Chapter Two: Regular and Semi-Regular Polyhedra

Section 1. Regular Polyhedra.

A **polyhedron** is a three-dimensional figure bounded by a finite number of polygonal faces. Its literal meaning is a many-faced figure because *poly* means many and *hedron* means face. Thus a tetrahedron is a four-faced figure, which can only be the triangular pyramid. The plural form of polyhedron is polyhedra. Some human beings are bihedra.

We will assume that the polyhedra we deal with are *convex*. In such a polyhedron, the line segment joining any of its two points lies entirely in the polyhedron. Most of the polyhedra we encounter, such as prisms and pyramids, are in fact convex.

The *skeleton* of a polyhedron consists of its vertices and edges only, and it contains all the essential information about the polyhedron. Thus we will represent any polyhedron by its skeleton.

We can facilitate the drawing of the skeleton of a polyhedron by the following process. Imagine that the edges are made of elastic strings. Choose a face as the base and stretch its edges so that the projection of every other vertex onto this face lies within its interior. For example, the skeleton of the tetrahedron with base BCD and opposite vertex A can be drawn as shown in the Figure 2.1. Such a representation is called the **Schlegel** diagram of the polyhedron.

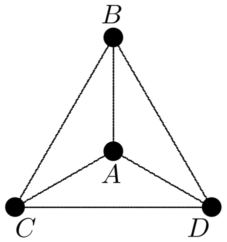

Figure 2.1

In his December, 1958 *Mathematical Games* column in *Scientific American* (see [2]), Martin Gardner wrote about the **Platonic** solids. They are named after the Greek philosopher Plato, and are the most pleasing of all polyhedra.

In a Platonic solid, all faces are the same kind of regular polygons and each vertex lies on the same number of faces. Thus there is perfect symmetry among the faces and among the vertices, both geometrically and combinatorially. It is not hard to see that there are only five Platonic solids. Suppose the faces are equilateral triangles. If we put three of them around each vertex, we have the regular tetrahedron as shown in Figure 2.1. If we put four of them around each vertex, we have the regular octahedron (double square pyramid) as shown in Figure 2.2.

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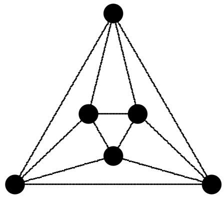

Figure 2.2

If we put five of them around each vertex, we have the regular icosahedron as shown in Figure 2.3. However, if we put six of them around a vertex, the configuration will be flat.

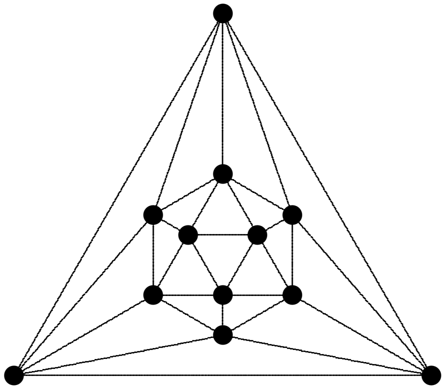

Figure 2.3

Suppose the faces are squares. If we put three of them around each vertex, we have the cube as shown in Figure 2.4. However, if we put four of them around a vertex, the configuration will be flat.

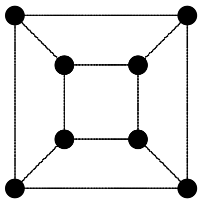

Figure 2.4

Suppose the faces are regular pentagons. If we put three of them around each vertex, we have the regular dodecahedron as shown in Figure 2.5. However, if we put four of them around a vertex, they will overlap.

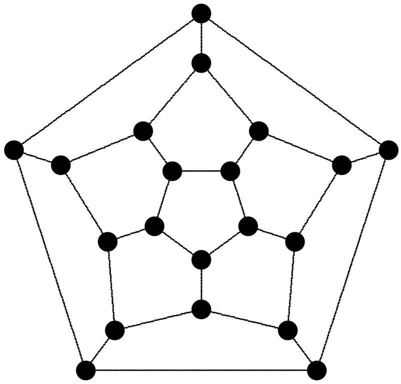

Figure 2.5

Suppose the faces are regular hexagons. Even if we place three of them around a vertex, the configuration will be flat. Thus there are indeed only five Platonic solids.

Schlegel diagrams are special cases of graphs. A **graph** is a collection of dots and lines, called **vertices** and **edges** respectively. Each edge connects two vertices. If these two vertices are identical, the edge is called a **loop**. If two edges connect the same two vertices, they are said to be **multiple edges**. The **degree** of a vertex is the number of edges which connects it to other vertices. Thus each edge contributes 2 to the total degree of a graph. A graph is said to be **connected** if any two vertices are accessible from each other via a sequence of edges.

If a graph can be drawn so that its edges meet only at the vertices, then it is called a **planar** graph. The Schlegel diagram of a polyhedron is also called a **polyhedral** graph, and must be planar. When drawn without crossing edges, a planar graph divides the plane unambiguously into regions. These regions are the **faces** of the graph. They correspond to the faces of the polyhedron. Clearly, non-planar graphs cannot be polyhedral graphs as the concept of a face is not well-defined. However, not all planar graphs are polyhedral graphs.

Note that in Figure 2.1, the base BCD is a face of the polyhedron, but seems to have disappeared as a face of the polyhedral graph. On the other hand, the graph has an infinite region which does not seem to be part of the polyhedron. To handle this apparent anomaly, imagine that the skeleton is drawn on the surface of a balloon, whose snout is contained in the face chosen as the base. If we stretch the balloon until the rim of its snout becomes a large circle enclosing a flat piece of rubber, we can see that the base has in fact become the infinite region.

Suppose a vertex of a polyhedron is surrounded in clockwise order by n faces, with x_i sides respectively for $1 \le i \le n$. We define its **vertex sequence** as $(x_1, x_2, \ldots, x_n)$. For a Platonic solid, all vertex sequences are identical, and it gives us a concise description of the solid. Thus the regular tetrahedron is (3,3,3), the regular octahedron is (3,3,3,3), the regular icosahedron is (3,3,3,3,3), the cube is (4,4,4) and the regular dodecahedron is (5,5,5).

Actually, a vertex sequence describes a whole class of polyhedra. For instance, (3,3,3) describes any tetrahedron, regular or otherwise.

A polyhedron is said to be **regular** if it satisfies the following two conditions:
(A) The sequences of all vertices are identical.
(B) All integers in the vertex sequence are identical.

How many regular polyhedra are there? There are at least five as all Platonic solids are regular polyhedra. Could there be other kinds? The geometric proof given earlier is no longer valid since we no longer require the faces to be regular polygons. Surprisingly, the answer is still five, but we need to give a combinatorial argument.

Our principal tool is a famous result due to the Swiss mathematician Leonhard Euler. He spent most of his active life in Prussia, under the patronage of Frederick the Great, and later in Russia, under the patronage of Catherine the Great. He had made contributions to many fields in mathematics, with an Euler's Formula in each of them. In particular, he was recognized as the father of graph theory.

Let V, E and F denote the numbers of vertices, edges and faces of a polyhedral graph. Then **Euler's Formula for Polyhedra** states that $V - E + F = 2$. It is also valid for connected planar graphs.

We shall prove Euler's Formula for Polyhedra in a slightly different form. A **component** of a graph is a connected subgraph which is not contained in any larger connected subgraph. In other words, each component is a connected piece of a graph. Denote by C the number of components. For connected graphs, we have $C = 1$.

We claim that for any planar graph, $V + F = E + C + 1$. Erase all the edges but retaining all the vertices. Initially, $V = C$, $F = 1$ and $E = 0$. We will reinstall the edges one at a time, so that E increases by 1 at each step. If the edge reinstalled in a particular step connects two vertices in different components, then C goes down by 1. If it connects two vertices in the same component, C remains unchanged but F goes up by 1 as an existing face is carved into two. Either way, the balance is maintained. At the end when all the edges have been reinstalled, we have $C = 1$ and $V + F = E + C + 1$ may be rewritten as $V - E + F = 2$.

As an application of Euler's Formula, we now prove that every planar graph without loops or multiple edges has a vertex of degree at least 5. Suppose to the contrary that every vertex has degree at least 6. Cut each edge into half-edges across its length. The total number of half-edges is exactly $2E$, and at least $6V$ by our assumption, so that $2E \geq 6V$. On the other hand, since there are no loops or multiple edges, each face is bounded by at least 3 edges, yielding $2E \geq 3F$. Substituting into Euler's Formula, we have $2 = V - E + F$
$le\frac{E}{3} - E + \frac{2E}{3} = 0$, and we have a contradiction.

Condition (A) in the definition of a regular polyhedra implies that all vertices of the polyhedron lies on the same number n of edges, and condition (B) implies that all faces of the polyhedron are bounded by the same number m of edges. Thus the vertex sequence of a regular polyhedra consists of n copies of m. Since every polyhedral graph has a vertex of degree at most 5, we see that $n \leq 5$. That $m \leq 5$ can be proved similarly. Thus we have nine cases, as shown in the chart below.

<table>
<tr><th></th><th>$m = 3$</th><th>$m = 4$</th><th>$m = 5$</th></tr>
<tr><td>$n = 3$</td><td>Standard Tetrahedron</td><td>Cuboid</td><td>Standard Dodecahedron</td></tr>
<tr><td>$n = 4$</td><td>Standard Octahedron</td><td colspan="2" rowspan="2">Impossible Cases</td></tr>
<tr><td>$n = 5$</td><td>Standard Icosahedron</td></tr>
</table>

To prove that the four cases marked impossible are indeed so, we need a preliminary result: $nV = 2E = kF$. Cut each edge in halves at its midpoint. Each of the V vertices is attached to n half-edges, so that the total number of half-edges is nV. On the other hand, the total number of half-edges is clearly $2E$ since each of the E edges is cut in halves. It follows that $nV = 2E$. Similarly, we can prove that $2E = kF$ but cutting each edge in halves along its midline and count the half-edges in two ways as before. Recall that we have used this argument in proving that K_5 and $K_{3,3}$ are non-planar.

Substituting into Euler's Formula, we have $\frac{2}{n}E + \frac{2}{m}E - E = 2$ or

$$\frac{1}{n} + \frac{1}{m} = \frac{1}{2} + \frac{1}{E} > \frac{1}{2}.$$

If $n = m = 4$, we have only $\frac{1}{n} + \frac{1}{m} = \frac{1}{2}$. In the other three cases, we have $\frac{1}{n} + \frac{1}{m} < \frac{1}{2}$. All contradict the above inequality.

Section 2. A Polyhedral Metamorphosis.

Circle members participate in the International Mathematics Competition, a sample paper of which is given in [5]. An important event in the competition is the Cultural Evening, during which each country presents a short performance that highlights their heritage. As a multi-cultural nation, Canada have had a hard time finding suitable things to do. Finally, in 2010 when the competition was in Inchon, South Korea, we decided to express ourselves in a universal language, namely, mathematics.

The performance is described in [4]. Ten students use six strings to construct the skeleton of each of the five Platonic solids in a sequence of continual transformation. Start with four students forming the tetrahedron. At some point, six students join in. After a while, the original four drop out. Eventually, the remaining six students form the octahedron. During this sequence, all of the other Platonic solids appear. At the end, the original four take over from the final six and restore the tetrahedron! This is adapted from the design by Karl Schaffer [6].

Step 1. Construction of the Tetrahedron

We start of with four students identified as N(orth), S(outh), E(ast) and W(est). Each designates one hand as the U(pper) hand and the other hand as the L(ower) hand. N and S hold out their U hands while E and W hold out their L hands. String 1 is held between UN and LW, string 2 between UN and LE, string 3 between LW and LE, string 4 between LW and US, string 5 between LE and US, and string 6 between UN and US. The completed tetrahedron is shown in Figure 2.6, with string 6 drawn in such a way to facilitate the next step.

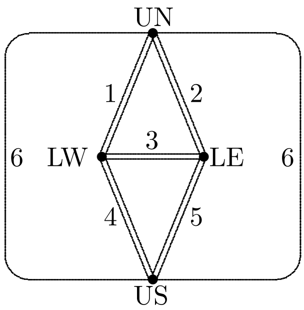

Figure 2.6

Step 2. Transformation into the Cube

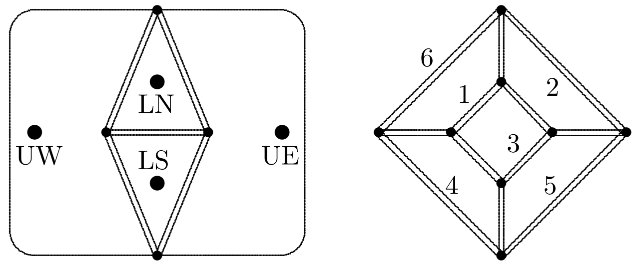

Figure 2.7

Each of the four students holds out the other hand and places it at the center of one of the four faces of the tetrahedron, as shown on the left side of Figure 2.7. Each of these hands will grab the three sides of the triangular face. The end result is a cube, as shown on the right side of Figure 2.7. Each string forms a face of the cube.

Step 3. Transformation into the Dodecahedron

We first redraw the cube as shown in Figure 2.8.

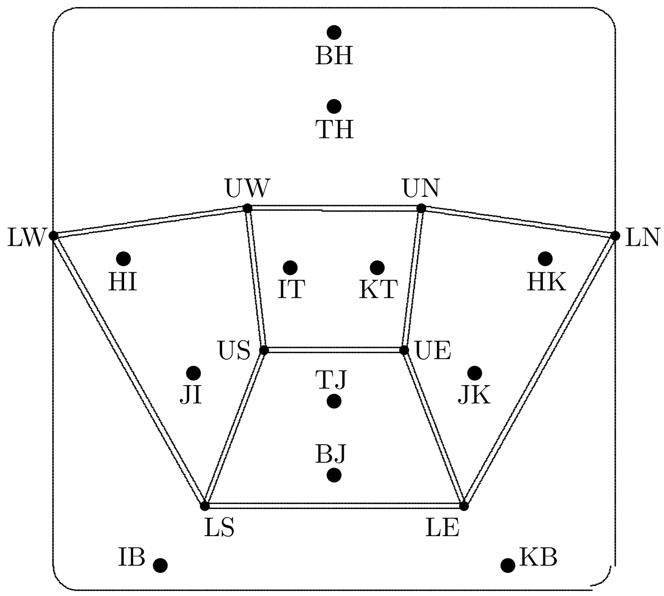

Figure 2.8

Now six other students enter the picture. They are identified as T(op face), B(ottom face), H (northwest face), I (southwest face), J (southeast face) and K (northeast face). Each of these students hold out both hands and places them symmetrically about the center of the assigned face of the cube. The line segment joining the two hands of each student is parallel to a side of the cube, and the segments on adjacent faces are perpendicular to each other. Each pair of these hands will grab the two sides of the square face parallel to the segment they form. Each hand will also grab the nearer one of the remaining two sides of the square face. The end result is a dodecahedron, as shown in Figure 2.9.

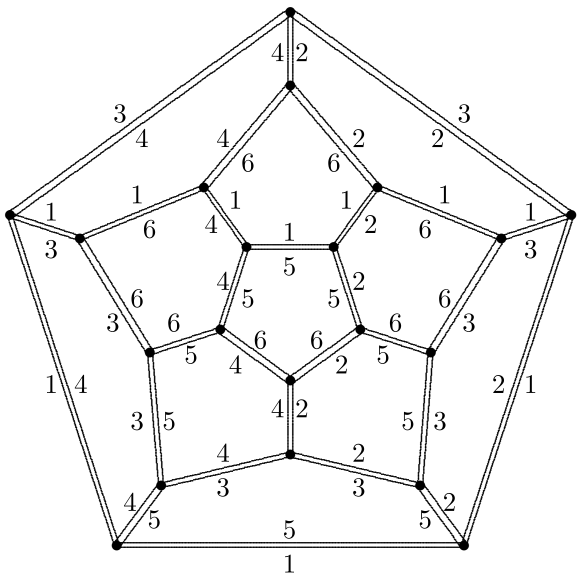

Figure 2.9

It should be emphasized that while each face of the cube is formed of one string, no part of this string is to be grabbed by the hands assigned to this face. Instead, the other four strings joining adjacent pairs of vertices of the face are grabbed, as illustrated in Figure 2.10. Failure to exercise the caution in the preceding paragraph will still produce a dodecahedron, but the whole structure will then fall apart in Step 4.

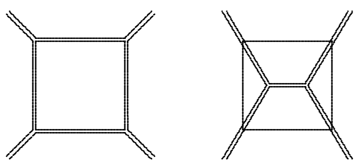

Figure 2.10

Step 4. Transformation into the Icosahedron

The original four students let go of their strings. The end result is an icosahedron, as shown in Figure 2.11.

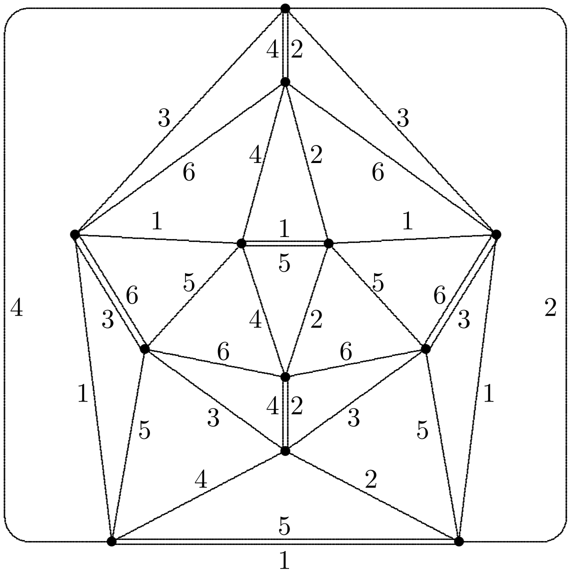

Figure 2.11

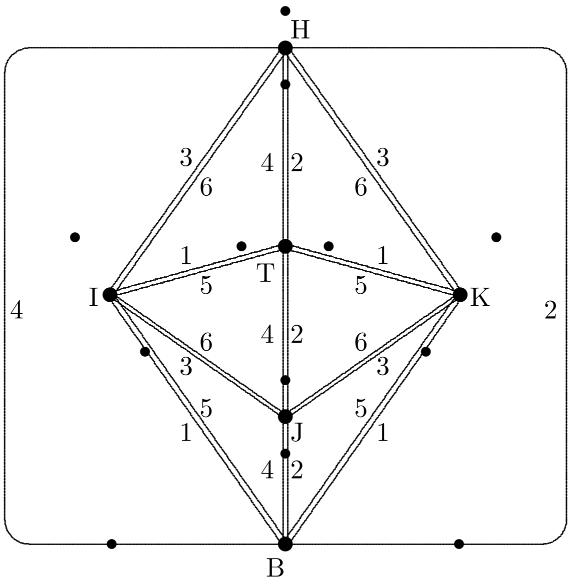

Figure 2.12

Step 5. Transformation into the Octahedron

Each of the remaining six students slides both hands together. The end result is an octahedron, as shown in Figure 2.12. Two strings which are opposite sides of the original tetrahedron now form the same square cross-section of the octahedron.

Step 6. Return to the Tetrahedron

The original four students N, S, E and W re-enter the picture. N puts the U hand in triangle HKT (north and top), S puts the U hand in triangle IJT (south and top), E puts the L hand in triangle JKB (bottom and east) and W puts the L hand in triangle HIB (bottom and west). This is shown in Figure 2.13.

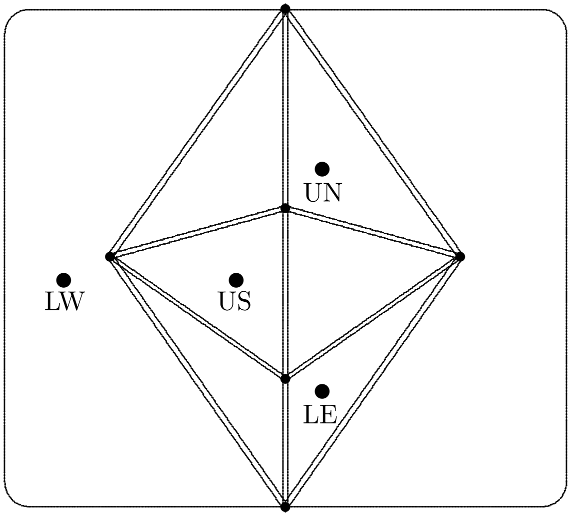

Figure 2.13

Each hands grabs the three strings it originally holds, and then the other six students let go of theirs. The end result is a tetrahedron, as shown in Figure 2.14.

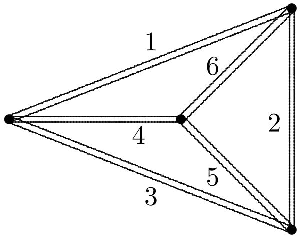

Figure 2.14

Section 3. Semi-Regular Polyhedra.

A polyhedron is said to be **semi-regular** if it satisfies condition (B) in the definition of a regular polyhedron. Clearly the Platonic solids are semi-regular, but they are now joined by infinitely many others. In Figure 2.15, we depict the case $n = 8$ from each of two infinite classes of semi-regular polyhedra — the prisms $(4, 4, n)$ and the antiprisms $(3, 3, 3, n)$, where $n \geq 3$. In particular, the cube is the order-4 prism, and the regular octahedron is the order-3 antiprism.

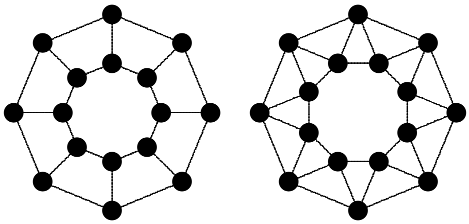

Figure 2.15

Apart from these three classes, there are other semi-regular polyhedra. Let us use a geometric approach to see if we can unearth some of them.

By slicing off, in a systematic manner, the corners of the regular tetrahedron, we obtain the truncated tetrahedron (3,6,6), as shown in Figure 2.16.

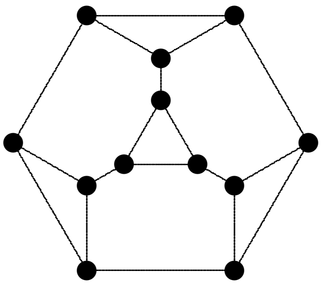

Figure 2.16

The same process applied to the cube, the regular dodecahedron, the regular octahedron and the regular icosahedron produces the truncated cube (3,8,8), the truncated dodecahedron (3,10,10), the truncated octahedron (4,6,6) and the truncated icosahedron (5,6,6), as shown in Figures 2.17, 2.18, 2.19 and 2.20.

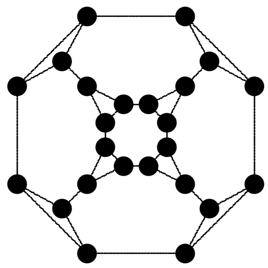

Figure 2.17

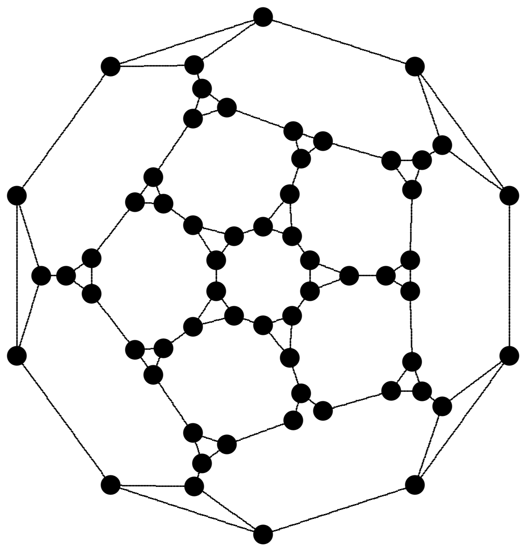

Figure 2.18

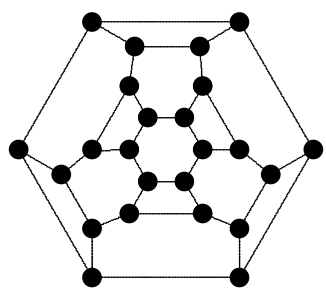

Figure 2.19

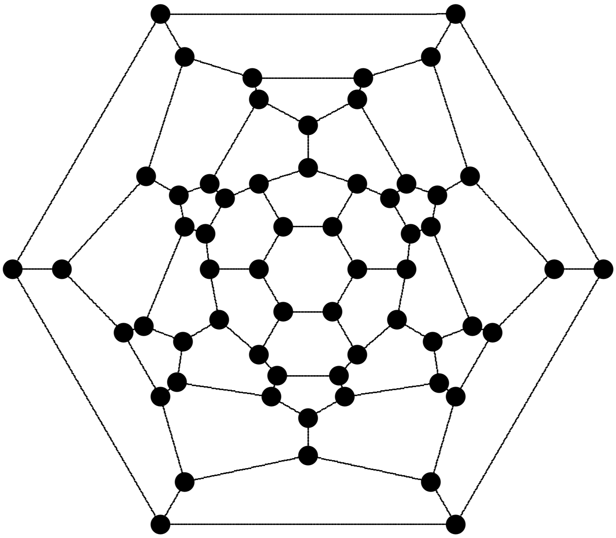

Figure 2.20

If we truncate the cube or the octahedron to the midpoints of the edges, we obtain the cuboctahedron (3,4,3,4). If we truncate the dodecahedron or the icosahedron to the midpoints of the edges, we obtain the icosadodecahedron (3,5,3,5). These are shown in Figures 2.21 and 2.22.

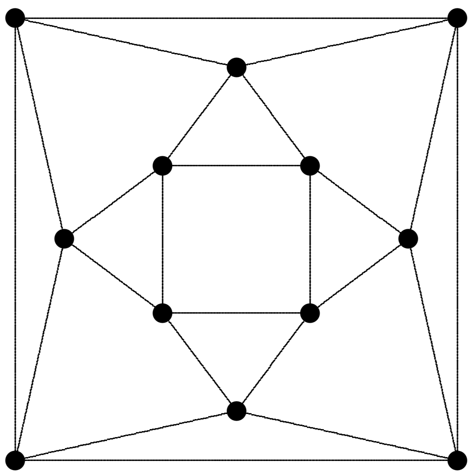

Figure 2.21

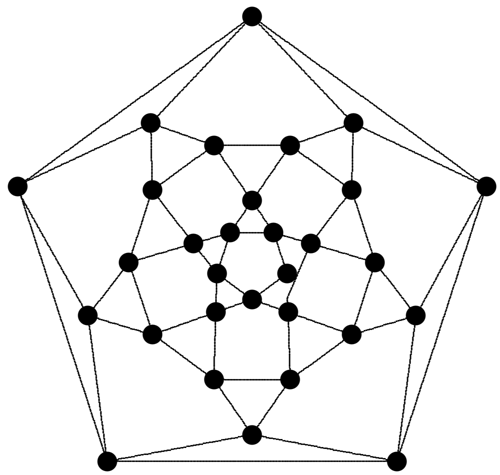

Figure 2.22

Figure 2.23 shows the great rhombicuboctahedron (4,6,8) which are obtained by truncation from the cuboctahedron.

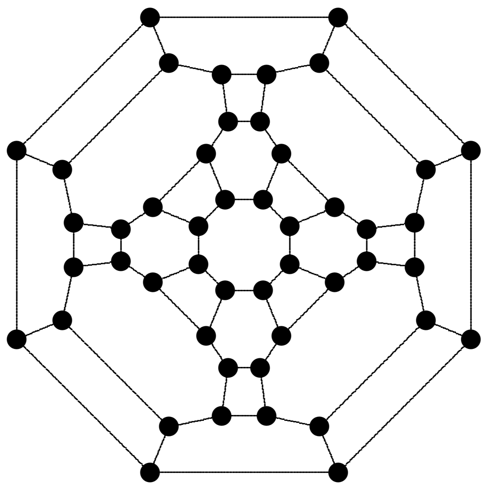

Figure 2.23

Figure 2.24 shows the great rhombicosadodecahedron (4,6,10) which is obtained by truncation from the icosadodecahedron.

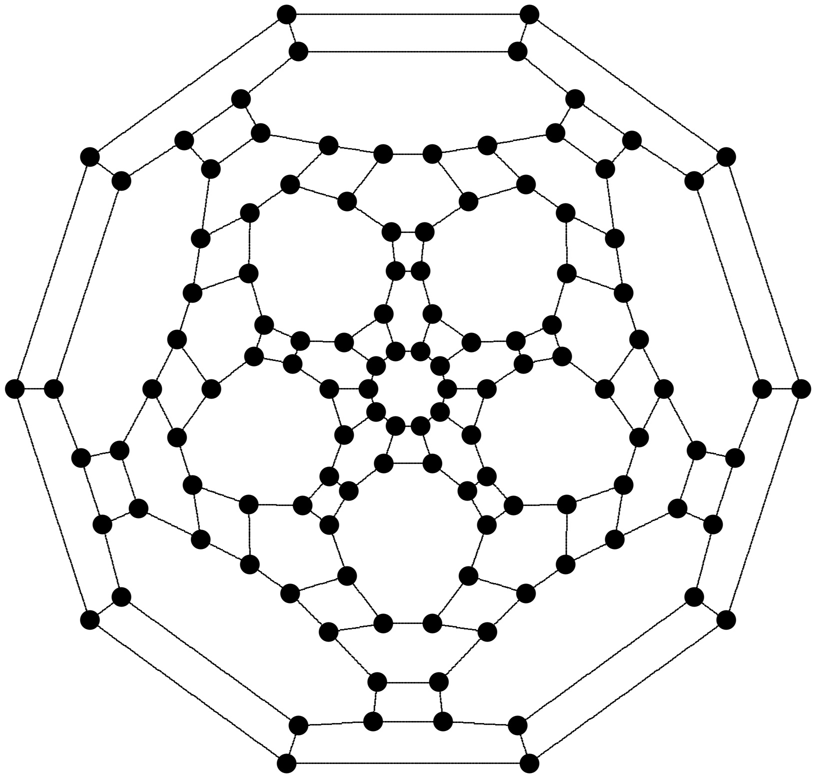

Figure 2.24

Figure 2.25 shows the small rhombicuboctahedron (3,4,4,4) which is obtained from the cuboctahedron by taking the truncation to the midpoints of the edges.

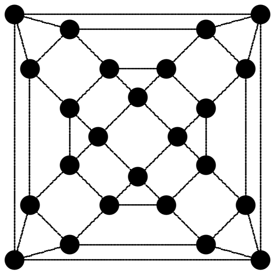

Figure 2.25

By rotating the octagon in Figure 2.25 45°, we obtain a polyhedron which has the same vertex sequence (3,4,4,4) as the small rhombicuboctahedron. It has much less geometric symmetry and is not considered to be a new semi-regular polyhedron. It is not obtained by truncation.

Figure 2.26 shows the small rhombicosadodecahedron (3,4,5,4) which is obtained from the icosadedecahedron by taking the truncation to the midpoints of the edges.

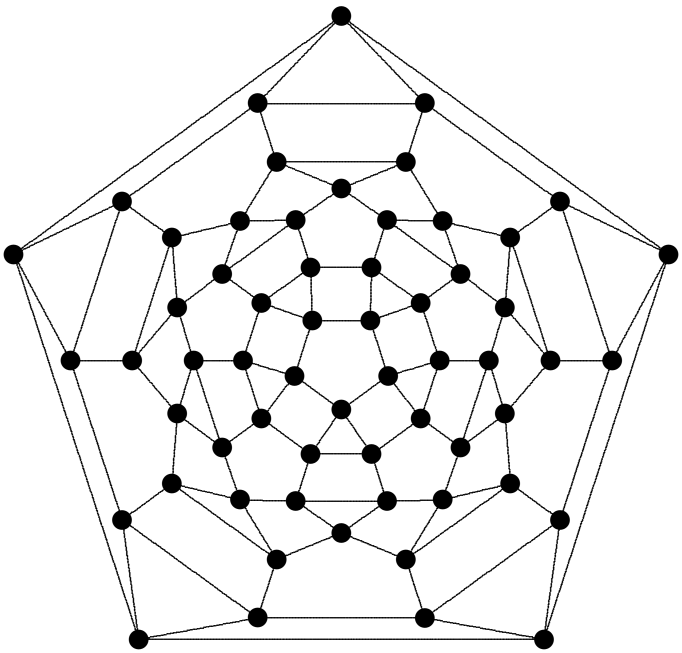

Figure 2.26

Are there some semi-regular polyhedra which are not obtained by truncation? In any case, how can we find all of them? To do so, we return to the combinatorial approach in Section 1 to prove our main result.

Theorem.
The vertex sequence of any semi-regular polyhedron is among the following:

(I) (3,3,3), (4,4,4), (5,5,5), (3,3,3,3), (3,3,3,3,3);

(II) $(4, 4, n), n \geq 3$;

(III) $(3, 3, 3, n), n \geq 3$;

(IV) (3,6,6), (3,8,8), (3,10,10), (4,6,6), (5,6,6), (4,6,8), (4,6,10), (3,4,3,4), (3,5,3,5), (3,4,4,4), (3,4,5,4), (3,3,3,3,4), (3,3,3,3,5).

The thirteen polyhedra in (IV) are called *sporadic* semi-regular polyhedra, and if their faces are regular polygons, they are called **Archimedean** solids, named after another Greek philosopher.

Note that the combinatorial approach yields two additional semi-regular polyhedra not obtained geometrically by truncation. They are the snub cube (3,3,3,3,4) and the snub dodecahedron (3,3,3,3,5). Both have two versions in opposite orientations, but they are not considered to be different semi-regular polyhedra.

To prove the Theorem, let there be t kinds of faces. For $1 \leq i \leq t$, let each face of the i-th kind be bounded by x_i edges, and let the number of such faces be F_i. Suppose the vertex sequence of the solid has length n and consists of λ_i copies of x_i for $1 \leq i \leq t$, with $\lambda_1 + \lambda_2 + \cdots + \lambda_t = n$.

Count the vertices of each face bounded by x_i edges. The total is $x_i F_i$. Each vertex has been counted λ_i times for a total of $\lambda_i V$. It follows that $F_i = \frac{\lambda_i}{x_i} V$. This is a generalization of our earlier result $nV = mF$. Moreover, we still have $nV = 2E$. Putting these into Euler's Formula, we have $V + \frac{\lambda_1}{x_1} V + \cdots + \frac{\lambda_t}{x_t} V - \frac{n}{2} V = 2$ or

$$\frac{\lambda_1}{x_1} + \cdots + \frac{\lambda_t}{x_t} = \frac{2}{V} + \frac{n-2}{2}.$$

We call this the characteristic equation.

Each of $n,\ x_1,\ \ldots,\ x_t$ is at least 3. We shall deduce from the characteristic equation that at least one of the x's is less than 6. Assuming the contrary, we have the following contradiction:

$$\frac{2}{V} + \frac{n-2}{2} = \frac{\lambda_1}{x_1} + \cdots + \frac{\lambda_t}{x_t} \leq \frac{n}{6} \leq \frac{n}{6} + \left(\frac{n}{3} - 1\right) = \frac{n-2}{2}.$$

Similarly, we can show that $n = 3$, 4 or 5. Assuming the contrary, we have the following contradiction:

$$\frac{2}{V} + \frac{n-2}{2} = \frac{\lambda_1}{x_1} + \cdots + \frac{\lambda_t}{x_t} \leq \frac{n}{3} \leq \frac{n}{3} + \left(\frac{n}{6} - 1\right) = \frac{n-2}{2},$$

We now divide the proof into three parts, for $n = 3,\ n = 4$ and $n = 5$ respectively.

Part One. $n = 3$.

Let the vertex sequence be (a, b, c) with $a \leq b \leq c$, where $3 \leq a \leq 5$. We consider four cases.

Case 1. $a = b = c$.

Since $3 \leq a \leq 5$, the only possibilities here are (3,3,3), (4,4,4) and (5,5,5).

Case 2. $a = b < c$.

Consider a typical a-sided face $ABCD\ldots$ (see Figure 2.27). Of the other two faces at the vertex B, one must be a-sided, and the other c-sided. Assume without loss of generality that the face alongside BC is c-sided. If we consider the vertex C, it follows that the face alongside CD is a-sided. Thus the neighbors of $ABCD\ldots$ have alternately a sides and c sides. Hence $ABCD\ldots$ must have an even number of neighbors, showing that a is even. Since $3 \le a \le 5$, we have $a = 4$. This gives rise to the infinite class $(4, 4, n), n \ge 5$.

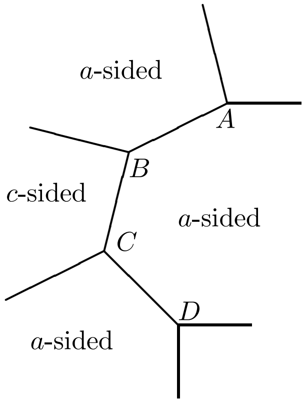

Figure 2.27

Case 3. $a < b = c$.

The characteristic equation in this case is

$$\frac{1}{a} + \frac{2}{b} = \frac{2}{V} + \frac{1}{2}.$$

Recall that $3 \le a \le 5$. As in Case 2, b must be even. Suppose $a = 3$. We have $V = \frac{12b}{12-b}$. Since V is a positive integer, the only meaningful values are $b = 4$, 6, 8 and 10. This gives rise to the vertex sequences (3,4,4), (3,6,6), (3,8,8) and (3,10,10). Suppose $a = 4$. We have $V = \frac{8b}{8-b}$. Hence $b = 6$, giving rise to (4,6,6). Finally, suppose $a = 5$. We have $V = \frac{20b}{20-3b}$. Hence $b = 6$, giving rise to (5,6,6).

Case 4. $a < b < c$.

The characteristic equation in this case is

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{2}{V} + \frac{1}{2}.$$

Here, as in Case 2, all of a, b and c must be even. Since $3 \leq a \leq 5$, we have $a = 4$, and the characteristic equation simplifies to

$$\frac{1}{b} + \frac{1}{c} = \frac{2}{V} + \frac{1}{4}.$$

Suppose $b \geq 8$. Recall $c > b$, so $\frac{2}{V} + \frac{1}{4} = \frac{1}{b} + \frac{1}{c} \leq \frac{1}{4}$. We have a contradiction, so $b = 6$. We have $V = \frac{24c}{12-c}$. Hence $c = 8$ or 10, giving rise to the vertex sequences (4,6,8) and (4,6,10).

Part Two. $n = 4$.

Let the vertex sequence be some permutation of $\{a, b, c, d\}$ such that $a \leq b \leq c \leq d$, where $3 \leq a \leq 5$. The characteristic equation is

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{2}{V} + 1.$$

Suppose $a \geq 4$. Then $\frac{2}{V} + 1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq 1$. This is a contradiction, so $a = 3$, and the characteristic equation simplifies to $\frac{1}{b} + \frac{1}{c} + \frac{1}{d} = \frac{2}{V} + \frac{2}{3}$. Suppose $b \geq 5$. Then $\frac{2}{V} + \frac{2}{3} = \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{3}{5}$. This is another contradiction, hence $b = 3$ or 4. We consider the two cases separately.

Case 1. $b = 3$.

Here, the characteristic equation simplifies further to

$$\frac{1}{c} + \frac{1}{d} = \frac{2}{V} + \frac{1}{3}.$$

Suppose $c \geq 6$. Then $\frac{2}{V} + \frac{1}{3} = \frac{1}{c} + \frac{1}{d} \leq \frac{1}{3}$. Again this is a contradiction, so $c = 3$, 4 or 5. If $c = 3$, we have the infinite class $(3, 3, 3, n), n \geq 3$.

We now have $a = b = 3$ and $c = 4$ or 5. We shall first show that the two 3's cannot be consecutive in the vertex sequence. Assuming the contrary, we let the vertex sequence be $(3, 3, c, d)$ and consider a typical triangular face ABC (see Figure 2.28). The vertex A must belong to a triangular face adjacent to ABC. Let this be ABD. Now the vertex C must also belong to a triangular face adjacent to ABC, so either A or B must belong to three triangular faces, which is a contradiction.

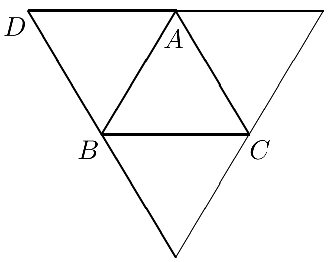

Figure 2.28

We shall now prove that $c = d$, so that the only possible vertex sequences are (3,4,3,4) and (3,5,3,5). Suppose $c < d$. Consider a typical triangular face ABC. The other triangular faces shown in Figure 2.29 are dictated by the form of the vertex sequence, which is $(3, c, 3, d)$. Of the three faces adjacent to ABC, two must contain the same number of edges, and one of A, B or C cannot have $(3, c, 3, d)$ as its sequence, a contradiction.

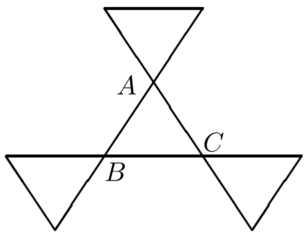

Figure 2.29

Case 2. $b = 4$.

Here, the characteristic equation becomes

$$\frac{1}{c} + \frac{1}{d} = \frac{2}{V} + \frac{5}{12}.$$

Suppose $c \geq 5$. Then $\frac{2}{V} + \frac{5}{12} = \frac{1}{c} + \frac{1}{d} \leq \frac{2}{5}$. This is a contradiction, so $c = 4$. We have $V = \frac{12d}{6-d}$. Hence $d = 4$ or 5. As in Case 1, it can be shown that (3,4,4,5) cannot be a vertex sequence, leaving (3,4,4,4) and (3,4,5,4) as the only possibilities.

Part Three. $n = 5$.

Let the vertex sequence be some permutation of $\{a, b, c, d, e\}$ such that $a \leq b \leq c \leq d \leq e$, where $3 \leq a \leq 5$. The characteristic equation is

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{2}{V} + \frac{3}{2}.$$

Suppose $d \geq 4$. Then

$$\frac{2}{V} + \frac{3}{2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} \leq \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{3}{2},$$

which shows the supposition to be untenable. Hence $a = b = c = d = 3$. We have $V = \frac{12e}{6-e}$. Hence $e = 3$, 4 or 5, giving rise to the vertex sequences (3,3,3,3,3), (3,3,3,3,4) and (3,3,3,3,5).

This completes the proof of the theorem.

The argument was due to Tom Boag, Charles Boberg and David Hughes [1]. They were junior high school students at the time, when the SMART Circle was not yet in existence. An earlier and different proof of the theorem was given by L. Lines in [3], a book which is now out of print.

The statistics for the Archimedean solids are summarized in the following chart, where F_i denotes the number of faces with i edges. These are obtained by expressing all other variables in terms of V and then substituting into Euler's Formula.

Vertex Sequences	Statistics							
	E	V	F_3	F_4	F_5	F_6	F_8	F_{10}
(3,6,6)	18	12	4	0	0	4	0	0
(3,8,8)	36	24	8	0	0	0	6	0
(3,10,10)	90	60	20	0	0	0	0	12
(4,6,6)	36	24	0	6	0	8	0	0
(5,6,6)	90	60	0	0	12	20	0	0
(3,4,3,4)	24	12	8	6	0	0	0	0
(3,5,3,5)	60	30	20	0	12	0	0	0
(4,6,8)	72	48	0	12	0	8	6	0
(4,6,10)	180	120	0	30	0	20	0	12
(3,4,4,4)	48	24	8	18	0	0	0	0
(3,4,5,4)	120	60	20	30	12	0	0	0
(3,3,3,3,4)	60	24	32	6	0	0	0	0
(3,3,3,3,5)	150	60	80	0	12	0	0	0

Exercises

1. The tetrahedron has 6 edges and the square pyramid has 8 edges. These are the simplest two polyhedra. Intuitively, no polyhedron can have exactly 7 edges. Prove this algebraically using Euler's Formula.
2. (a) With two students and one string, form the skeleton of a tetrahedron.
   (b) With three students and one string, form the skeleton of a standard octahedron.
   (c) With four students and one string, form the skeleton of a cuboid.
3. (a) Draw the Schlegel diagram of one orientation of the snub cube.
   (b) Draw the Schlegel diagram of one orientation of the snub dodecahedron.

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