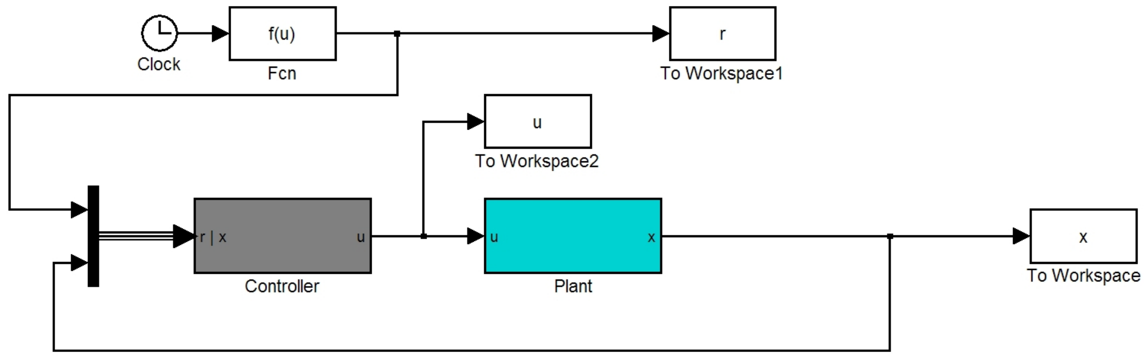
$$u = K_x x + K_r r$$

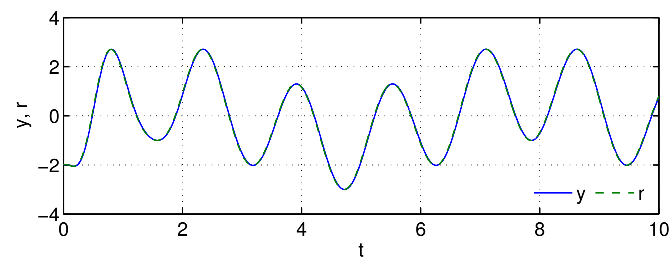
where

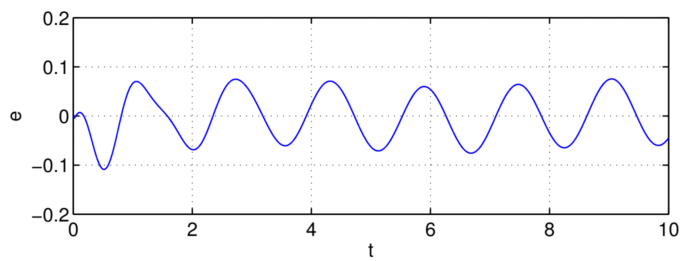
$$K_x = -R^{-1}B^\top W$$

$$K_r = -R^{-1}B^\top \left(A^\top - WBR^{-1}B^\top\right)^{-1} C^\top Q$$

The Simulink model is as shown.
By trial-and-error, $q = 59$ results in $\sqrt{\frac{1}{t_f}\int_0^{t_f}(y-r)^2\,dt} = 0.0499 \le 0.05$ for $t \in [0, 10]$ sec. The corresponding gain matrices are $K_x = \begin{bmatrix} -59.5053 & -0.9979 \end{bmatrix}$ and $K_r = 58.9952$. The response of $y(t)$and the tracking error $e(t) = y(t) - r(t)$ are shown in the following plot.







5. Consider a time delay second-order SISO system

$$\ddot{y} - \dot{y} + y = u\left(t - t_d\right)$$

where $t_d = 0.1$ sec is a time delay.
The unstable open-loop plant is stabilized with an adaptive controller

$$u = K_x x$$

where $x(t) = \begin{bmatrix} y(t) & \dot{y}(t) \end{bmatrix}^\top \in \mathbb{R}^2$ and $K_x(t) = \begin{bmatrix} k_p(t) & k_d(t) \end{bmatrix}$, to achieve an ideal reference model

$$\ddot{y}_m + 6\dot{y}_m + y_m = 0$$

a. Express the optimal control modification adaptive law for $K_x(t)$. Let $\Gamma \to \infty$ and $Q = I$, calculate the equilibrium values of $K_x(t)$ as a function of the modification parameter ν.
b. Determine numerically the value of the modification parameter ν to achieve the maximum time delay margin to within 0.001. Compute the equilibrium values of $K_x(t)$ corresponding to this modification parameter ν. Implement the adaptive controller in Simulink with this modification parameter using the following information: $\Gamma = 10I$, $y(0) = 1$, $\dot{y}(0) = 0$, and $K_x(0) = 0$ with a time step $\Delta t = 0.001$ sec. Plot the time histories of $x(t)$, $u(t)$, and $K_x(t)$ for $t \in [0, 10]$ sec.
c. Increase the adaptation rate to $\Gamma = 10000I$. Repeat the simulations with a time step $\Delta t = 0.0001$ sec. Compare the steady-state values of $K_x(t)$ at 10 sec with those results computed in part (b).

Solution:

a. The optimal control modification adaptive law is

$$\dot{K}_x^\top = -\Gamma_x x x^\top \left(P - \nu K_x^\top B^\top P A_m^{-1}\right) B$$

As $\Gamma \to \infty$, the equilibrium value of K_x can be found by setting $\dot{K}_x(t) = 0$

$$PB - \nu \bar{K}_x^\top B^\top P A_m^{-1} B \Rightarrow \bar{K}_x^\top = \frac{1}{\nu}\left(B^\top P A_m^{-1} B\right)^{-1} PB = \frac{1}{\nu}\begin{bmatrix} -1 & -\frac{1}{3} \end{bmatrix}^\top$$

b. The controller with the equilibrium value of $K_x(t)$ is

$$u = -\frac{1}{\nu} y - \frac{1}{3\nu}\dot{y} + k_r r$$

The closed-loop transfer function is

$$\frac{y}{r} \triangleq G(s) = \frac{k_r r}{s^2 - s + 1 + \frac{1}{\nu} e^{-t_d s} + \frac{1}{3\nu} s e^{-t_d s}}$$

The characteristic equation with $s = j\omega$ is

$$-\omega^2 - j\omega + 1 + \frac{1}{\nu}\left(\cos\omega t_d - j\sin\omega t_d\right) + \frac{1}{3\nu} j\omega\left(\cos\omega t_d - j\sin\omega t_d\right) = 0$$

which results in two equations

$$-\omega^2 + 1 + \frac{1}{\nu}\cos\omega t_d + \frac{1}{3\nu}\omega\sin\omega t_d = 0$$

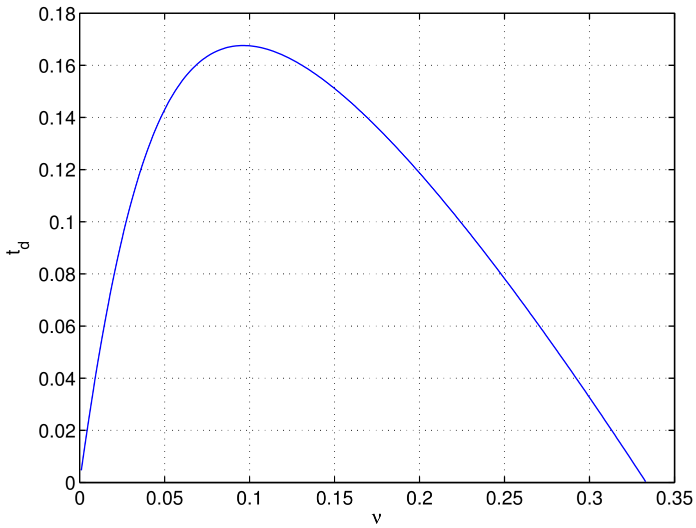
$$-\omega - \frac{1}{\nu}\sin\omega t_d + \frac{1}{3\nu}\omega\cos\omega t_d = 0$$

The cross-over frequency and time delay margin are determined from the following equations

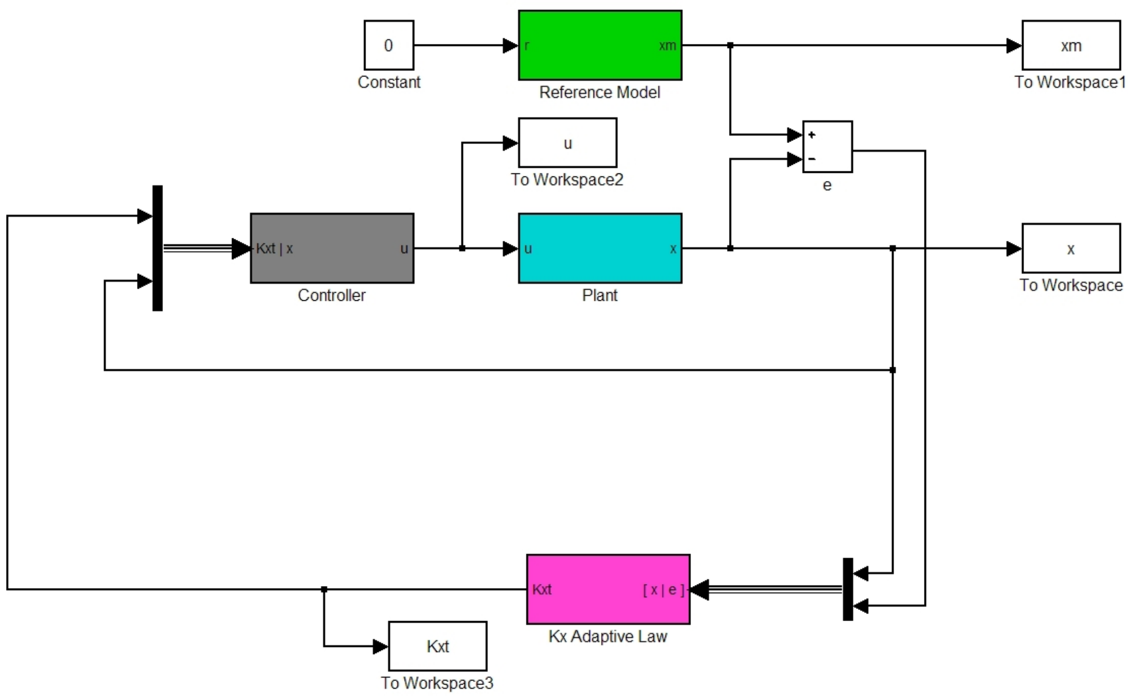
$$\omega^4 - \left(1 + \frac{1}{9\nu^2}\right)\omega^2 + \left(1 - \frac{1}{\nu^2}\right) = 0$$

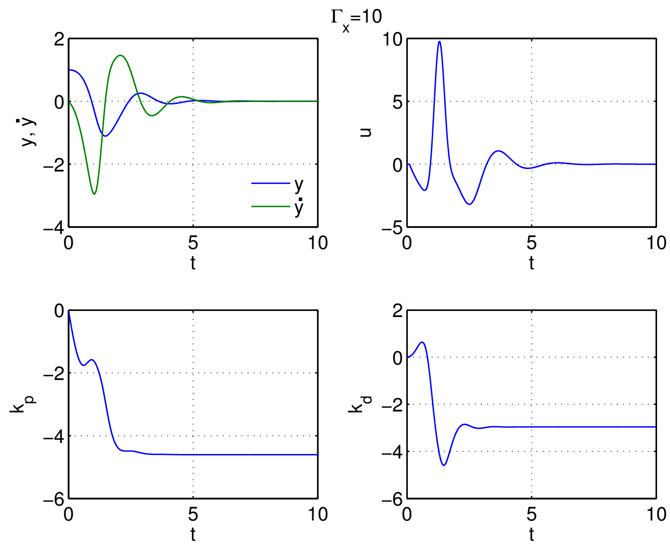
$$t_d = \frac{1}{\omega}\cos^{-1}\left[\frac{3\nu\left(4\omega^2 - 3\right)}{\omega^2 + 9}\right]$$

t_d varies as a function of ν as shown in the following figure.
The maximum time delay margin is $t_d = 0.1676$ sec for $\nu = 0.096$. The ideal equilibrium values of $K_x(t)$ for the maximum time delay margin is $\bar{K}_x = \begin{bmatrix} -10.4167 & -3.4722 \end{bmatrix}$.

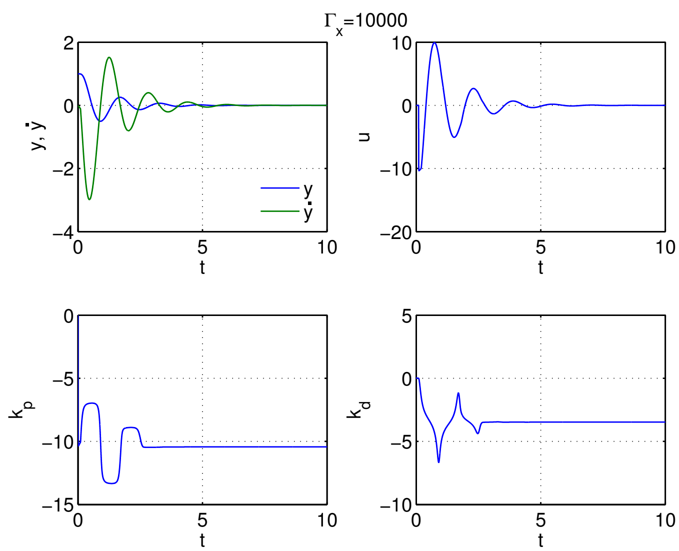


The Simulink model is as shown.
The closed-loop is completely stable with $K_x(t)$ converging to the equilibrium values $\bar{K}_x = \begin{bmatrix} -4.6046 & -2.9598 \end{bmatrix}$.





c. If Γ increases to a larger value, say $10000I$, $K_x(t)$ tends to the equilibrium values of $\bar{K}_x = \begin{bmatrix} -10.4356 & -3.4727 \end{bmatrix}$ which agree very well with the analytical results in part (b). It is also noted that the closed-loop system with a large adaptation rate for fast adaptation exhibits no high frequency response as shown.



6. Consider a first-order SISO plant as

$$\dot{x} = ax + b\left(u + \theta^* x + w\right)$$

with $a = -1$, $b = 1$, $\theta^* = 2$, and

$$w = \cos t + 4\sin t - 4e^{-t}\sin t - (\cos 2t + 2\sin 2t)\sin t$$

This disturbance will cause a parameter drift when the standard MRAC is used in a regulator design. An adaptive controller is designed as

$$u = k_r r - \theta(t)x - \hat{w}(t)$$

to enable the plant to follow a reference model

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -2$, $b_m = 2$, and $r(t) = 1$.

a. Calculate k_r. Express the adaptive loop recovery modification adaptive laws for $\theta(t)$ and $\hat{w}(t)$ using a modification parameter $\eta = 0.1$.
b. Implement the adaptive controller in Simulink using the following information: $x(0) = 0$, $\theta(0) = 0$, $\hat{w}(0) = 0$, and $\gamma = \gamma_w = 100$ with a time step $\Delta t = 0.001$ sec. Plot the time histories of $x(t)$, $u(t)$, $\theta(t)$, and $w(t)$ and $\hat{w}(t)$ together on the same plot for $t \in [0, 100]$ sec.

Solution:

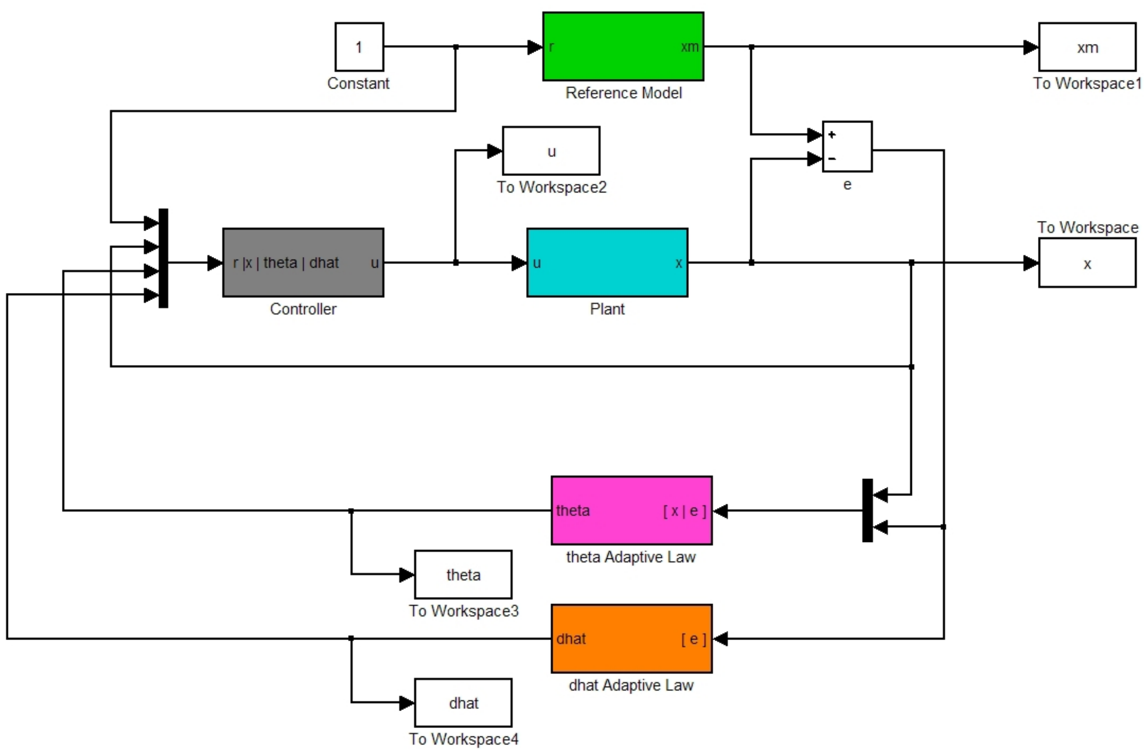
a. The adaptive controller with the adaptive loop recovery modification adaptive laws is given by

$$u = k_r - \theta(t)x - \hat{w}(t)$$

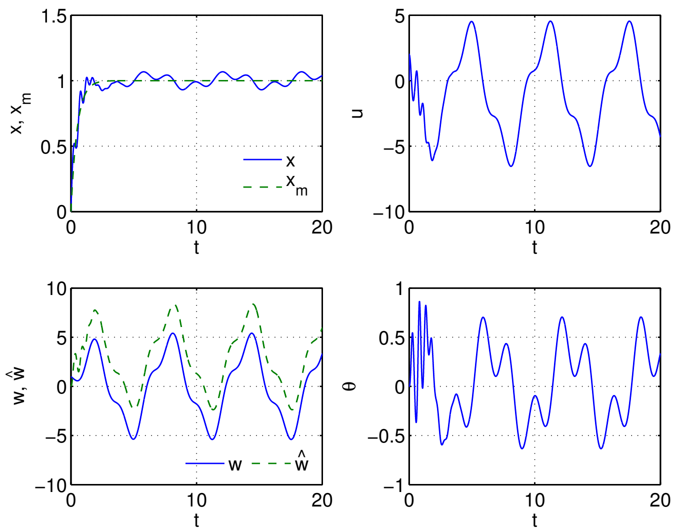
$$\dot{\theta} = -\gamma(xeb + \eta\theta)$$

$$\dot{\hat{w}} = -\gamma_w eb$$

where $k_r = \frac{b_m}{b} = 1$.
Note that for the adaptive law for $\hat{w}(t)$, $\phi(x) = 1$ and $\phi_x(x) = 0$. So it is simply just the standard MRAC. The adaptive law for $\theta(t)$ is just the σ modification since $\phi(x) = x$ and $\phi_x(x) = 1$.
b. The Simulink model is as shown.



The response of the closed-loop system with the adaptive loop recovery modification is stable as shown. $\hat{w}(t)$ approximates the disturbance $w(t)$ fairly well. $\theta(t)$ is bounded but does not converge to a steady-state value.



7. Consider a second-order SISO plant

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = bu\,(t - t_d)$$

where $\zeta = -0.5$ and $\omega_n = 1$ rad/sec are unknown, $b = 1$ is known, and t_d is a known time delay. Design an adaptive controller using the normalized MRAC without the projection method to allow the plant to follow a reference model

$$\ddot{y}_m + 2\zeta_m\omega_m\dot{y}_m + \omega_m^2 y_m = b_m r\,(t)$$

where $\zeta_m = 3$, $\omega_m = 1$, $b_m = 1$, and $r\,(t) = r_0 \sin t$.

a. Implement the adaptive controller in Simulink using the following information: $t_d = 0$, $x\,(0) = 0$, $K_x\,(0) = 0$, and $\varGamma_x = 100I$ with a time step $\varDelta t = 0.001$ sec for the standard MRAC by setting $R = 0$ with $r_0 = 1$ and $r_0 = 100$. Plot the time histories of $y\,(t)$ and $y_m\,(t)$, $e_1\,(t) = y_m\,(t) - y\,(t)$, $u\,(t)$, and $K_x\,(t)$ for $t \in [0, 100]$ sec. Comment on the effect of the amplitude of the reference command signal on MRAC.

b. Repeat part (a) for the normalized MRAC with $R = I$ and $r_0 = 100$ for $t_d = 0$ and $t_d = 0.1$ sec. Comment on the effect of normalization on the amplitude of the reference command signal and time delay.

Solution:

a. The plant is expressed as

$$\dot{x} = Ax + Bu\,(t - t_d)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The reference model is expressed as

$$\dot{x}_m = A_m x_m + B_m r$$

where $r\,(t) = a \sin t$ and

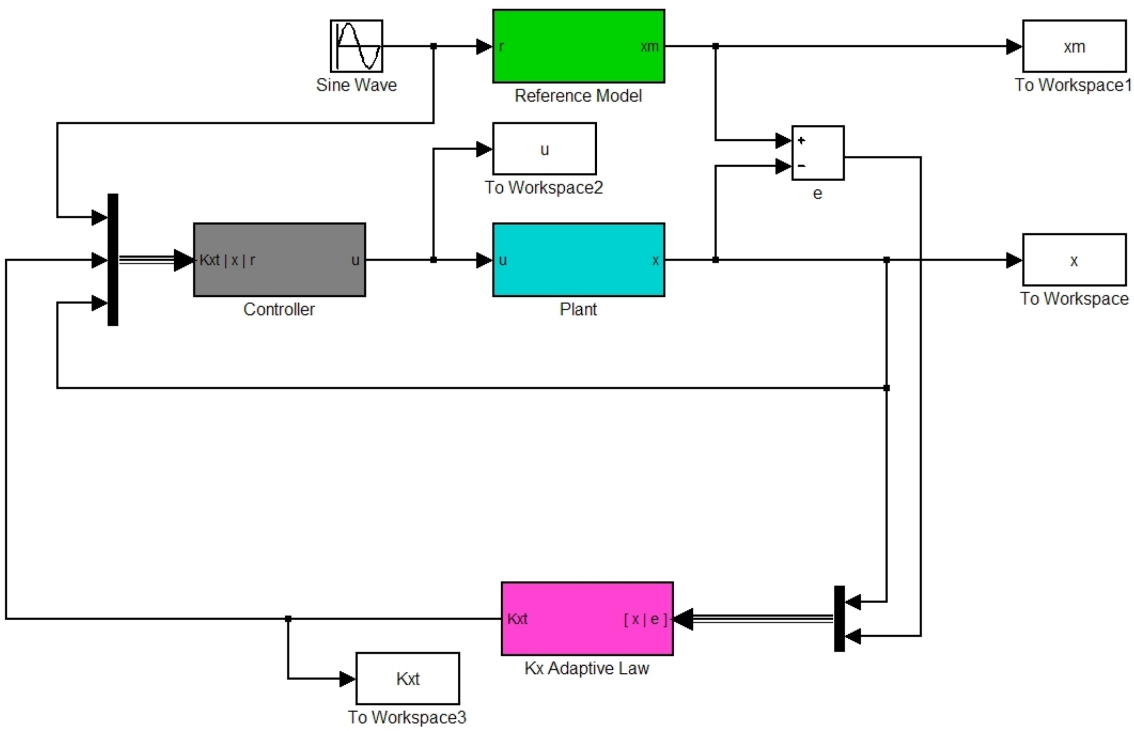
$$A_m = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m\omega_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -6 \end{bmatrix}, \; B_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The standard MRAC adaptive controller with normalization is given by

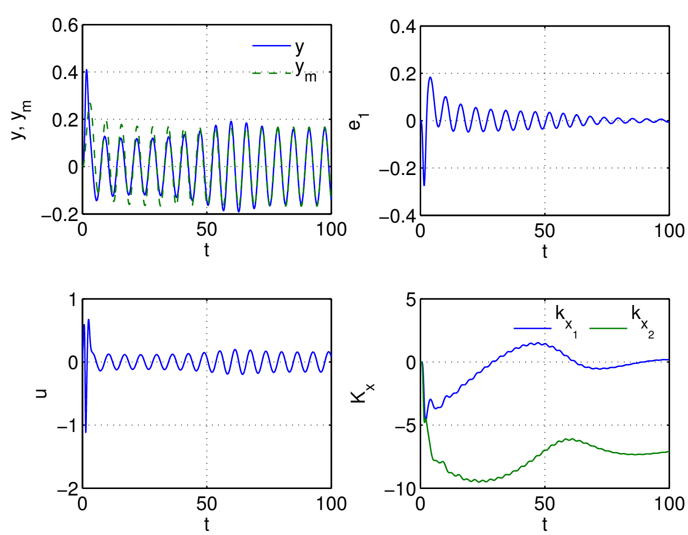
$$u = K_x(t)\,x + k_r r$$

$$\dot{K}_x^\top = \frac{\Gamma_x x e^\top P B}{1 + x^\top R x}$$

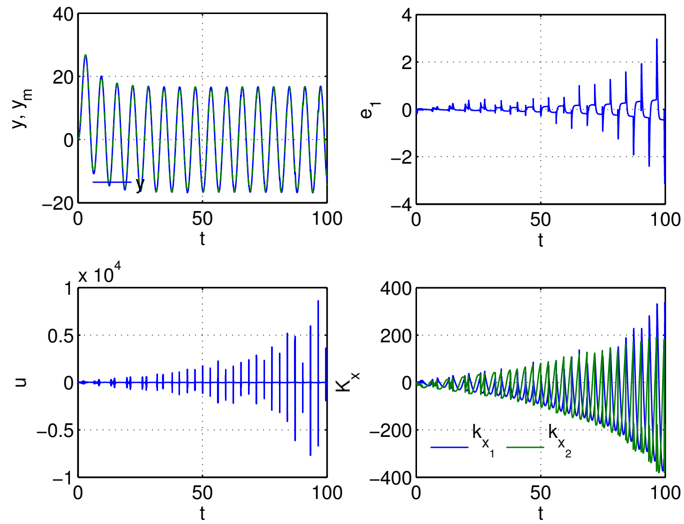
where $k_r = 1$.
The Simulink model is as shown.



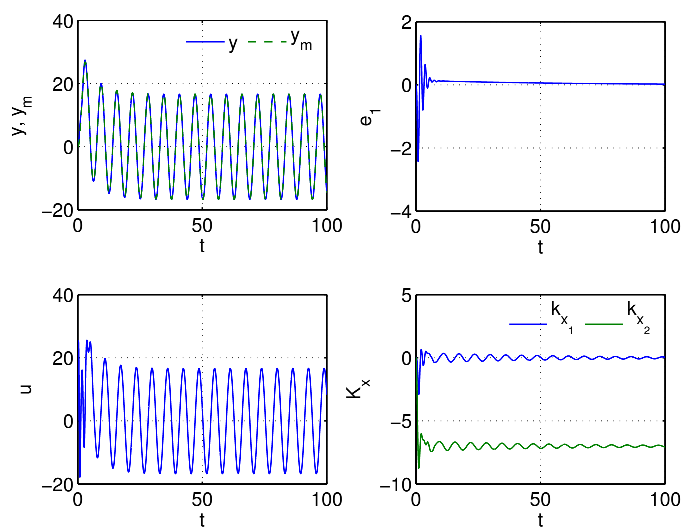
The response of the closed-loop system with the standard MRAC for $r_0 = 1$ and $t_d = 0$ is as shown. The closed-loop plant does not seem to track the reference model very well.



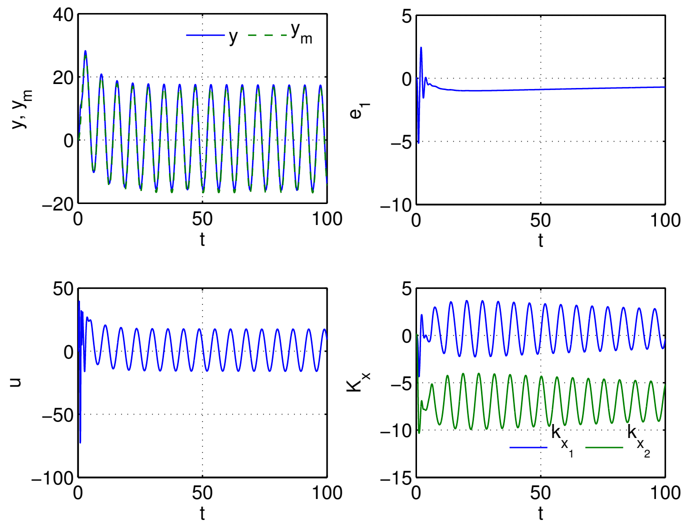
The response of the closed-loop system with the standard MRAC for $r_0 = 100$ and $t_d = 0$ is as shown. The closed-loop is unstable. Thus, unlike linear systems, the amplitude of the reference command signal does affect the closed-loop stability.



b. The response of the closed-loop system with the normalized MRAC ($R = I$) for $r_0 = 100$ and $t_d = 0$ is as shown. The closed-loop plant is completely stable with the tracking error tending to zero asymptotically. The adaptive parameters are bounded.



The response of the closed-loop system with the normalized MRAC ($R = I$) for $r_0 = 100$ and $t_d = 0.1$ sec is as shown. The closed-loop plant is still stable but the tracking error does not tend to zero. The adaptive parameters are more oscillatory but still bounded.



In conclusion, the normalization seems to be able to eliminate the effect of the amplitude of the reference command signal as well as the time delay on the closed-loop stability. It is more robust than the standard MRAC.

8. For the Rohrs' counterexample, design a standard MRAC with the covariance adjustment method without the projection method.

   a. Implement the adaptive controller in Simulink using the following information: $y(0) = 0$, $k_y(0) = -0.65$, $k_r(0) = 1.14$, $\gamma_y(0) = \gamma_r(0) = 1$, and $\eta = 5$ with a time step $\Delta t = 0.01$ sec. Plot the time histories of $k_y(t)$, $k_r(t)$, $\gamma_y(t)$, and $\gamma_r(t)$ for $t \in [0, 100]$ sec. Note: plot $\gamma_y(t)$ and $\gamma_r(t)$ with the logarithmic scale in the y axis for better visualization.
   b. Repeat part (a) with $t \in [0, 1000]$ sec. Do $k_y(t)$ and $k_r(t)$ reach their equilibrium values or do they exhibit a parameter drift behavior?

Solution:

a. The standard MRAC adaptive laws for $k_y(t)$ and $k_r(t)$ with the covariance adjustment method are

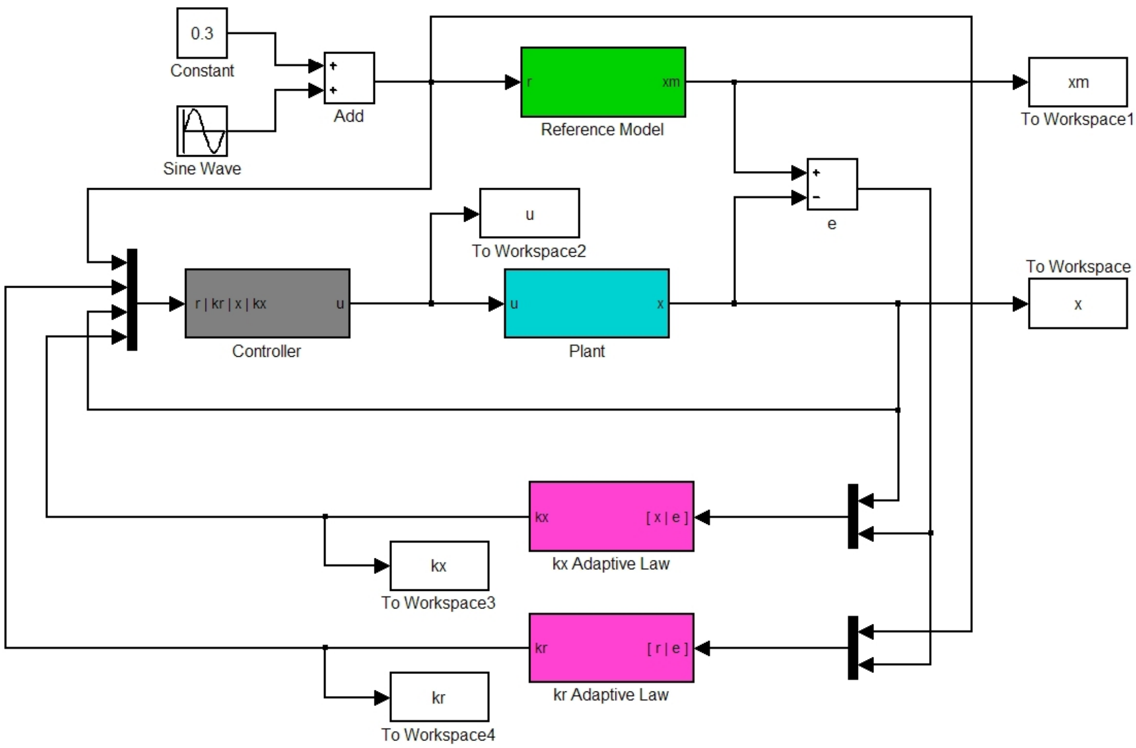
$$\dot{k}_y = \gamma_y(t)\, ye$$

$$\dot{k}_r = \gamma_r(t)\, re$$

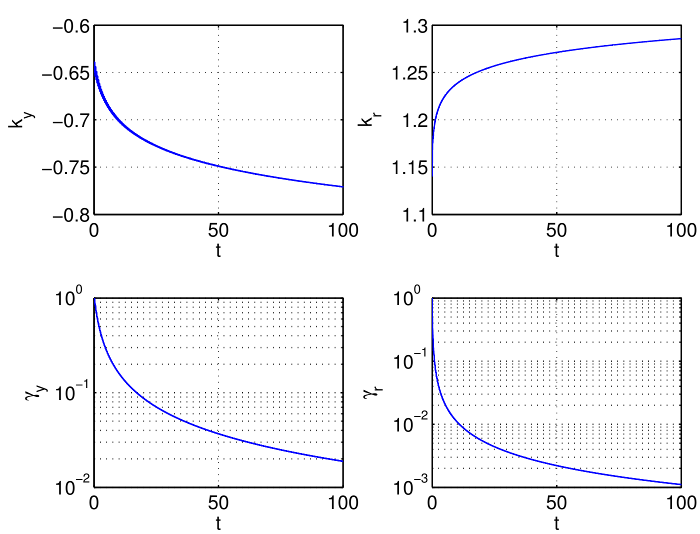
$$\dot{\gamma}_y = -\eta y^2 \gamma_y^2$$

$$\dot{\gamma}_r = -\eta r^2 \gamma_r^2$$

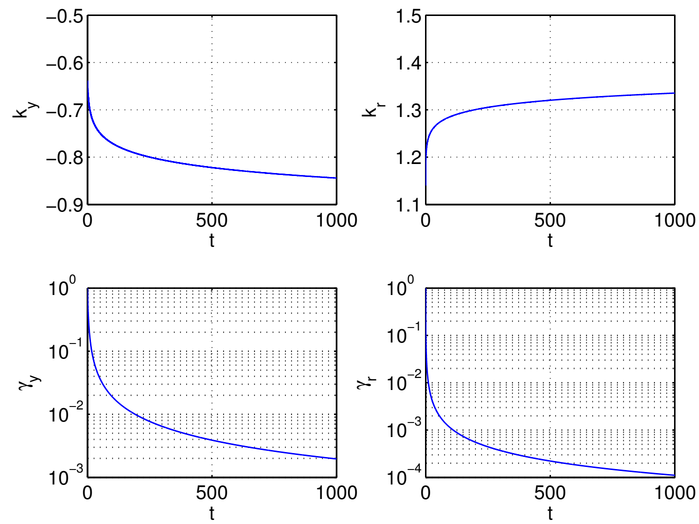
The Simulink model is as shown.



The response of the closed-loop system for $t \in [0, 100]$ sec is as shown. The closed-loop plant is stable up to $t = 100$ sec, but $k_x(t)$ and $k_r(t)$ do not appear to reach their equilibrium values. The adaptation rates at $t = 100$ sec are $\gamma_y(100) = 0.0189$ and $\gamma_r(100) = 0.0011$ which are quite small.



b. The response of the closed-loop system for $t \in [0, 1000]$ sec is as shown. The closed-loop plant is still stable up to $t = 1000$ sec. $k_x(t)$ and $k_r(t)$ still do not appear to reach their equilibrium values. The adaptation rates at $t = 1000$ sec are $\gamma_x(1000) = 0.0020$ and $\gamma_r(1000) = 0.0001$ which are very small. Nonetheless, $\gamma_x(t)$ and $\gamma_r(t)$ will always be positive as they tend to zero as $t \to \infty$. Therefore, $k_x(t)$ and $k_r(t)$ will continue to drift forever eventually when $k_x(t)$ reaches a limiting value at $k_x = -17.0306$ whereupon the closed-loop system becomes unstable.



9. Consider a first-order SISO plant

$$\dot{x} = ax + b\lambda \left[u\left(t - t_d\right) + \theta^* \phi\left(x\right)\right] + w$$

where $a = -1$ and $b = 1$ are known, $\lambda = -1$ and $\theta^* = 0.5$ are unknown, but the sign of λ is known, $\phi\left(x\right) = x^2$, $t_d = 0.1$ sec is a known time delay, and $w\left(t\right) = 0.02 + 0.01\cos 2t$
The reference model is given by

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -2$, $b_m = 2$, and $r\left(t\right) = \sin t$.

a. Design an adaptive controller using the standard tracking-error based optimal control modification method. Express the adaptive laws.
b. Implement the adaptive controller in Simulink using the following information: $x\left(0\right) = k_x\left(0\right) = k_r\left(0\right) = \theta\left(0\right) = 0$ and $\gamma_x = \gamma_r = \gamma_\theta = 20$ with a time step $\Delta t = 0.001$ sec for the standard MRAC with $\nu = 0$ and for the optimal control modification with $\nu = 0.2$. Plot the time histories of $x\left(t\right)$ and $x_m\left(t\right)$ on the same plot, $u\left(t\right)$, $k_x\left(t\right)$, $k_r\left(t\right)$, and $\theta\left(t\right)$ for $t \in [0, 60]$ sec.

Solution:

a. The adaptive controller is

$$u = k_x\left(t\right)x + k_r\left(t\right)r - \theta\left(t\right)\phi\left(x\right)$$

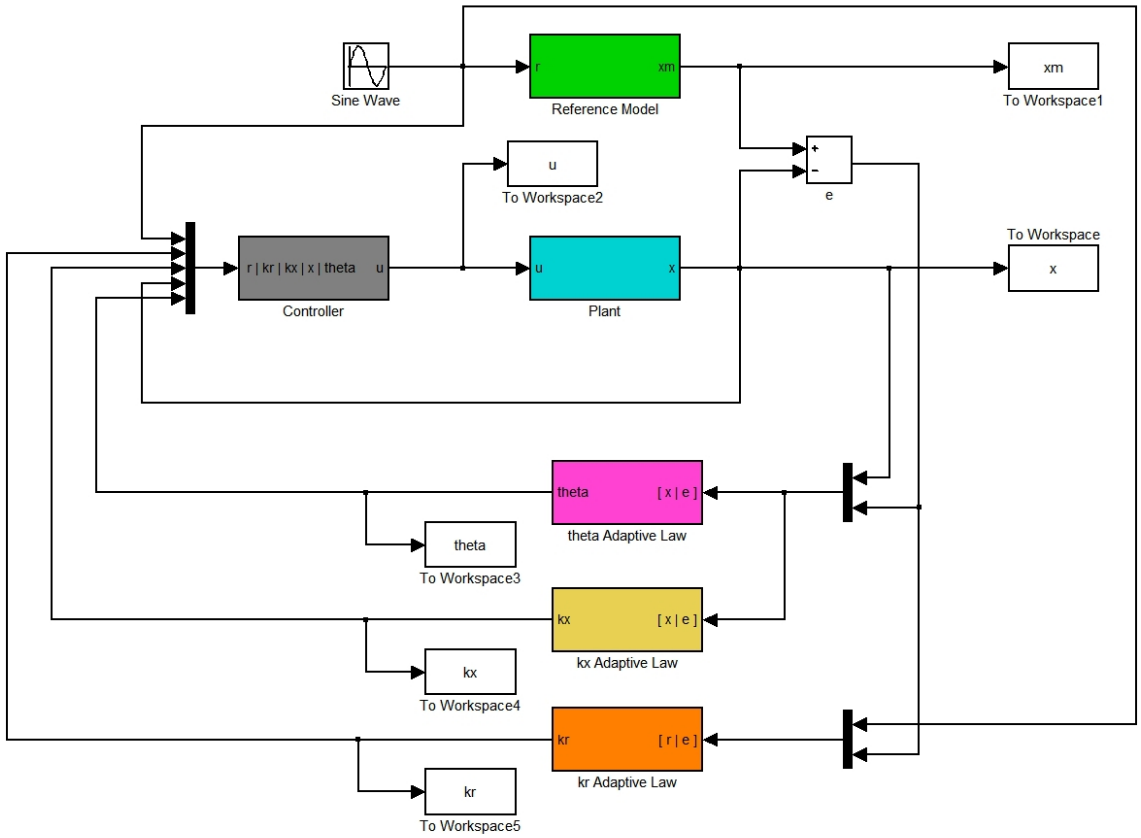
where the optimal control modification adaptive laws are given by

$$\dot{k}_x = \gamma_x x\left(e\mathrm{sgn}\lambda + \nu x k_x b a_m^{-1}\right)b$$

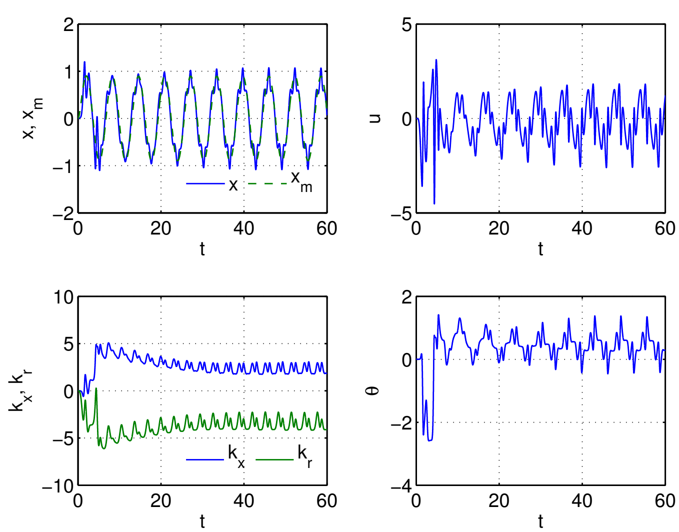
$$\dot{k}_r = \gamma_r r\left(e\mathrm{sgn}\lambda + \nu r k_r b a_m^{-1}\right)b$$

$$\dot{\theta} = -\gamma_\theta \phi\left(x\right)\left[e\mathrm{sgn}\lambda - \nu\phi\left(x\right)\theta b a_m^{-1}\right]b$$

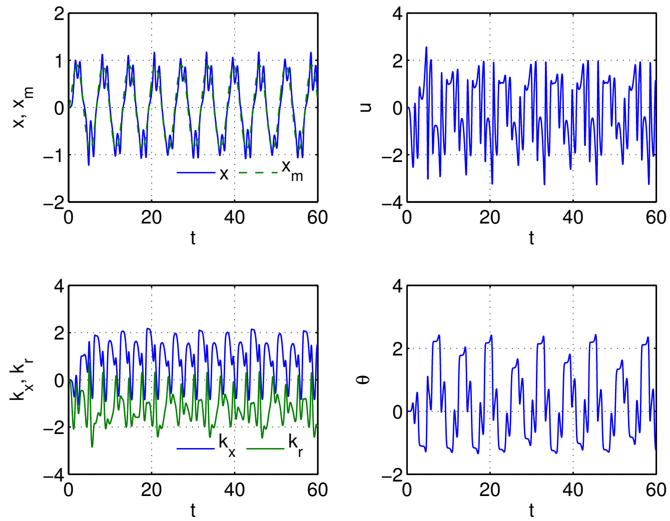
b. The Simulink model is as shown.



The response of the closed-loop system with MRAC is as shown. The closed-loop plant begins to diverge. Note that reducing the adaptation rate will improve performance.



The response of the closed-loop system with the optimal control modification is as shown. The closed-loop plant is bounded, but the tracking is not as good. This is expected since the adaptive law achieves robustness at the expense of tracking. The optimal control modification is more robust than the standard MRAC.



10. For Exercise 9.9, suppose λ is completely unknown.

a. Design an adaptive controller using the bi-objective optimal control modification method. Express the adaptive laws.
b. Implement the adaptive controller in Simulink using $\gamma_\lambda = \gamma_w = 20$, $\eta = 0$, and the rest of the information in Exercise 9.9 along with the initial conditions $\hat{\lambda}(t) = 1$ and $\hat{w}(t) = 0$. Plot the time histories of $x(t)$ and $x_m(t)$ on the same plot, $u(t)$, $k_x(t)$, $k_r(t)$, $\theta(t)$, $\hat{\lambda}(t)$ and $\hat{w}(t)$ and w on the same plot for $t \in [0, 60]$ sec.
c. Comment on the results of Exercise 9.9 and Exercise 9.10. Which method seems to work better?

Solution:

a. The bi-objective optimal control modification adaptive laws are

$$\dot{k}_x = \gamma_x x \left(e + \nu u \hat{\lambda} b a_m^{-1}\right) b\hat{\lambda}$$

$$\dot{k}_r = \gamma_r r \left(e + \nu u \hat{\lambda} b a_m^{-1}\right) b\hat{\lambda}$$

$$\dot{\theta} = -\gamma_\theta x^2 \left(e + e_p + \nu u \hat{\lambda} b a_m^{-1} - \eta \left\{[u + 2\theta\phi(x)]\hat{\lambda} b + \hat{w}\right\} a_m^{-1}\right) b\hat{\lambda}$$

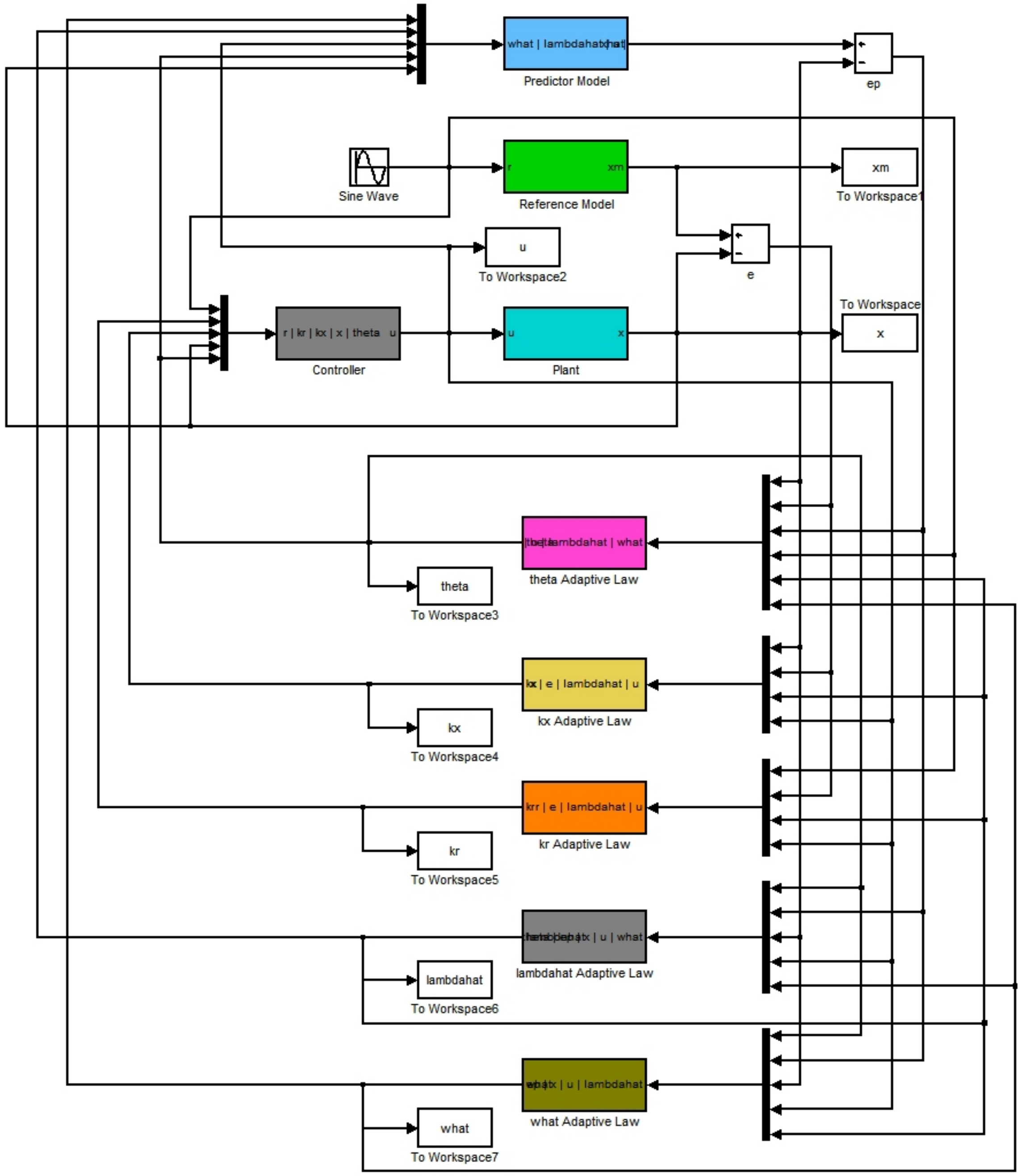
$$\dot{\hat{\lambda}} = -\gamma_\lambda [u + \theta\phi(x)] \left(e_p - \eta \left\{[u + 2\theta\phi(x)]\hat{\lambda} b + \hat{w}\right\} a_m^{-1}\right) b$$

$$\dot{\hat{w}} = -\gamma_w \left(e_p - \eta \left\{[u + 2\theta\phi(x)]\hat{\lambda} b + \hat{w}\right\} a_m^{-1}\right)$$

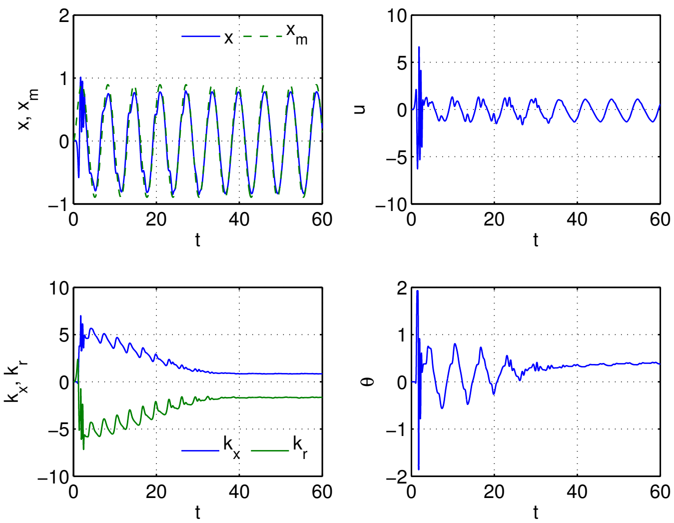
where the predictor error $e_p(t) = \hat{x}(t) - x(t)$ is computed from the predictor model

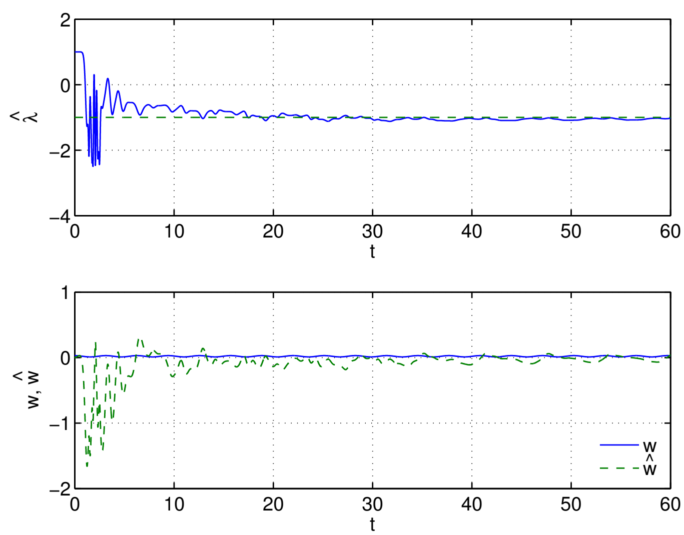
$$\dot{\hat{x}} = a_m \hat{x} + (a - a_m) x + b\hat{\Lambda}[u(t - t_d) + \theta\phi(x)] + \hat{w}$$

b. The Simulink model is as shown.

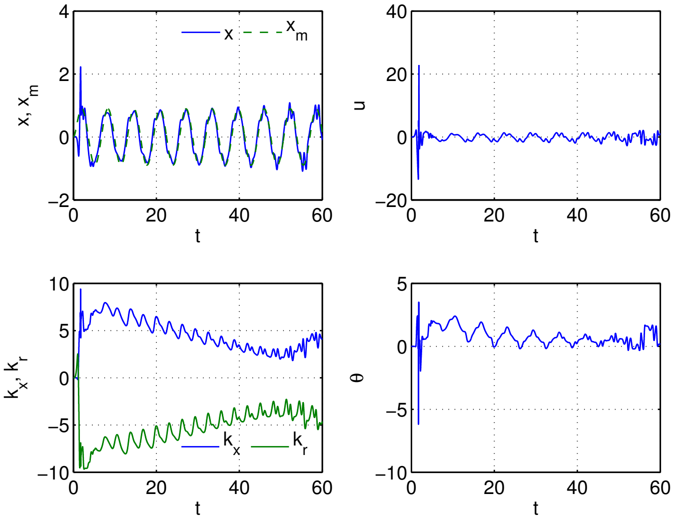


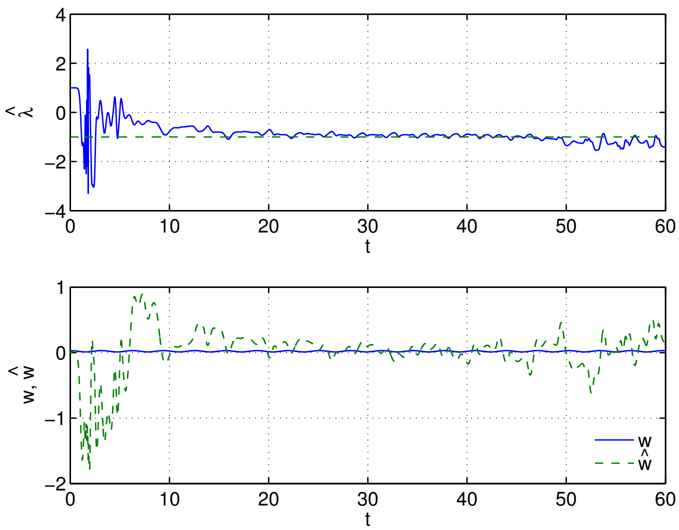
The response of the closed-loop system with bi-objective optimal control modification is as shown. There are large initial transients as expected due to the control reversal. After 30 sec, the closed-loop plant begins to follow the reference model. Note that $\hat{\lambda}(t)$ converges to the true value. The other adaptive parameters $k_x(t)$, $k_r(t)$, and $\theta(t)$ also tend to their true values of 1, -2, and 0.5, respectively.





The response of the closed-loop system with bi-objective MRAC is as shown. The closed-loop plant begins to diverge.





c. In general, the bi-objective optimal control modification works better than all the other methods in Exercise 9.9 and Exercise 9.10 in the presence of input uncertainty. The parameter convergence of the bi-objective optimal control modification is much better than all the rest.

Chapter 10 Exercises

1. Consider the equation of motion of an inverted pendulum

$$\frac{1}{3}mL^2\ddot{\theta} - \frac{1}{2}mgL\sin\theta + c\dot{\theta} = u\left(t - t_d\right)$$

a. Expand $\sin\theta$ using the Taylor series expansion about $\theta\left(t\right) = 0$ for the first two terms. Then, express the equation of motion in the form of

$$\dot{x} = Ax + B\left[u\left(t - t_d\right) + \Theta^{*\top}\Phi\left(x\right)\right]$$

where $x_1\left(t\right) = \theta\left(t\right)$, $x_2\left(t\right) = \dot{\theta}\left(t\right)$, $x\left(t\right) = \left[x_1\left(t\right)\ x_2\left(t\right)\right]^\top$, $\Phi\left(x\right)$ is comprised of the function in the nonlinear term of the Taylor series expansion of $\sin\theta$ and the function in the damping term, and Θ^* is a vector of parameters associated with $\Phi\left(x\right)$ which are assumed to be unknown.

b. Given $m = 0.1775$ slug, $L = 2$ ft, $c = 0.2$ slug-ft^2/sec, $t_d = 0.05$ sec, and $\theta\left(0\right) = \dot{\theta}\left(0\right) = 0$. Using the equation of motion in part (a), design an adaptive controller using the optimal control modification to enable the closed-loop plant to track a reference model specified by

$$\ddot{\theta}_m + 2\zeta_m\omega_m\dot{\theta}_m + \omega_m^2\theta_m = \omega_m^2 r$$

where $\zeta_m = 0.5$, $\omega_m = 2$, and $r = \frac{\pi}{12}$. Calculate K_x and k_r.

c. Implement the adaptive controller in Simulink using the nonlinear plant with $\Theta^\top\left(0\right) = \left[\theta_1^*\ 0\right]$ and a time step $\Delta t = 0.001$ sec for the standard MRAC with $\Gamma = 100$ and the optimal control modification with $\Gamma = 100$ and $\nu = 0.5$. Plot the time histories of $x\left(t\right)$ and $x_m\left(t\right)$ on the same plot, $u\left(t\right)$, and $\Theta\left(t\right)$ for $t \in [0, 10]$ sec. Compare the closed-loop response with the optimal control modification to that in Example 10.1. Does the linear nominal controller design in this problem appear to work as well as the nonlinear nominal controller design in Example 10.1?

Solution:

a. The Taylor series expansion of $\sin\theta$ about $\theta = 0$ for the first two terms is

$$\sin\theta = \theta - \frac{\theta^3}{3!}$$

Then, the equation of motion is then approximated as

$$\ddot{\theta} = \frac{3g}{2L}\left(\theta - \frac{1}{3!}\theta^3\right) + \frac{3}{mL^2}u\left(t - t_d\right) - \frac{3c}{mL^2}\dot{\theta}$$

or

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ \frac{3g}{2L} & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{3}{mL^2} \end{bmatrix}}_{B} \left\{ u\left(t - t_d\right) + \underbrace{\begin{bmatrix} -\frac{mgL}{12} & -c \end{bmatrix}}_{\Theta^{*\top}} \underbrace{\begin{bmatrix} x_1^3 \\ x_2 \end{bmatrix}}_{\Phi(x)} \right\}$$

Note that the actual plant is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{3g}{2L}\sin x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{3}{mL^2} \end{bmatrix} \left[u\left(t - t_d\right) - cx_2\right]$$

b. The reference model is expressed as

$$\dot{x}_m = A_m x_m + B_m r$$

where

$$A_m = \begin{bmatrix} 0 & 0 \\ -\omega_m^2 & -2\zeta_m\omega_m \end{bmatrix}, \; B_m = \begin{bmatrix} 0 \\ \omega_m^2 \end{bmatrix}$$

The adaptive controller is designed using the approximated equation of motion as

$$u = K_x x + k_r r - \Theta^\top(t)\,\Phi(x)$$

where

$$\dot{\Theta} = -\Gamma\Phi(x)\left[e^\top P - \nu\Phi^\top(x)\,\Theta B^\top P A_m^{-1}\right] B$$

$$A + BK_x = A_m$$

$$Bk_r = B_m$$

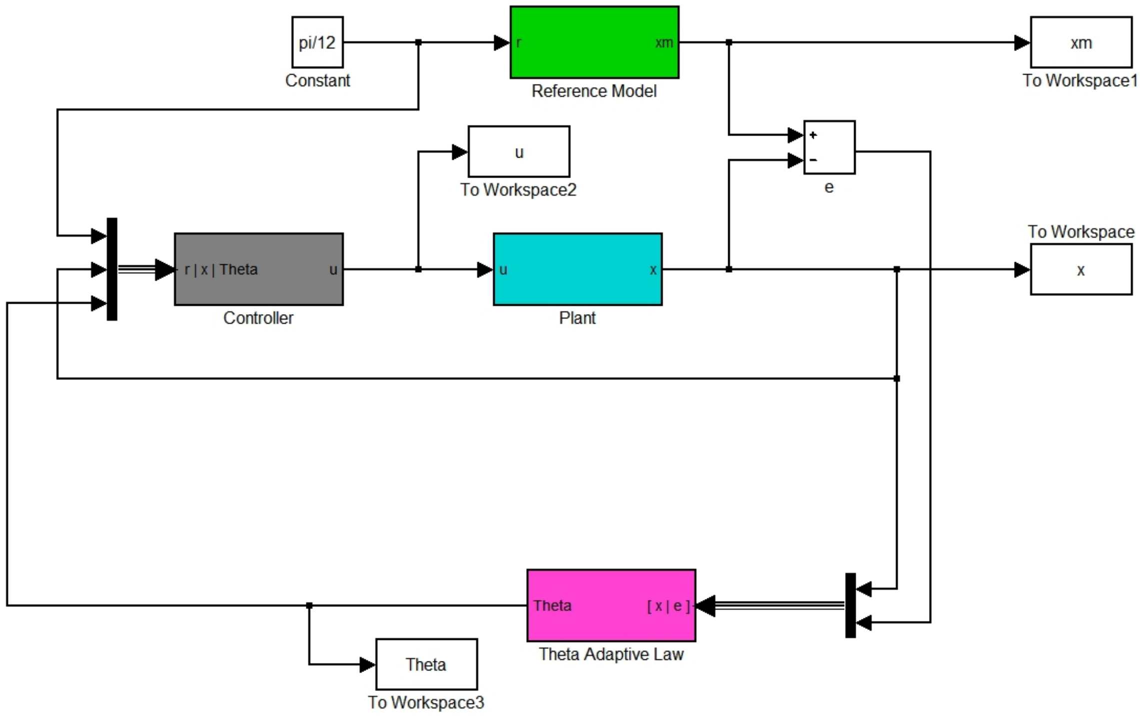
which yield

$$K_x = \left(B^\top B\right)^{-1} B^\top \left(A_m - A\right) = \begin{bmatrix} 0 & \frac{mL^2}{3} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -\omega_m^2 - \frac{3g}{2L} & -2\zeta_m\omega_m \end{bmatrix}$$

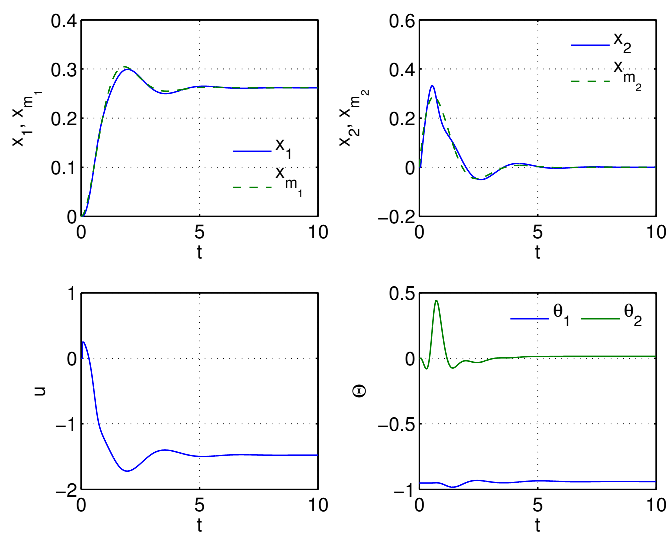
$$= -\frac{mL^2}{3}\begin{bmatrix} \omega_m^2 + \frac{3g}{2L} & 2\zeta_m\omega_m \end{bmatrix} = \begin{bmatrix} -6.6576 & -0.4733 \end{bmatrix}$$

$$k_r = \left(B^\top B\right)^{-1} B^\top B_m = \begin{bmatrix} 0 & \frac{mL^2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ \omega_m^2 \end{bmatrix} = \frac{mL^2\omega_m^2}{3} = 0.9467$$

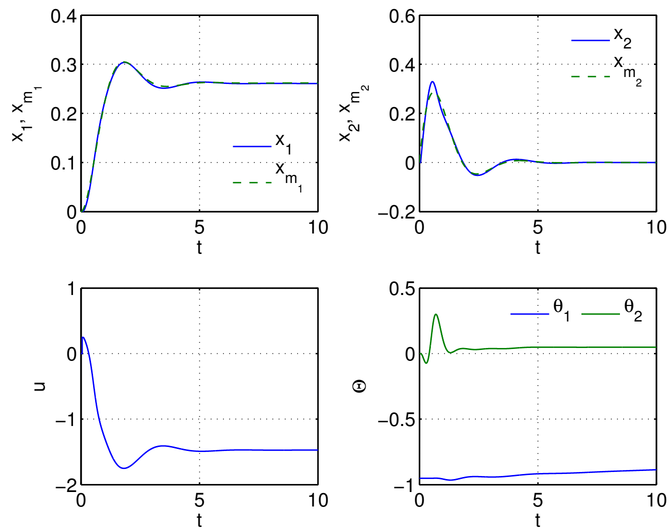
c. The Simulink model is as shown.



The response of the closed-loop system with MRAC is as shown. The closed-plant tracks the reference model quite well.



The response of the closed-loop system with the optimal control modification is as shown. The tracking is slightly better than that with MRAC.



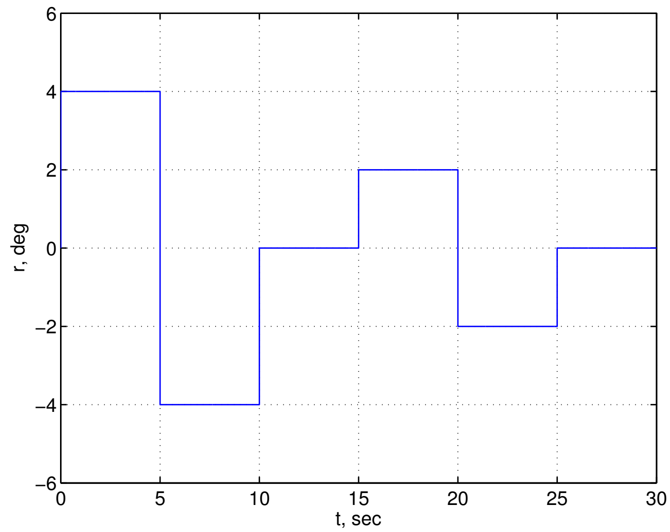
The linear nominal controller with adaptive control works as well as the nonlinear nominal controller in Example 10.1.

2. Implement a longitudinal dynamic model of an aircraft.

$$\dot{x} = Ax + B\left[u\left(t - t_d\right) + \Theta^{*\top} x\right]$$

where $x(t) = \left[\alpha(t)\ \theta(t)\ q(t)\right]^\top$, $u(t) = \delta_e(t)$, and $\Theta^* = \left[\theta_\alpha^*\ 0\ \theta_q^*\right]^\top$, with the following information: $\bar{V} = 795.6251$ ft/sec, $\bar{\gamma} = 0$, $Z_\alpha = -642.7855$ ft/sec^2, $Z_{\delta_e} = -55.3518$ ft/sec^2, $M_\alpha = -5.4898$ sec^{-2}, $M_{\delta_e} = -4.1983$ sec^{-2}, $M_q = -0.6649$ sec^{-1}, $M_{\dot{\alpha}} = -0.2084$ sec^{-1}, $\theta_\alpha^* = 0.5$, $\theta_q^* = -0.5$, and $t_d = 0.1$ sec.

a. Design an adaptive pitch attitude controller using the optimal control modification to enable the closed-loop plant to follow a second-order reference model of the pitch attitude specified by $\zeta_m = \frac{1}{\sqrt{2}}$ and $\omega_m = 2$ rad/sec. Express the adaptive controller with the feedback gain values and the reference model.
b. Implement the adaptive controller in Simulink using the following information: $x(0) = 0$ and $\Theta(0) = 0$ with a time step $\Delta t = 0.01$ sec for: 1) the nominal controller, 2) the standard MRAC with $\Gamma = 500$, and 3) the optimal control modification with $\Gamma = 500$ and $\nu = 0.5$. The reference command signal $r(t)$ is a pitch attitude doublet specified in the following plot.



For each controller, plot the time histories of each of the elements of $x(t)$ and $x_m(t)$ on the same plot, and $u(t)$ for $t \in [0, 30]$ sec. Plot in units of deg for $\alpha(t)$, $\theta(t)$, and $\delta_e(t)$, and deg/sec for $q(t)$. Comment on the simulation results.

Solution:

a. The longitudinal dynamic model of the aircraft is evaluated numerically as

$$\dot{x} = Ax + B\left[u(t-t_d) + \Theta^{*\top} x\right]$$

$$A = \begin{bmatrix} -0.8079 & 0 & 1 \\ 0 & 0 & 1 \\ -5.3214 & 0 & -0.8733 \end{bmatrix},\ B = \begin{bmatrix} -0.0696 \\ 0 \\ -4.1838 \end{bmatrix},\ \Theta^* = \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \end{bmatrix}$$

where $x(t) = \begin{bmatrix} \alpha(t) & \theta(t) & q(t) \end{bmatrix}^\top$, $u(t) = \delta_e(t)$, and $\Theta^* = \begin{bmatrix} \theta_\alpha & 0 & \theta_q \end{bmatrix}^\top$.
The adaptive pitch attitude controller is designed as

$$u = K_x x + k_r r - \Theta^\top(t)\, x$$

where

$$K_x = \begin{bmatrix} k_\alpha & k_\theta & k_q \end{bmatrix} = \begin{bmatrix} -1.2719 & 0.9561 & 0.4673 \end{bmatrix}$$

$$k_r = -k_\theta = -0.9561$$

The optimal control modification adaptive law for $\Theta(t)$ is

$$\dot{\Theta} = -\Gamma\Phi(x)\left[e^\top P - \nu\Phi^\top(x)\,\Theta B^\top P A_m^{-1}\right] B$$

Then, the reference model is established as

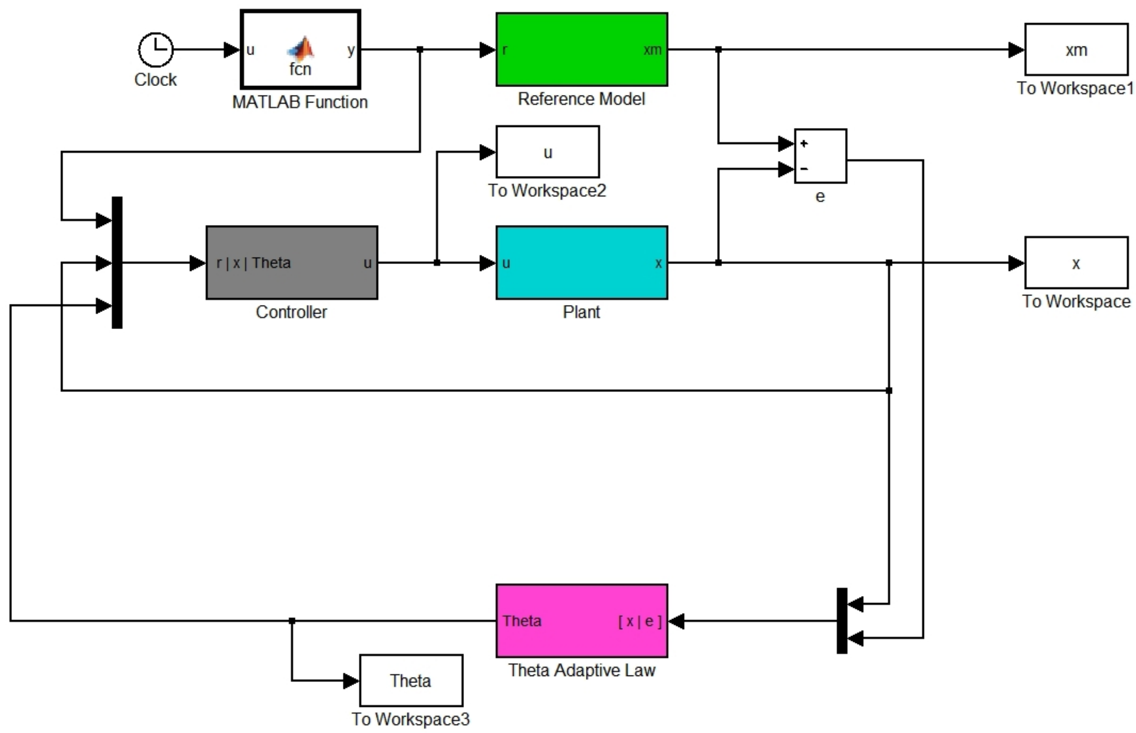
$$\dot{x}_m = A_m x_m + B_m r$$

where

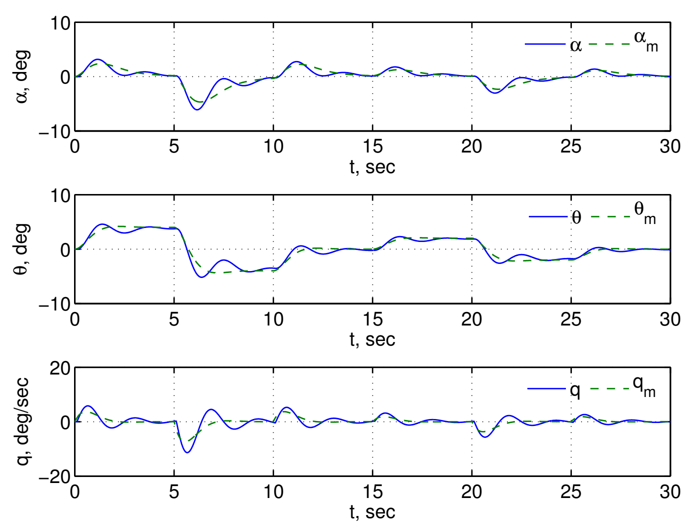
$$A_m = A + BK_x = \begin{bmatrix} -0.7194 & -0.0665 & 0.9675 \\ 0 & 0 & 1 \\ 0 & -4.0000 & -2.8284 \end{bmatrix}$$

$$B_m = Bk_r = \begin{bmatrix} 0.0665 \\ 0 \\ 4.0000 \end{bmatrix}$$

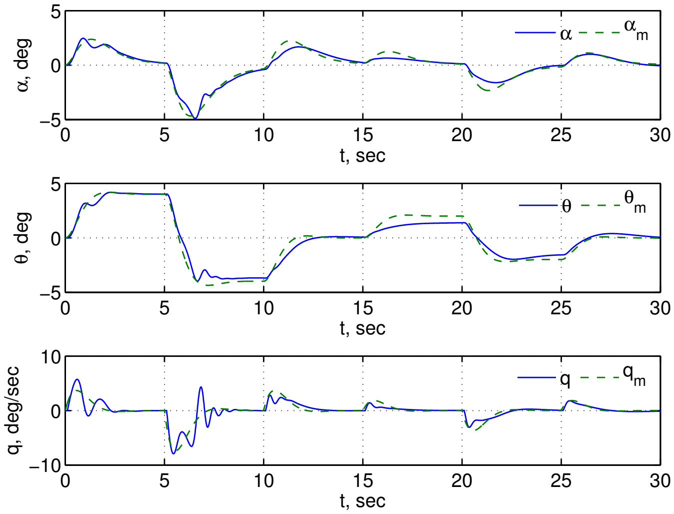
b. The Simulink model is as shown.



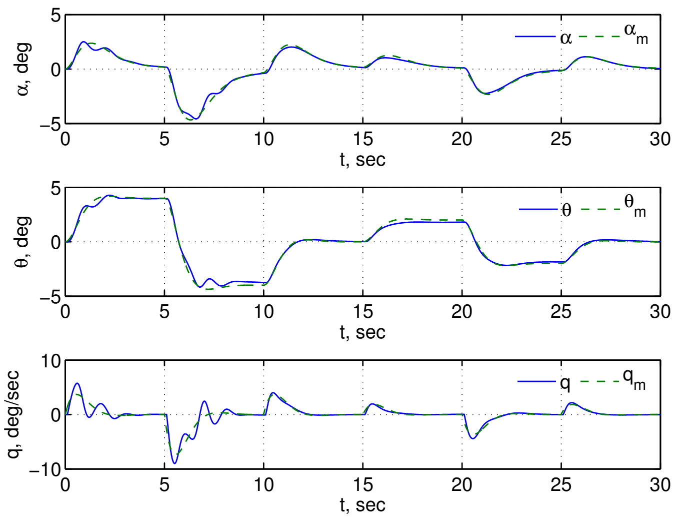
The response of the closed-loop system with the nominal controller is as shown.



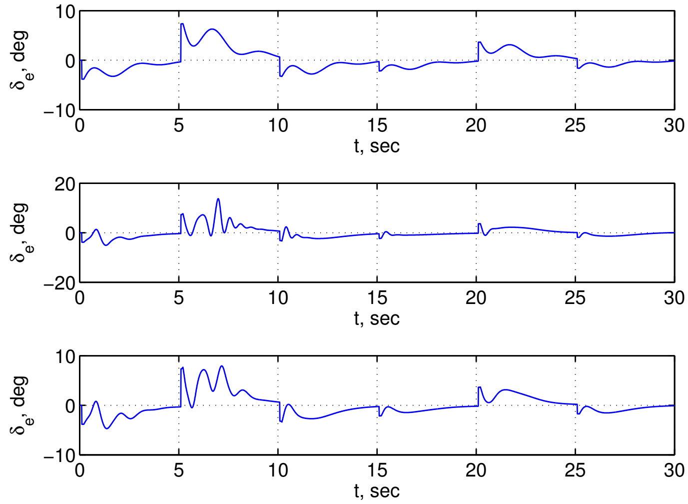
The response of the closed-loop system with MRAC is as shown. The tracking is somewhat improved over that with the nominal controller, but there are some oscillations in the pitch rate.



The response of the closed-loop system with the optimal control modification is as shown. The tracking is better than that with MRAC.



The control signals produced by the nominal controller, MRAC, and optimal control modification are as shown. The standard MRAC produces the largest elevator control surface deflection, whereas the amplitude of the control signal due to the optimal control modification is nominally the same as that due to the nominal controller. The large amplitude of the control signal due to MRAC can lead to robustness issues.



10
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Suggested Exam Questions and Solutions

1. For the following systems, determine the equilibrium points. Use the Lyapunov's direct method to determine the type of Lyapunov stability for each of the equilibrium points. Determine all the invariant sets and the values of the Lyapunov function on the sets. If an equilibrium point is stable, conclude if it is asymptotically stable, and if so, show whether or not it is also exponentially stable.

   a.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)\left(x_1^2 + x_2^2 - 1\right) \\ -(x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right) \end{bmatrix}$$

   b.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^2 - x_2 \\ -x_1 - x_2 + 1 \end{bmatrix}$$

   c.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - (1 + \sin x_1)\, x_2 \end{bmatrix}$$

   Solution:

   a.
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)\left(x_1^2 + x_2^2 - 1\right) \\ -(x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right) \end{bmatrix}$$

   The only equilibrium point is the origin at $x_1^* = 0$ and $x_2^* = 0$.
   Choose a Lyapunov candidate function

$$V(x) = x_1^2 + x_2^2$$

   Then,

$$\dot{V}(x) = 2x_1(x_2 - x_1)\left(x_1^2 + x_2^2 - 1\right) - 2x_2(x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right) = -2\left(x_1^2 + x_2^2\right)\left(x_1^2 + x_2^2 - 1\right)$$

   $\dot{V}(x) < 0$ for all $x(t) \in \mathcal{S}$ where

$$\mathcal{S} = \left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 - 1 > 0\right\}$$

   which does not include the origin. Thus, the equilibrium point is unstable in the sense of Lyapunov.
   Let $\mathcal{R}_1$ be

$$\mathcal{R}_1 = \left\{x \in \mathbb{R}^2 : \dot{V}(x) = 0 \Rightarrow g_1(x) = x_1^2 + x_2^2 = V(x) = 0\right\}$$

   Then,

$$\dot{g}_1(x) = \dot{V}(x) = 0$$

   if and only if $x_1(t) = 0$ and $x_2(t) = 0$. Thus, $\mathcal{R}_1$ is also an invariant set which only contains the origin.
   Then, all trajectories in $\mathcal{R}_1$ must remain in $\mathcal{R}_1$ at all times. Therefore,

$$V\left(x\left(t\right) \in \mathcal{R}_1\right) = 0$$

Let $\mathcal{R}_2$ be

$$\mathcal{R}_2 = \left\{ x \in \mathbb{R}^2 : \dot{V}\left(x\right) = 0 \Rightarrow g_2\left(x\right) = x_1^2 + x_2^2 - 1 = V\left(x\right) - 1 = 0 \right\}$$

Then,

$$\dot{g}_2\left(x\right) = \dot{V}\left(x\right) = 0$$

Thus, $\mathcal{R}_2$ is also an invariant set and

$$\lim_{t \to \infty} V\left(x\left(t\right) \in \mathcal{R}_2\right) = 1$$

b.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^2 - x_2 \\ -x_1 - x_2 + 1 \end{bmatrix}$$

The equilibrium point is found by

$$x_2^2 - x_2 = 0 \Rightarrow x_2^* = 0 \text{ or } x_2^* = 1$$

$$x_1 + x_2 - 1 = 0 \Rightarrow x_1^* = 1 \text{ or } x_1^* = 0$$

So, there are two equilibrium points: one at $x_1^* = 1$ and $x_2^* = 0$, and the other at $x_1^* = 0$ and $x_2^* = 1$.

i. Stability of equilibrium point $(1, 0)$ - First, the system must be transformed so that the equilibrium is at the origin. Let $y_1\left(t\right) = x_1\left(t\right) - 1$ and $y_2\left(t\right) = x_2\left(t\right)$. Then,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2^2 - y_2 \\ -y_1 - 1 - y_2 + 1 \end{bmatrix} = \begin{bmatrix} y_2^2 - y_2 \\ -y_1 - y_2 \end{bmatrix}$$

Choose a Lyapunov candidate function

$$V\left(y\right) = y_1^2 + y_2^2$$

Then,

$$\dot{V}\left(y\right) = 2y_1\left(y_2^2 - y_2\right) + 2y_2\left(-y_1 - y_2\right) = 2y_1y_2^2 - 4y_1y_2 - 2y_2^2 = 2y_2\left(y_1y_2 - 2y_1 - y_2\right)$$

$\dot{V}\left(y\right) \le 0$ for all $y\left(t\right) \in \mathcal{S}$ where

$$\mathcal{S} = \left\{ y\left(t\right) \in \mathbb{R}^2 : \dot{V}\left(y\right) \le 0 \Rightarrow y_1\left(y_2 - 2\right) \ge y_2 \text{ and } y_2 \le 0, \text{ or } y_1\left(y_2 - 2\right) \le y_2 \text{ and } y_2 \ge 0 \right\}$$

Since the boundary of $\mathcal{S}$ goes through the origin, therefore the equilibrium is a saddle point and is unstable.
Let $\mathcal{R}_1$ be

$$\mathcal{R}_1 = \left\{ y \in \mathbb{R}^2 : \dot{V}\left(y\right) = 0 \Rightarrow g_1\left(x\right) = y_2 = 0 \right\}$$

Then,

$$\dot{g}_1\left(y\right) = \dot{y}_2 = -y_1 - y_2 = 0$$

if and only if $y_1\left(t\right) = 0$. Thus, $\mathcal{M}_1 \subset \mathcal{R}_1 = \left\{ y\left(t\right) \in \mathbb{R}^2 : y_1\left(t\right) = 0, \, y_2\left(t\right) = 0 \right\}$ is an invariant set which contains only the origin, and

$$V\left(y\left(t\right) \in \mathcal{M}_1\right) = 0$$

Let $\mathcal{R}_2$ be

$$\mathcal{R}_2 = \left\{ y\left(t\right) \in \mathbb{R}^2 : \dot{V}\left(y\right) = 0 \Rightarrow g_2\left(y\right) = y_1y_2 - 2y_1 - y_2 = 0 \right\}$$

Then,

$$\dot{g}_2\left(y\right) = \left(y_2 - 2\right)\dot{y}_1 + \left(y_1 - 1\right)\dot{y}_2 = \left(y_2 - 2\right)\left(y_2^2 - y_2\right) - \left(y_1 - 1\right)\left(y_1 + y_2\right) \ne 0$$

So $\mathcal{R}_2$ is not an invariant set.

ii. Stability of equilibrium point $(0,1)$- Let $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t) - 1$. Then,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2^2 + y_2 \\ -y_1 - y_2 \end{bmatrix}$$

Choose a Lyapunov candidate function

$$V(y) = y_1^2 + y_2^2$$

Then,

$$\dot{V}(y) = 2y_1\left(y_2^2 + y_2\right) + 2y_2(-y_1 - y_2) = 2y_1y_2^2 - 2y_2^2 = -2y_2^2(1 - y_1)$$

$\dot{V}(y) \le 0$ for all $y(t) \in \mathcal{S}$ where

$$\mathcal{S} = \left\{y(t) \in \mathbb{R}^2 : \dot{V}(y) \le 0 \Rightarrow y_1 \le 1\right\}$$

Since $\mathcal{S}$ includes the origin, therefore the equilibrium is stable in the sense of Lyapunov. $\mathcal{S}$ is also the region of attraction.

Let $\mathcal{R}_1$ be

$$\mathcal{R}_1 = \left\{y(t) \in \mathbb{R}^2 : \dot{V}(y) = 0 \Rightarrow g_1(y) = y_2 = 0\right\}$$

Then,

$$\dot{g}_1(y) = \dot{y}_2 = -y_1 - y_2 = 0$$

if and only if $y_1(t) = 0$. Thus, $\mathcal{M}_1 \subset \mathcal{R}_1 = \left\{y(t) \in \mathbb{R}^2 : y_1(t) = 0,\, y_2(t) = 0\right\}$ is an invariant set which contains only the origin, and

$$V(y(t) \in \mathcal{M}_1) = 0$$

According to the corollary of LaSalle's invariant theorem, the equilibrium point is asymptotically stable.

Let $\mathcal{R}_2$ be

$$\mathcal{R}_2 = \left\{y(t) \in \mathbb{R}^2 : \dot{V}(y) = 0 \Rightarrow g_2(y) = 1 - y_1 = 0\right\}$$

Then,

$$\dot{g}_2(y) = -\dot{y}_1 = -y_2^2 - y_2 = 0$$

if and only if $y_2(t) = 0$ or $y_2(t) = -1$. Since $\mathcal{M}_1$ already includes the origin, therefore $\mathcal{M}_2 \subset \mathcal{R}_2$ is another invariant set that contains only the point $y_1(t) = 1$ and $y_2(t) = -1$, which in fact is the first equilibrium point. Then,

$$V(y(t) \in \mathcal{M}_2) = 1^2 + (-1)^2 = 2$$

The equilibrium is not exponentially stable because

$$\dot{V}(y) = -2y_2^2(1 - y_1) \nleq -\beta V(y),\ \beta > 0$$

c.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - (1 + \sin x_1)\, x_2 \end{bmatrix}$$

The origin is the only equilibrium point. Choose a Lyapunov candidate function

$$V(x) = x_1^2 + x_2^2$$

Then,

$$\dot{V}(x) = 2x_1x_2 + 2x_2\left[-x_1 - (1 + \sin x_1)\, x_2\right] = -2x_2^2(1 + \sin x_1) \le 0$$

The equilibrium is stable in the sense of Lyapunov.

Let $\mathcal{R}$ be

$$\mathcal{R}_1 = \left\{x(t) \in \mathbb{R}^2 : \dot{V}(x) = 0 \Rightarrow g_1(x) = x_2 = 0\right\}$$

Then,

$$\dot{g}_1(x) = \dot{x}_2 = -x_1 - (1 + \sin x_1)\, x_2 = 0$$

if and only if $x_1 = 0$. Thus, $\mathcal{M} \subset \mathcal{R} = \left\{x(t) \in \mathbb{R}^2 : x_1 = 0,\, x_2 = 0\right\}$ is the only invariant set which contains only the origin, and

$$V(x \in \mathcal{M}) = 0$$

From the corollary of the LaSalle's invariant theorem, the origin is asymptotically stable.
At first, it is tempted to conclude that the equilibrium is not exponentially stable because

$$\dot{V}(x) = -2x_2^2(1 + \sin x_1) \leq -2x_2^2 \nleq -\beta V(y),\ \beta > 0$$

However, this nonlinear system needs to be examined further. Notice that the nonlinear system is bounded from below and above by stable linear systems in all cases where

$$\begin{bmatrix} x_2 \\ -x_1 - 2x_2 \end{bmatrix} \leq \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - (1 + \sin x_1)\, x_2 \end{bmatrix} \leq \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}, \forall x_2 > 0$$

$$\begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \leq \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - (1 + \sin x_1)\, x_2 \end{bmatrix} \leq \begin{bmatrix} x_2 \\ -x_1 - 2x_2 \end{bmatrix}, \forall x_2 < 0$$

So, the stability of the original nonlinear system can be determined by the stability of the bounded linear system. For a linear system

$$\dot{x} = Ax$$

where A is Hurwitz, choose a Lyapunov candidate function

$$V(x) = x^\top P x > 0$$

where $P = P^\top > 0$ solves the Lyapunov equation

$$PA + A^\top P = -Q$$

where $Q = Q^\top > 0$.
Then,

$$\dot{V}(x) = \dot{x}^\top P x + x^\top P \dot{x} = x^\top \left(PA + A^\top P\right) x = -x^\top Q x < 0$$

Now, for any positive definite quadratic function, the following relationships apply

$$\lambda_{min}(P)\, \|x\|^2 \leq x^\top P x \leq \lambda_{max}(P)\, \|x\|^2$$

$$-\lambda_{max}(P)\, \|x\|^2 \leq -x^\top P x \leq -\lambda_{min}(P)\, \|x\|^2$$

$$-\lambda_{max}(Q)\, \|x\|^2 \leq -x^\top Q x \leq -\lambda_{min}(Q)\, \|x\|^2$$

Therefore,

$$\dot{V}(x) \leq -\lambda_{min}(Q)\, \|x\|^2 \leq -\lambda_{min}(Q)\, \frac{\lambda_{max}(P)\, \|x\|^2}{\lambda_{max}(P)} \leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} x^\top P x = -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} V(x)$$

The solution of $V(x)$ as an explicit function of t is

$$V(t) = V(t_0) \exp\left[-\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} t\right]$$

Therefore, the equilibrium point of a linear system with A Hurwitz is exponentially stable with a rate of convergence of $\frac{\lambda_{min}(Q)}{\lambda_{max}(P)}$. As a result, the equilibrium point of the original nonlinear system is also exponentially stable.

2. Linearize the systems in problem 2 and determine the types of equilibrium points. Plot phase portraits.

Solution:

a.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)\left(x_1^2 + x_2^2 - 1\right) \\ -(x_1 + x_2)\left(x_1^2 + x_2^2 - 1\right) \end{bmatrix}$$

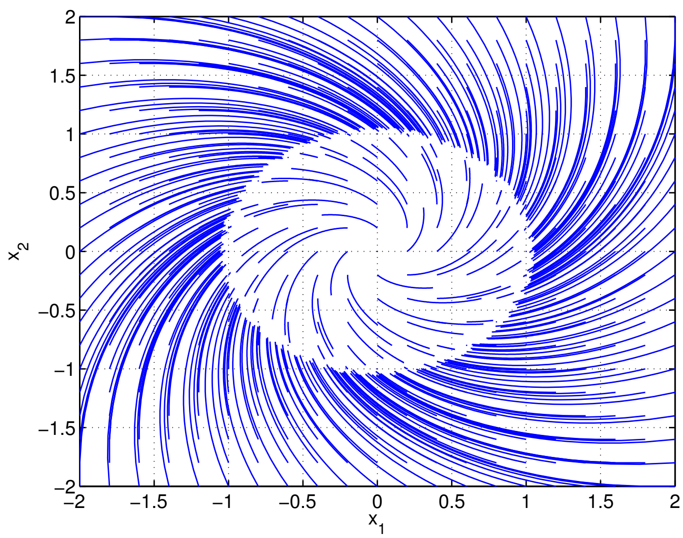
The Jacobian matrix is

$$J(x) = \begin{bmatrix} -\left(x_1^2 + x_2^2 - 1\right) + 2x_1(x_2 - x_1) & \left(x_1^2 + x_2^2 - 1\right) + 2x_2(x_2 - x_1) \\ -\left(x_1^2 + x_2^2 - 1\right) + 2x_1(x_1 + x_2) & -\left(x_1^2 + x_2^2 - 1\right) + 2x_2(x_1 + x_2) \end{bmatrix}$$

$$J(x_1^* = 0, x_2^* = 0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_{1,2} = 1 \pm i$$

The equilibrium is an unstable focus.



b.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^2 - x_2 \\ -x_1 - x_2 + 1 \end{bmatrix}$$

The Jacobian matrix is

$$J(x) = \begin{bmatrix} 0 & 2x_2 - 1 \\ -1 & -1 \end{bmatrix}$$

i. Stability of equilibrium point $(1, 0)$

$$J(x_1^* = 1, x_2^* = 0) = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

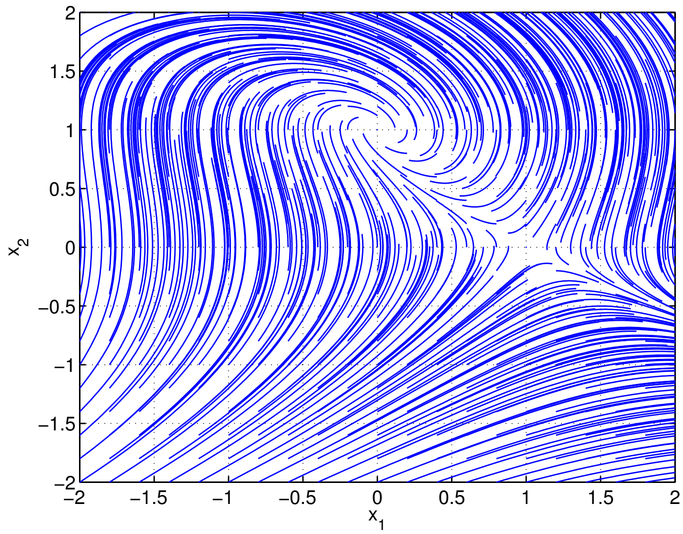
The equilibrium point is a saddle point.

ii. Stability of equilibrium point $(0,1)$

$$J\left(x_1^*=0, x_2^*=1\right)=\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\lambda_{1,2}=\frac{-1 \pm i\sqrt{3}}{2}$$

The equilibrium is a stable focus.



c.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}=\begin{bmatrix} x_2 \\ -x_1-\left(1+\sin x_1\right) x_2 \end{bmatrix}$$

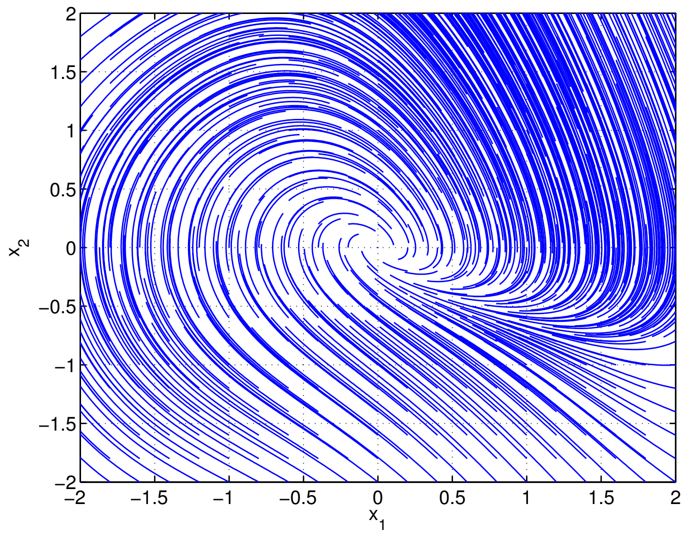
The Jacobian matrix is

$$J\left(x\right)=\begin{bmatrix} 0 & 1 \\ -1-\cos x_1 x_2 & -1-\sin x_1 \end{bmatrix}$$

$$J\left(x_1^*=0, x_2^*=0\right)=\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\lambda_{1,2}=\frac{-1 \pm i\sqrt{3}}{2}$$

The equilibrium is a stable focus.



3. Given the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2+\sin^2 x_1 & 1-\sin x_1 \cos x_2 \\ -1+\sin x_1 \cos x_2 & -2-\cos^2 x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Determine the stability of this system about the origin using the Lyapunov candidate function

$$V(x) = \frac{1}{2} x^\top x$$

If asymptotically stable, determine if the origin is exponentially stable and find the rate of convergence of $\|x\|$ where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^\top$.

Solution:
Choose a Lyapunov candidate function

$$V(x) = \frac{1}{2} x^\top x = \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

Then,

$$\begin{aligned} \dot{V}(x) = x^\top \dot{x} &= -\left(2-\sin^2 x_1\right) x_1^2 + \left(1-\sin x_1 \cos x_2\right) x_1 x_2 - \left(1-\sin x_1 \cos x_2\right) x_1 x_2 - \left(2+\cos^2 x_2\right) x_2^2 \\ &= -\left(2-\sin^2 x_1\right) x_1^2 - \left(2+\cos^2 x_2\right) x_2^2 \le -x_1^2 - 2x_2^2 \le -x_1^2 - x_2^2 = -2V(x) < 0 \end{aligned}$$

Thus, the origin is asymptotically stable. It is also exponentially stable.

$$V(t) \le V(0)\, e^{-2t} \Leftrightarrow \frac{1}{2} x^\top x = \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x(0)\|^2 e^{-2t}$$

So the rate of convergence is 1.

4. Given a linear system

$$\dot{x} = Ax + Bh(t)$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^\top \in \mathbb{R}^2$ and

$$A = \begin{bmatrix} -1 & 2 \\ -4 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; h(t) = \left(1 + e^{-t}\right)(\sin t + \cos t)$$

a. Compute P that solves the Lyapunov equation

$$PA + A^\top P = -I$$

and also compute the eigenvalues of P to verify that P is positive definite.

b. Use the following Lyapunov candidate function

$$V(x) = x^\top P x$$

to compute $\dot{V}(x)$. Establish an upper bound on $\dot{V}(x)$ in terms of $\|x\|$, and then determine a lower bound on $\|x\|$ that satisfies $\dot{V}(x) \le 0$ using the $\mathcal{L}_\infty$ norm and the Cauchy-Schwartz inequality

$$\|CD\| \le \|C\| \|D\|$$

c. Find an analytical solution of an upper bound of the Lyapunov function $V(t)$ as an explicit function of t from $\dot{V}(x)$ in part (b), given $V(0) = 2$, by utilizing the following relationship for a positive definite function

$$\lambda_{min}(P)\|x\|^2 \le V(x) = x^\top P x \le \lambda_{max}(P)\|x\|^2$$

and the following variable transformation

$$W(t) = \sqrt{V(t)}$$

d. Find the ultimate bound of $\|x\|$ by finding the limit of $V(t)$ as $t \to \infty$. If an ultimate bound exists, then the solution of $x(t)$ is uniformly ultimately bounded.

Solution:

a.

$$PA + A^\top P = -I$$

The solution yields

$$P = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$\lambda_{1,2}(P) = \frac{1}{2}, \frac{1}{4} > 0$$

So, $P > 0$.

b. Choose a Lyapunov candidate function

$$V(x) = x^\top P x = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^2$$

Then,

$$\dot{V}(x) = x^\top\left(PA + A^\top P\right)x + x^\top PBh(t) = -x^\top x + 2x^\top PBh(t) \le -\|x\|^2 + 2\|x\|\|PB\|\|h(t)\|$$

But

$$\|PB\| = \left\|\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\| = \left\|\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}\right\| = \frac{1}{2}$$

$$\|h\| = \left\|\left(1 + e^{-t}\right)(\sin t + \cos t)\right\| = 2\sqrt{2}$$

So,

$$\dot{V}(x) \le -\|x\|^2 + 2\sqrt{2}\|x\| = -\|x\|\left(\|x\| - 2\sqrt{2}\right)$$

$\dot{V}(x) \leq 0$ if

$$\|x\| \geq 2\sqrt{2}$$

c. Since

$$\lambda_{min}(P)\|x\|^2 \leq V(x) = x^\top P x \leq \lambda_{max}(P)\|x\|^2 \Leftrightarrow \frac{1}{4}\|x\|^2 \leq V(x) \leq \frac{1}{2}\|x\|^2$$

Then,

$$\|x\|^2 \leq 4V(x) \Rightarrow \|x\| \leq 2\sqrt{V(x)}$$

Also,

$$-\lambda_{max}(P)\|x\|^2 \leq -V(x) \leq -\lambda_{min}(P)\|x\|^2 \Leftrightarrow -\frac{1}{2}\|x\|^2 \leq -V(x) \leq -\frac{1}{4}\|x\|^2$$

Then,

$$-\|x\|^2 \leq -2V(x)$$

Then, $\dot{V}(x)$ can be expressed as

$$\dot{V}(x) \leq -\|x\|^2 + 2\sqrt{2}\|x\| \leq -2V(x) + 4\sqrt{2V(x)}$$

Let

$$W(x) = \sqrt{V(x)} > 0$$

Then,

$$\dot{W}(x) = \frac{\dot{V}(x)}{2\sqrt{V(x)}} = \frac{\dot{V}(x)}{2W(x)}$$

Upon substitution, one gets

$$2W(x)\dot{W}(x) \leq -2W^2(x) + 4\sqrt{2}W(x)$$

Dividing both sides by $2W(x) > 0$ yields

$$\dot{W}(x) \leq -W(x) + 2\sqrt{2}$$

The solution of $W(x)$ is

$$W(t) \leq W(0)e^{-t} + 2\sqrt{2}$$

Substituting back in terms of $V(x)$ gives

$$\sqrt{V(t)} \leq \sqrt{V(0)}e^{-t} + 2\sqrt{2}$$

Given $V(0) = 2$, then

$$\sqrt{V(t)} \leq \sqrt{2}\left(e^{-t} + 2\right)$$

or

$$V(t) \leq 2\left(e^{-t} + 2\right)^2$$

d. The limit of $V(t)$ is

$$\lim_{t\to\infty} V(t) \leq 8$$

So the largest of $V(t)$ is equal to

$$V_{max} = 8$$

But

$$V_{max} = \lambda_{max}(P)\max_x \|x\|^2 = \frac{1}{2}\max_x \|x\|^2$$

Therefore,

$$\frac{1}{2}\max_x \|x\|^2 = 8$$

or

$$\max_x \|x\| = \sqrt{16} = 4$$

Thus, the ultimate bound of $\|x\|$ is 4. Then, $\|x\|$ is upper- and lower-bounded by

$$2\sqrt{2} \le \|x\| \le 4$$

5. Consider a first-order nonlinear SISO system with a matched uncertainty

$$\dot{x} = ax + b\left(u + \theta^* x^2\right)$$

where a is unknown but b is known, and θ^* is unknown.
A reference model is specified by

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m < 0$ and b_m are known, and $r(t)$ is a bounded command signal.

a. Design a direct adaptive controller that enables the plant output $x(t)$ to track the reference model signal $x_m(t)$. Show by Lyapunov stability analysis that the tracking error is asymptotically stable; i.e., $e(t) \to 0$ as $t \to \infty$.
b. Implement the adaptive controller in Simulink, given $b = 2$, $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$. For adaptation rates, use $\gamma_x = 1$ and $\gamma = 1$. For simulation purposes, assume $a = 1$ and $\theta^* = 0.2$ for the unknown parameters. Plot $e(t)$, $x(t)$, $x_m(t)$, $u(t)$, and $\theta(t)$ for $t \in [0, 50]$.
c. Repeat part (b) for $\gamma_x = 10$ and $\gamma = 10$. Plot the same sets of data as in part (b). Comment on the simulation results for parts (b) and (c) regarding the tracking of the reference model, the quality of the signal in terms of the relative frequency content, and the convergence of $k_x(t)$ and $\theta(t)$ as the adaptation rates increase.
d. Repeat part (b) for $r(t) = 1(t)$ where $1(t)$ is the unit-step function. Plot the same sets of data as in part (b). Comment on the convergence of $k_x(t)$ and $\theta(t)$ to the ideal values k_x^* and θ^*.

Solution:

a. Define the ideal gain k_x^* that satisfies one of the model matching conditions

$$a + bk_x^* = a_m$$

and the known gain k_r that satisfies the other model matching condition

$$bk_r = b_m$$

since b is known.
The adaptive controller is given by

$$u = k_x(t)\, x + k_r r - \theta(t)\, x^2$$

Let $\tilde{k}_x(t) = k_x(t) - k_x^*$ and $\tilde{\theta}(t) = \theta(t) - \theta^*$ be the estimation errors. Then, the closed-loop plant model is

$$\dot{x} = \left(\underbrace{ax + bk_x^*}_{a_m} + b\tilde{k}\right) x + \underbrace{bk_r}_{b_m} r - b\tilde{\theta}x^2$$

The closed-loop tracking error equation is obtained as

$$\dot{e} = \dot{x}_m - \dot{x} = a_m e - b\tilde{k}_x x + b\tilde{\theta}x^2$$

Choose a Lyapunov candidate function

$$V\left(e,\tilde{k}_x,\tilde{\theta}\right)=e^2+\frac{\tilde{k}_x^2}{\gamma_x}+\frac{\tilde{\theta}^2}{\gamma}$$

Then,

$$\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)=2e\left(a_m e-b\tilde{k}_x x+b\tilde{\theta}x^2\right)+\frac{2\tilde{k}_x\dot{\tilde{k}}_x}{\gamma_x}+\frac{2\tilde{\theta}\dot{\tilde{\theta}}}{\gamma}=2a_m e^2-2\tilde{k}_x\left(xeb-\frac{\dot{\tilde{k}}_x}{\gamma_x}\right)+2\tilde{\theta}\left(x^2eb+\frac{\dot{\tilde{\theta}}}{\gamma}\right)$$

The adaptive laws are then obtained as

$$\dot{k}_x=\gamma_x xeb$$
$$\dot{\theta}=-\gamma x_\theta^2 eb$$

Then,

$$\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)=2a_m e^2\leq 0$$

Since $\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)$ is negative semi-definite, $e(t)\in\mathcal{L}_\infty$, $k_x(t)\in\mathcal{L}_\infty$, and $\theta(t)\in\mathcal{L}_\infty$, i.e., they are bounded. Also,

$$V(t\to\infty)-V(t_0)=\int_{t_0}^{\infty}\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)dt=2a_m\int_{t_0}^{\infty}e^2(t)\,dt=2a_m\|e\|_2^2$$

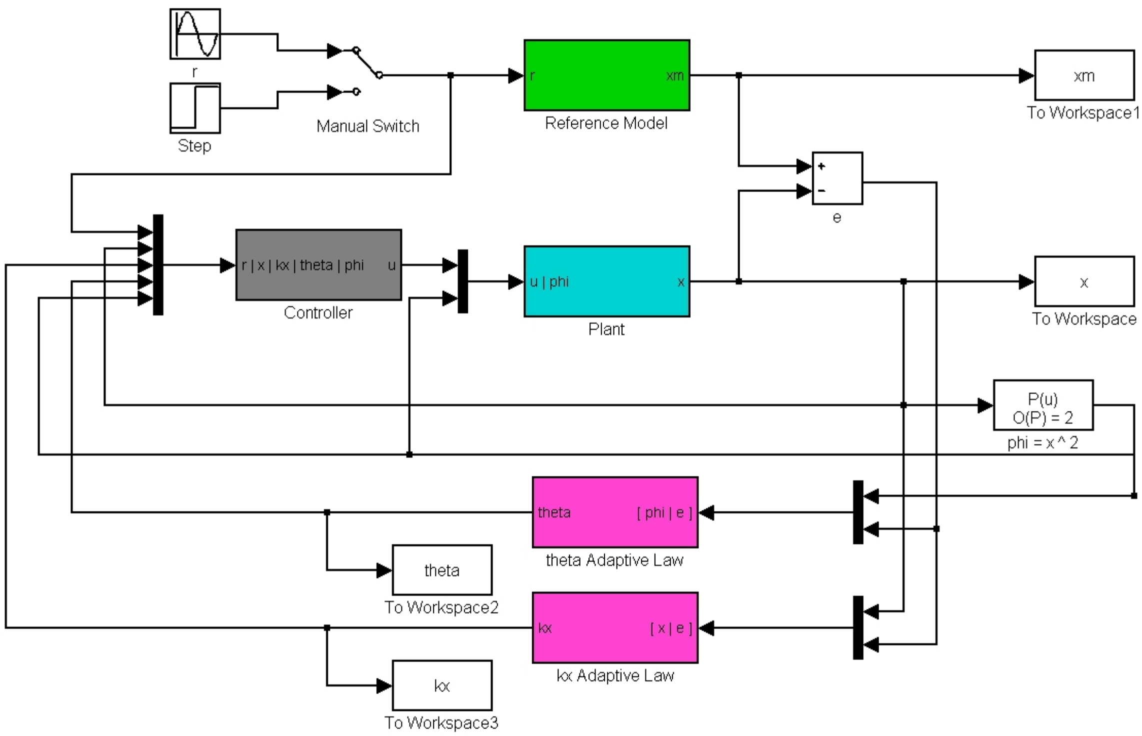
So, $V\left(e,\tilde{k}_x,\tilde{\theta}\right)$ has a finite limit as $t\to\infty$. Since $\|e\|_2$ exists, therefore $e(t)\in\mathcal{L}_2\cap\mathcal{L}_\infty$.

Differentiating $\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)$ yields

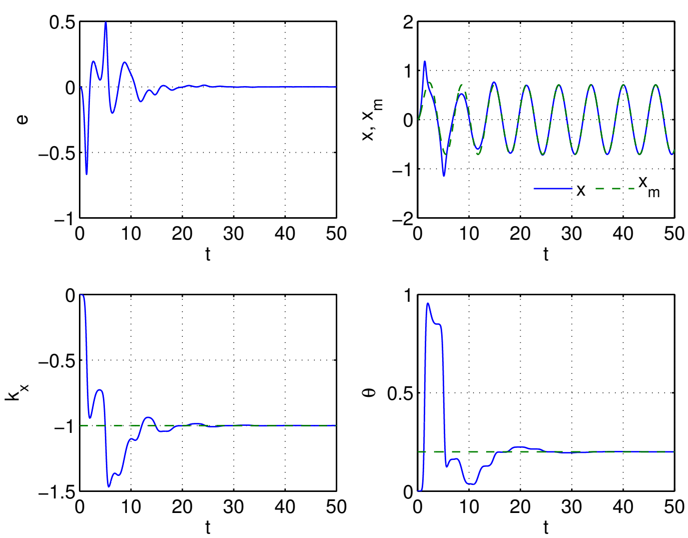
$$\ddot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)=4a_m e\left(a_m e-b\tilde{k}_x x+b\tilde{\theta}x^2\right)$$

Since $e(t)\in\mathcal{L}_2\cap\mathcal{L}_\infty$, $k_x(t)\in\mathcal{L}_\infty$, and $\theta(t)\in\mathcal{L}_\infty$ by the virtue that $\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)\leq 0$, and $x(t)\in\mathcal{L}_\infty$ since $e(t)\in\mathcal{L}_2\cap\mathcal{L}_\infty$ and $x_m(t)\in\mathcal{L}_\infty$ because $r(t)\in\mathcal{L}_\infty$, therefore $\ddot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)\in\mathcal{L}_\infty$. Thus, $\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)$ is uniformly continuous. It follows from the Barbalat's lemma that $\dot{V}\left(e,\tilde{k}_x,\tilde{\theta}\right)\to 0$ which implies $e(t)\to 0$ as $t\to\infty$. The tracking error is asymptotically stable.

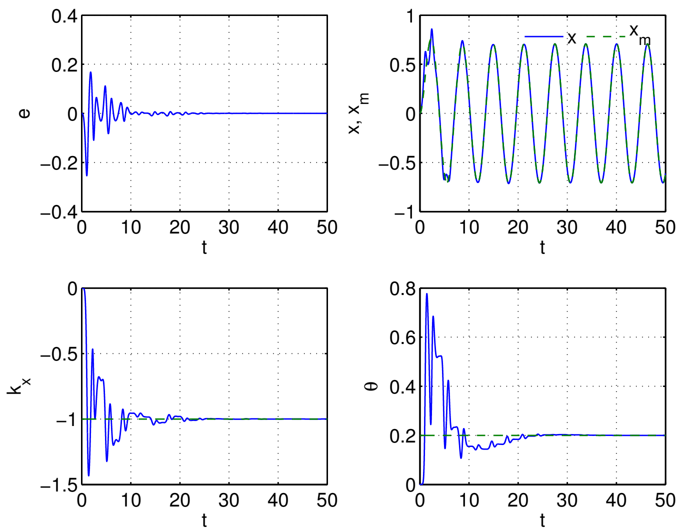
b. The Simulink model of the adaptive controller is as shown.



The ideal, unknown feedback gain is $k_x^* = -1$.
The response of the adaptive controller for $\gamma_x = 1$ and $\gamma_\theta = 1$ is as shown.

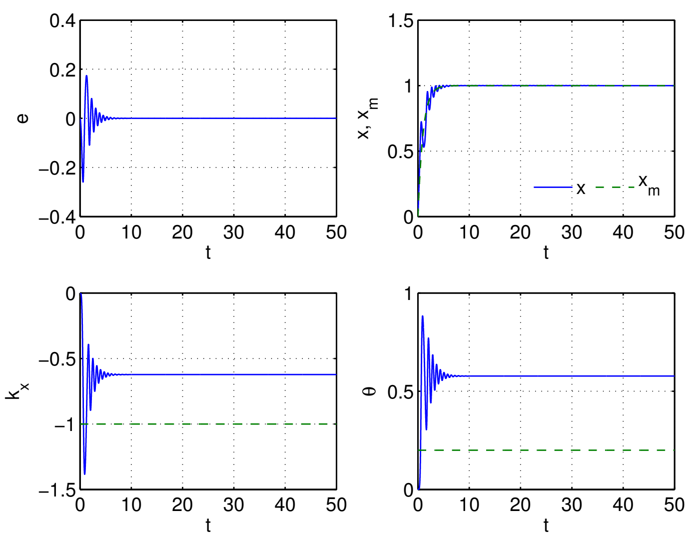


c. The response of the adaptive controller for $\gamma_x = 10$ and $\gamma_\theta = 10$ is as shown.



As the adaptation rates γ_x and γ_θ increase, the tracking error reduces more rapidly and the plant model tends to the reference model at a faster rate. However, there is an increase in the frequency content as the adaptation rates increase. The parameters $k_x(t)$ and $\theta(t)$ also converge more rapidly to their corresponding ideal values as the adaptation rates increase.

d. The response of the adaptive controller for $\gamma_x = 10$ and $\gamma_\theta = 10$ is as shown.



The parameters $k_x(t)$ and $\theta(t)$ do not converge to their corresponding ideal values in this case. This is due to the input signal $r(t)$ not possessing a quality known as "persistent excitation", which is required for parameter convergence. A persistently exciting signal possesses a sufficient frequency content in order to excite the plant, so that there is a sufficient response of the plant output for the adaptive law to correctly estimate the plant parameters.

6. Given a first-order nonlinear system

$$\dot{x} = ax + Bu + cx^2$$

where $x(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}^2$, a is an unknown constant, $B = \begin{bmatrix} 1 & 2 \end{bmatrix}$ is known, and c is an unknown constant. The reference model is specified as

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$.
Express the system in the form of a matched uncertainty

$$\dot{x} = ax + B\left[u + \Theta^{*\top}\Phi(x)\right]$$

Determine K_x^*, K_r^*, and Θ^*. Write down the adaptive laws for $K_x(t)$ and $\Theta(t)$. Implement the controller in Simulink. Use $\gamma_x = \gamma_\Theta = 1$. Assume all initial conditions to be zero and $a = 1$, $c = 0.2$ for simulation purpose. Plot $e(t)$, $x(t)$ versus $x_m(t)$, $K_x(t)$, and $\Theta(t)$ for $t \in [0, 40]$.

Solution:
The plant can be written as

$$\dot{x} = ax + Bu + BB^\top\left(BB^\top\right)^{-1} cx^2 = ax + B\left[u + \Theta^{*\top}x^2\right]$$

where

$$\Theta^{*\top} = B^\top\left(BB^\top\right)^{-1} c = \begin{bmatrix} 0.04 \\ 0.08 \end{bmatrix}$$

The ideal control gains can be computed from the model matching conditions as

$$a + BK_x^* = a_m \Rightarrow K_x^* = B^\top\left(BB^\top\right)^{-1}(a_m - a) = \begin{bmatrix} -0.4 \\ -0.8 \end{bmatrix}$$

$$BK_r = b_m \Rightarrow K_r^* = B^\top\left(BB^\top\right)^{-1} b_m = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$$

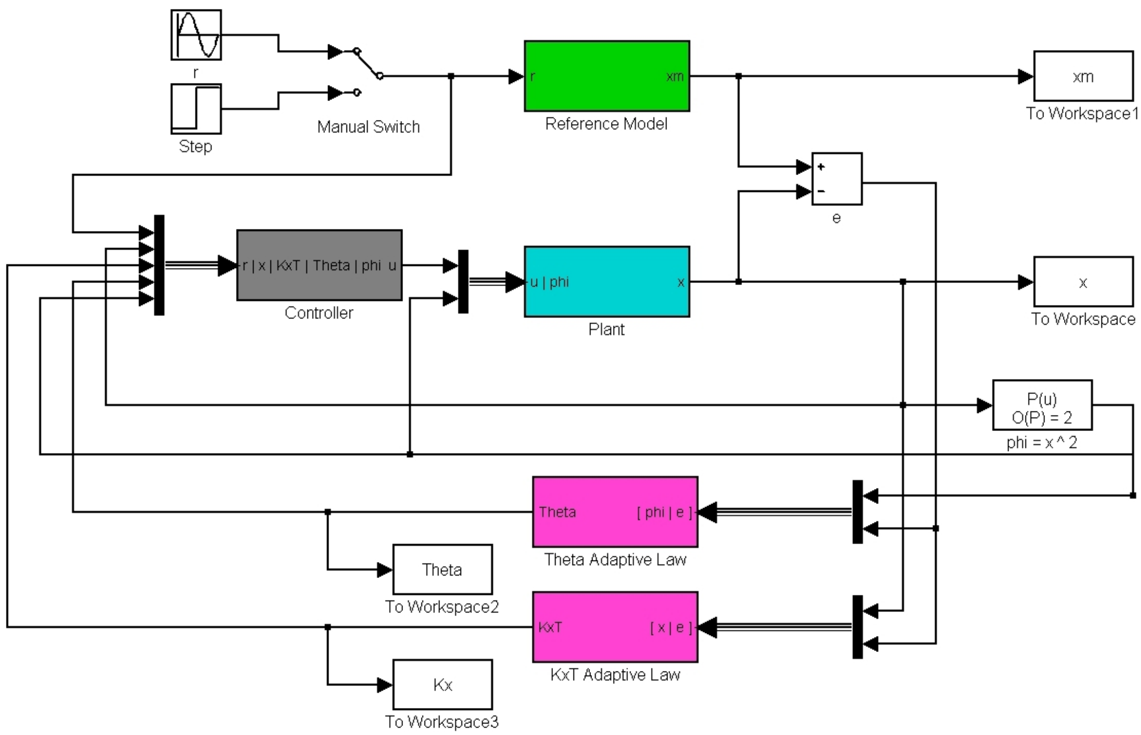
The adaptive controller is given by

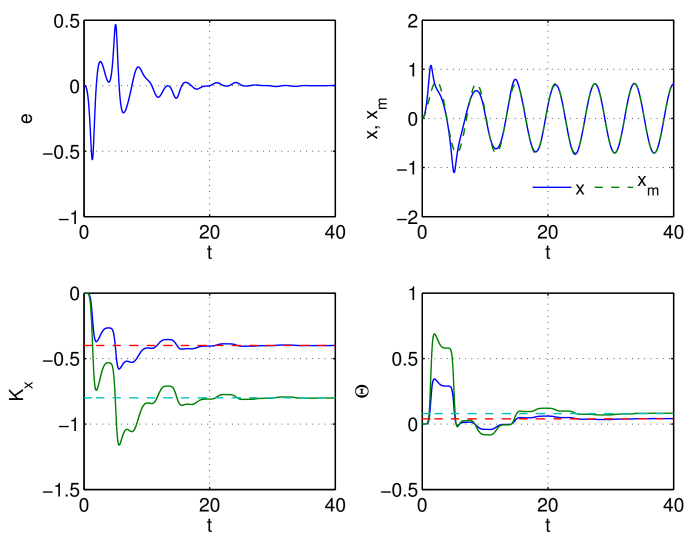
$$u = K_x(t)\,x + K_r r - \Theta^\top(t)\,\Phi(x)$$

The adaptive laws are

$$\dot{K}_x^\top = \gamma_x x e B$$
$$\dot{\Theta} = -\gamma_\Theta x^2 e B$$

The Simulink model and simulation results are as shown.

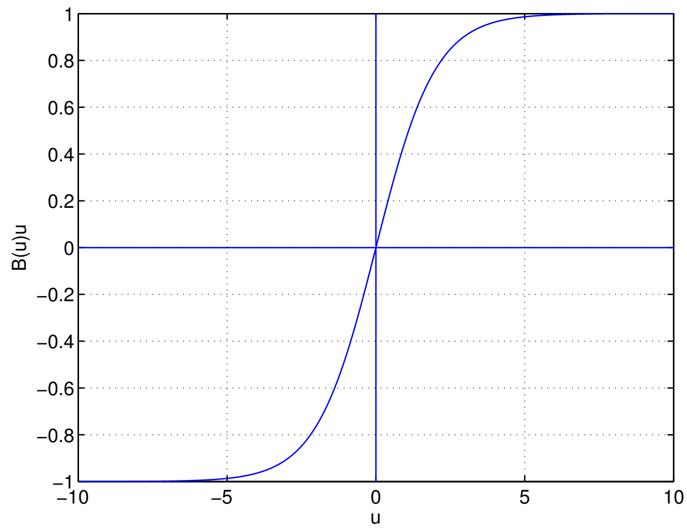




7. The symmetric sigmoidal function

$$\sigma(x) = \frac{1 - e^{-x}}{1 + e^{-x}}$$

can be used to model a control actuator saturation, which frequently exists in real systems.



Saturation occurs when a control actuator ceases to be effective. When there are more inputs than commands, a control allocation strategy should be developed to allocate redundant control effectors in an optimal manner so as to produce an output that tracks a command. Define $y(u)$ as the output of a control allocator as

$$y = V^\top \sigma\left(W^\top u\right)$$

where $y(u) \in \mathbb{R}^n$, $V \in \mathbb{R}^m \times \mathbb{R}^n$, $W \in \mathbb{R}^p \times \mathbb{R}^m$, and $u \in \mathbb{R}^p$, $p \geq n$.
V can be used to specify a saturation limit, while $V^\top W^\top$ plays the role of a nonlinear $B(u)$ matrix. Develop an optimal control allocation strategy by computing the gradient of the following cost function with respect to u, i.e, ∇J_u

$$J(u) = \frac{1}{2}\epsilon^\top \epsilon$$

where $\epsilon = y - r$ and $r \in \mathbb{R}^n$ is a command vector for which an optimal control vector u is to be found to minimize the cost function.
Given $r = 1$ and

$$V = \begin{bmatrix} 0.75 \\ 0.5 \end{bmatrix}, \; W = \begin{bmatrix} 1.2 & 0.8 \\ 0.5 & 1.5 \end{bmatrix}$$

Write a Matlab code to compute u using the steepest descent method with an adaptation rate $\varepsilon = 0.1$ and a number of iteration of $n = 1000$. Indicate the final value of u and plot u.

Solution:
The cost function is expressed as

$$J(u) = \frac{1}{2}\epsilon^\top \epsilon = \frac{1}{2}\left[V^\top \sigma\left(W^\top u\right) - r\right]^\top \epsilon$$

Evaluating the gradient of the cost function gives

$$\nabla J_u = \frac{\partial J}{\partial u} = W\sigma'\left(W^\top u\right) V\epsilon$$

where

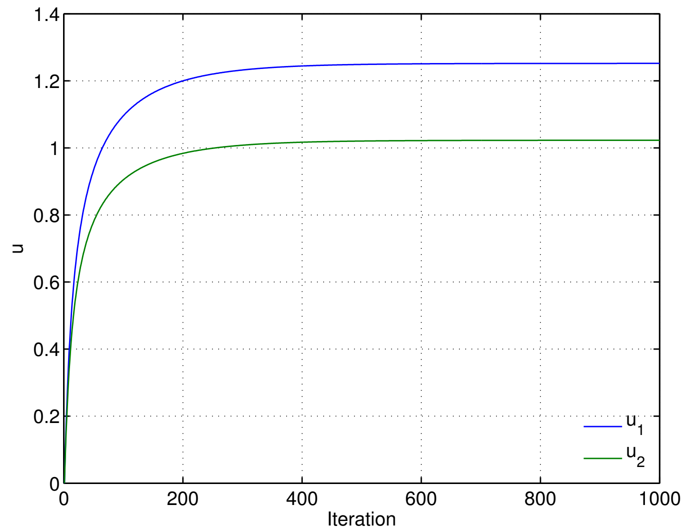
$$\sigma(x) = \frac{1 - e^{-x}}{1 + e^{-x}}$$

$$\sigma'(x) = \frac{2e^{-x}}{\left(1 + e^{-x}\right)^2}$$

The steepest descent method is expressed as

$$u_{i+1} = u_i - \varepsilon \nabla J_{u_i} = u_i - \varepsilon W \sigma^{'} \left(W^\top u_i\right) V \epsilon$$

The solution is as shown and the final value of u is $u = \left[\, 1.2521 \; 1.0228 \,\right]^\top$.



8. Adaptive control can be used for disturbance rejection. Disturbances are usually time signals that may have multiple frequency contents. Unlike unstructured uncertainty in the form of an unknown function $f(x)$, an unknown function of time $f(t)$ should be approximated by a bounded function. This prevents adaptive signals from blowing up in time. Both the sigmoidal and radial basis functions are bounded functions, but a polynomial function is not. Consider a first-order system with an unknown disturbance

$$\dot{x} = ax + b\left[u + f(t)\right]$$

where a and $f(t)$ are unknown, but $b = 2$. For simulation purpose, $a = 1$ and $f(t) = 0.1 \sin 2.4t - 0.3 \cos 5.1t + 0.2 \sin 0.7t$.
The reference model is given by

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r(t) = \sin t$.
Implement in Simulink a direct adaptive control using the least-squares gradient method to approximate $f(t)$ by a sigmoidal neural network with $\Theta(t) \in \mathbb{R}^5$, $W(t) \in \mathbb{R}^2 \times \mathbb{R}^4$ using the activation function $\sigma(x) = \frac{1}{1+e^{-x}}$. Write down the neural net adaptive laws for $k_x(t)$, $\Theta(t)$, and $W(t)$. All initial neural net weights are randomized between 0 and 1. The initial condition for $k_x(t)$ is zero. Use $\Gamma_x = 10I$. Plot $e(t)$, $\epsilon(t)$, $x(t)$ versus $x_m(t)$ with disturbance rejection, $x(t)$ versus $x_m(t)$ without disturbance rejection, $k_x(t)$, $\Theta(t)$, and $W(t)$ for $t \in [0, 40]$.

Solution:
The adaptive controller is given by

$$u = k_x(t)\, x + k_r r - \Theta^\top(t)\, \Phi\left(W^\top \bar{t}\right)$$

The adaptive laws are

$$\dot{k}_x = \frac{\gamma_x x \epsilon}{b}$$

$$\dot{\Theta} = -\frac{\Gamma_{\Theta}\Phi\left(W^{\top}\bar{t}\right)\epsilon}{b}$$

$$\dot{W} = -\frac{\Gamma_{W}\bar{t}\epsilon V^{\top}\sigma^{'}\left(W^{\top}\bar{t}\right)}{b}$$

where

$$\bar{t} = \left[\,1\;t\,\right]^{\top}$$

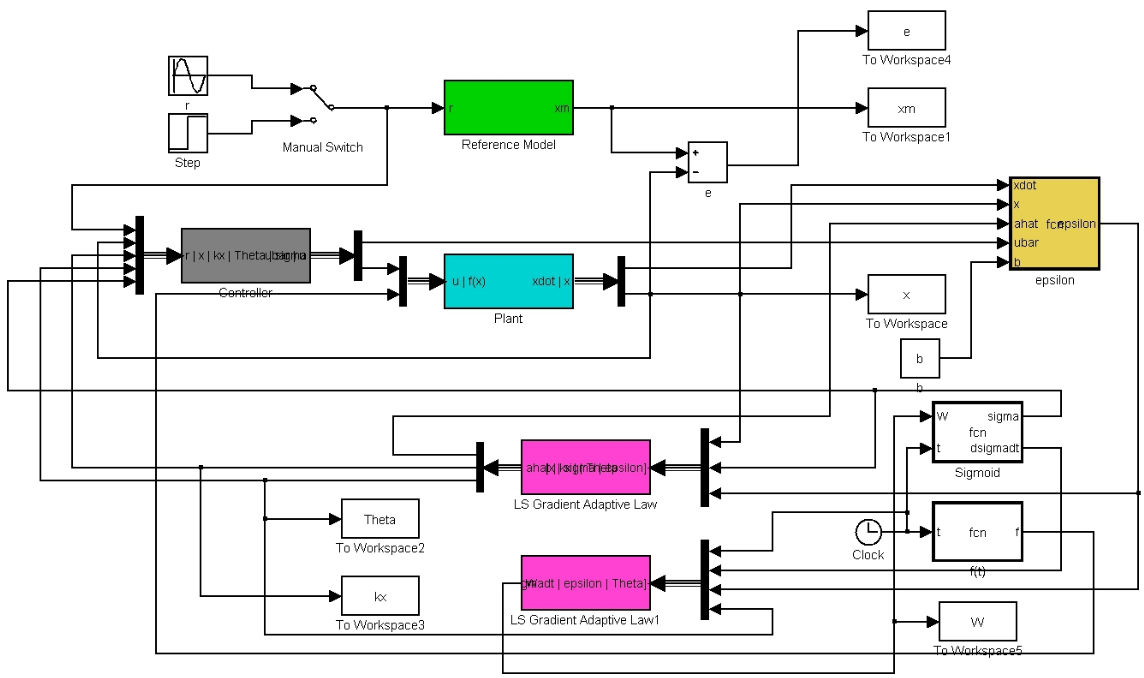
$$\epsilon = \hat{a}x + b\bar{u} - \dot{x}$$

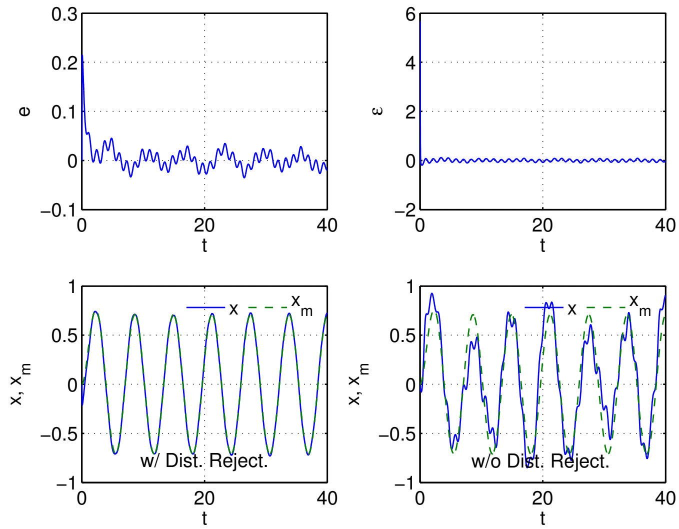
$$\hat{a} = a_m - bk_x$$

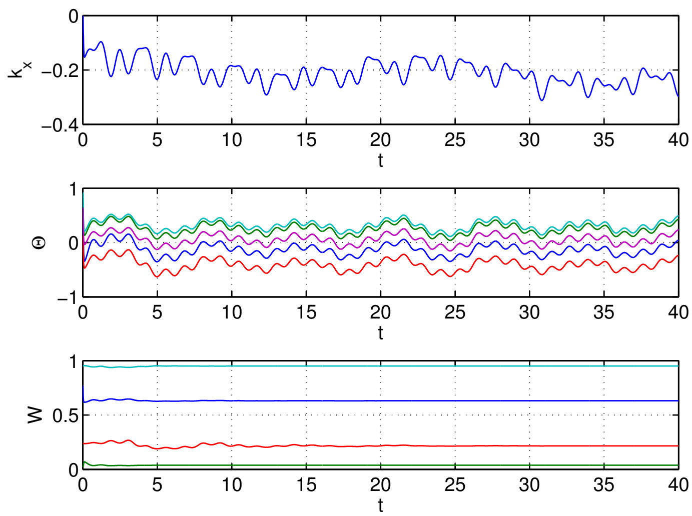
$$\bar{u} = k_x x + k_r r$$

$$k_r = \frac{b_m}{b}$$

The Simulink model and simulation results are as shown.







9. Given the following plant

$$\dot{x} = -2x - z + u + w$$

$$\dot{z} = -3z + 4u$$

$$y = x$$

where $x(t)$ is the plant output, $z(t)$ is an internal state, and $w(t) = 1$ is a constant disturbance

a. If a linear controller $u(t) = k_x x(t)$ is used, where k_x is constant, express the transfer function from $w(t)$ to $x(t)$. Find all values of k_x for which the closed-loop plant is stable.

b. Find the equilibrium state $\bar{x}$ as a function of k_x from part (a). Suppose an adaptive regulator controller is designed with the σ modification

$$u = k_x(t)\, x$$

$$\dot{k}_x = -\gamma\left(x^2 + \sigma k_x\right)$$

Find the minimum value of the modification parameter σ_{min} to within 0.01 by finding the roots of a polynomial in terms of $\bar{k}_x$ for which one or more roots satisfies the values of k_x in part (a). Calculate $\bar{k}_x$ and $\bar{x}$.

c. Implement the adaptive controller in Simulink with $\sigma = \sigma_{min} - 0.05$ and $\sigma = 0.5$ using the following information: $x(0) = 0$, $z(0) = 0$, $k_x(0) = 0$, and $\gamma = 10$ using a time step $\Delta t = 0.001$ sec. Plot the time histories of $x(t)$ and $\theta(t)$ for $t \in [0, 10]$ sec for both values of σ. Comment on the two responses. Calculate $\bar{k}_x$ and $\bar{x}$ for $\sigma = 0.5$ analytically and compare them with the simulation results.

Solution:

a. The open-loop plant is expressed as

$$x = \frac{-z + u + w}{s + 2}$$

$$z = \frac{4u}{s + 3}$$

$$x = \frac{(s-1)\,u + (s+3)\,w}{s^2 + 5s + 6}$$

Let $u(t) = k_x x(t)$. Then, the transfer function from $w(t)$ to $x(t)$ is obtained as

$$x = \frac{(s-1)\,k_x x + (s+3)\,w}{s^2 + 5s + 6}$$

$$\frac{x}{w} = \frac{s+3}{s^2 + (5 - k_x)\,s + 6 + k_x}$$

The closed-loop plant is stable for $-6 \le k_x \le 5$.

b. The equilibrium state $\bar{x}$ is found by setting $s = 0$ as $t \to \infty$

$$\bar{x} = \frac{3}{6 + k_x}$$

The equilibrium value of $k_x(t)$ of the σ modification is found by setting $\dot{k}_x(t) = 0$

$$\bar{k}_x = -\frac{\bar{x}^2}{\sigma} = -\frac{9}{\sigma\left(6 + \bar{k}_x\right)^2}$$

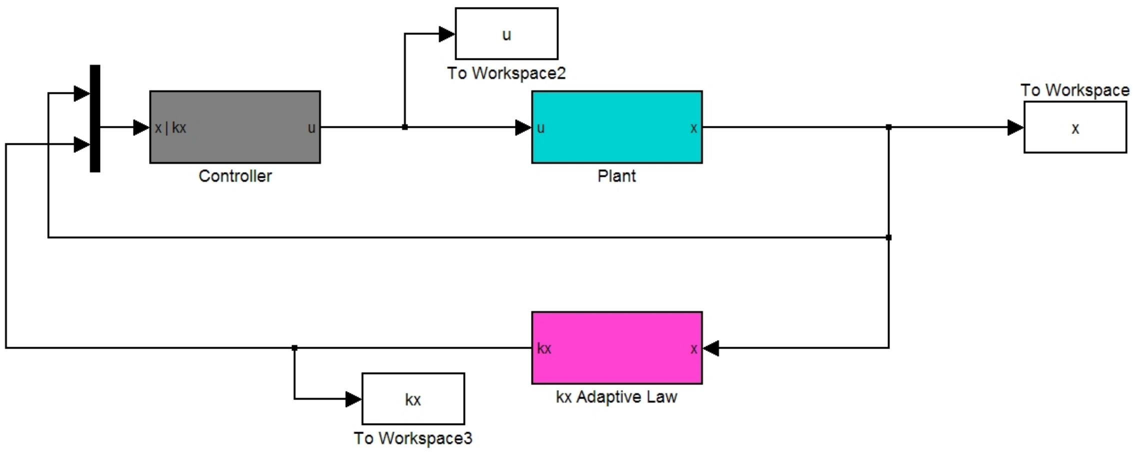
This results in a third-degree polynomial

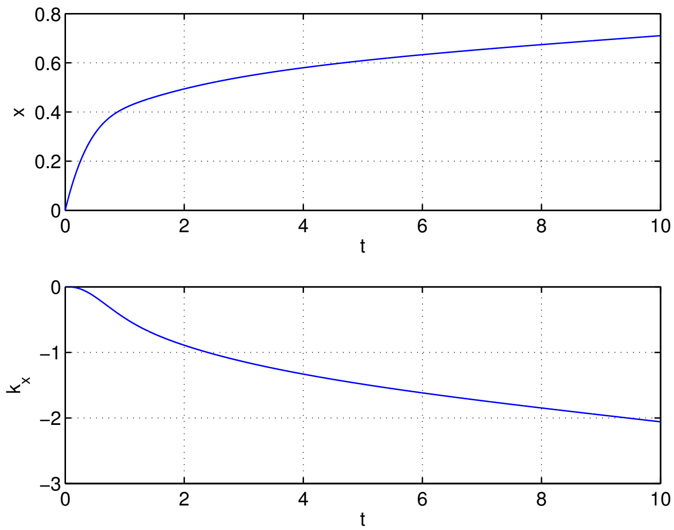
$$\bar{k}_x^3 + 12\bar{k}_x^2 + 36\bar{k}_x + \frac{9}{\sigma} = 0$$

For $\sigma = 0.29$, the roots are -7.9729, -2.4158, and -1.6112. For $\sigma < 0.29$, the real roots are less than -6. Thus, $\sigma_{min} = 0.29$, $\bar{k}_x = -1.6112$ and $\bar{x} = 0.6836$.

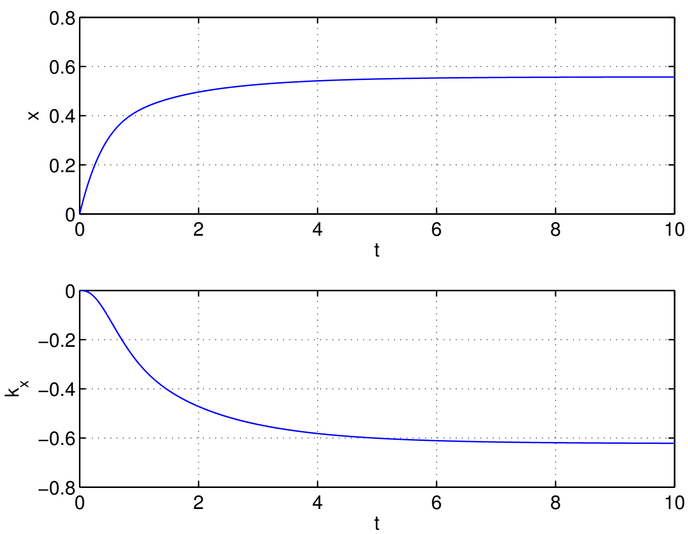
c. The Simulink model is as shown.

The closed-loop plant with $\sigma = \sigma_{min} - 0.05 = 0.24$ is unstable due to parameter drift with $k_x \to -\infty$ as $t \to \infty$. This validates the analytical result of $\sigma_{min} = 0.29$, for which the system begins to be stable.





The closed-loop plant with $\sigma = 0.5$ is completely stable as shown.



The roots of the polynomial in part (b) are -7.5446, -3.8329, -0.6224. The feasible solution is $\bar{k}_x = -0.6224$ which gives $\bar{x} = 0.5579$ since the first root would result in an unstable closed-loop plant, and the second root would result in $\bar{x} = 1.3844$ that tends away from zero. The simulation results are $\bar{k}_x = -0.6215$ and $\bar{x} = 0.5575$ which agree very well with the analytical results.

10. Given a first-order SISO system with a matched uncertainty

$$\dot{x} = ax + b\left(u + \theta^* x^2\right)$$

subject to $x(0) = x_0$, where $a = 1$ and $b = 1$ are known, and $\theta^* = 2$ is unknown.
An adaptive controller is designed using the optimal control modification adaptive law to enable the plant to follow a reference model

$$\dot{x}_m = a_m x_m + b_m r$$

where $a_m = -1$, $b_m = 1$, and $r(t) = 1$
The adaptive controller is given by

$$u = k_x x + k_r r - \theta(t) x^2$$

a. Express the closed-loop system with the nominal (non-adaptive) controller $u = k_x x$ in terms of the reference model parameters a_m and b_m. Determine whether or not the closed-loop system with the nominal controller is unconditionally (globally) stable by explicitly integrating the plant model to find the solution of $x(t)$. If the closed-loop plant is not globally stable, find the stability condition imposed on x_0.

b. Express the optimal control modification adaptive law for $\theta(t)$. Use Section 9.5.3 to estimate the limiting value of the modification parameter ν_{max} to within 0.001. If applicable, express $\varphi(\|x\|, \|x_m\|, \nu, \theta^*)$. Then, ν_{max} can be found by trial and error to be the largest value for which $\varphi(\|x\|, \|x_m\|, \nu_{max}, \theta^*) = 0$ such that $\|x\| > \|x_m\|$. Express the ultimate bound of $\|e\|$ and $\left\|\tilde{\theta}\right\|$ as a function $\|x\|$, ν, and γ. Evaluate them for $\gamma = 500$.

c. Implement the adaptive controller in Simulink with MRAC for which $\nu = 0$ and the optimal control modification with $\nu = \nu_{max}$ determined from part (b) using the following information: $x(0) = 1$, $\theta(0) = 0$, and $\gamma = 500$ with a time step $\Delta t = 0.001$ sec. Plot the time histories of $x(t)$, $u(t)$, and $\theta(t)$ for $t \in [0, 10]$ sec for both MRAC and the optimal control modification. Comment on the responses of the two adaptive controllers and compare the maximum tracking error $\|e\|$ and maximum parameter estimation error $\left\|\tilde{\theta}\right\|$ due to the optimal control modification to those determined from part (b).

Solution:

a. The closed-loop plant with the nominal controller is

$$\dot{x} = a_m x + b\theta^* x^2$$

This equation can be integrated as

$$\int \frac{dx}{a_m x + b\theta^* x^2} = t + c$$

Using partial fraction, then

$$\frac{1}{a_m x + b\theta^* x^2} = \frac{1}{a_m x} - \frac{b\theta^*}{a_m\left(b\theta^* x + a_m\right)}$$

Upon integration, we get

$$\frac{1}{a_m} \ln \frac{b\theta^* x}{b\theta^* x + a_m} = t + c$$

Using the initial condition, c is determined to be

$$c = \frac{1}{a_m} \ln \frac{b\theta^* x_0}{b\theta^* x_0 + a_m}$$

The solution of $x(t)$ is then obtained as

$$x = \frac{a_m x_0}{(a_m + b\theta^* x_0) e^{-a_m t} - b\theta^* x_0}$$

If $b\theta^* x_0 > 0$ and $a_m + b\theta^* x_0 > 0$, then $(a_m + b\theta^* x_0) e^{-a_m t}$ will grow until it is equal to $b\theta^* x_0$ at which time the solution is unbounded. The system has a finite escape time at

$$t_e = -\frac{1}{a_m} \ln \left(\frac{b\theta^* x_0}{a_m + b\theta^* x_0} \right)$$

If $b\theta^* x_0 < 0$ and $a_m + b\theta^* x_0 < 0$, then the system also has the same finite escape time.
Therefore, the closed-loop system with the nominal controller is not globally stable. The closed-loop system is stable for $b\theta^* x_0 > 0$ and $a_m + b\theta^* x_0 < 0$ or $b\theta^* x_0 < 0$ and $a_m + b\theta^* x_0 > 0$ or $a_m + b\theta^* x_0 = 0$ and $b\theta^* x_0 \neq 0$.

b. The optimal control modification adaptive law is

$$\dot{\theta} = -\gamma \left(x^2 e b - \nu x^4 \theta b^2 a_m^{-1} \right)$$

Note that we implicitly choose $p = 1$ in the adaptive law. This implies that $q = -2pa_m = 2$ in the Lyapunov equation. Since the closed-loop system with the nominal controller is not globally stable, then we determine ν_{max} from

$$\varphi \left(\|x\|, \|x_m\|, \nu, \theta^* \right) = -c_1 \|x\|^2 + 2 \left(c_1 c_2 + c_5 \|x_m\| \right) \|x\| + 2c_1 c_2 \|x_m\| - c_1 \|x_m\|^2 + \nu c_3 c_4^2 \|\Phi(x)\|^2$$

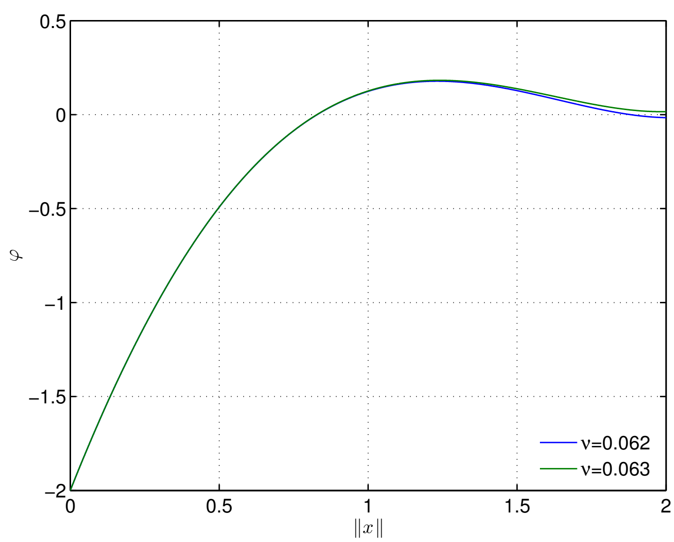
where $c_1 = q = 2$, $c_2 = 0$, $c_5 = q = 2$, $c_3 = b^2 a_m^{-2} q = 2$, and $c_4 = \frac{pb^2 \left| a_m^{-1} \right| |\theta^*|}{b^2 a_m^{-2} q} = \frac{|\theta^*|}{2} = 1$.
$\|x_m\|$ is determined by

$$\|x_m\| = \left\| -\frac{b_m}{a_m} \right\| \|r\| = 1$$

Then,

$$\varphi \left(\|x\|, \nu \right) = -2 \|x\|^2 + 4 \|x\| - 2 + 2\nu \|x\|^4$$

The limiting value is determined by trial and error to be $\nu_{max} = 0.062$ which corresponds to $\|x\| = 1.8809$ (see plot).



The ultimate bounds are obtained as

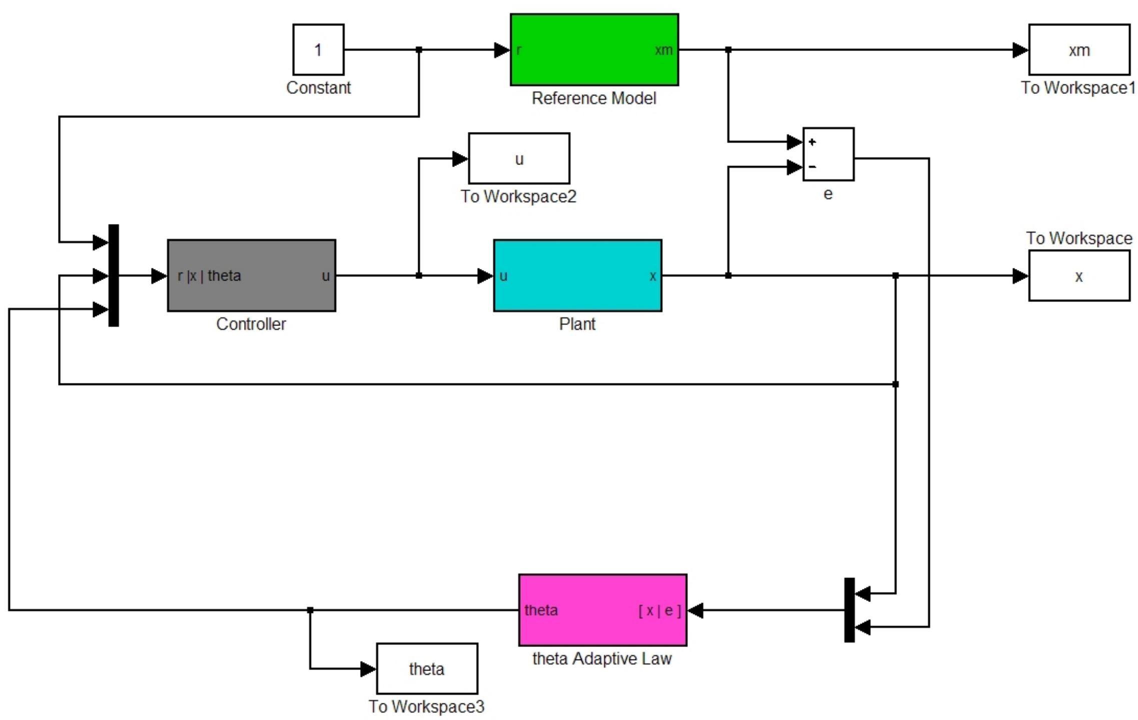
$$\|e\| \geq r = \sqrt{\frac{\nu c_3 c_4^2 \|x\|^4}{c_1}} = \sqrt{\nu}\,\|x\|^2 = 0.8809$$

$$\left\|\tilde{\theta}\right\| \geq \alpha = 2c_4 = 2$$

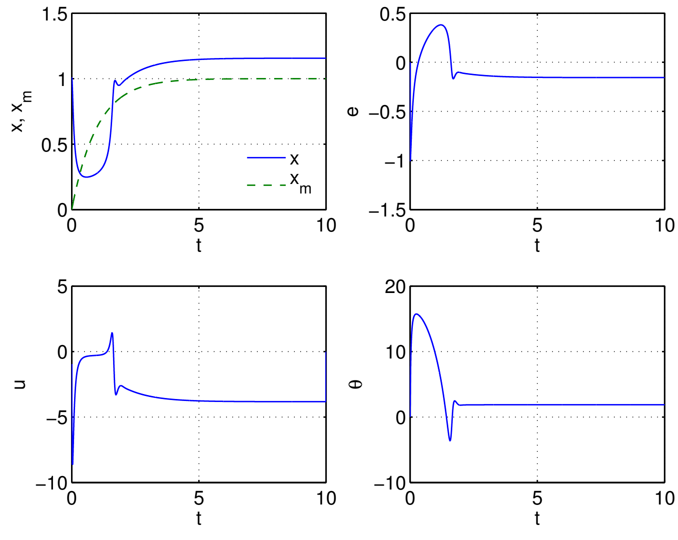
$$\|e\| \leq \rho = \sqrt{r^2 + \frac{\alpha^2}{\gamma}} = 0.8854$$

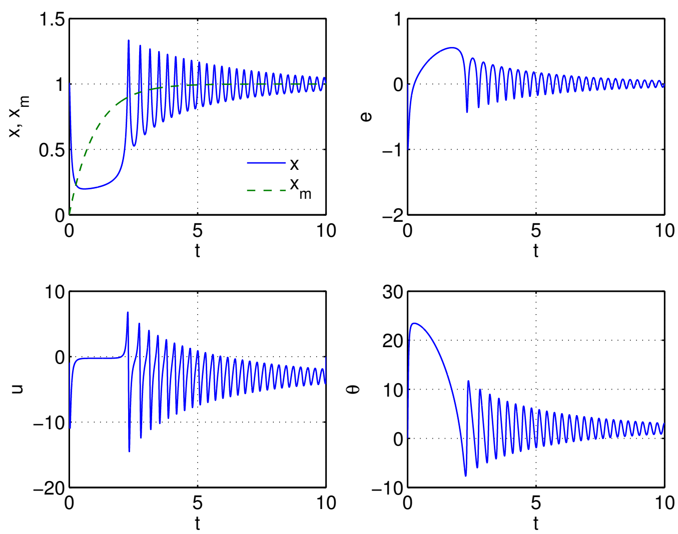
$$\left\|\tilde{\theta}\right\| \leq \beta = \sqrt{\gamma r^2 + \alpha^2} = 19.7993$$

c. The Simulink model is as shown.

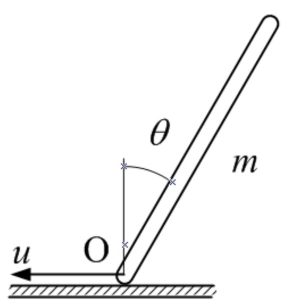


The response of the closed-loop system with the optimal control modification is as shown. The system is guaranteed to be stable with $\nu = 0.062$ whereas the closed-loop system with the nominal controller is unstable since $b\theta^* x_0 > 0$ and $a_m + b\theta^* x_0 > 0$. Numerical evidence shows that the system is stable up to a value of $\nu = 0.142$.
The maximum tracking error is 0.3803. The Lyapunov stability analysis gives 0.8809 which is conservative. The maximum value of $\theta(t)$ is 15.7238. So the maximum parameter estimation error is 13.7238. The estimate from the Lyapunov stability analysis is 19.7993 which is also conservative.
The response of the closed-loop system with MRAC is highly oscillatory. This is a well-known behavior of MRAC which acts as a nonlinear integral control that causes the crossover frequency to increase as γ increases.





11. Consider the equation of motion of an inverted pendulum constrained to move horizontally by a control force $u(t)$



$$\frac{1}{12}mL^2\left(4-3\cos^2\theta\right)\ddot{\theta}-\frac{1}{2}mgL\sin\theta+\frac{1}{8}mL^2\dot{\theta}^2\sin 2\theta+c\dot{\theta}=\frac{1}{2}L\cos\theta u\left(t-t_d\right)$$

where m is the mass of the pendulum, L is the length, g is the gravity constant, c is the damping coefficient which is assumed to be unknown, $\theta(t)$ is the angular position, $u(t)$ is the control input which represents the horizontal force at point O, and t_d is a time delay which represents the motor actuator dynamics.

a. Let $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^\top$. Derive the expressions for the nonlinear dynamic inversion adaptive controller and the σ modification adaptive law to estimate the unknown coefficient c in order to enable the closed-loop plant to track a reference model specified by

$$\ddot{\theta}_m + 2\zeta_m\omega_m\dot{\theta}_m + \omega_m^2\theta_m = \omega_m^2 r$$

which can be expressed in general as

$$\dot{x}_m = A_m x_m + B_m r$$

b. Given $m = 0.1775$ slug, $g = 32.174$ ft/sec, $L = 2$ ft, $c = 0.2$ slug-ft^2/sec, $\zeta_m = 0.75$, $\omega_m = 2$, and $r = \frac{\pi}{12}\sin 2t$. Implement the adaptive controller in Simulink with the following information: $x(0) = 0$, $\hat{c}(0) = 0$, $\gamma = 100$ and a time step $\varDelta t = 0.001$ sec for the following cases: 1) the standard MRAC with $t_d = 0$, 2) the standard MRAC with $t_d = 0.01$ sec, and 3) the σ modification with $\sigma = 0.1$. For each case, plot the time histories of $x(t)$ and $x_m(t)$ on the same plot, $u(t)$, and $\hat{c}(t)$ for $t \in [0, 10]$ sec. Plot in the units of deg for $x_1(t)$, deg/sec for $x_2(t)$, lb for $u(t)$, lb-ft-sec for $\hat{c}(t)$.

Solution:

a. The equation of motion can be expressed as

$$\ddot{\theta} = \frac{12g\sin\theta - 3L\dot{\theta}^2\sin 2\theta}{2L\left(4 - 3\cos^2\theta\right)} + \frac{6\cos\theta}{mL\left(4 - 3\cos^2\theta\right)}\left(u - \frac{2c\dot{\theta}}{L\cos\theta}\right)$$

Let $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$, and $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^\top$. Then,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_2 \\ \frac{12g\sin x_1 - 3Lx_2^2\sin 2x_1}{2L(4-3\cos^2 x_1)} \end{bmatrix}}_{f(x)} + \underbrace{\begin{bmatrix} 0 \\ \frac{6\cos x_1}{mL(4-3\cos^2 x_1)} \end{bmatrix}}_{p(x)}\left(u - \underbrace{\frac{2cx_2}{L\cos x_1}}_{ch(x)}\right)$$

$$\dot{x} = f(x) + p(x)\left[u - ch(x)\right]$$

The reference model is specified by

$$\dot{x}_m = A_m x_m + B_m r$$

Then, the dynamic inversion control is obtained as

$$u = \left[p^\top(x)\,p(x)\right]^{-1} p^\top(x)\left[A_m x + B_m r - f(x)\right] + \hat{c}(t)\,h(x)$$

where

$$\left[p^\top(x)\,p(x)\right]^{-1} p^\top(x) = \begin{bmatrix} 0 & \frac{mL\left(4-3\cos^2 x_1\right)}{6\cos x_1} \end{bmatrix}$$

The closed-loop plant becomes

$$\dot{x} = A_m x + B_m r + \tilde{c}(t)\,p(x)\,h(x)$$

Let

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\phi(x) = \frac{12x_2}{mL^2(4 - 3\cos^2 x_1)}$$

Then,

$$p(x)\,h(x) = B\phi(x)$$

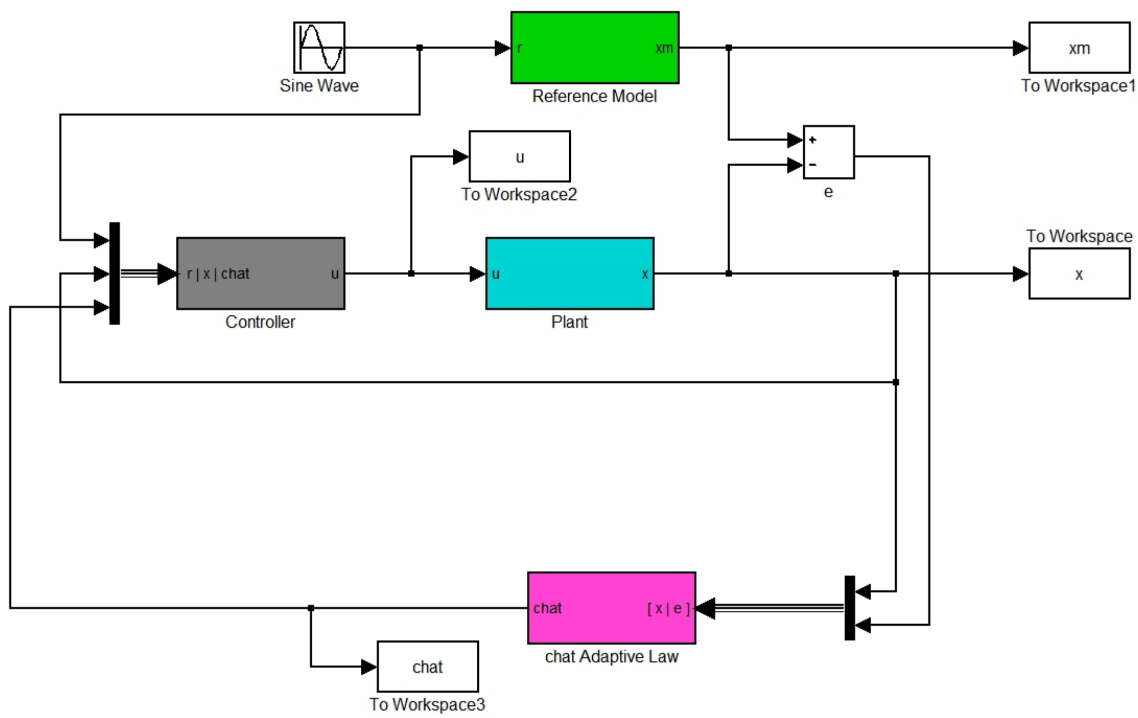
The tracking error equation is obtained as

$$\dot{e} = A_m e - B\tilde{c}\phi(x)$$

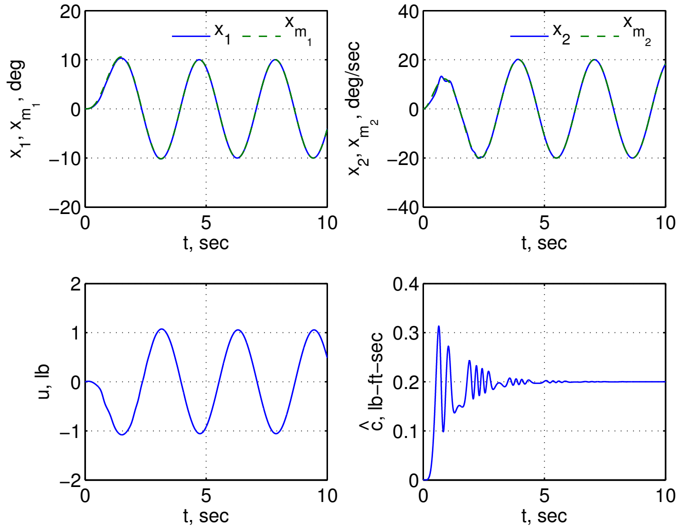
Therefore, the adaptive law is obtained as

$$\dot{\hat{c}} = \gamma\left[\phi(x)\,e^\top PB - \sigma\hat{c}\right]$$

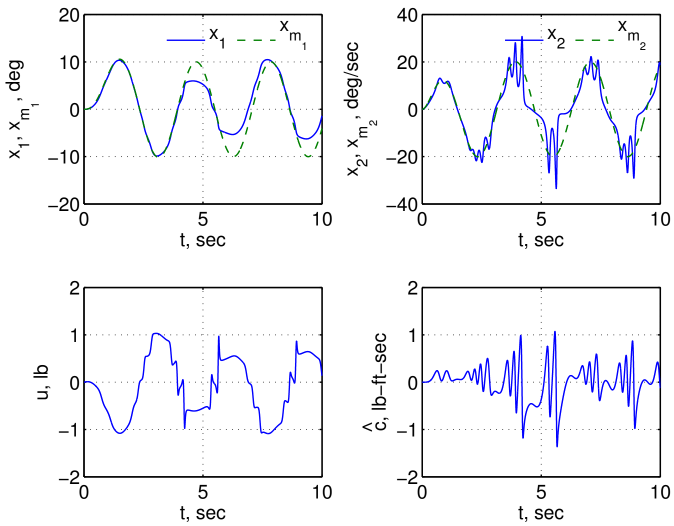
b. The Simulink model is as shown.



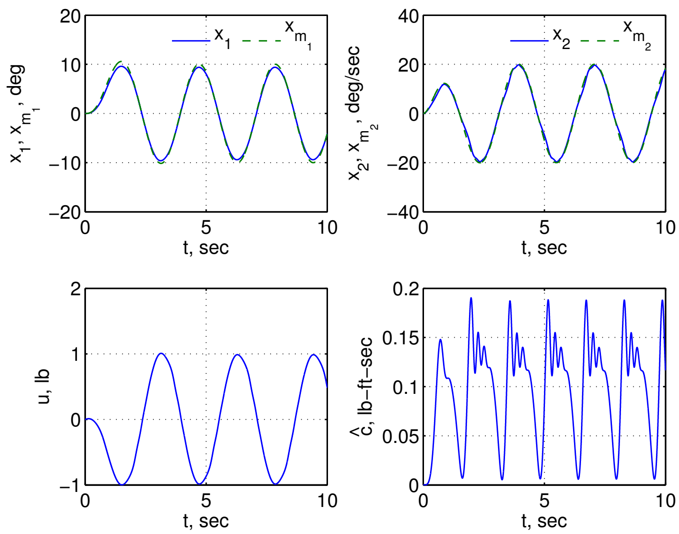
The response of the closed-loop system with MRAC with $t_d = 0$ is as shown. The closed-loop plant asymptotically tracks the reference model exactly as expected. The estimate $\hat{c}(t)$ converges to the true value.



The response of the closed-loop system with MRAC and $t_d = 0.01$ sec is as shown. The tracking of $\theta(t)$ improves, but the closed-loop plant is on the verge of instability, as seen by the high frequency response of $\dot{\theta}(t)$.



The response of the closed-loop system with the σ modification and $t_d = 0.01$ sec is as shown. The tracking is much better than that with MRAC.



12. Given a longitudinal dynamic model of an aircraft with a matched uncertainty

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{\bar{V}} & 1 \\ M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{\bar{V}} & M_q + M_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \alpha \\ q \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{\bar{V}} \\ M_{\delta_e} + \frac{M_{\dot{\alpha}} Z_{\delta_e}}{\bar{V}} \end{bmatrix} \left(\delta_e \left(t - t_d\right) + \begin{bmatrix} \theta_\alpha^* & \theta_q^* \end{bmatrix} \begin{bmatrix} \theta \\ q \end{bmatrix} \right)$$

with the following information: $\bar{V} = 795.6251$ ft/sec, $\bar{\gamma} = 0$, $Z_\alpha = -642.7855$ ft/sec^2, $Z_{\delta_e} = -55.3518$ ft/sec^2, $M_\alpha = -5.4898$ sec^{-2}, $M_{\delta_e} = -4.1983$ sec^{-2}, $M_q = -0.6649$ sec^{-1}, $M_{\dot{\alpha}} = -0.2084$ sec^{-1}, $\theta_\alpha^* = -5.4$, $\theta_q^* = -0.3$, and $t_d = 0.01$ sec.

a. Design a nominal proportional-integral control

$$\delta_e = k_p \alpha + k_i \int_0^t (\alpha - r)\, d\tau + k_q q$$

by finding the general expressions and the numerical values for k_p, k_i, and k_q to enable the aircraft to track a reference model of the angle of attack

$$\ddot{\alpha}_m + 2\zeta_m \omega_m \dot{\alpha}_m + \omega_m^2 \alpha_m = \omega_m^2 r$$

where $\zeta_m = 0.75$ and $\omega_m = 1.5$ rad/sec.

b. Let $z(t) = \int_0^t (\alpha(t) - r(t))\, d\tau$, provide the general expression and the numerical value for the reference model of the aircraft as

$$\dot{x}_m = A_m x_m + B_m r$$

where $x(t) = \begin{bmatrix} z(t) & \alpha(t) & q(t) \end{bmatrix}^\top$.

c. Let $\Theta^* = \begin{bmatrix} 0 & \theta_\alpha^* & \theta_q^* \end{bmatrix}^\top$. Design an adaptive angle-of-attack controller using the optimal control modification to enable the closed-loop plant to track the reference model. Express the adaptive controller and the adaptive law. Given $Q = 100I$, select the modification parameter to guarantee stability of the closed-loop plant by using the linear asymptotic property of the optimal control modification and the following formulas to compute the crossover frequency and time delay margin for MIMO systems. Plot ν versus t_d for $\nu \in [0, 5]$ and determine ν to within 0.01 for $t_d = 0.01$ sec

For a general time-delay system

$$\dot{x} = Ax + Bu(t - t_d)$$

with a linear controller

$$u = K_x x$$

the crossover frequency and time delay margin can be estimated as

$$\omega = \overline{\mu}\left(-jA\right) + \left\|BK_x\right\|$$

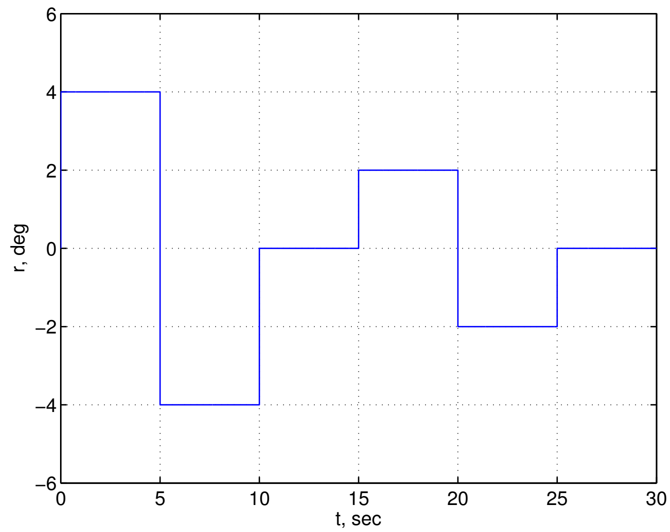
$$t_d = \frac{1}{\omega}\cos^{-1}\frac{\overline{\mu}\left(A\right)}{\overline{\mu}\left(-BK_x\right)}$$

where $\bar{\mu}$ is a matrix measure quantity defined as

$$\overline{\mu}\left(C\right) = \max_{1\le i\le n}\lambda_i\left(\frac{C + C^*}{2}\right)$$

for a general complex-value matrix C with its conjugate transpose C^*.

d. Implement the adaptive controller in Simulink using the following information: $x\left(0\right) = 0$, $\varTheta\left(0\right) = 0$, $\varGamma = 1000I$, and ν determined from part (c) with a time step $\varDelta t = 0.001$ sec. The reference command signal $r\left(t\right)$ is a pitch attitude doublet specified in the following plot.



Plot the time histories of each of the elements of $x\left(t\right)$ and $x_m\left(t\right)$ on the same plot, and $u\left(t\right)$ for $t \in [0, 30]$ sec. Plot in the units of deg-sec for $z\left(t\right)$, deg for $\alpha\left(t\right)$ and $\delta_e\left(t\right)$, and deg/sec for $q\left(t\right)$.

Solution:

a. Given

$$\delta_e = k_p\alpha + k_i\int_0^t\left(\alpha - r\right)d\tau + k_q q$$

Substituting $\delta_e\left(t\right)$ into the equation for $\dot{\alpha}\left(t\right)$ yields

$$\dot{\alpha} = \frac{Z_\alpha}{\bar{V}}\alpha + q + \frac{Z_{\delta_e}}{\bar{V}}\left[k_p\alpha + k_i\int_0^t\left(\alpha - r\right)d\tau + k_q q\right]$$

Differentiating $\dot{\alpha}\left(t\right)$ results in

$$\ddot{\alpha} = \frac{Z_\alpha}{\bar{V}}\dot{\alpha} + \dot{q} + \frac{Z_{\delta_e}}{\bar{V}}k_p\dot{\alpha} + \frac{Z_{\delta_e}}{\bar{V}}k_i(\alpha - r) + \frac{Z_{\delta_e}}{\bar{V}}k_q\dot{q}$$
$$= \left(\frac{Z_\alpha}{\bar{V}} + \frac{Z_{\delta_e}}{\bar{V}}k_p\right)\dot{\alpha} + \left(1 + \frac{Z_{\delta_e}}{\bar{V}}k_q\right)\dot{q} + \frac{Z_{\delta_e}}{\bar{V}}k_i(\alpha - r)$$

To track the reference model, the angle-of-attack dynamics must be

$$\ddot{\alpha} = -2\zeta_m\omega_m\dot{\alpha} - \omega_m^2(\alpha - r)$$

Therefore, the control gains can be computed by equating terms which results in

$$\frac{Z_\alpha}{\bar{V}} + \frac{Z_{\delta_e}}{\bar{V}}k_p = -2\zeta_m\omega_m \Rightarrow k_p = \frac{-2\zeta_m\omega_m - \frac{Z_\alpha}{\bar{V}}}{\frac{Z_{\delta_e}}{\bar{V}}}$$

$$1 + \frac{Z_{\delta_e}}{\bar{V}}k_q = 0 \Rightarrow k_q = -\frac{1}{\frac{Z_{\delta_e}}{\bar{V}}}$$

$$\frac{Z_{\delta_e}}{\bar{V}}k_i = -\omega_m^2 \Rightarrow k_i = -\frac{\omega_m^2}{\frac{Z_{\delta_e}}{\bar{V}}}$$

Numerically, $k_p = 20.7287$, $k_i = 32.3414$, and $k_q = 14.3740$

b. Let $z(t) = \int_0^t (\alpha(t) - r(t))\,d\tau$, then the plant model becomes

$$\begin{bmatrix}\dot{z}\\ \dot{\alpha}\\ \dot{q}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & \frac{Z_\alpha}{\bar{V}} & 1\\ 0 & M_\alpha + \frac{M_{\dot{\alpha}}Z_\alpha}{\bar{V}} & M_q + M_{\dot{\alpha}}\end{bmatrix}\begin{bmatrix}z\\ \alpha\\ q\end{bmatrix}$$
$$+ \begin{bmatrix}0\\ \frac{Z_{\delta_e}}{\bar{V}}\\ M_{\delta_e} + \frac{M_{\dot{\alpha}}Z_{\delta_e}}{\bar{V}}\end{bmatrix}\left(\delta_e(t - t_d) + \begin{bmatrix}0 & \theta_\alpha^* & \theta_q^*\end{bmatrix}\begin{bmatrix}z\\ \alpha\\ q\end{bmatrix}\right) + \begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}r$$

which is expressed in general as

$$\dot{x} = Ax + B\left[u(t - t_d) + \Theta^{*\top}x\right] + Cr$$

where

$$A = \begin{bmatrix}0 & 1 & 0\\ 0 & -0.8079 & 1\\ 0 & -5.3214 & -0.8733\end{bmatrix},\ B = \begin{bmatrix}0\\ -0.0696\\ -4.1838\end{bmatrix},\ \Theta^* = \begin{bmatrix}0\\ -5.4\\ -0.3\end{bmatrix},\ C = \begin{bmatrix}-1\\ 0\\ 0\end{bmatrix}$$

Let $u(t) = \delta_e(t)$. Then, the nominal controller is

$$u = \begin{bmatrix}k_i & k_p & k_q\end{bmatrix}\begin{bmatrix}z\\ \alpha\\ q\end{bmatrix} = K_x x$$

The nominal closed-loop plant becomes

$$\dot{x} = (A + BK_x)x + Cr$$

The reference model is then chosen as the nominal closed-loop plant. Therefore,

$$\dot{x}_m = A_m x_m + B_m r$$

where $B_m = C$ and

$$
\begin{aligned}
A_m &= A + BK_x \\
&= \begin{bmatrix} 0 & 1 & 0 \\ -\omega_m^2 & -2\zeta_m\omega_m & 0 \\ \left(M_{\delta_e} + \frac{M_{\dot\alpha} Z_{\delta_e}}{\bar V}\right) k_i & M_\alpha + \frac{M_{\dot\alpha} Z_\alpha}{\bar V} + \left(M_{\delta_e} + \frac{M_{\dot\alpha} Z_{\delta_e}}{\bar V}\right) k_p & M_q + M_{\dot\alpha} + \left(M_{\delta_e} + \frac{M_{\dot\alpha} Z_{\delta_e}}{\bar V}\right) k_q \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ -2.25 & -2.25 & 0 \\ -135.3101 & -92.0462 & -61.0111 \end{bmatrix}
\end{aligned}
$$

c. The adaptive controller is

$$u = K_x x - \Theta^\top (t)\, x$$

The closed-loop plant becomes

$$\dot x = A_m x + B_m r - B\tilde\Theta^\top x$$

The error equation is

$$\dot e = A_m e + B\tilde\Theta^\top x$$

Therefore, the optimal control modification adaptive law is

$$\dot\Theta = -\Gamma x \left(e^\top P - \nu x^\top \Theta B^\top P A_m^{-1}\right) B$$

The asymptotic solution of the optimal control modification adaptive law is

$$B\Theta^\top x = \frac{1}{\nu} P^{-1} A_m^\top P e = \frac{1}{\nu} P^{-1} A_m^\top P \left(x_m - x\right)$$

Then, the asymptotic closed-loop plant with time delay is

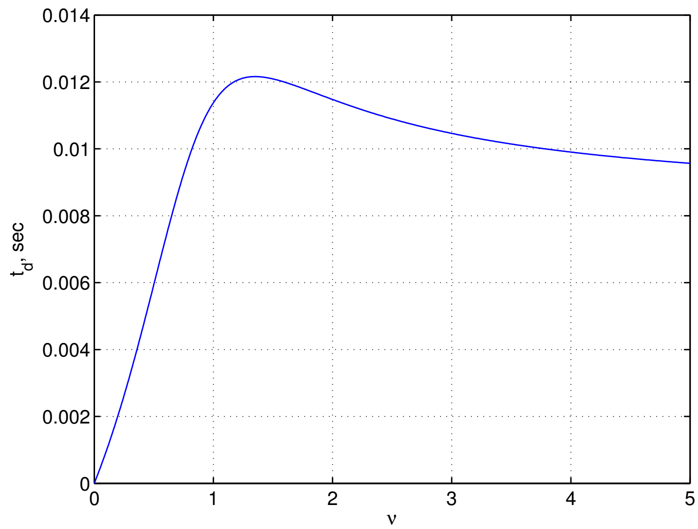
$$
\begin{aligned}
\dot x &= Ax + BK_x x\left(t - t_d\right) - B\Theta^\top \left(t - t_d\right) x \left(t - t_d\right) + B\Theta^{*\top} x + B_m r\left(t\right) \\
&= \left(A + B\Theta^{*\top}\right) x + \left[BK_x + \frac{1}{\nu} P^{-1} A_m^\top P\right] x\left(t - t_d\right) - \frac{1}{\nu} P^{-1} A_m^\top P x_m \left(t - t_d\right) + B_m r\left(t\right)
\end{aligned}
$$

The cross-over frequency and time delay margin are computed as

$$\omega = \overline{\mu}\left(-j\left(A + B\Theta^{*\top}\right)\right) + \left\|BK_x + \frac{1}{\nu} P^{-1} A_m^\top P\right\|$$

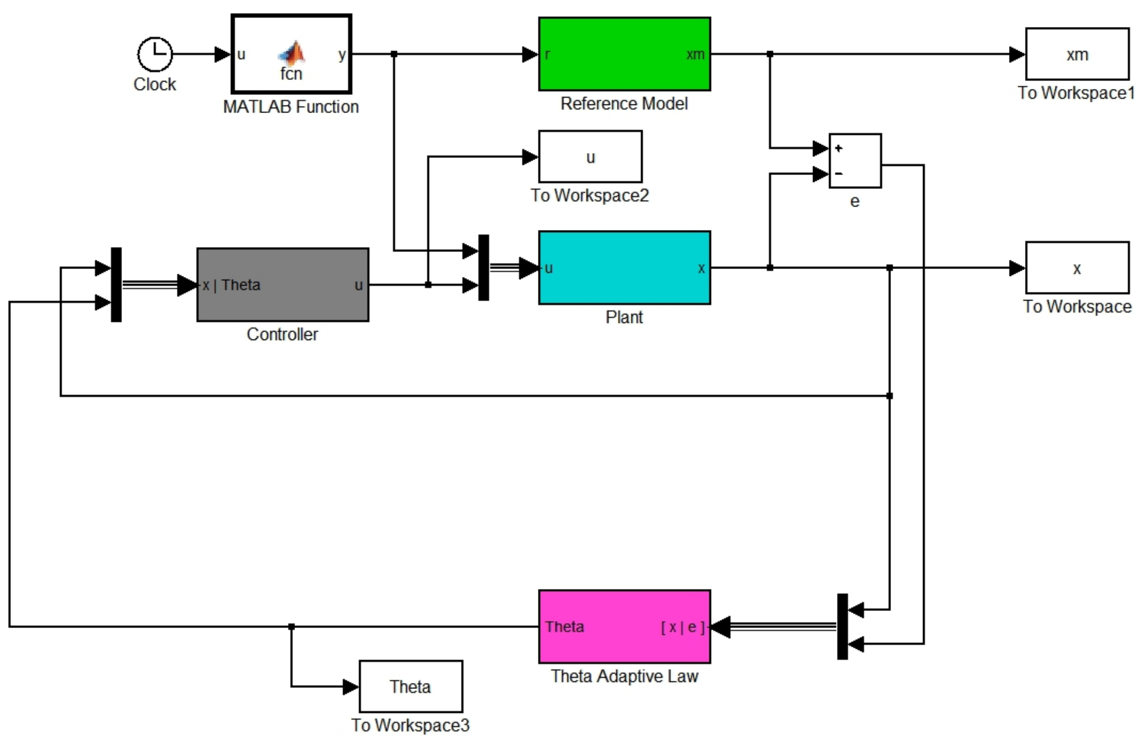
$$
\begin{aligned}
t_d &= \frac{1}{\omega} \cos^{-1} \frac{\overline{\mu}\left(A + B\Theta^{*\top}\right)}{\overline{\mu}\left(-BK_x - \frac{1}{\nu} P^{-1} A_m^\top P\right)} \\
&= \frac{\nu}{\nu\overline{\mu}\left(-j\left(A + B\Theta^{*\top}\right)\right) + \left\|\nu BK_x + P^{-1} A_m^\top P\right\|} \cos^{-1} \frac{\nu\overline{\mu}\left(A + B\Theta^{*\top}\right)}{\overline{\mu}\left(-\nu BK_x - P^{-1} A_m^\top P\right)}
\end{aligned}
$$

The plot of ν versus t_d is as shown.

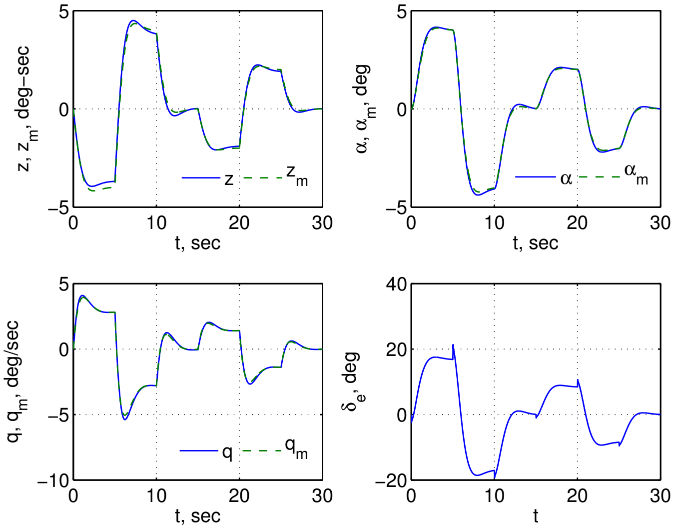


For $t_d = 0.01$ sec, $\nu = 0.82$. Note that this value is a conservative estimate, but it does guarantee stability of the closed-loop plant for any $t_d \leq 0.01$ sec. Also note that a conservative estimate of the maximum time delay margin that the closed-loop system can tolerate is 0.0122 sec.

d. The Simulink model is as shown.



The response of the closed-loop system with the adaptive controller is as shown. The closed-loop plant tracks the reference model very well.



13. Given a longitudinal dynamic model of a damaged aircraft

$$\dot{\alpha} = \left(\frac{Z_\alpha}{\bar{V}} + \Delta A_{\alpha\alpha}\right)\alpha + q + \left(\frac{Z_{\delta_e}}{\bar{V}} + \Delta B_\alpha\right)\delta_e\left(t - t_d\right)$$

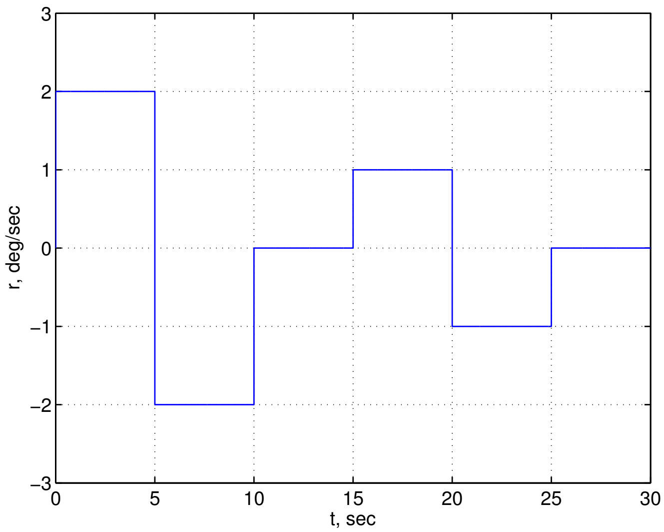
$$\dot{q} = \left(M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{\bar{V}} + \Delta A_{q\alpha}\right)\alpha + \left(M_q + M_{\dot{\alpha}} + \Delta A_{qq}\right)q + \left(M_{\delta_e} + \frac{M_{\dot{\alpha}} Z_{\delta_e}}{\bar{V}} + \Delta B_q\right)\delta_e\left(t - t_d\right)$$

a. Design an ACAH hybrid adaptive flight controller for the pitch axis to enable the aircraft to follow a reference model

$$\dot{q}_m = -\omega_q\left(q_m - r\right)$$

by providing the expressions for the hybrid adaptive controller, the least-squares gradient parameter estimation of $\Delta A_{q\alpha}$, ΔA_{qq}, ΔB_q, and the optimal control modification adaptive law to handle the residual tracking error.

b. Implement the hybrid adaptive flight controller in Simulink using the same aircraft parameters from Exam Problem 12 and the following additional information: $t_d = 0.02$ sec, $\zeta_q = 0.75$, $\omega_q = 2.5$ rad/sec, $\Delta A_{\alpha\alpha} = 0.1616$ /sec, $\Delta A_{q\alpha} = 2.1286$ /sec^2, $\Delta A_{qq} = 0.5240$ /sec, $\Delta B_\alpha = -0.0557$ /sec, $\Delta B_q = -2.5103$ /sec^2, $\alpha(0) = 0$, $q(0) = 0$, $\Delta\hat{A}_{q\alpha}(0) = 0$, $\Delta\hat{A}_{\alpha\alpha}(0) = 0$, $\Delta\hat{B}_q(0) = 0$, $R = 1000I$, $\Gamma = 1000I$, and $\nu = 0.1$ with a time step $\Delta t = 0.001$ sec. The reference command signal $r(t)$ is a pitch rate doublet specified in the following plot.



c. Simulate three cases: 1) nominal controller, 2) only direct MRAC, and 3) hybrid adaptive control with both direct MRAC and indirect least-squares gradient adaptive control. For each case, plot the time histories of each of the elements of $\alpha(t)$, $\theta(t)$, $q(t)$ and $q_m(t)$ on the same plot, and $u(t)$ for $t \in [0, 30]$ sec. In addition, plot $\Theta(t)$ for case 2; and $\Delta\hat{A}_{q\alpha}(t)$, $\Delta\hat{A}_{qq}(t)$, $\Delta\hat{B}_q(t)$ for case 3. Plot in the units of deg for $\alpha(t)$, $\theta(t)$, and $\delta_e(t)$, and deg/sec for $q(t)$.

Solution:

a. The pitch rate equation is used for the hybrid adaptive controller design as

$$\dot{q} = \left(M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{\bar{V}} + \Delta A_{q\alpha}\right)\alpha + \left(M_q + M_{\dot{\alpha}} + \Delta A_{qq}\right) q + \left(M_{\delta_e} + \frac{M_{\dot{\alpha}} Z_{\delta_e}}{\bar{V}} + \Delta B_q\right)\delta_e\left(t - t_d\right)$$

The estimated plant model is expressed as

$$\dot{\hat{q}} = \left(M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{\bar{V}} + \Delta\hat{A}_{q\alpha}\right)\alpha + \left(M_q + M_{\dot{\alpha}} + \Delta\hat{A}_{qq}\right) q + \left(M_{\delta_e} + \frac{M_{\dot{\alpha}} Z_{\delta_e}}{\bar{V}} + \Delta\hat{B}_q\right)\delta_e$$

The desired pitch acceleration is

$$\dot{q}_d = \dot{q}_m + \bar{u} - u_{ad}$$

where

$$\bar{u} = k_p\left(q_m - q\right) + k_i \int_0^t \left(q_m - q\right) d\tau$$

$$u_{ad} = \Delta\Theta^\top \Phi\left(q, \alpha, \delta_e\right)$$

$$\Delta\dot{\Theta} = -\Gamma\Phi\left(q, \alpha, \delta_e\right)\left[e^\top PB - \nu\Phi^\top\left(q, \alpha, \delta_e\right)\sigma\Delta\Theta BPA_m^{-1}\right]B$$

$$k_p = 2\zeta_q\omega_q$$

$$k_i = \omega_q^2$$

$$A_m = \begin{bmatrix} 0 & 1 \\ -k_i & -k_p \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Delta\Theta = \begin{bmatrix} \epsilon_{A_{qq}} \\ \epsilon_{A_{q\alpha}} \\ \epsilon_{Bq} \end{bmatrix}, \; \Phi\left(q, \alpha, \delta_e\right) = \begin{bmatrix} q \\ \alpha \\ \delta_e \end{bmatrix}$$

The dynamic inversion controller is then obtained as

$$\delta_e = \frac{\dot{q}_d - \left(M_q + M_{\dot{\alpha}} + \Delta\hat{A}_{qq}\right) q - \left(M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{\bar{V}} + \Delta\hat{A}_{q\alpha}\right) \alpha}{M_{\delta_e} + \frac{M_{\dot{\alpha}} Z_{\delta_e}}{\bar{V}} + \Delta\hat{B}_q}$$

where $\Delta\hat{A}_{qq}(t)$, $\Delta\hat{A}_{q\alpha}(t)$, and $\Delta\hat{B}_q(t)$ are estimated by an indirect least-squares gradient adaptive law as

$$\dot{\Theta} = -R\Phi(q, \alpha, \delta_e)\,\epsilon^\top$$

where

$$\epsilon = \dot{q}_d - \dot{q}$$

$$\Theta = \begin{bmatrix} \Delta\hat{A}_{qq} \\ \Delta\hat{A}_{q\alpha} \\ \Delta\hat{B}_q \end{bmatrix}$$

b. Let $x_m(t) = \begin{bmatrix} \theta_m(t) \; q_m(t) \end{bmatrix}^\top$, then the reference model can be expressed as

$$\dot{x} = A_{m_q} x + B_{m_q} r$$

$$\dot{q}_m = C\left(A_{m_q} x + B_{m_q} r\right)$$

where

$$A_{m_q} = \begin{bmatrix} 0 & 1 \\ 0 & -\omega_q \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2.5 \end{bmatrix}$$

$$B_{m_q} = \begin{bmatrix} 0 \\ \omega_q \end{bmatrix} = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 \; 1 \end{bmatrix}$$

Let $z(t) = \begin{bmatrix} \alpha(t) \; x(t) \end{bmatrix}^\top$, then

$$\dot{z} = (A_z + \Delta A_z) z + (B_z + \Delta B_z) u(t - t_d)$$

$$\dot{q} = D\left[(A_z + \Delta A_z) z + (B_z + \Delta B_z) u(t - t_d)\right]$$

where

$$A_z = \begin{bmatrix} \frac{Z_\alpha}{\bar{V}} & 0 & 1 \\ 0 & 0 & 1 \\ M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{\bar{V}} & 0 & M_q + M_{\dot{\alpha}} \end{bmatrix} = \begin{bmatrix} -0.8079 & 0 & 1 \\ 0 & 0 & 1 \\ -5.3214 & 0 & -0.8733 \end{bmatrix}$$

$$\Delta A_z = \begin{bmatrix} \Delta A_{\alpha\alpha} & 0 & 0 \\ 0 & 0 & 0 \\ \Delta A_{q\alpha} & 0 & \Delta A_{qq} \end{bmatrix} = \begin{bmatrix} 0.1616 & 0 & 0 \\ 0 & 0 & 0 \\ 2.1286 & 0 & 0.5240 \end{bmatrix}$$

$$B_z = \begin{bmatrix} \frac{Z_{\delta_e}}{\bar{V}} \\ 0 \\ M_{\delta_e} + \frac{M_{\dot{\alpha}} Z_{\delta_e}}{\bar{V}} \end{bmatrix} = \begin{bmatrix} -0.0696 \\ 0 \\ -4.1838 \end{bmatrix}$$

$$\Delta B_z = \begin{bmatrix} \Delta B_\alpha \\ 0 \\ \Delta B_q \end{bmatrix} = \begin{bmatrix} -0.0557 \\ 0 \\ -2.5103 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \; 0 \; 1 \end{bmatrix}$$

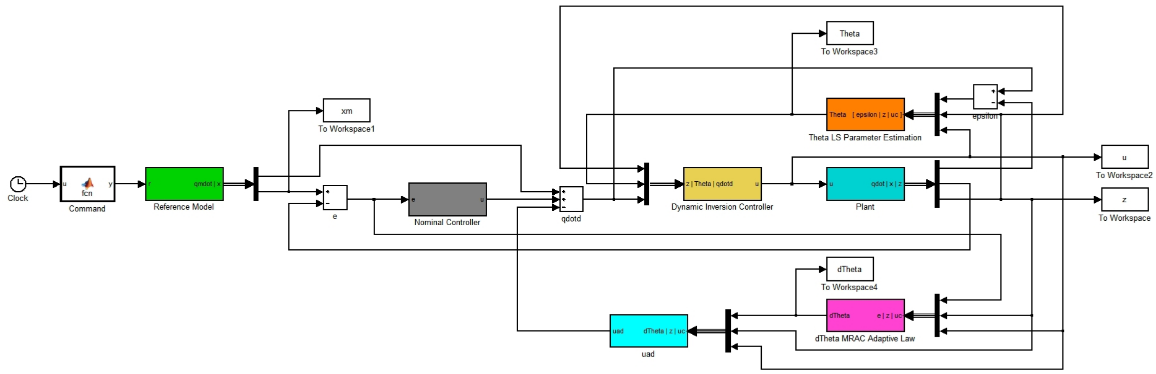
The nominal controller is expressed as

$$\bar{u} = K_e e$$

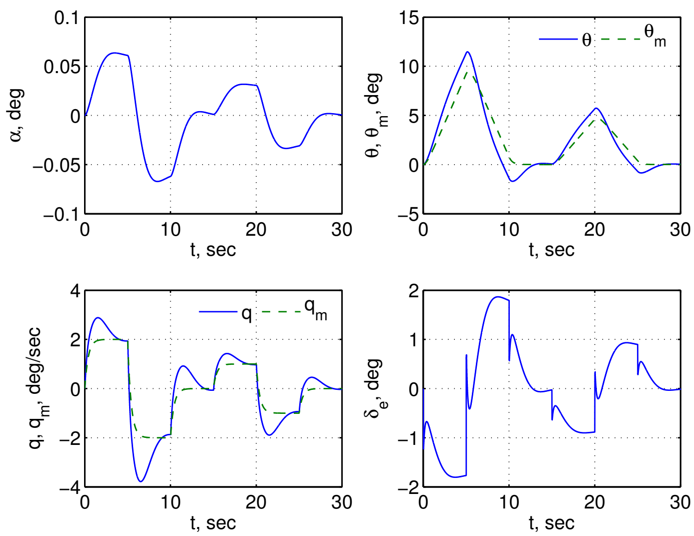
where $e(t) = x_m(t) - x(t)$ and

$$K_e = \begin{bmatrix} k_i & k_p \end{bmatrix} = \begin{bmatrix} 6.25 & 3.75 \end{bmatrix}$$

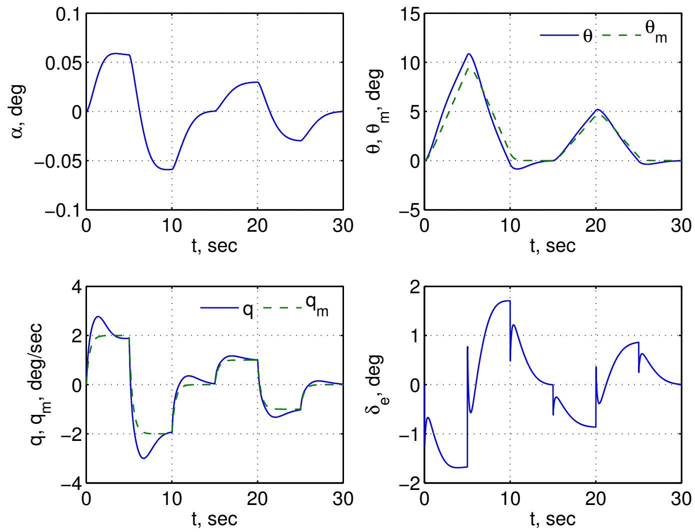
c. The Simulink model is as shown.



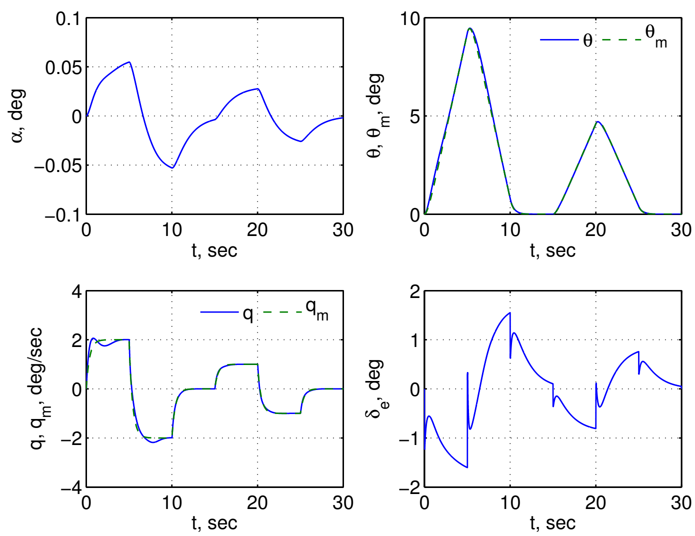
The response of the closed-loop system with the nominal controller is as shown. The closed-loop plant does not follow the reference model well. The pitch rate response oveshoots the reference pitch rate significantly.



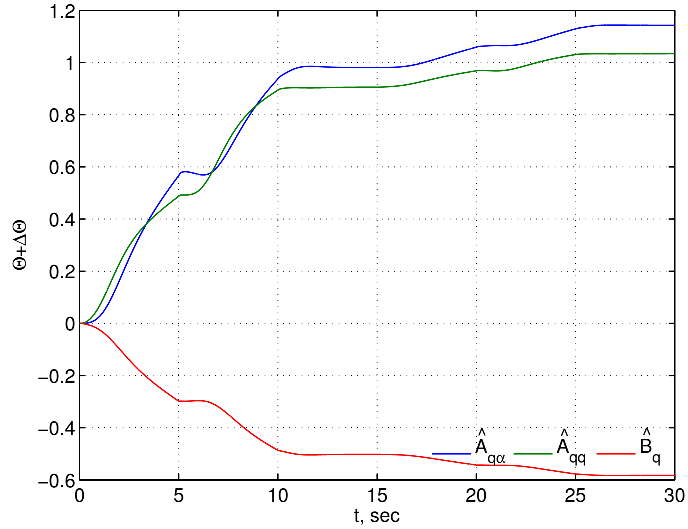
The response of the closed-loop system with the direct MRAC is as shown. The pitch rate response improves, but there are still some overshoots.



The response of the closed-loop system with the hybrid adaptive control is as shown. The tracking improves significantly with the pitch rate response tracking the reference pitch rate very well.



The parameter convergence of the estimates of ΔA_{qq}, $\Delta A_{q\alpha}$, and ΔB_q due to of the direct MRAC is as shown. The estimates of ΔA_{qq}, $\Delta A_{q\alpha}$, and ΔB_q do not seem to converge to the true values.



The parameter convergence of the estimates of ΔA_{qq}, $\Delta A_{q\alpha}$, and ΔB_q due to of the indirect least-squares gradient adaptive control is as shown. The estimates of ΔA_{qq}, $\Delta A_{q\alpha}$, and ΔB_q converge to their steady state values which are close to the true values. The error in the parameter convergence is due to the presence of the time delay in the system.