

Chapter 2

Plane Waves: Time Domain Solutions

Now that we have identified the significant properties of an acoustic signal we may address two fundamental questions: How was the signal generated at its source? How was the signal modified as it traveled from its source to the location where it was observed? These questions often are closely related. Generation of sound is usually associated with oscillation of another medium that shares a surface with the fluid in which the sound is observed. Usually the other body is a solid object like a loudspeaker or an automobile or our vocal cords, but it also could be a region of fluid, such as the gases exhausted by a jet engine. Determination of how one can induce an object to vibrate in the manner required to generate a specific sound is the realm of acoustic specialties such as engineering acoustics, musical acoustics, and structural acoustics and vibrations.

Our study in this chapter is devoted to the simplest kind of acoustic signal, in which the pressure depends only on time and position along a straight line. We say that it is a *plane wave* because all points on a plane perpendicular to the straight line experience the same pressure at every instant. Despite their relative simplicity, plane waves are extremely important to acoustics. They are frequently observed; many other types of signals can be represented as superpositions of plane waves; and some other types of waves tend to become planar at long propagation distances. In addition, the relative simplicity of plane waves makes it easier to identify underlying physical phenomena, and to perform analyses in both the time and frequency domains.

Our first task is to derive the equations whose solution will tell us what pressure will be observed at a specified position and time. We will do so by adapting the basic equations of fluid mechanics to account for the relative smallness of acoustic signals. The result will be the wave equation. A progression of time-domain solutions eventually will demonstrate several acoustical phenomena that also occur for multidimensional signals. These time domain solutions will have certain limitations whose removal will require that we shift in the next chapter to frequency domain

analysis. In this respect the treatment of planar waves is like our approach for later topics: Find out what we can learn in the time domain, then try to learn more with frequency domain formulations.

2.1 Continuum Equations in One Dimension

Field equations are partial differential equations that govern how variables describing the state of a fluid or solid evolve in time and are distributed spatially. Some assumptions, which usually are quite valid, are made for our initial studies. One is that energy dissipation due to viscosity and other processes is ignored. Another is a restriction to situations where the fluid is homogeneous, which means that its properties in the undisturbed state are the same everywhere. We also consider the fluid to be at rest in the absence of an acoustic disturbance.

These assumptions parallel those that often are invoked in the study of the mechanics of a compressible, inviscid fluid. The transition to the specialized equations of acoustics comes when the small-signal restriction is invoked. This entails simplifying the equations to account for the fact that acoustical signals have an amplitude that is much less than some reference value, such as the ambient pressure. (We have already seen that a very loud sound corresponds to a rather small pressure amplitude.) When these idealizations apply, the field equation will be the linear wave equation for a single state variable, usually pressure. Each of the idealizations is examined in later chapters, but failure to invoke them at the outset of our exploration would obscure learning the core concepts, and it would make any investigation much more difficult.

2.1.1 Conservation of Mass

The particles of a solid have an identifiable position when the solid is undisturbed by outside forces. In contrast, if we look at a fluid in motion, and then remove the causes of that motion, the fluid particles will not return to some reference position. Thus derivation of basic equations requires the concept of a *control volume*, which is merely a region that we artificially isolate. This control volume may be fixed in space, or it may move with the fluid in order to follow a specific group of particles.

We begin with a derivation of the basic equation that governs how a fluid may move without creating or destroying mass. The equation that results is usually referred to as the *continuity equation*, because it enforces the requirement that there may be no voids in which there are no fluid particles. One way to derive the continuity equation is to examine the mass that enters and leaves a *fixed control volume* over some time interval.

A common system in which one might encounter a planar wave is a long straight tube having constant cross-sectional area $\mathcal{A}$. In acoustics such a system usually is

said to be a *duct*, or a *waveguide*, both of which convey the fact that the propagation is confined to a specific direction. Position along the tube is specified by the spatial coordinate x . All state variables at a specified x and t in the one-dimensional model have the same value, regardless of where they are situated on a cross section. Figure 2.1 depicts a control volume $\mathcal{V}$ that is formed by isolating the region inside the tube between positions x_1 and x_2 . The illustration to the left describes the state at some instant t . It shows $\mathcal{V}$ and an adjacent region in $x < x_1$ that is the portion of fluid that will enter $\mathcal{V}$ in an infinitesimal time interval dt . The illustration to the right depicts the state at time $t + dt$. Here we see $\mathcal{V}$ and a region in $x > x_2$ that contains the fluid that has emerged from the control volume.

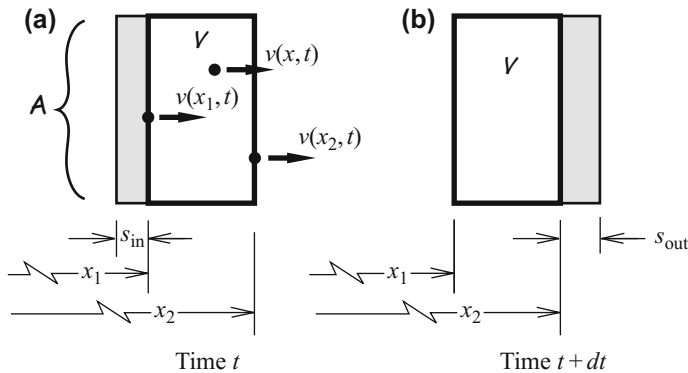


Fig. 2.1 Control volume for formulation of the continuity equation

All particles on a cross section advance in unison at velocity $v(x, t)$. Consequently, the domains that enter and leave are prismatic, as is the control volume. The volume of each region is the product of $\mathcal{A}$ and its length in the x direction. For $\mathcal{V}$ this length is $x_2 - x_1$, but what are the lengths s_{in} and s_{out} for the domains that enter and leave? To answer this we observe that the distance traveled by a cross-section at position x in an infinitesimal time interval is $v(x, t) dt$. It might seem problematic that the movement changes the x and t values on which v depends, but these changes are infinitesimal. For this reason we may use the velocity at x_1 at time t to evaluate s_{in} and the velocity at x_2 at time t to evaluate s_{out} . Thus, we have

$$s_{in} = v(x_1, t) dt, \quad s_{out} = v(x_2, t) dt \quad (2.1.1)$$

The mass of the two domains depicted in Fig. 2.1a for time t must be the same as the mass of the two domains in Fig. 2.1b for $t + dt$. The density ρ between x_1 and x_2 is not necessarily constant at any instant, so the mass contained in $\mathcal{V}$ must be evaluated as an integral over the length of $\mathcal{V}$. The fact that s_{in} and s_{out} are infinitesimal allows us to consider the density of the entering domain to be $\rho(x_1, t)$, and the density of the exiting domain to be $\rho(x_2, t)$, so conservation of mass reduces to

$$\int_{x_1}^{x_2} \rho(x, t + dt) \mathcal{A} dx + \rho(x_2, t) \mathcal{A} v(x_2, t) dt = \int_{x_1}^{x_2} \rho(x, t) \mathcal{A} dx + \rho(x_1, t) \mathcal{A} v(x_1, t) dt \quad (2.1.2)$$

Division of this conservation equation by $\mathcal{A} dt$ leads to

$$\int_{x_1}^{x_2} \left[\frac{\rho(x, t + dt) - \rho(x, t)}{dt} \right] dx + \rho(x_2, t) v(x_2, t) - \rho(x_1, t) v(x_1, t) = 0 \quad (2.1.3)$$

The integrand is the definition of the partial derivative of density with respect to time. To progress further we consider the terms at x_1 and x_2 to be the limits of an integral over $x_1 < x < x_2$. Specifically, we use the property that

$$f(x_2) - f(x_1) \equiv \int_{x_1}^{x_2} \frac{\partial f}{\partial x} dx \quad (2.1.4)$$

To some extent it might seem like a trick to introduce this identity, but it actually is the one-dimensional version of the divergence theorem. When Eq. (2.1.4) is used to replace the limit terms in the continuity equation, the result is

$$\int_{x_1}^{x_2} \left[\frac{\partial \rho}{\partial t} dx + \frac{\partial}{\partial x} (\rho v) \right] dx = 0 \quad (2.1.5)$$

This integral relation must be satisfied for any choice of x_1 and x_2 , which can only happen if the integrand vanishes. The result is the *one-dimensional continuity equation*

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v) = 0} \quad (2.1.6)$$

The first term represents an accumulation of fluid particles within $\mathcal{V}$, while ρv is the mass flux, that is, the net rate per unit surface area at which mass flows through a cross section.

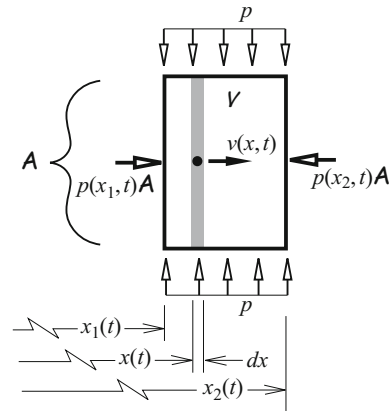
2.1.2 Momentum Equation

The next question is how do fluid particles react to the forces acting on them? The forces are those associated with the pressure. (Gravity is not included here because it has a static role for a homogeneous fluid.) The basic laws that relate motion and forces are those of Newton. We have adopted a continuum model in which an aggregate of fluid particles constitute a differential element. Correspondingly we enforce Newton's second law for a system of particles. It states that the time rate at

which the total momentum of all particles in a system changes equals the resultant force acting on that system. This statement is often said to be the momentum-impulse principle, even though we use it in its time derivative form.

Formulation of this principle requires that we track a specific set of particles, so the control volume for the derivation *moves* in unison with the along with the particles. Thus, the control volume is taken to be the prismatic domain extending over $x_1(t) \leq x \leq x_2(t)$, with the stipulation that the lower and upper limits move in unison with the fluid particles at the respective cross-sections. This control volume is depicted as a free body diagram in Fig. 2.2.

Fig. 2.2 Control volume for formulation of the momentum equation



Along the prismatic surface (the side), there is a pressure distribution that is the reaction to the fluid pressing on the wall of the tube. This effect acts perpendicularly to the axis of the tube, and the pressure is the same for all particles at a specified x . It can be proven that the resultant of this force system at each x is zero, which is consistent with the restriction of the fluid to movement in the x direction. Consequently, the pressure exerted by the tube wall can be ignored for the formulation of the momentum equation. There is a resultant force acting in the axial direction. The pressure pushes on the exposed cross-sections faces, and its value is constant along each face. Thus, the force at $x_1(t)$ is $p(x_1, t)A$ acting in the direction of increasing x , while the force at $x_2(t)$ is $p(x_2, t)A$ in the direction of decreasing x .

All fluid particles situated at the same x have the same velocity, so the momentum of a differential element of mass is $v(x, t) dm$. The control volume has been defined to move with the fluid. Correspondingly, we identify $dm = \rho A dx$ as the mass contained between cross-sections at x and $x + dx$ that move in unison with the particles. Because mass is conserved, dm is invariant in time, so only $v(x, t)$ needs to be differentiated. Thus, the impulse-momentum principle for a system of particles states that

$$\int_{x_1(t)}^{x_2(t)} \left[\frac{d}{dt} v(x, t) \right] dm = p(x_1, t)A - p(x_2, t)A \quad (2.1.7)$$

A subtlety arises at this stage, at the heart of which is what $v(x, t)$ means. Suppose the velocity field is independent time, but varies spatially. This means that a particle's velocity is $v(x)$. Consider a particle that is at $x = \xi$ at time t . In an infinitesimal time interval dt , the particle would displace by distance $v(\xi) dt$, so its new position would be $x = \xi + v(\xi) dt$. Thus the velocity of the particle would change from $v(\xi)$ at time t to $v(\xi + v(\xi) dt)$ at time $t + dt$. In other words, even though the velocity field does not depend on time, the particle accelerates because it has moved to a location where the velocity is different. This effect is incorporated in the evaluation of acceleration by recognizing that x is the position of the particle at time t , and that $dx/dt = v$, so that the general functional dependence of the velocity should be written as $v(x(t), t)$. The total time derivative of velocity, which is the acceleration, is found by the chain rule, which gives

$$\frac{d}{dt}v(x, t) = \frac{\partial}{\partial t}v(x, t) + \frac{\partial}{\partial x}v(x, t) \frac{\partial}{\partial t}x(t) = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \quad (2.1.8)$$

where the functional dependence has been omitted because there no longer is a question of interpretation.

The last step applies Eq. (2.1.4) to the boundary terms in Eq. (2.1.7), in order to bring them into the integral. When all terms are collected, and dm is replaced by ρdV , the result is

$$\int_{x_1(t)}^{x_2(t)} \left[\rho \mathcal{A} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial x} (\mathcal{A} p) \right] dx = 0 \quad (2.1.9)$$

As was true for the derivation of the continuity equation, we argue that this integral must vanish for any x_1 and x_2 , which will only be true if the integrand is zero. The cross section area $\mathcal{A}$ is constant, so factoring it out leads to the final form of the momentum equation for one-dimensional motion,

$$\boxed{\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) = - \frac{\partial p}{\partial x}} \quad (2.1.10)$$

Although this equation was derived by Euler in 1752, we will reserve the name “Euler’s equation” for a simpler version that results from linearization on the next section.

2.2 Linearization and the One-Dimensional Wave Equation

Three variables describing the instantaneous state of the fluid appear in the continuity equation, Eq. (2.1.6), and the momentum equation (2.1.10): pressure p , velocity v , and density ρ , each of which is a function of x and t . Only two equations are available

at this juncture, so we are not yet ready to consider solving for the state variables. The missing element is the *equation of state*, which relates pressure and density. For gases, the equation of state is derived from thermodynamic considerations, which introduces temperature T into the mix of variables. Thermodynamic assumptions that have been substantiated by measurements and difficult analyses will lead in the next section to removal of T from the set of unknowns. The consequence is that the equation of state gives a unique function relating pressure and density,

$$p = \tilde{p}(\rho) \quad (2.2.1)$$

Liquids have the same functional representation for their equation of state, but the actual form usually is determined experimentally.

Both the continuity and momentum equations are nonlinear. This means that some of their terms are a product of the state variables, specifically ρv for continuity, and $\rho(\partial v/\partial t + v\partial v/\partial x)$ for momentum. Nonlinear partial differential equations are quite difficult to solve. Fortunately, most phenomena of everyday interest feature acoustic pressure levels that are very small relative to the ambient pressure, as we saw in the discussion of sound pressure level.

The notion that the effects associated with the acoustical response are small is quantified by replacing each state variable by its value in a reference state and a *perturbation*, which is the amount by which the presence of the signal disturbs the reference state. For our initial studies the fluid is homogeneous in the reference state, which means that there is no spatial variation of the density and temperature. Also, we shall restrict our attention to situations where the fluid is at rest in the absence of the signal. The reference state in these conditions consists of ambient pressure p_0 , ambient density ρ_0 , and ambient temperature T_0 , each of which is the same at all locations.

Broad classes of systems and phenomena fit these specifications, but several applications are excluded. For example, a study of propagation to the ground of the sound generated by aircraft must account for the variation of the temperature and density with altitude, and the effect of wind velocity sometimes needs to be addressed. The situation is even more complicated in many aeroacoustic applications where the acoustic signal represents a small perturbation of a very complicated fluid flow that features viscous effects, often including turbulence.

A prime will be used temporarily to indicate the amount by which a variable is perturbed from its value at the reference state. Thus, the absolute pressure, density, and particle velocity are represented as

$$p = p_0 + p', \quad \rho = \rho_0 + \rho', \quad v = v' \quad (2.2.2)$$

The continuity equation that results from this change of variables is

$$\frac{\partial}{\partial t} (\rho_0 + \rho') + \frac{\partial}{\partial x} [(\rho_0 + \rho') v'] = 0 \quad (2.2.3)$$

Some individuals say the approximation we shall invoke considers the signal to be infinitesimal, because it takes the variables representing the acoustic disturbance to be sufficiently small that their products are negligible. A better description is that we are linearizing the governing equations, because only terms that are linear in the state variables are retained. Accordingly, we drop $\rho' v'$ from the preceding continuity equations. Furthermore, the reference state corresponds to p_0 and ρ_0 being constants. The remaining terms are the *linearized continuity equation*,

$$\boxed{\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} = 0} \quad (2.2.4)$$

A linearized momentum equation is derived in a similar manner. Substitution of Eq. (2.2.2) into (2.1.7) yields

$$(\rho_0 + \rho') \left(\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial x} \right) = - \frac{\partial}{\partial x} (p_0 + p') \quad (2.2.5)$$

Products of primed variables are negligible, and p_0 is taken to be constant. Thus, the linearized *one-dimensional momentum equation* is

$$\boxed{\rho_0 \frac{\partial v'}{\partial t} = - \frac{\partial p'}{\partial x}} \quad (2.2.6)$$

We will employ this relation in many situations, on which occasions we will refer to it as the *Euler equation* in recognition of his contribution.

The variables in the continuity equation are ρ' and v' , while those in the momentum equation are v' and p' . We need another equation, which is the equation of state, Eq. (2.2.1). Replacing the absolute pressure and density with their perturbation representations produces

$$p_0 + p' = \tilde{p}(\rho_0 + \rho') \quad (2.2.7)$$

Because $|\rho'|$ is restricted to being small relative to ρ_0 , it is permissible to linearize the equation of state with the aid of a Taylor series. In the following, a subscript “0” indicates that the quantity is evaluated at the reference state.

$$p_0 + p' = \tilde{p}(\rho_0) + \left(\frac{d\tilde{p}}{d\rho} \right)_0 \rho' + \frac{1}{2} \left(\frac{d^2 \tilde{p}}{d\rho^2} \right)_0 (\rho')^2 + \dots \quad (2.2.8)$$

The term $\tilde{p}(\rho_0)$ is the pressure corresponding to density ρ_0 , which is p_0 , so the leading terms cancel. The first derivative of $\tilde{p}(\rho)$ evaluated at the reference state is a constant. We shall use c^2 to represent it,

$$\boxed{c = \sqrt{\left(\frac{d\tilde{p}}{d\rho} \right)_0}} \quad (2.2.9)$$

The dimensionality of c is $(\text{pressure}/\text{density})^{1/2} = ((\text{force}/\text{length}^2)/(\text{mass}/\text{length}^3))^{1/2} = \text{length}/\text{time}$, in other words, speed. It will be apparent in the general solution of the wave equation two sections forward from here that c is the *speed of sound*.

The linear equation of state that results when the quadratic term in Eq. (2.2.8) is dropped is

$$p' = c^2 \rho' \quad (2.2.10)$$

Substitution of this expression into the linearized continuity equation, Eq. (2.2.4), converts it to

$$\frac{\partial p'}{\partial t} + \rho_0 c^2 \frac{\partial v'}{\partial x} = 0 \quad (2.2.11)$$

In combination with the linearized momentum equation, Eq. (2.2.6), we now have two differential equations for two state variables, p and v . It is easier to solve a single field equation, and pressure is the primary quantity for an acoustical analysis. To eliminate v' we observe that the term containing this variable in the continuity equation is $\rho_0 c^2 \partial v' / \partial x$, while it shows up as $\rho_0 \partial v' / \partial t$ in the momentum equation. To make v' appear similarly in both equations, we differentiate the momentum equation with respect to x , then divide the continuity equation by c^2 and differentiate it with respect to t . This gives

$$\begin{aligned} \rho_0 \frac{\partial^2 v'}{\partial x \partial t} &= - \frac{\partial^2 p'}{\partial x^2} \\ \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} + \rho_0 \frac{\partial^2 v'}{\partial t \partial x} &= 0 \end{aligned} \quad (2.2.12)$$

The result of subtracting the second equation into the first is the *one-dimensional wave equation*,

$$\frac{\partial^2 p'}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p'}{\partial t^2} = 0 \quad (2.2.13)$$

When we investigate multidimensional systems, we will generalize this field equation, but the result will still be called the wave equation. In addition to its great importance for acoustics, the one-dimensional version is a key element for other subjects. For example, in structural vibration it governs the transverse displacement of a tensioned string, as well as the axial displacement of an elastic bar. For this reason the one-dimensional wave equation has been the subject of much attention for applied mathematicians. It is fortunate for us that we will not require a great deal of mathematical expertise to begin our studies.

EXAMPLE 2.1 A modification that has been suggested as a way to incorporate the effects of viscous friction at the wall of a narrow tube is to add a longitudinal shear force that acts in the direction that is opposite the particle velocity. The model we consider here is restricted to situations where the size of the cross section is sufficiently large that the velocity profile may be considered to be a uniform (planar) distribution that transitions to a narrow boundary layer at the wall. If this is true, the shear stress exerted between the wall and the fluid will be proportional to the mainstream velocity. In addition, this stress is proportional to the shear strain rate, with the dynamic viscosity μ being the factor of proportionality. Multiplying the shear stress by the circumference C of the cross section gives f_v , the viscous force per unit length along the waveguide. Thus, it must be that $f_v = \chi\mu Cv$. The factor χ , whose dimensionality is reciprocal length, is described in Sect. 3.3. Derive the analogs of the one-dimensional linear wave equation and Euler's equation corresponding to this model.

Significance

The ability to adapt field equations to address special situations is a vital skill. Doing so for the effect of wall friction reveals that the basic wave equation is ever present, although it is modified.

Solution

The linearized continuity equation remains valid, but the momentum equation requires modification. The wall friction force acting on the section of length dx is $-\chi\mu Cv dx$, where the negative sign applies because the force acts oppositely to the sense of positive v . The total friction force exerted on the control volume is the accumulation of these differential contributions, which leads to addition of another integral term to the force equation for the control volume, Eq. (2.1.7),

$$\int_{x_1(t)}^{x_2(t)} \frac{d}{dt} [v(x, t) dm] = p(x_1, t) \mathcal{A} - p(x_2, t) \mathcal{A} - \int_{x_1(t)}^{x_2(t)} \chi\mu Cv dx$$

We convert the pressure terms to an integral representation by invoking the divergence theorem, Eq. (2.1.4), and bring all terms to the left side of the equality, which gives

$$\int_{x_1(t)}^{x_2(t)} \left[\rho \mathcal{A} \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial x} (\mathcal{A} p) + \chi\mu Cv \right] dx = 0$$

The integral must vanish for any pair of limits x_1 and x_2 , so we set the integrand to zero.

Our interest is in a linear set of equations, which can be obtained directly without introducing primes to represent the acoustical disturbance. Thus, we replace ρ with

ρ_0 , and drop the nonlinear acceleration terms. The result is a modified version of the linearized momentum equation,

$$\rho_0 \frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} + \chi \mu \frac{C}{A} v = 0 \quad (1)$$

The linearized continuity equation is unaltered,

$$\frac{\partial p}{\partial t} + \rho_0 c^2 \frac{\partial v}{\partial x} = 0 \quad (2)$$

The task now is to eliminate v from Eqs. (1) and (2). This can be done in a heuristic manner by observing that v appears in Eq. (2) as $\partial v / \partial x$, and differentiating Eq. (1) with respect to x would produce an equation in which particle velocity only occurs as $\partial v / \partial x$ and $\partial^2 v / \partial x \partial t$, that is,

$$\rho_0 \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 p}{\partial x^2} + \chi \mu \frac{C}{A} \frac{\partial v}{\partial x} = 0$$

Substitution of $\partial v / \partial x$ from Eq. (2) into this equation yields

$$c^2 \frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial t^2} - \chi \frac{\mu}{\rho_0} \frac{C}{A} \frac{\partial p}{\partial t} = 0 \quad (3)$$

Hence, the model of wall friction adds a first derivative term to the wave equation.

This heuristic approach to eliminating v might be difficult to implement in more complicated situations. A more formal approach is to write Eqs. (1) and (2) in differential operator form as

$$\begin{aligned} \mathcal{L}_{v1}(v) + \mathcal{L}_{p1}(p) &= 0 \\ \mathcal{L}_{v2}(v) + \mathcal{L}_{p2}(p) &= 0 \end{aligned}$$

where

$$\begin{aligned} \mathcal{L}_{v1} &= \rho_0 \frac{\partial}{\partial t} + \mu \frac{C}{A}, & \mathcal{L}_{p1} &= \frac{\partial}{\partial x} \\ \mathcal{L}_{v2} &= \rho_0 c^2 \frac{\partial}{\partial x}, & \mathcal{L}_{p2} &= \frac{\partial}{\partial t} \end{aligned}$$

We treat the operators like coefficients in a pair of simultaneous equations, which we solve for p ,

$$(\mathcal{L}_{p1} \mathcal{L}_{v2} - \mathcal{L}_{p2} \mathcal{L}_{v1})(p) \equiv \left(\rho_0 c^2 \frac{\partial^2}{\partial x^2} - \rho_0 \frac{\partial^2}{\partial t^2} - \chi \mu \frac{C}{A} \frac{\partial}{\partial t} \right)(p) = 0$$

Division of this equation by ρ_0 yields Eq. (3).

We will see in Chap. 3 that the term containing $\partial p / \partial t$ is associated with dissipation, which is not surprising because it describes a friction effect. It will be noticed that the magnitude of this effect depends on the kinematic viscosity μ / ρ_0 , which is a property of the fluid. It also is proportional to $C / \mathcal{A}$, which is a property of the cross-section. This ratio becomes smaller as the cross-section's size increases (with the shape held fixed). For example for a circle $C / \mathcal{A} = 2 / r$. Thus, wall friction decreases in importance as the width of the waveguide is increased.

2.3 Equation of State and the Speed of Sound

The only property of the fluid that appears in the wave equation is the speed of sound c . One way of determining it is to measure it experimentally based on a variety of acoustical phenomena. The value of c does not vary by a great deal for liquids in the usual range of ambient conditions, so experimental measurement usually is sufficient, possibly supplemented by empirical observations regarding the dependence of c on ambient conditions, notably temperature. It is less satisfactory to rely on experiments in the case of gases because c is strongly influenced by the ambient conditions.

It is possible to derive an equation of state for an ideal gas solely from thermodynamic considerations. Even then, some knowledge derived from experimental observations will be required. Pressure p , density ρ , and absolute temperature T (degrees K) for an infinitesimal volume of gas are related by the ideal gas law,

$$p = R\rho T \quad (2.3.1)$$

The parameter R is the ratio of the universal gas constant $R_0 = 8314 \text{ J/(kg}\cdot\text{K)}$ to the average molecular weight of the gas. The value for air is $R = 287.1 \text{ J/(kg}\cdot\text{K)}$.

Early researchers, starting with Newton in the *Principia* (1687), invoked a plausible assumption to apply the ideal gas law to acoustics. It was argued that the pressure changes extremely rapidly, even at the lowest audible frequencies. If that is so, there should be little time for the fluid to heat up due to an increase in the pressure, or to cool when the pressure decreases. In other words they assumed that acoustical processes are isothermal, so that T is always the ambient value T_0 . Because p_0 , ρ_0 , and T_0 must satisfy the ideal gas law, values of R and T_0 cancel when Eq. (2.3.1) is used to form p / p_0 . The result is that

$$p = p_0 \frac{\rho}{\rho_0} \quad (2.3.2)$$

The speed of sound corresponding to this equation of state is found from Eq. (2.2.9) to be

$$c_{\text{isothermal}} = \left(\frac{p_0}{\rho_0} \right)^{1/2} \quad (2.3.3)$$

Despite the plausibility of the isothermal argument, it could not be confirmed experimentally. Measurements generally indicated that the predicted sound speed was 16% lower than the actual value. Many individuals tried to resolve this conflict between theory and experiment, but the fault was not identified until 1816, when Laplace introduced an equally plausible assumption. Based on the same notion that audible signals constitute a rapid change of state, he said that the change is so rapid that there is insufficient time for heat to flow into or out of a volume of gas before the state reverses. A process in which there is no heat flow is said to be **adiabatic**. In such a process the temperature within the volume must change to maintain a fixed heat content when the pressure and density change.

To obtain the adiabatic equation of state consider the differential increase in temperature obtained by application of the chain rule to the ideal gas law,

$$T = \frac{p}{R\rho} \implies dT = dp \frac{\partial}{\partial p} \left(\frac{p}{R\rho} \right) + d\rho \frac{\partial}{\partial \rho} \left(\frac{p}{R\rho} \right) \quad (2.3.4)$$

By definition a partial derivative holds all other variables constant. Hence, the above differential may be considered to describe a pair of processes: Process *A* increases the pressure with the density held constant, while process *B* increases the density with the pressure held constant. The associated contributions to the temperature increment are

$$\begin{aligned} dT_A &= dp \left. \frac{d}{dp} \left(\frac{p}{R\rho} \right) \right|_{\text{constant } \rho} = \frac{dp}{R\rho} \\ dT_B &= d\rho \left. \frac{d}{d\rho} \left(\frac{p}{R\rho} \right) \right|_{\text{constant } p} = -\frac{p}{R\rho^2} d\rho \end{aligned} \quad (2.3.5)$$

Specific heat is the heat gain per unit mass resulting from a one degree increase in temperature. For the mass we use a differential mass dm at an arbitrary location. Two kinds of specific heat are used. One is C_v , which applies if density is held constant. (The subscript v in C_v is the specific volume, which is the reciprocal of density.) Process *A* fits this specification, so we have

$$dq_A = (C_v dT_A) dm = \left(C_v \frac{p}{R\rho} dp \right) dm \quad (2.3.6)$$

The other specific heat is C_p , which describes the behavior if the pressure is held constant. This is process *B*, so the corresponding heat gain is

$$dq_B = (C_p dT_B) dm = \left(-C_p \frac{p}{R\rho^2} d\rho \right) dm \quad (2.3.7)$$

where the negative sign merely says that the heat content decreases.

The adiabatic condition requires that the combination of processes *A* and *B* not change the heat content of dm . In other words it must be that $dq_A + dq_B = 0$. We use the above relations to describe each increment, and cancel the common factor

dm , which leads to the requirement that

$$C_v \frac{1}{R\rho} dp - C_p \frac{p}{R\rho^2} d\rho = 0 \quad (2.3.8)$$

This relation may be solved for $dp/d\rho$. The speed of sound, which was defined in Eq. (2.2.9), features this derivative at the ambient state, so we replace p and ρ with p_0 and ρ_0 , respectively. The *ratio of specific heats* is typically denoted as γ , so we find that

$$c = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad \gamma = \frac{C_p}{C_v} \quad (2.3.9)$$

Thus the isothermal sound speed in Eq. (2.3.3) is too small by a factor of $1/\gamma^{1/2}$. For air $\gamma = 1.4$, so $1/\gamma^{1/2} = 0.845$, which matches the 16% error that had been observed. A significant aspect is that γ typically depends solely on the molecular composition of the gas, but there are gases for which the specific heats are temperature dependent. Even if γ is temperature dependent, Eq. (2.3.9) remains valid because the relation was derived by considering only infinitesimal changes in temperature.

For linearized acoustics analyses we only need to know c , but studies of nonlinear effects requires knowing the actual equation of state. Separating variables converts Eq. (2.3.8) to a perfect differential,

$$\frac{dp}{p} = \gamma \frac{d\rho}{\rho} \quad (2.3.10)$$

The indefinite integral is

$$\ln(p) = \gamma \ln(\rho) + \ln(\alpha) \quad (2.3.11)$$

The constant of integration $\ln(\alpha)$ must be such that $p = p_0$ when $\rho = \rho_0$. We impose this condition, then take the antilogarithm of each side, with the result that

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0} \right)^\gamma, \quad \gamma = \frac{C_p}{C_v}$$

An alternative form in terms of the acoustic perturbation quantities, $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, is obtained from the preceding by applying the identity that $p_0 = \rho_0 c^2 / \gamma$. This leads to

$$\frac{p'}{\rho_0 c^2} = \frac{1}{\gamma} \left[\left(1 + \frac{\rho'}{\rho_0} \right)^\gamma - 1 \right] \quad (2.3.12)$$

The quantity $\rho_0 c^2$ is the *bulk modulus* K ,

$$K = \rho_0 c^2 \quad (2.3.13)$$

The reciprocal of K is the *adiabatic compressibility*. The bulk modulus is analogous to Young's modulus for an elastic solid. To see why consider an elastic bar having constant cross section $\mathcal{A}$ and length L_0 in the stress-free state. A standard static test measures the increase in length $\Delta L = L - L_0$, when an axial tensile force F is applied slowly. The axial stress is $\sigma = F/\mathcal{A}$ and the axial strain is $\varepsilon = \Delta L/L$. These two quantities are related by $\sigma = E\varepsilon$, where E is Young's modulus. In other words

$$\sigma = E \frac{\Delta L}{L} \quad (2.3.14)$$

To construct the analogy for a fluid replace E with K and σ with the pressure perturbation $-p'$, where the minus sign accounts for the definition of positive stress is tensile, whereas positive p' is compressive. The analog to strain is the ratio of the volume change to the original volume. This is the *condensation* $\Delta\mathcal{V}/\mathcal{V}_0$, where $\mathcal{V}_0$ is the volume in the ambient state. Thus, the analogy is that

$$p' \equiv -K \frac{\Delta\mathcal{V}}{\mathcal{V}_0} \quad (2.3.15)$$

where the negative sign indicates that a positive pressure decreases the volume. Conservation of mass relates the condensation to density. The mass in the ambient state is $\rho_0\mathcal{V}_0$, while it is $(\rho_0 + \rho')\mathcal{V}$ when p' is applied. When we set $\mathcal{V} = \mathcal{V}_0 + \Delta\mathcal{V}$, and equate the mass for each state, we obtain

$$(\rho_0 + \rho')(\mathcal{V}_0 + \Delta\mathcal{V}) = \rho_0\mathcal{V}_0 \implies \frac{\Delta\mathcal{V}}{\mathcal{V}_0} \approx -\frac{\rho'}{\rho_0} \quad (2.3.16)$$

where the approximate equality results from dropping $\rho'\Delta\mathcal{V}$ as a nonlinear effect. The quantity ρ'/ρ_0 is referred to as the dilatation in fluid mechanics. In the linearized approximation it is the negative of the condensation. We have established that $p' = K(\rho'/\rho_0)$. However, the linearized equation of state, Eq. (2.2.10), indicates that $p' = c^2\rho'$. Equating these alternate description of p' leads to Eq. (2.3.13), which confirms the analogy.

The equation of state for a liquid may be described as a Taylor series expansion of Eq. (2.2.1),

$$p_0 + p' = \tilde{p}(\rho_0 + \rho') = \tilde{p}(\rho'_0) + \left(\frac{d\tilde{p}}{d\rho}\right)_0 \rho' + \frac{1}{2} \left(\frac{d^2\tilde{p}}{d\rho^2}\right)_0 (\rho')^2 + \dots \quad (2.3.17)$$

The ambient pressure corresponding to ρ_0 is p_0 , so the leading terms cancel. Furthermore, we do not know the actual functional dependence, so the derivatives may be replaced by coefficients. The result is that the equation of state for a liquid is a simple power series given by

$$p' = A \frac{\rho'}{\rho_0} + \frac{1}{2} B \left(\frac{\rho'}{\rho_0}\right)^2 + \dots \quad (2.3.18)$$

We know that $c^2 = dp'/d\rho'$ at ambient conditions, so a comparison of this representation to the Taylor series shows that

$$A = \rho_0 \left(\frac{d\tilde{p}}{d\rho} \right)_0 = \rho_0 c^2 = K, \quad B = \rho_0^2 \left(\frac{d^2 \tilde{p}}{d\rho^2} \right)_0 \quad (2.3.19)$$

The values of c and B/A for liquids usually come from measurements.

The equations of state for a liquid and an ideal gas may be placed in a common form. To do so Eq. (2.3.12) is expanded in a binomial series that is truncated at terms that are quadratic in ρ' ,

$$\begin{aligned} \frac{p'}{\rho_0 c^2} &= \frac{1}{\gamma} \left[1 + \gamma \frac{\rho'}{\rho_0} + \frac{1}{2} \gamma (\gamma - 1) \left(\frac{\rho'}{\rho_0} \right)^2 + \cdots - 1 \right] \\ &= \frac{\rho'}{\rho_0} + \frac{1}{2} (\gamma - 1) \left(\frac{\rho'}{\rho_0} \right)^2 + \cdots \end{aligned} \quad (2.3.20)$$

Because $A = \rho_0 c^2$, this representation matches that in Eq. (2.3.18) if $B/A = \gamma - 1$. Values of B/A above 10 have seldom been reported for liquids or gases, with $B/A = 0.4$ for air, and B/A ranges from 4.2 to 6.1 for water.¹ (Some research papers have said that B/A is of the order of 1000 for rock, but rock is not a liquid.)

It might seem that halting the equation of state for a liquid at quadratic terms is a limitation. For example, doing so leads to the anomaly that p' has a minimum of $-0.5 (\rho_0 c^2) / (B/A)$ at $\rho'/\rho_0 = -1/(B/A)$. This does not limit the utility of Eq. (2.3.18). The bulk modulus is very large for a gas, $\rho_0 c^2 \approx 140$ kPa for air, and it is enormous for liquids, $\rho_0 c^2 \approx 2$ GPa for water. (For comparison, Young's modulus of steel is approximately 200 GPa.). This means that this minimum value is well beyond the range of what is considered to be an acoustic signal. Only in explosions might there be a pressure pulse that is comparable to the bulk modulus.

A further consequence of the largeness of the bulk modulus is that not much error in the density-pressure relation will arise if the equation of state is linearized. Dropping the quadratic term in justifiable if

$$\frac{B}{2A} \left(\frac{\rho'}{\rho_0} \right)_{\max} \ll 1 \quad (2.3.21)$$

A reasonable upper bound for common liquids and gases is $B/A = 10$, which implies that the linearized equation of state is accurate if $5 (\rho'/\rho_0)_{\max} \ll 1$. Let us conservatively say that $\rho/\rho_0 = 0.002$ is the maximum that would be considered to satisfy this condition. The linear equation corresponding to this value of the condensation gives $p = 0.002 \rho_0 c^2$. This corresponds to 140 dB/20 μ Pa in air, which is a level that causes permanent auditory damage. In water it is 250 dB/1 μ Pa, which is far

¹An extensive tabulation of B/A for several fluids are provided by R.T. Beyer, "The parameter B/A ," in *Nonlinear Acoustics*, M.F. Hamilton and D.T. Blackstock, eds., Academic Press, pp. 34–37 (1998).

above the level at which water near the surface will cavitate because the total pressure $p_0 + p'$ is negative. In other words, it is justifiable to linearize the equation of state for the vast majority of applications.

The primary properties for an acoustical analysis are c and ρ_0 , supplemented by B/A if nonlinear effects are to be addressed. Even if the fluid medium has been set, each property depends to some extent on environmental conditions. For air the key factor affecting c is the absolute temperature. This becomes evident when the ideal gas law is used to replace p_0/ρ_0 in Eq. (2.3.9), which leads to

$$c = \sqrt{\gamma RT_0} \quad (2.3.22)$$

The ambient temperature, as well as the ambient pressure, are the primary factors affecting the density of air. In addition, the vapor content (that is, humidity) plays a role by altering the molecular composition on which R depends. (The next example examines standard atmospheric conditions.) For water the important parameters affecting c are temperature, salinity, and ambient pressure, but the density varies little with depth. A comprehensive discussion is provided by Pierce.² Unless there is need for precision, one may use as representative values $\rho_0 = 1.2 \text{ kg/m}^3$ and $c = 340 \text{ m/s}$ for air, and $\rho_0 = 1000 \text{ kg/m}^3$ and $c = 1480 \text{ m/s}$ for water.

The developments thus far have shown that the only variables requiring determination in a typical situation are the acoustic perturbations of pressure and particle velocity. The ambient state is taken to be known, which means that we know the density ρ_0 , pressure p_0 , and sound speed c . For this reason, we may discontinue the use of primes to denote acoustic perturbations, which will serve to simplify the notation. Thus, the general relation between the acoustic pressure and particle velocity in any field is the Euler (momentum) equation,

$$\frac{\partial p}{\partial x} = -\rho_0 \frac{\partial v}{\partial t} \quad (2.3.23)$$

The one-dimensional wave equation correspondingly is now written as

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \quad (2.3.24)$$

It might be puzzling that the linearized Euler momentum equation is the primary relation, rather than the continuity equation. This choice is made because a common task entails determining the pressure associated with a known motion of a surface. In such situations we know the history of velocity at a spatial location, so may evaluate the acceleration. The continuity equation requires knowledge of the spatial dependence of particle velocity within the fluid, which is not known a priori.

²A.D. Pierce, *Acoustics*, pp. 28–34, McGraw-Hill (1981), reprinted by ASA Press (1989).

EXAMPLE 2.2 The U. S. Standard Atmosphere is a government recognized model for the dependence of various thermodynamic properties on altitude. The ambient properties relevant to acoustics are tabulated below. (a) Determine whether the entries for pressure, density, and temperature at each altitude are consistent with the ideal gas law. (b) Compare the speed of sound obtained from Eqs. (2.3.9) and (2.3.22) (c) Graph the adiabatic bulk modulus as a function of altitude. (d) When the surface of a sphere of radius a is made to vibrate harmonically as $v_s \sin(\omega t)$, the steady-state pressure at radial distance r from the center of the sphere is

$$p(r, t) = \rho_0 c v_s \frac{a}{r} \sin\left(\omega\left(t - \frac{r - a}{c}\right)\right)$$

Consider a case where $v_s = 20$ m/s, $a = 40$ mm, and $\omega = 1$ kHz. Evaluate the sound pressure level at $r = 10$ m for locations that are at each of the tabulated altitudes. It may be assumed that reflections from the ground are unimportant at any altitude.

Altitude above Sea Level (m)	Temperature (°C)	Absolute Pressure (kPa)	Density (kg/m ³)
−1000	21.50	113.9	1.347
0	15.00	101.3	1.225
1000	8.50	89.88	1.112
2000	2.00	79.50	1.007
3000	−4.49	70.12	0.9093
4000	−10.98	61.66	0.8194
5000	−17.47	54.05	0.7364
6000	−23.96	47.22	0.6601
7000	−30.45	41.11	0.5900
8000	−36.94	35.65	0.5258
9000	−43.42	30.80	0.4671
10000	−49.90	26.50	0.4135
15000	−56.50	12.11	0.1948
20000	−56.50	5.529	0.08891
25000	−51.60	2.549	0.04008
30000	−46.64	1.197	0.01841
40000	−22.80	0.287	0.003996
50000	−2.5	0.07978	0.001027
60000	−26.13	0.02196	0.0003097
70000	−53.57	0.0052	0.00008283
80000	−74.51	0.0011	0.00001846

Significance

In addition to illustrating how to evaluate basic acoustical properties, this example also will lead to some interesting observations regarding acoustics in outer space.

Solution

Because the ambient pressure P_0 and density ρ_0 cover an enormous range, they are plotted on a logarithmic scale in the plot in Fig. 1. The plots of both variables are linear, which corresponds to a power law dependence on altitude H . A fit to the data that uses the values at $H = 0$ and $H = 80$ km to construct a straight line leads to

$$\begin{aligned}\rho_0 &= 1.225 (10)^{-0.0603H/1000} \text{ kg/m}^3 \\ P_0 &= 101.03 (10)^{-0.0621H/1000} \text{ kPa}\end{aligned}$$

As shown in Fig. 1, each empirical fit closely matches the given data.

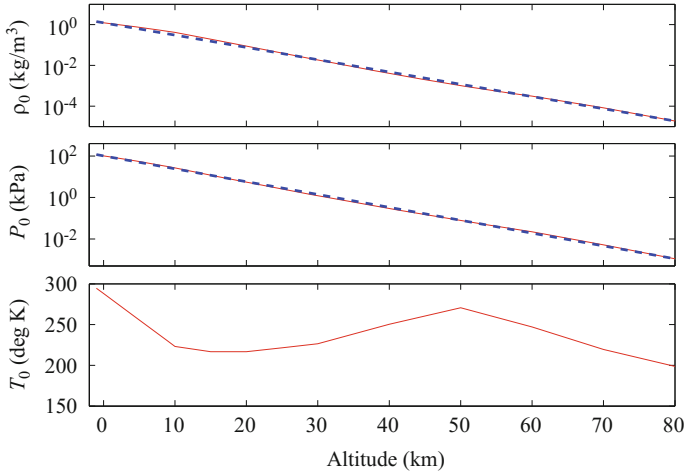


Figure 1.

If the tabulated entries are consistent with the ideal gas law, then each computed value of $p_0/(\rho_0 T_0)$ will equal R , which is $287.1 \text{ J/(kg}\cdot\text{K)}$ for air. Another consistence check is obtained by comparing the alternative computations of the sound speed, which are

$$c_{\text{density}} = \left(\frac{\gamma p_0}{\rho_0} \right)^{1/2}, \quad c_{\text{temperature}} = (\gamma R T_0)^{1/2}$$

Both would be the same if the ideal gas law applies. Errors for each quantity are

$$\varepsilon_R = \frac{R - p_0/(\rho_0 T_0)}{R}, \quad \varepsilon_c = \frac{\sqrt{\gamma R T_0} - \sqrt{\gamma p_0/\rho_0}}{\sqrt{\gamma R T_0}}$$

Note that the temperature is measured in degrees Kelvin, so 273.13 must be added to the tabulated temperatures. As implied in the definition of ε_c , the speed of sound as a function of temperature is taken as the reference value because that form only depends on one state variable.

The first graph in Fig.2 compares the altitude dependence of c_T and c_ρ . Also plotted there is a cubic polynomial approximation of the given data obtained by the method of linear least squares. The approximate relation is

$$c = 338.6 - 4.525 \left(\frac{H}{1000} \right) + 0.14421 \left(\frac{H}{1000} \right)^2 - 0.001212 \left(\frac{H}{1000} \right)^3 \text{ m/s}$$

The values of c derived from the tabulated properties are barely different up to an altitude of 60 km, and only differ significantly at 80 km (and above). The discrepancy of the computed value of R , and between the two evaluations of sound speed, are seen in the second graph in Fig. 2 to be small, except at the highest altitude.

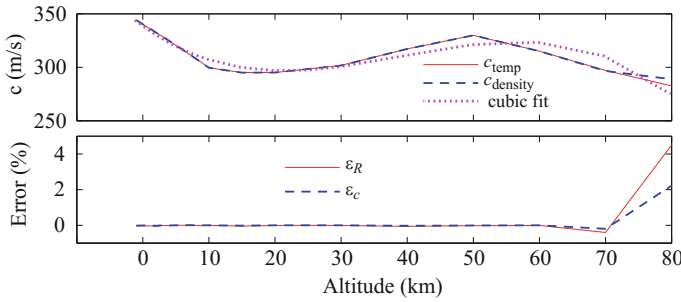


Figure 2.

The variability of the sound speed profile is important because it informs us that we should not consider the atmosphere to be homogeneous if we are interested in propagation over a wide range of altitudes. The fact that c has a maximum and a minimum affects how sound is channeled as it propagates. Chapter 11 introduces the study of propagation through heterogeneous fluids.

Figure 3 describes the bulk modulus. The fact that it too appears to be a straight line in a log-linear scale stems from the fact $K = \rho_0 c^2$, and c is essentially constant relative to the enormous range of ρ_0 .

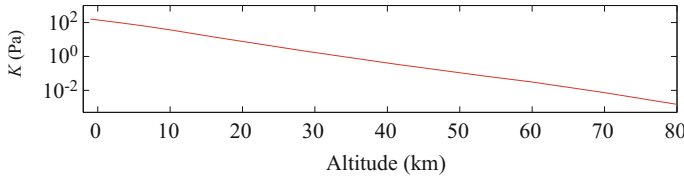


Figure 3.

The amplitude of the sound emitted by the vibrating sphere is $|p| = \rho_0 c v_s (a/r)$. Because the signal is a pure harmonic, we may compute the sound pressure level as

$$\mathcal{L}_p = 20 \log_{10} \left(\frac{|p|/\sqrt{2}}{20 (10^{-6})} \right) = 20 \log_{10} (v_s) + 20 \log_{10} \left(\frac{a}{r} \right) + 93.1 + 20 \log_{10} (\rho_0 c)$$

The altitude dependence of this quantity is describe in Fig.4. The sound pressure level covers an enormous range, from a maximum of 125 dB/20 μ Pa to a minimum of 26 dB. The part of $\mathcal{L}_p$ that depends on the altitude is $\rho_0 c$, which is called the *characteristic impedance* of the fluid. We will encounter this property in the next section.

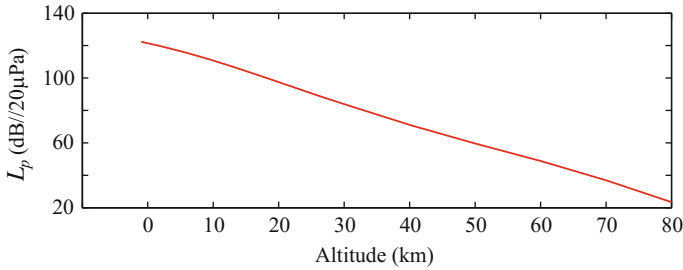


Figure 4.

The linear decrease of $\mathcal{L}_p$ is a consequence of ρ_0 being proportional to a power of H , whereas c changes comparatively little. Continued increase in altitude would yield even smaller values of ρ_0 . It is commonly stated that sound cannot be produced in outer space, and that is true. The reason is usually stated that it is a vacuum, which is not true. The density there is variable, but one hydrogen atom per cubic centimeter is a representative value. The scale of an acoustic signal is its wavelength. There are not enough particles in that distance to justify application of the kinetic theory of gases. Hence, it would not be appropriate to consider there to be an acoustic signal in which a disturbance is passed from particle to particle. However, if we consider signals whose wavelength is comparable to the size of planets and stars, there might be a sufficient number of particles to justify a continuum view. If such is the case, c , which depends only on the ambient temperature, would be finite. Nevertheless, unless c is enormous, the nearly infinitesimal density would correspond to an extremely small characteristic impedance. Generating a detectable sound in such an environment would be impossible.

This brings to the fore an interesting question: What is the behavior near a stellar object? If we assume that the ideal gas law is correct there, at least for order of magnitude evaluations, then c will be much larger than it is in the Earth's atmosphere because the temperature is high. Also, the density of the gases will not be small. This would suggest that the sound pressure level near a star will be very high. Should we send a probe containing a microphone to the Sun in order to measure the sound field?

2.4 The d'Alembert Solution

Several paths leading to the general solution of the (linear) wave equation are available. At one end of the spectrum, we could verify that the known solution satisfies the equation. At the other end we could pursue a formal line of mathematical analysis

known as the method of characteristics, which also can be used to study signals in inhomogeneous media, as well as nonlinear effects. The approach we follow here is intermediate to those extremes.

2.4.1 Derivation

Combinations of derivatives may be grouped as operators. In this view the one-dimensional wave equation may be factorized into a product of first order operators,

$$\left(\frac{\partial}{\partial x} + \frac{1}{c} \frac{\partial}{\partial t} \right) \left(\frac{\partial}{\partial x} - \frac{1}{c} \frac{\partial}{\partial t} \right) p = 0 \quad (2.4.1)$$

Each of the operator factors is reminiscent of a chain rule derivative associated with changing variables. Specifically, suppose instead of describing the wave equation in terms of derivatives with respect to x and t , we wish to change to a pair of variables, ξ and η . Assuming that we know the transformations, $x = x(\xi, \eta)$ and $t = t(\xi, \eta)$, the chain rule would state that

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{\partial x}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \xi} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial \eta} &= \frac{\partial x}{\partial \eta} \frac{\partial}{\partial x} + \frac{\partial t}{\partial \eta} \frac{\partial}{\partial t} \end{aligned} \quad (2.4.2)$$

We want these derivatives to match the operators in the factorized wave equation, Eq. (2.4.1). At the same time, it is useful for later developments to multiply that equation by $1/4$. Then the two operator factors reduce to derivatives with respect to a single variable if

$$\begin{aligned} \frac{\partial x}{\partial \xi} &= \frac{1}{2}, \quad \frac{\partial x}{\partial \eta} = \frac{1}{2} \\ \frac{\partial t}{\partial \xi} &= -\frac{1}{2c}, \quad \frac{\partial t}{\partial \eta} = \frac{1}{2c} \end{aligned} \quad (2.4.3)$$

In a partial differentiation with respect to a variable, other independent variables are held constant. Thus, if we integrate the first equation we obtain $x = \xi/2 + r_x(\eta)$, whereas integrating the second equation gives $x = \eta/2 + s_x(\xi)$. If each equation is considered in isolation from the other, $r_x(\eta)$ and $s_x(\xi)$ are arbitrary. However, the alternative equations for x must be consistent, which requires that $r_x(\eta) = \eta/2$ and $s_x(\xi) = \xi/2$. Similar treatment of the second pair of equations leads to $t = -\xi/(2c) + r_t(\eta)$ and $t = \eta/(2c) + s_t(\xi)$, from which it follows that $r_t(\eta) = \eta/(2c)$ and $s_t(\xi) = -\xi/(2c)$. It follows that a change of variables given by

$$x = \frac{1}{2}(\xi + \eta), \quad t = \frac{1}{2c}(\eta - \xi) \quad (2.4.4)$$

will transform the wave equation to

$$\frac{\partial^2 p}{\partial \xi \partial \eta} = 0 \quad (2.4.5)$$

Because other variables are held constant in a partial differentiation, there are only two possibilities: either $\partial p / \partial \xi$ is an arbitrary function of ξ or $\partial p / \partial \eta$ is an arbitrary function of η . Integrating each possibility leads to

$$p = F(\xi) + G(\eta) \quad (2.4.6)$$

where F and G are arbitrary functions.

Our interest is in how p depends in x and t , which requires that we determine the inverse of the transformation in Eq. (2.4.4). It is

$$\boxed{\xi = x - ct, \quad \eta = x + ct} \quad (2.4.7)$$

The variables ξ and η are the *characteristics*. (The reason for this terminology will soon be apparent.) When the characteristic variables in Eq. (2.4.6) are replaced by the physical variables, the result is the *d'Alembert solution* (1747) of the wave equation,

$$\boxed{p = F(x - ct) + G(x + ct)} \quad (2.4.8)$$

This constitutes the most general solution possible.

No restrictions are placed on the arbitrary functions, other than finiteness and continuity. The latter condition applies because a discontinuity of either function would correspond to dp/dx being infinite. According to the Euler equation, this would require an infinite acceleration, which clearly is impossible. (Occasionally, it will be convenient to consider a discontinuous function, such as a step or square wave. It is justifiable to do so provided that we recognize that the discontinuity is a simplified description of a continuous function that changes greatly over a very small interval).

An important feature of many excitations is that they impose a particle velocity. Hence, we need to know what the particle velocity is according to the d'Alembert solution. Throughout our studies, we will turn to Euler's equation, Eq. (2.2.6), when we need to relate pressure and velocity. Here it requires that

$$\rho_0 \frac{\partial v}{\partial t} = -\frac{\partial}{\partial x} [F(x - ct) + G(x + ct)] \quad (2.4.9)$$

The quantities inside parentheses for F and G are the respective functions arguments, or equivalently, the characteristics ξ and η , respectively. The formally correct way of carrying out the derivative with respect to x is to differentiate each function with respect to its characteristic, then multiply the result by the derivative of the

characteristic with respect to x , which gives

$$\begin{aligned}\rho_0 \frac{\partial v}{\partial t} &= - \left(\frac{\partial \xi}{\partial x} \frac{\partial F}{\partial \xi} \right) \Big|_{\xi=x-ct} - \left(\frac{\partial \eta}{\partial x} \frac{\partial F}{\partial \eta} \right) \Big|_{\eta=x+ct} \\ &= -(1)F'(x-ct) - (1)G'(x+ct)\end{aligned}\quad (2.4.10)$$

where a prime denotes the derivative of a function with respect to its argument. For example, if $F = \sin(kx - \omega t)$, then $F' = \cos(kx - \omega t)$. A simpler way of arriving at the same result without introducing the characteristics is to differentiate each function with respect to its argument, then multiply that derivative by the derivative of the argument with respect to x .

The next step is to integrate Eq. (2.4.10) with respect to t . To do so, we observe that the time derivatives of the F and G functions are described by

$$\begin{aligned}\frac{\partial}{\partial t} F(x-ct) &= F'(x-ct) \left[\frac{\partial}{\partial t} (x-ct) \right] = -cF'(x-ct) \\ \frac{\partial}{\partial t} G(x+ct) &= G'(x+ct) \left[\frac{\partial}{\partial t} (x+ct) \right] = +cG'(x+ct)\end{aligned}\quad (2.4.11)$$

We solve these for F' and G' , and substitute the result into the above form of Euler's equation, Eq. (2.4.10), which gives

$$\rho_0 \frac{\partial v}{\partial t} = \frac{1}{c} \left[\frac{\partial}{\partial t} F(x-ct) - \frac{\partial}{\partial t} G(x+ct) \right] \quad (2.4.12)$$

This relation may be integrated directly to find that

$$\boxed{v = \frac{1}{\rho_0 c} [F(x-ct) - G(x+ct)]} \quad (2.4.13)$$

2.4.2 Interpretation

The portion of the d'Alembert solution that is $p = F(x-ct)$ constitutes a *wave* that propagates in the direction of increasing x . The term “wave” suggests that it is analogous to an ocean waves, in which the elevated water crest seems to be transported along the surface. Here the information that is transported is the pressure value. Recognition that this wave propagates in the direction of increasing x follows from the property that the same F is found at all space-time locations for which $\xi = x - ct$ is constant value, so increasing t must be compensated by an increase in x . Suppose we observe that F has a certain value at location x_1 and time t_1 , and we wish to know where that same value of F will be observed at time $t_2 > t_1$. We know that the constant value of the characteristic at x_1, t_1 is $\xi = x_1 - ct_1$, which must also be the value of the characteristic associated with x_2, t_2 . It follows that

$$\xi = x_1 - ct_1 = x_2 - ct_2 \implies x_2 = x_1 + c(t_2 - t_1) \quad (2.4.14)$$

In other words the information contained in a value of F is observed at locations whose coordinate x increases proportionally to the elapsed time. The factor of proportionality in such a relation is velocity, and the value of ξ represents the phase of the pressure signal, so $+c$ is the *phase velocity* of the wave described by $p = F(x - ct)$. Standard terminology is to say that ξ is the *forward characteristic* because the associated wave transmits the pressure value in the positive x direction with increasing time.

The other part of the d'Alembert solution has a similar interpretation. In this case a specific value of G is observed at two pairs x_1, t_1 and x_2, t_2 if

$$\eta = x_1 + ct_1 = x_2 + ct_2 \implies x_2 = x_1 - c(t_2 - t_1) \quad (2.4.15)$$

In other words the location at which a specific G is observed decreases proportionally to elapsed time, with the factor of proportionality being $-c$. Thus, this solution represents a wave that propagates in the direction of decreasing x , and its phase velocity is $-c$. Correspondingly $\eta = x + ct$ is the *backward characteristic*. The term *phase speed* omits the directionality of the phase velocity. For both waves this quantity is the speed of sound, c .

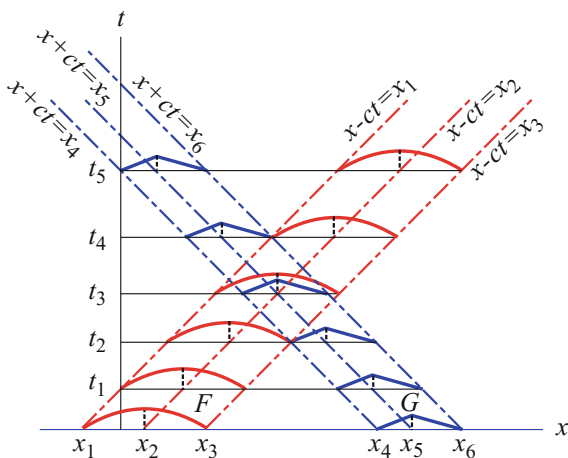
Sometimes, $p = F(x - ct)$ and $p = G(x + ct)$ are said to be *simple waves*, because any other wave is more complicated. A more descriptive name is that each is a *progressive plane wave*. This term conveys the fact that they propagate in a single direction, and that there is no variation perpendicular to the direction of propagation. The most specific terminology refers to the two solutions as forward and backward propagating plane waves.

If we were to plot $F(x - ct)$ and $G(x + ct)$ as functions of x at a specific instant t , each plot would have a shape whose position along the x axis depends on the selected value of t . Such a plot represents a *spatial profile*. In the case of F this shape would translate in the direction of increasing x as time elapses (forward propagation), while the shape of the G plot would translate in the direction of decreasing x (backward propagation). Waves that propagate without changing shape are said to be *nondispersive*. The d'Alembert solution states that plane waves are a superposition (that is, addition) of two nondispersive waves, but the resulting sum does not maintain its shape as time evolves.

One way to visualize the manner in which the waves propagate is to plot lines in an x, t graph that are the locus of all points along which ξ and η are constant. The profiles $F(x)$ and $G(x)$ of each nondispersive wave at the initial instant, usually $t = 0$, are plotted along the x axis. A three-dimensional plot could use p as the third axis, but we will overlay the profiles onto the x, t plane. Figure 2.3 shows the result when F is parabolic and G is a triangle.

The construction of this picture began by plotting $F(x)$ and $G(x)$ along the x axis in order to describe the simple waves at $t = 0$. Six values of x are identified as being important. The extent of the forward propagating wave initially is $x_1 \leq x \leq x_3$ and the maximum value of $F(x)$ occurs at x_2 . The backward propagating wave is nonzero

Fig. 2.3 Propagation of spatial pressure profiles along forward and backward characteristics in x, t space



for $x_4 \leq x \leq x_6$, with the maximum $G(x)$ occurring at x_5 . (In the sketch, only x_1 is negative, but the whole picture could be shifted farther to the left because the physical domain is considered to be unbounded.) The next step was to construct the characteristics that pass through each of these points. The characteristics are straight lines defined by Eq. (2.4.7), so their slope is $dt/dx = \pm 1/c$, and the constant values of ξ or η for each is the x value at the intersection with the x axis. As shown in the figure, the signals at any subsequent instant are depicted by constructing a line parallel to the x axis at that t , then translating the plot of $F(x)$ and $G(x)$ up to that line along the associated characteristic.

Much insight can be gained from a diagram like Fig. 2.3. At time t_1 the forward propagating wave has passed $x = 0$; after this instant the region $x < 0$ will not be disturbed by this wave. At time t_2 the leading edges (that is, the phase of each wave that is farthest in the direction the wave is propagating) have met. Because the total pressure is the sum of the contribution of each simple wave, we can identify t_2 as the instant when the pressure will be enhanced by their overlap. This is said to be *constructive interference* of the waves. (If F and G have opposite signs, the result would be a *destructive interference*.) We also can see that the waves will no longer interfere when $t > t_4$, because their tails intersect at $t = t_4$. At instant t_3 the characteristics marking the maximum of each simple wave intersect. This is the instant when the maximum total pressure will occur. The last instant indicated in the figure is t_5 , which marks the time when the backward propagating wave enters $x < 0$. Thus, $x < 0$ is quiet only in the time interval $t_1 \leq t \leq t_5$.

Figure 2.3 describes the solution to an *initial value problem* for an infinite (one-dimensional) domain. This term stems from the observation that the functions $F(x)$ and $G(x)$ plotted along the x axis represent the simple waves at the initial instant. This is $t = 0$ for the Figure, but it could be any t at which the simple waves are known.

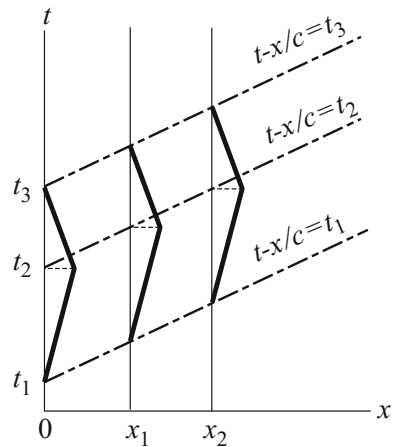
Boundary excitation is another class of problems that can be described by a diagram like Fig. 2.3. This is the situation in which the acoustical response is specified at some location as a function of t . A physical system that would give rise to this condition is a tube in which one end is terminated by a vibrating piston. The medium extends from the boundary to infinity in the x direction. We may define the location of the boundary as $x = 0$, so the fluid's domain is $0 \leq x < \infty$. This domain is said to be semi-infinite because it only extends to infinity in one direction. Suppose we know that $p = f(t)$ at $x = 0$. We plot this function along the t axis in the manner of Fig. 2.4. The pressure at each instant propagates into the region $x > 0$ along the forward characteristics that emanates from that point the t axis. There are no backward characteristics emanating from the t axis because they represent propagation into the region $x < 0$, but the fluid does not exist there.

The instants t_1 and t_3 in Fig. 2.4 mark the start and termination of the excitation, and t_2 marks the instant when it is maximum. The diagram shows that the characteristic through t_1 is the front of the forward propagating simple wave, while t_3 marks the tail. The equation of a forward characteristics is $\xi = x - ct$, where the value of ξ is now set by its intersection with the t axis, that is $\xi = -ct_1, -ct_2$, and $-ct_3$. It is convenient to rewrite the equation of these characteristics as $t - x/c = t_1, t_2$, or t_3 . Correspondingly, the one-dimensional signal resulting from boundary excitation of a semi-infinite system is

$$p = f\left(t - \frac{x}{c}\right) \quad (2.4.16)$$

A plot of pressure as a function of time at a specific location is said to be a *waveform*. Figure 2.4 shows that the waveform at any positive x has the same shape as the excitation $f(t)$, but it is delayed by x/c , which is the time required for a specific pressure to propagate there for $x = 0$. This is another way of stating that the waves are nondispersive. But how can we extract from this information a *spatial*

Fig. 2.4 Pressure waveform propagating along the forward characteristic in x, t space



profile, which is p as a function of x at fixed t ? A related question for an initial value problem is how can we identify a waveform directly from Fig. 2.3? These questions could be answered graphically by considering the characteristics in those figures to be equal pressure contours in a map. Profiles would be slices parallel to the x axis at any t , and waveforms would be slices parallel to the t axis at any x . Another approach would be to simply evaluate analytical solutions. Section 2.5 will introduce an alternative graphical representation of the signal that simplifies the evaluation of profiles and waveforms.

The best way to understand the description of the particle velocity, Eq. (2.4.13), as well as the overall d'Alembert solution, is to use subscript 1 to designate properties associated with the forward propagating wave (increasing x) and subscript 2 for the backward propagating wave. Then the total acoustic pressure and particle velocity are described by

$$\begin{aligned} p &= p_1 + p_2, & v &= v_1 + v_2 \\ p_1 &= F(x - ct), & p_2 &= G(x + ct) \\ v_1 &= \frac{p_1}{\rho_0 c}, & v_2 &= -\frac{p_2}{\rho_0 c} \end{aligned} \quad (2.4.17)$$

Recall that v is the particle velocity component in the x direction, so negative v means that a particle is moving in the sense of decreasing x . Thus, the particle velocity corresponding to a positive pressure in each simple wave is in the same sense as the direction in which that wave propagates. The magnitude of that particle velocity is proportional to the magnitude of the pressure. The factor of proportionality, $\rho_0 c$, is the *characteristic impedance* of the fluid. (The ratio of a force variable to a velocity variable is generally referred to as an impedance.) The dimensionality of the characteristic impedance are $\text{F-T/L}^3 = \text{M}/(\text{L}^2\text{T})$. In SI units $1 \text{ kg}/(\text{m}^2\text{s})$ is defined to be 1 Rayl in recognition of Rayleigh's many contributions to the science of acoustics.³

EXAMPLE 2.3 The pressure field in an infinite domain for $t < 0$ consists of a forward propagating sinusoidal wave in $x < 0$ and a backward propagating sinusoidal wave in $x > 0$. They are timed such that their leading edges both arrive at $x = 0$ when $t = 0$. They are described as

$$p = \begin{cases} -B \sin\left(\frac{2\pi\xi}{L}\right) h(-\xi), & \xi = x - ct, \text{ if } x < 0 \text{ \& } t < 0 \\ B \sin\left(\frac{2\pi\eta}{L}\right) h(\eta), & \eta = x + ct, \text{ if } x > 0 \text{ \& } t < 0 \end{cases}$$

³His name is John William Strutt, third Baron Lord Rayleigh. The numerous references herein are a small sample of his significant contributions to acoustics. He also was a pioneer in optics, and won the Nobel Prize in 1924 as the codiscoverer of argon.

where $h(z)$ is the step: one if $z > 0$ and zero if $z < 0$. The first task is to construct spatial profiles of pressure at $t = 0$, $1.5L/c$, and $2.25L/c$. Then use those profiles to determine the maximum possible pressure, as well as the instants and locations where that maximum will occur.

Significance

Using characteristic lines in x, t space as the framework will be seen here to be quite useful for tracking how waves evolve.

Solution

The field in $x < 0$ for $t < 0$ is p_1 in Eq. (2.4.17), and the field in $x > 0$ for $t < 0$ is p_2 . For $t > 0$ these waves overlap, so we must evaluate their interference pattern. The first step is to plot in an x, t diagram p_1 and p_2 at $t = 0$. At that instant $\xi = \eta = x$, and $h(-x) = 1$ if $x < 0$. Thus, the pressure profile at $t = 0$ for $x > 0$ is a sine curve with amplitude B and wavelength L that starts at $x = 0$. It is a negative sine function for $x < 0$ with the same amplitude and period. As can be seen in Fig. 1, the combination is an even function of x with a slope discontinuity at $x = 0$. This is the location of both leading edges.

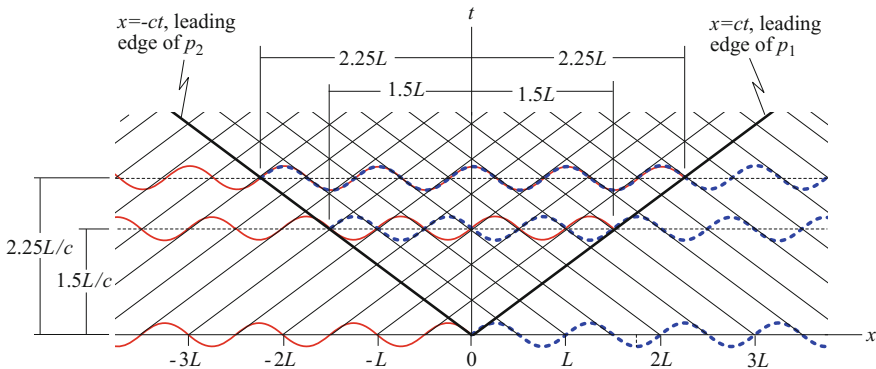


Figure 1. Spatial profiles of pressure showing interference between forward and backward propagating waves.

For $t > 0$, both waves continue without modification. We draw the characteristics that intersect the origin in order to track the leading edge of each wave. Other features that we could track with characteristic lines are the zeros, maxima and minima. Showing all of these would lead to a rather dense set of lines; showing only the characteristics along which p_1 or p_2 is zero is adequate. Each of these emanates from a location on the x axis at which their profile has a zero.

The next step is to construct the profile of each simple wave at $t = 1.5L/c$. This is done by drawing a line parallel to the x axis at $t = 1.5L/c$. The initial p_1

profile is translated to this line and shifted in the direction of increasing x by $1.5L$, which is the distance the wave propagates in that interval. Thus, the p_1 profile at this instant extends backward from the leading edge at $x = 1.5L$. The p_2 profile is treated similarly, with the only difference being that it is shifted in the direction of decreasing x by $-1.5L$. Figure 1 shows that the zeros of p_1 and p_2 overlap for $-1.5L < x < 1.5L$. In this interval p_1 and p_2 have equal magnitude, but opposite sign. Outside this interval, there is only p_2 for $x > 1.5L$, and only p_1 for $x < -1.5L$. The total pressure is the superposition of the two simple waves, $p = p_1 + p_2$, so $p = 0$ in the region of overlap, and p_1 or p_2 is the pressure outside that interval.

The profile at $t = 2.25L/c$ is treated in the same way. We construct a line parallel to the x axis at $t = 2.25L/c$ and shift the p_1 and p_2 profiles up to that line. Relative to their positions at $t = 1.5L/c$, the p_1 profile is shifted in the positive x direction by $0.75L$, which is the distance p_1 travels in the elapsed time subsequent to $t = 1.5L/c$. Similarly, the p_2 profile is shifted relative to its earlier position by $0.75L$ in the direction of decreasing x . The region in which p_1 and p_2 overlap is the interval between the leading edges. Now it is $-2.25L < x < 2.25L$. The profiles of p_1 and p_2 are the same in this interval, so the total pressure is twice that in either profile. Inspection of the graph shows that each profile at this instant fits $B \cos(2\pi x/L)$. For x values left of the leading edge of p_2 , there is only the p_1 profile, while for x values to the right of the leading edge of p_1 there is only the p_2 profile. These considerations allow us to construct profiles of the total pressure at both instants, which are shown Fig. 2.

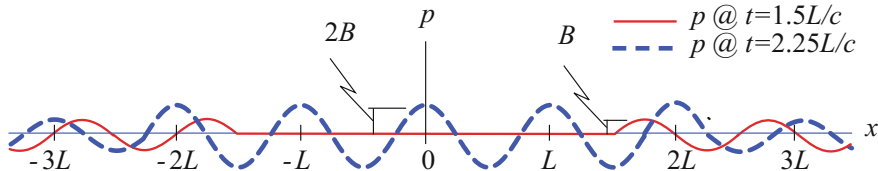


Figure 2. Selected pressure profiles.

The profile at $t = 2.25L/c$ gives a hint regarding when and where the pressure attains a maximum value. The maximum value of either simple wave is B , so the maximum possible value of their superposition is $2B$. It follows that $p = 2B$ whenever and wherever a ξ characteristic tracking $p_1 = B$ intersects an η characteristic tracking $p_2 = B$. At $t = 0$, the peak values of p_2 occur at $x = L/4, 5L/4, \dots$, and the peak values of p_1 occur at $x = -L/4, -5L/4, \dots$. Furthermore, at $t = 0$, we know that $\xi = \eta = x$, so we are led to the conclusion that $p = 2B$ if

$$\eta = x + ct = \left(n + \frac{1}{4}\right)L \text{ and } \xi = x - ct = -\left(m + \frac{1}{4}\right)L$$

The simultaneous solution of these conditions is

$$t = \left(\frac{n+m}{2} + \frac{1}{4} \right) \frac{L}{c}, \quad x = \left(\frac{n-m}{2} \right) L$$

Both n and m are arbitrary and non-negative, so $n+m$ may be any integer $j \geq 0$. The value of t is set by assigning j , in which case $n = j - m$. It follows from the preceding conditions that

$$p = 2B \text{ when } t = \left(\frac{2j+1}{4} \right) \frac{L}{c} \text{ at } x = \left(\frac{j-2m}{2} \right) L$$

Thus we are led to the realization that the maximum pressure is observed at odd multiples of $L/(4c)$. The locations at which $p = 2B$ at these instants are separated by L in both directions, alternating between starting at $x = 0$ and at $x = \pm L/2$, depending on j . In any case, the location at which a maxima occurs must fall in the range where p_1 and p_2 overlap, which always is $-ct < x < ct$.

It should be evident from Figure 1 that the minimum value of p is $-2B$, and that it occurs at the same instants that the maxima occur. Furthermore, the minima occur at positions that are halfway between the maxima. It is left to the reader to verify that $p = 0$ in the interval $-ct < x < ct$ when t is an integer multiple of $L/(2c)$. Another interesting observation, which is somewhat more difficult to prove by viewing profiles and characteristics, is that $p = p_1 + p_2 = 0$ at any x that is an odd multiple of $L/4$ in the region of overlap, regardless of t .

It actually is easier to demonstrate each of these features mathematically with the aid of a trigonometric identity. In the region where p_1 and p_2 overlap, the total pressure is

$$\begin{aligned} p = p_1 + p_2 &= B \left[\sin \left(\frac{2\pi(x+ct)}{L} \right) - \sin \left(\frac{2\pi(x-ct)}{L} \right) \right] \\ &\equiv 2B \cos \left(\frac{2\pi x}{L} \right) \sin \left(\frac{2\pi ct}{L} \right), \quad -ct \leq x \leq ct \end{aligned}$$

Thus, $|p| = 2B$ when $2\pi ct/L = \pi/2, 5\pi/2, \dots$ at $2\pi x/L = 0, \pm 2\pi, \dots$ and when $2\pi ct/L = 3\pi/2, 7\pi/2, \dots$ at $2\pi x/L = \pm\pi, 3\pi, \dots$. We also see that $p = 0$ when $2\pi ct/L = 0, \pi, \dots$ regardless of x , and at $2\pi x/L = \pm\pi/2, \pm 3\pi/2, \dots$ regardless of t . Each of these conditions is the same as the one identified from the graphical construction.

The dependence of pressure on x and t has many attributes that we will again encounter in our later studies of waveguides that are closed. The signal in such systems will be found to be a standing wave field.

2.4.3 Harmonic Waves

A large part of Chap. 1 was devoted to signals that vary harmonically in time, so it is helpful to consider a plane wave that varies harmonically. Because the d'Alembert

solution is a sum of forward and backward propagating waves, it must be that a harmonic signal can only be obtained if each simple wave also is harmonic. Let us consider a forward propagating wave by itself. We know that it must fit $p = F(\xi)$ with $\xi = x - ct$, but the coefficient of t must be the frequency ω . The appropriate form is obtained by multiplying ξ by a constant k , so a sine wave is described as

$$p = A \sin(k(x - ct)) \quad (2.4.18)$$

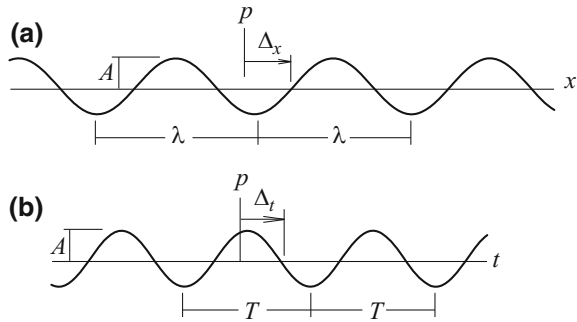
The coefficient of t is the frequency if $kc = \omega$, in other words,

$$k = \frac{\omega}{c} \quad (2.4.19)$$

This parameter is called the *wavenumber* for reasons that will be discussed. The ratio ω/c arises in the analysis of every type of harmonically varying wave. Its designation as the symbol k is nearly universal.

The profile of a sine wave at an arbitrarily selected t appears in Fig. 2.5a, while the waveform in Fig. 2.5b views the same signal as it would be measured at an arbitrary x . The amplitude A is the maximum absolute pressure in the signal. The spatial interval over which the pattern repeats is the *wavelength* λ , in contrast to the period T , which gives the amount of time that must elapse for the pattern to repeat.

Fig. 2.5 A typical harmonic wave; **a** Spatial profile, **b** Temporal waveform



Neither the wavelength nor the period nor the frequency appear explicitly in the basic representation of Eq. (2.4.18), but substitution of the definition of k gives an alternate form

$$p = A \sin(kx - \omega t) \quad (2.4.20)$$

The period T is the time required for the phase of the signal to change by 2π , so

$$T = \frac{2\pi}{\omega} \quad (2.4.21)$$

The wavelength is the distance by which the spatial location x must change (at a fixed t) for a 2π phase change. Thus, we set $k\lambda = 2\pi$.

$$\lambda = \frac{2\pi}{k} \quad (2.4.22)$$

It can be seen that k plays the same role for the spatial dependence of a harmonic wave as that played by ω for frequency dependence. Just as $\omega/(2\pi) \equiv 1/T$ gives the number of cycles that occur in a unit time interval, $k/(2\pi) \equiv 1/\lambda$ is the number of cycles per unit length. Indeed, a useful alternative form relates the wavelength to the cyclical frequency as

$$\lambda = \frac{c}{f} \quad (2.4.23)$$

Although the number of wavelengths per unit length is the one that is meaningful from a physical viewpoint, the quantity k is the one that is used for mathematics. For that reason we refer to k , rather than $k/(2\pi)$, as the wavenumber.

The other parameters that appear in Fig. 2.5 but not in Eq. (2.4.18) are the spatial shift Δ_x and time shift Δ_t . To interpret these parameters, we observe that the profile at $t = 0$ is $p(x, 0) = A \sin(kx)$, whereas $p(x, t) = A \sin(k(x - \Delta_x))$ in Fig. 2.5a. Similarly, the waveform at $x = 0$ is $p(0, t) = -A \sin(\omega t)$ whereas $p(x, t) = -A \sin(\omega(t - \Delta_t))$. A comparison of each form to Eq. (2.4.20) might seem to imply that $k\Delta_x = \omega t$ and that $\omega\Delta_t = kx$. Both relations are erroneous because they fail to account for the periodic nature of the signal. In the case of a profile, any point on the curve where the sine function passes through zero and is increasing might correspond to a profile that was a positive sine function at $t = 0$. Similarly, any point on the waveform at which the sine is zero and decreasing could correspond to a waveform that was a negative sine function at $x = 0$. Thus, we only can deduce from Fig. 2.5 that

$$\begin{aligned} \Delta_x &= ct \pm m\lambda \\ \Delta_t &= x/c \pm nT \end{aligned} \quad (2.4.24)$$

where m and n are integers.

There would be no ambiguity about the amount by which the profile at $x = 0$ is advanced at $t > 0$, and by how much the waveform at $t = 0$ is retarded at $x > 0$ if we see the leading edge of the wave when we begin to observe it. If the leading edge is at the origin when $t = 0$, then we could say that the shifts are $\Delta_x = ct$ and $\Delta_t = x/c$.

2.5 The Method of Wave Images

Although we have obtained the most general form of a plane wave, it is of little use until we can determine the F and G functions. Here we will adapt the d'Alembert

solution to fit several situations in which initial and/or boundary conditions are imposed. We will begin by establishing how to determine F and G in the case of an initial value problem for an infinite domain. The next step will develop a relatively simple way to identify spatial profiles at an arbitrary instant, and waveforms at an arbitrary location. This capability will be the fundamental tool for extension to systems with one boundary (a semi-infinite domain), and then two boundaries (a finite domain). In each case the development will lead to an equivalent representation of the basic solution as the result of an initial value problem in an infinite domain. Each solution will exemplify basic acoustical phenomena, including reflections, dissipation, reverberation time, normal modes, standing waves, and resonance. The phenomena occur in a variety of systems, some of which are multidimensional. The only limitation is that the analyses assume that the boundary has special mechanical properties. Despite this shortcoming, the ability to follow the evolution of a signal, and the relatively low level of mathematics required to implement an analysis, enable us to understand at a fundamental level why various phenomena occur.

2.5.1 Initial Value Problem in an Infinite Domain

Determination of the Wave Functions

Our analyses using characteristics began with the assumption that the F and G functions are known. The key observation here is that there are two functions to identify, so knowledge of the initial pressure would be inadequate. However, if we know F and G we can determine the initial velocity. The corollary is that it should be possible to determine $F(x)$ and $G(x)$ if we know the initial pressure and particle velocity profiles. Let $P(x)$ and $V(x)$ respectively denote these profiles. It is convenient to let $t = 0$ be the initial instant.

There are several situations in which a plane wave is encountered. The most common is a straight tube whose cross section does not change along its length. An initial temperature distribution can be generated by differential heating along the length of the tube. If such a distribution is maintained sufficiently long, thermal equilibrium over the cross section at any x could be achieved, thereby setting up an initial pressure field $P(x)$. Generation of an initial velocity field $V(x)$ is much more difficult. One possibility is that an electromagnetic field induces movement of a magnetic fluid. However, a much greater range of options is obtained by changing our perspective. Suppose that some excitation had been applied well before $t = 0$, at which time the excitation ended. If we begin our observation of such a system at $t = 0$, we would only know the current physical state, which is the pressure and particle velocity.

Equations (2.4.17) describe the response at any instant. Matching p and v at $t = 0$ to the initial conditions requires that

$$F(x) + G(x) = P(x), \quad F(x) - G(x) = \rho_0 c V(x) \quad (2.5.1)$$

which leads to

$$\boxed{\begin{aligned} F(x) &= \frac{1}{2} [P(x) + \rho_0 c V(x)], \quad G(x) = \frac{1}{2} [P(x) - \rho_0 c V(x)] \\ p_1 &= F(x - ct), \quad p_2 = G(x + ct), \quad p = p_1 + p_2 \end{aligned}} \quad (2.5.2)$$

An interesting aspect of this result is that it is possible to set up initial conditions that will induce only one of the two simple waves. For example, if $\rho_0 c V(x) = -P(x)$, then only p_2 will be generated. The reason for this is that the condition $p = -\rho_0 c v$ is exactly how the backward propagating wave behaves, so the initial conditions in this case are essentially a snapshot of a backward propagating wave that was generated prior to $t = 0$.

Evaluation of Spatial Profiles and Temporal Waveforms

It is assumed henceforth that the functions $F(x)$ and $G(x)$ have been determined. Now the task is to convert the mathematical description of plane wave in Eqs. (2.4.17) and (2.5.2) into spatial profiles of p as a function of x at fixed time, and temporal waveforms depicting p versus t at a specified x . One approach is to evaluate Eq. (2.5.2) directly with mathematical software. To evaluate profiles a sequence of x_n would be defined. The corresponding set of characteristics at any instant are $\xi_n = x_n - ct$ and $\eta_n = x_n + ct$. Then the pressure at each location is $p_n = F(\xi_n) + G(\eta_n)$. Similarly a waveform at any x may be found by defining a set of time values t_n then evaluating the corresponding characteristic values. Alternatively, computer graphic capabilities can create a three-dimensional view that shows spatial profiles and waveforms. If x_n and t_j are a set of position and time values, then matrices of characteristic values have elements $\xi_{j,n} = x_n - ct_j$ and $\eta_{j,n} = x_n + ct_j$. Evaluation of the pressure at these values of the characteristics gives a matrix $p_{j,n} = F(\xi_{j,n}) + G(\eta_{j,n})$ in which column n is the waveform at x_n and row j is the profile at t_j . Figure 2.6 is typical of the pictures that can be constructed.

The initial pressure distribution described by Fig. 2.6 is parabolic over an interval of length L . If the initial velocity was such that $\rho_0 c V(x)$ equals $P(x)$, then only a forward propagating wave would be generated, whereas waves in both direction at equal amplitude would be generated if $V(x)$ is zero. The function described here is half way between these alternatives. The result is that the forward propagating wave is three times stronger than the backward wave.

A picture like Fig. 2.6 is useful when the forward and backward waves are distinct. However, it is less apparent that there are distinct waves in the region of x, t where the simple waves overlap, which is $0 < x < L, 0 < ct < L/2$ in the case of the Figure. In each of the topics that we will consider there will be significant contribution from oppositely traveling simple waves. We need a better way of viewing the details of mutual contributions. The general concept to be developed here entails propagating each simple wave separately to identify how each contributes to a profile or waveform. The method will describe procedures in terms of graphical constructions, because such views give a “hands on” experience, but it also provides a foundation for computations.

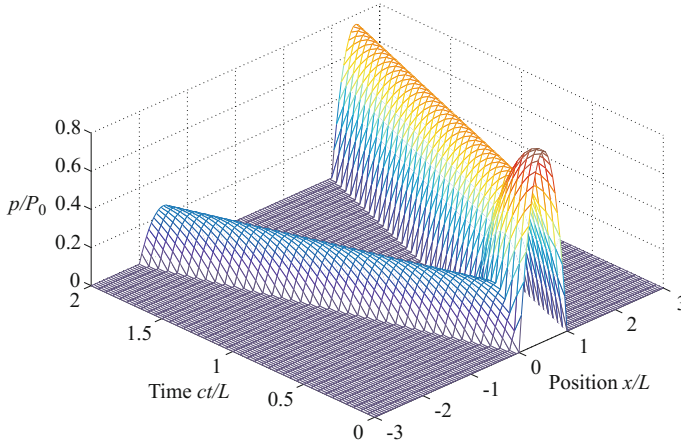


Fig. 2.6 Evolution of a plane wave generated by initial conditions in which $P(x) = 4P_0(x/L)(1 - x/L)$ for $0 < x < L$ and $\rho_0 c V(x) = P(x)/2$

We begin with spatial profiles. A graphical construction begins by drawing $p_1 = F(x)$ and $p_2 = G(x)$. As we saw when the characteristics were plotted in an x, t diagram, a plot of p_1 versus x at some $t > 0$ shows the function $F(x)$ translated, without changing shape, in the positive x direction by the propagation distance ct . Similarly p_2 versus x at that t would be the plot of $G(x)$ translated in the negative x direction by the same propagation distance ct . Thus, the range of x over which the initial profiles are sketched must be sufficiently large that shifting them for a later time does not leave a gap in which no value was computed. After the p_1 and p_2 profiles have been determined, the total pressure is the sum of p_1 and p_2 at a specific x . This sum may be done graphically by adding the height of each profile at a specific x if both waves have the same sign, or taking their difference if their values have opposite signs. The graphical construction is illustrated in Fig. 2.7.

In Fig. 2.7b for time t' , the plot of p_1 is $F(x)$ shifted to the right (increasing x) by ct' , and the plot of p_2 is $G(x)$ shifted to the left (decreasing x) by ct' . Both $F(x)$ and $G(x)$ were selected as connected straight lines because doing so facilitates the graphical addition of p_1 and p_2 . The sum of two functions that are linear in x also is linear in x . Consequently, p also is a set of connected straight lines. The locations where these lines change slope are those where p_1 and/or p_2 change slope. Drawing the profile of p requires that we sum the heights of p_1 and p_2 at these discontinuities. The profile (or waveform) then is obtained by connecting the plotted points with straight lines.

Graphical construction of waveforms at a specific x begins by evaluating the waveform at $x = 0$. At this location, the d'Alembert solution states that

$$p_1(0, t) = F(-ct), \quad p_2(0, t) = G(ct) \quad (2.5.3)$$

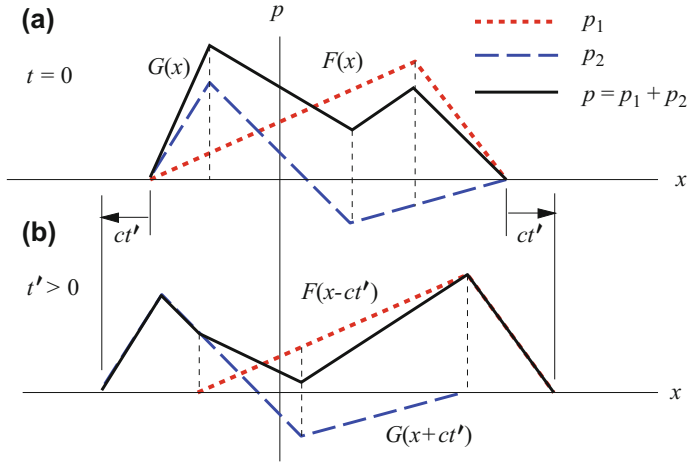


Fig. 2.7 Propagation of pressure profiles as time elapses according to the method of wave images

Let us scale the time axis by c , so that a plotted point is p_1 or p_2 along the ordinate and ct along the abscissa. Then the waveform of p_2 at $x = 0$ has the same appearance as the profile of p_2 at $t = 0$. In contrast, the argument of $F(-ct)$ has the opposite sign from that of $F(x)$. This means that a waveform plot of p_1 at $x = 0$ would assign at each point ct on the abscissa the value of the F function when x is replaced by $-ct$. Consequently, the waveform of p_1 at $x = 0$ appears to be the result of swapping the left and right sides of $F(x)$ relative to the origin. An alternative view is that the p_1 waveform at $x = 0$ appears to be the mirror image of the p_1 profile at $t = 0$ when the mirror is placed at $t = 0$. For this reason the construction methodology shall be referred to as the *method of wave images*. Figure 2.8a shows the construction for the same functions as those in the Fig. 2.7.

To construct a waveform at some location $x_1 > 0$, we recall that constancy of $F(x - ct)$ is obtained if the value of ct is increased by the value of x_1 . In other words the waveform of $p_1(t)$ at $x_1 > 0$ is the waveform of p_1 at $x = 0$, shifted forward along the ct axis by distance x_1 . An equivalent description is that p_1 is delayed by x_1/c relative to the waveform at $x = 0$. For p_2 , we track a constant value of $G(x + ct)$, which requires that the value of ct be decreased by x . Thus, the waveform of p_2 at $x_2 > 0$ is the p_2 waveform at $x = 0$, shifted backward along the ct axis by distance $-x_2$. This corresponds to an advancement in time by x_2/c . The construction is depicted in Fig. 2.8b. It will be noticed that although the waveforms of p_1 and p_2 are depicted for $t < 0$, they have been added only for $t \geq 0$, because $t = 0$ is the initial instant when we began to observe the signal.

The evaluation of a waveform for $x_2 < 0$ follows the same line of reasoning. Constancy of $x - ct$ for p_1 requires that ct be decreased by the amount that x_2 is negative. Thus, the waveform of p_1 at $x_2 < 0$ is obtained by translating the p_1 waveform at $x = 0$ backward along the ct axis by distance $-x_2$. Similarly, the p_2

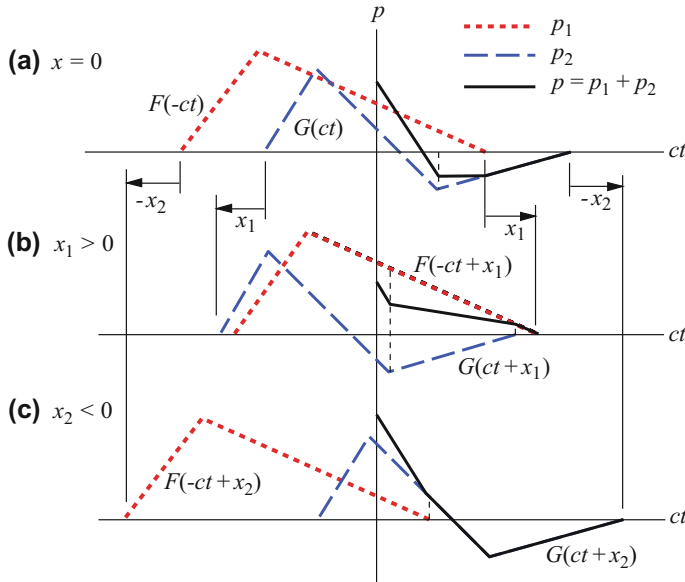


Fig. 2.8 Construction of pressure waveforms at various locations according to the method of wave images

profile at $x_2 > 0$ is shifted forward along the ct axis by $-x_2$. This construction is depicted in Fig. 2.8c, where the total pressure is evaluated only for $t \geq 0$.

EXAMPLE 2.4 Consider the initial forward and backward waves specified in Example 2.3. (a) Construct the spatial profile of the pressure and particle velocity at $t = 0.9L/c$. (b) Construct the waveform of pressure versus time at x locations where the maximum pressure is observed.

Significance

Construction of spatial profiles and temporal waveforms from the initial conditions of a wave will always be a cornerstone of the wave image method, even after it has been extended to treat systems with boundaries. One benefit of using the method of images to solve a problem that was previously analyzed by the method of characteristics is enhanced understanding of both approaches.

Solution

The initial profiles of p_1 and p_2 shown in the first graph of Fig. 1 are the same as in Example 2.3. At $t = 0.9L/c$, the p_1 wave will have propagated forward by $ct = 0.9L$, so its profile is obtained by shifting the initial p_1 profile in the positive

x direction by $0.9L$. Similarly, the p_2 wave is shifted in the negative x direction by $0.9L$. The individual profiles at this instant are shown in the second graph of Fig. 1.

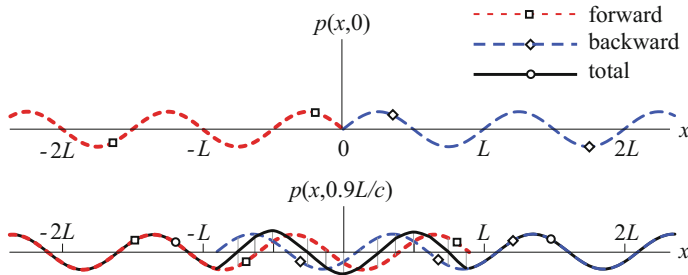


Figure 1. Spatial profiles of pressure at two instants.

A graphical construction was used to describe $p = p_1 + p_2$. Vertical lines were drawn from the x axis to p_2 at a sequence of x values where p_1 and p_2 overlap. Then these lines were translated to the p_1 curve, above it if p_2 is positive or below it if p_2 is negative. The ends of these vertical lines mark p at each location. Before these x, p points were connected, a little mathematics was used. First, it can be seen that because p_1 and p_2 are mirror images, and they translate by equal distances in opposite directions as time elapses, it must be that their sum is an even function of x , that is, $p(-x, t) = p(x, t)$. It follows that after the leading edges move away from $x = 0$, it must be that $\partial p / \partial x = 0$ at $x = 0$. At this juncture, the second basic property comes into play. Both p_1 and p_2 are spatially periodic in an interval L . Furthermore, both p_1 and p_2 are harmonic functions, so their sum must also be harmonic. A harmonic function having zero slope at the origin is a cosine. (This property was identified in Example 2.3 by the application of a trigonometric identity.) Correspondingly, the x, p points that were constructed were connected to resemble a cosine curve. This curve extends over the region where p_1 and p_2 overlap, which is the interval between their leading edges at this instant, $-0.9L < x < 0.9L$. For $x < -0.9L$, the total pressure consists of only p_1 because p_2 has not yet propagated back to that region. Similarly, for $x > 0.9L$ the pressure is p_2 because p_1 has yet to reach that region. Notice that because p_1 and p_2 are both zero at their leading edge, the pressure profile is continuous everywhere.

Now let us turn our attention to waveforms at locations where the maximum pressure is observed. Even without having considered these waves in a previous example, it should be apparent that if lobes of p_1 and p_2 versus x overlap, the maximum pressure $2B$ will be attained. We could draw a few diagrams to identify all possible distances s by which the p_1 and p_2 profiles should be shifted in opposite directions to bring them into alignment. Instead we shall pursue a more formal method. Let us denote as x_j , $j = 1, 2, \dots$, the locations at which $p_1 = B$ when $t = 0$. Similarly, x_n , $n = 1, 2, \dots$ denotes locations where $p_2 = B$ at $t = 0$. Both sets of locations are readily identified from the first plot in the preceding diagram.

At an arbitrary time t , the x_j values will have increased by ct , and the x_n values will have decreased by that amount. Thus, these locations are given by

$$x_j = -\frac{L}{4} - (j-1)L + ct$$

$$x_n = \frac{L}{4} + (n-1)L - ct$$

The maximum pressure occurs when *any* x_j equals *any* x_n , which occurs when

$$ct = \frac{L}{4} (2j + 2n - 3)$$

Because j and n are any positive integer, the maxima occur when $ct = L/4, 3L/4, 5L/4, \dots$. Because ct is an odd multiple of $L/4$, the x_j and x_n values are integer multiples of $L/2$. Thus, we conclude that the maxima occur at $x = 0, \pm L/2, \pm L, \dots$

Construction of the waveform at $x = 0$, from which the other waveforms are derived, is described in the first graph of Fig. 2. The waveform of p_1 is obtained by reflecting the initial profile of p_1 about the vertical axis through $x = 0$, then relabeling the axis from x to ct . The waveform of p_2 at $x = 0$ is the same as the initial p_2 profile, with the axis relabeled from x to ct . The total pressure is the sum of p_1 and p_2 , which can be seen to be twice the contribution of either simple wave.

The second graph in Fig. 2 describes the construction for $x = L/2$ which is the next location where a maximum pressure is observed. The p_1 profile is retarded by shifting it to the right by $\Delta(ct) = L/2$, and the p_2 profile is advanced by shifting it to the left by the same amount. The leading edge for each simple wave is marked by the symbol $\otimes$. These are added to obtain p as a function of t at $x = L/2$.

The construction for $x = -L/2$, which appears in the third graph of Fig. 2, is reversed. Because p_1 is traveling in the positive x direction, it arrives earlier at negative x locations, so we shift the initial profile of p_1 backward (left) along the ct axis by distance $\Delta(ct) = |x|$. Similarly, p_2 is shifted forward (right) along the ct axis by the same distance. It will be noted that the total pressure is the same at $x = -L/2$ and $x = L/2$. This is a consequence of the fact that the initial profiles are symmetric relative to $x = 0$, and the propagation of each wave is a symmetric image of the other. Thus, the same response should be observed at $+x$ and $-x$, so only waveforms at positive x need be considered. The last two graphs of Fig. 2 show the waveforms at larger x where a pressure maximum occurs. Each is obtained by the same procedure that constructed the waveform at $x = L/2$, except that the time by which the p_1 and p_2 profiles are advanced is greater. It is worth noting that the locations at which the maximum pressure is observed also are the locations of minimum (that is, largest negatively) pressure. This is a direct consequence of the overlapping of p_1 and p_2 .

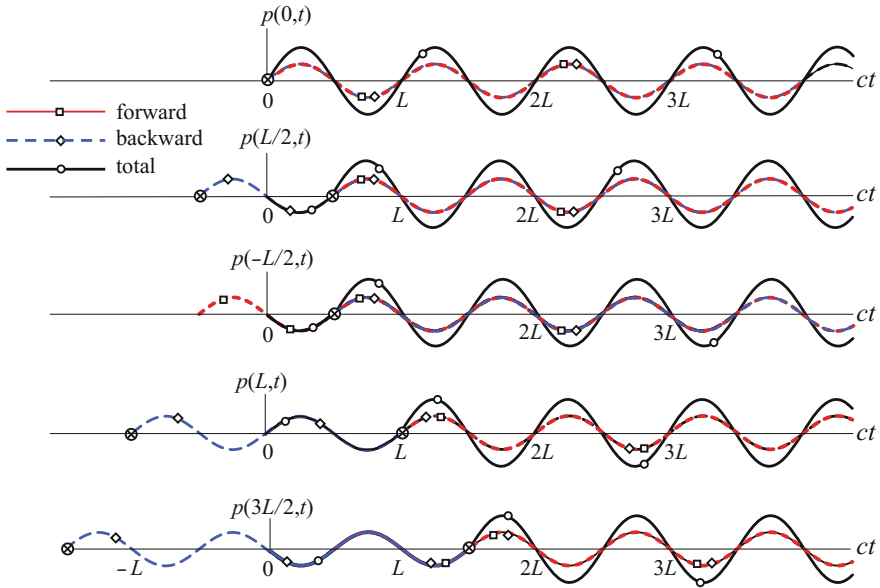


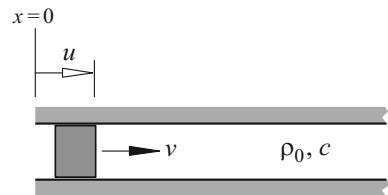
Figure 2. Waveforms at several locations.

2.5.2 Plane Waves in a Semi-infinite Domain

Velocity Excitation

We now turn our attention to a semi-infinite domain in which there is an excitation at the near end, which is defined to be $x = 0$. It is possible that the excitation is a specification of the pressure as a function of time. This case was analyzed by the method of characteristics, which led to Fig. 2.4. However, the usual situation is an imposed velocity at the boundary. A prototype for this condition appears in Fig. 2.9, where there is a piston at one end of a tube.

Fig. 2.9 Continuity of particle velocity at a moving boundary



No gaps can open up between the face of the piston and the fluid because the gap would be a vacuum that violates the continuity condition. In the Figure the face of the piston that is exposed to the fluid is at $x = 0$ when the system is in its reference position. The displacement of the piston relative to this position is $u(t)$, which is a

specified function of time. Thus, the instantaneous position of the exposed face is $x = u$, and the velocity of the piston is du/dt . Continuity at the face of the piston requires that du/dt equal the fluid's particle velocity v at the interface between the media, so

$$v(x, t)|_{x=u} = \frac{du}{dt} \quad (2.5.4)$$

This statement constitutes a *moving boundary condition*. This type of boundary condition is difficult to satisfy because the location of the boundary is not constant. Let us consider simplifying it based on the notion that u probably is small. A Taylor series gives

$$v(x, t)|_{x=0} + u(t) \left[\frac{\partial}{\partial x} v(x, t) \right] \Big|_{x=0} + \cdots = \frac{du}{dt} \quad (2.5.5)$$

We have seen that a linearized analysis requires that $|p| / (\rho_0 c^2)$ be small, and we know that for each of the simple plane waves $|p_n| = \rho c |v_n|$, so it follows that $|v|/c$ and $|p| / (\rho_0 c^2)$ have comparable small orders of magnitude. The above boundary condition then indicates that v and du/dt have comparable orders of magnitude. It follows that u is indeed very small, so the second term in the Taylor series expansion of the boundary condition may be dropped. **An overdot is standard notation for a time derivative**, so the linearized boundary condition is

$$\boxed{v(x, t)|_{x=0} = \dot{u}} \quad (2.5.6)$$

In other words, because the displacement of the boundary is small, it is adequate to consider fluid particles to contact the wall at the wall's reference position. This is an approximation that will be seen to be fundamental to any linearized description of the way a signal in a fluid is affected by a solid boundary.

Implicit to the definition of $\dot{u}$ is the specification that it is zero for $t < 0$, which may be enforced with a factor $h(t)$. It is not difficult to determine the one-dimensional plane wave induced by the movement of the boundary. The discussion of Fig. 2.4 explained that only forward propagating waves are permissible. A backward propagating wave would have to originate a long time earlier at a great distance from the boundary, in violation of the principle of causality. Thus, we discard the p_2 wave in Eq. (2.4.17), so that $v = p_1 / \rho_0 c$, and $p_1 = F(x - ct)$. Continuity of velocity at the fluid-wall interface requires that

$$F(-ct) = \rho_0 c \dot{u}(t) h(t) \quad (2.5.7)$$

The step function, which is $h(t) = 0$ if $t < 0$ and $h(t) = 1$ if $t > 0$, as been appended to $\dot{u}$ to emphasize that the piston motion is zero prior to $t = 0$, which is when we began to observe the system.

Although the description of the F function has been derived by considering the boundary condition at $x = 0$, it must be true everywhere because ξ is the only

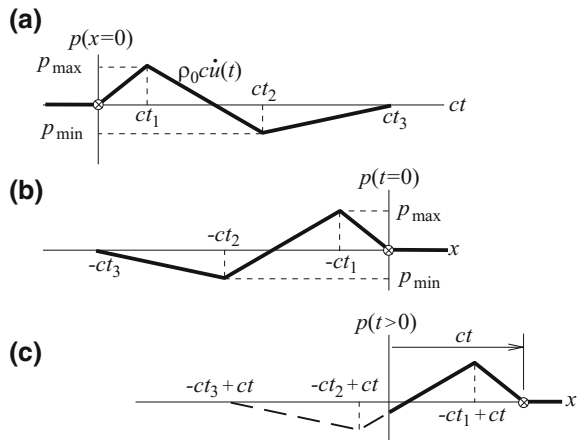
variable on which F can depend. At $x = 0$, we have $\xi = -ct$, so we replace t on the right side with $-\xi/c \equiv t - x/c$. The result is that the pressure wave is given by

$$p = \rho_0 c \dot{u} \left(t - \frac{x}{c} \right) h \left(t - \frac{x}{c} \right) \quad (2.5.8)$$

This expression states that the pressure is like the velocity of the wall multiplied by the characteristic impedance of the fluid, $\rho_0 c$, and delayed by the propagation time x/c .

Let us consider constructing spatial profiles of this wave. We temporarily ignore the fact that the region $x < 0$ is not physically meaningful because it is behind the wall. At $t = 0$, the signal is $p = \rho_0 c \dot{u}(-x/c) h(-x/c)$. The quantity $\dot{u}(-x/c)$ is the function $\dot{u}(t)$ evaluated at time $-x/c$. This suggests that the initial profile may be constructed by a procedure that is reminiscent of the one used for the initial value problem in an infinite domain. First we plot the waveform at $x = 0$ as p versus ct , that is, we evaluate p at a specified t and use ct as the abscissa. This is Fig. 2.10a. The initial profile is then obtained by swapping this waveform to the other side of $t = 0$ and changing the abscissa's label from ct to x . Doing so produces Fig. 2.10b. In Fig. 2.10c the waveform at $t > 0$ is constructed. This entails translating to the right the profile in Fig. 2.10b by ct . The portion of the profile that lies in $x < 0$ is blocked out because the fluid is situated in $x > 0$ only.

Fig. 2.10 Conversion of a velocity versus time boundary condition to a spatial pressure profile



From one perspective it is unnecessary to consider a graphical construction because Eq. (2.5.8) is not difficult to evaluate. The reason we have done so is that the overall objective is to convert all plane wave solutions to an equivalent form defined by an initial profile for an infinite domain. Such a description allows us to employ the

techniques in Figs. 2.7 and 2.8 to construct spatial profiles and temporal waveforms in any situation.

Reflection of a Simple Wave

A boundary that has an imposed motion is said to be active, because some outside energy source, like a motor, is required to force that motion to occur. A passive boundary also can move, but it does so in response to the pressure that is applied to it. Thus, the piston in Fig. 2.9 was active, because it was forced to move. If we replace that force with a mechanical spring or dashpot (a device that generates a resistance force that is proportional to velocity), or consider the inertia of the piston, we would have a situation where the piston's velocity $\dot{u}$ depends on the pressure. Our consideration here is limited to the dashpot model of resistance, which states that

$$p|_{x=0} = -\rho_0 c \zeta \dot{u} \quad (2.5.9)$$

The quantity ζ is called the *specific impedance* because it describes a ratio of pressure to velocity, with “specific” connoting that it is a factor multiplying some reference value, in this case the characteristic impedance of the fluid. The minus sign is inserted into the boundary condition because a positive pressure acting on the boundary pushes the surface in the direction of decreasing x , so it should produce a negative value of $\dot{u}$. The last element of the boundary condition is the continuity requirement, which is the same as the condition in the previous section, where $\dot{u}$ was an imposed motion. In both cases no gap can open up at the interface between the fluid and the boundary. Previously $\dot{u}$ was specified, but it must equal the particle velocity. Thus, the condition to be satisfied is

$$\boxed{p|_{x=0} = -\rho_0 c \zeta v|_{x=0}} \quad (2.5.10)$$

This relation is extremely idealized, because all solid surfaces have some resistance features that are elastic (proportional to displacement u) and inertial (proportional to acceleration $\ddot{u}$). Those situations are excluded because the model of a velocity resistance we have adopted will lead to algebraic boundary conditions, whereas any generalization beyond that would lead to boundary conditions that are ordinary differential equations. It is quite difficult, (but not impossible) to obtain time domain solutions for such general conditions. In contrast, restriction to a velocity resistance makes accessible many phenomena that are also encountered in more general cases.

Before we address the response of specific systems, it is useful to examine generally how the presence of an impedance boundary affects a simple wave. We consider a domain $x > 0$ in which a p_2 wave propagates backward until it encounters the boundary at $x = 0$. We say that this wave is *incident* on the boundary. As is true for light waves, a wave is reflected from the surface. This is a p_1 wave because it propagates away from the wall in the direction of increasing x . It follows that p_1 and p_2 must behave according to Eq. (2.4.17), but in the present scenario only $G(x)$ is known, and we must determine $F(x)$.

Substitution of the descriptions of pressure and particle velocity in Eq. (2.4.17) into (2.5.10) gives

$$F(-ct) + G(ct) = -\zeta [F(-ct) - G(ct)], \quad t > 0 \quad (2.5.11)$$

We began with the assumption that we know $G(x)$, but this applies only for $x > 0$ because $x < 0$ is not part of the domain. Hence we know $G(ct)$ for positive t only, which is the reason this range has been specified in the above. We solve this relation for $F(-ct)$, which yields

$$F(-ct) = RG(ct), \quad t > 0 \quad (2.5.12)$$

where R is the reflection coefficient,

$$R = \frac{\zeta - 1}{\zeta + 1} \quad (2.5.13)$$

To understand Eq. (2.5.12) recall that the time dependencies of p_1 and p_2 at $x = 0$ are described by $F(-ct)$ and $G(ct)$, respectively. Thus, the reflection coefficient is a factor of proportionality that gives the instantaneous pressure in the reflected wave at the boundary as a fraction of the incident wave's pressure at that instant.

There are three values of the specific impedance for which the reflection coefficient takes on a special value. Suppose the resistance to motion is enormous, so that the wall barely moves for any pressure. The limiting condition is $\zeta \rightarrow \infty$, in which case $R \rightarrow 1$. This is the case of a *rigid* boundary, which is a commonly used model, even though it is unattainable in reality. According to Eq. (2.5.12), if $R = 1$, the incident and reflected waves at the boundary have equal pressure at any instant. The consequence is that the total pressure at the wall is double that of the incident wave. The opposite condition is a wall that is so soft that application of a very small pressure induces a very large $\dot{u}$. The limit is $\zeta = 0$, which leads to $R = -1$. The total pressure at the boundary is zero, so this condition is said to represent a *pressure-release surface*. The third special case is a wall whose impedance matches the characteristic impedance of the fluid, which corresponds to $\zeta = 1$. Because $R = 0$ in this case, we say that is a *nonreflecting boundary*. Any value of ζ that is neither zero nor infinite leads to a reflection coefficient whose magnitude is less than one, and therefore attenuates the reflected wave. When R is positive, the pressure in the reflected wave leaves the boundary with the sign of the pressure that was incident, whereas a negative R reverses the sign of the reflected wave relative to the incident one. It is worth noting that the limitation of the reflection coefficient to $-1 \leq R < 1$ applies to a passive boundary. A feedback system is active in the sense that it exerts a controlled force. A feedback system could have a gain factor that leads to $|R| > 1$.

A description of the pressure at any location follows from Eq. (2.5.12). In general, F can depend solely on the forward characteristic ξ , and $\xi = -ct$ at $x = 0$. The result of replacing $-ct$ with ξ is $F(\xi) = RG(-\xi)$, that is,

$$F(x - ct) = RG(ct - x) \quad (2.5.14)$$

Equation (2.5.14) is quite general. Let us use it to describe explicitly the field associated with reflection of a backward propagating wave that is incident at the boundary when $t = 0$. The first step is to specialize the G function to account for the transient nature of the incident wave. The interval prior to arrival of the wave at the boundary corresponds to negative t , so the location of the wavefront of p_2 in that interval is $x = -ct$. Furthermore, $p_2 = 0$ ahead of the wavefront, which is the region $x < -ct$, once again, with $t < 0$. The general concept of the method of wave images is to extend the spatial profiles to $-\infty < x < \infty$, which will require that we distinguish between the extended wave and the actual wave. To that end we denote the function describing the incident wave behind its wavefront as $g(\eta)$, whereas $G(\eta)$ denotes the extended wave. To make p_2 be zero ahead of its wavefront we multiply $g(\eta)$ by a step function $h(\eta)$. Thus, we describe the incident wave by setting $G(x + ct) = g(x + ct)h(x + ct)$. To apply Eq. (2.5.14) we change the argument of the G function from $x + ct$ to $ct - x$, and multiply by the reflection coefficient. The result is that the pressure field at any x, t is given by

$$\begin{aligned} p_2 &= g(x + ct)h(x + ct) \\ p_1 &= Rg(ct - x)h(ct - x) \\ p &= p_1 + p_2 \end{aligned} \quad (2.5.15)$$

It would not be difficult to program these relations, but they also can be interpreted via a graphical construction. As we did in Fig. 2.10, let us temporarily ignore the fact that there is no fluid in $x < 0$. Then the initial pressure profiles of p_1 and p_2 are the distributions obtained from evaluation of Eq. (2.5.15) for $-\infty < x < \infty$ when $t = 0$. The profile of p_2 at this instant is $g(x)$ for $x > 0$ and zero for $x < 0$. The profile of p_1 is zero for $x > 0$, and it is the value of $g(-x)$ multiplied by R for $x < 0$. A different view is that we should take a value of g at some distance from the origin in the positive x direction, multiply that value by R , and assign it to p_1 at the same distance from the origin in the negative x direction. This is the construction on Fig. 2.11a.

In this graph the g function has a maximum $p_a > 0$ at x_a and a minimum $p_b < 0$ at x_b . The situation that is depicted corresponds to $R = -0.6$, so the minimum value of p_1 in the initial profile is Rp_a at $x = -x_a$ and the maximum is Rp_b at $x = -x_b$. If R were plus one, then p_1 would be the mirror image of p_2 with respect to $x = 0$. If R were minus one, p_1 would be the inverted mirror image of p_2 . Even though R is neither special value in Fig. 2.11a, we nevertheless refer to the initial p_1 as the image of p_2 .

Now that the initial profiles of both simple wave have been defined, they may be propagated in the same manner as we have thus far for initial and boundary value problems. Thus, in Fig. 2.11b the incident wave's profile shifts in the negative x by direction by distance ct , and the reflected wave's profile shifts in the direction of positive x by the same amount. The profile of the total pressure at this instant is the

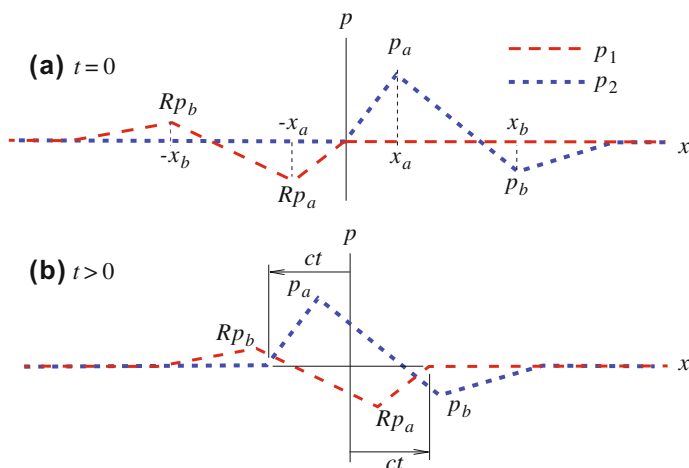


Fig. 2.11 Description by the method of wave images of reflection at a boundary. The incident backward propagating plane wave arrives at $t = 0$ and R is the reflection coefficient

sum of p_1 and p_2 for $x > 0$. In other words, we construct the image of the incident wave as an aid to the study and description of reflection, but ignore any portion of that image that lies behind the boundary when we evaluate the actual pressure.

Just as the construction of spatial profiles follows the procedure developed previously, so too does the construction of waveforms. First the waveforms of both simple waves is created as described by Fig. 2.12a. In this Figure the initial profile of p_2

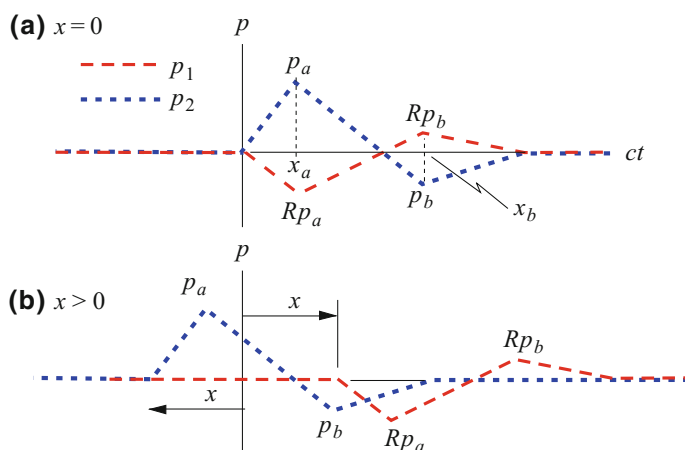


Fig. 2.12 Construction of waveforms of the incident and reflected waves. The reflection coefficient is $R = -0.6$

is replicated from Fig. 2.11a along the ct axis, whereas the initial profile of p_1 is flipped about the vertical axis through the origin. The result is that the waveform of p_1 at $x = 0$ is the waveform of p_2 multiplied by R . The waveforms at any location $x > 0$ are obtained by following Fig. 2.12b. The waveform of p_2 at the origin is advanced by shifting it to the left (decreasing ct). The amount of this shift is x . The p_1 waveform at the same location is delayed by shifting it to the right (increasing ct) relative to its appearance at the origin. The amount of this shift also is x . Once the individual waveforms are identified, the total pressure may be obtained by adding p_1 and p_2 in $ct \geq 0$.

The General Initial Value Problem

The preceding analysis of reflection was based on knowing the $g(x)$ function for the incident wave. Such information is available in an initial value problem. Like the case of an infinite waveguide, suppose that the initial pressure distribution is a given function $P(x)$ and the initial particle velocity is known to be $V(x)$. This means that Eq. (2.5.2) may be used, but only for $x > 0$, that is,

$$\begin{aligned} F(x) &= f(x), \quad G(x) = g(x), \quad x > 0 \\ f(x) &\equiv \frac{1}{2} [P(x) + \rho_0 c V(x)], \quad g(x) \equiv \frac{1}{2} [P(x) - \rho_0 c V(x)] \end{aligned} \quad (2.5.16)$$

The $G(x)$ function belongs to the backward propagating wave. Therefore, it is incident at the boundary $x = 0$ starting at time $t = 0$. The previous section described how this wave is reflected. In contrast, the $F(x)$ function belongs to the forward propagating wave. It is unimpeded by the boundary, so it propagates in the sense of increasing x as though the domain had no boundaries.

According to the principle of superposition, the total field within the waveguide is the sum of the field resulting from the incident p_2 wave and the contribution associated with $f(x)$. The former is described by Eq. (2.5.15) and the latter is $p_1 = f(\xi)$, but only for $\xi > 0$. This restriction on p_1 can be enforced with a step function. Thus, adding the two contributions yields the general solution of the initial value problem,

$$\begin{aligned} p_2 &= g(x + ct) h(x + ct) \\ p_1 &= Rg(ct - x) h(ct - x) + f(x - ct) h(x - ct) \\ f(x) &\equiv \frac{1}{2} [P(x) + \rho_0 c V(x)], \quad g(x) \equiv \frac{1}{2} [P(x) - \rho_0 c V(x)] \\ p &= p_1 + p_2 \end{aligned} \quad (2.5.17)$$

This form is readily described by wave images. Compared to the case where there is only an incident wave, the initial profile of the backward wave is unchanged. The only modification of the forward wave is addition of the f function for $x > 0$, where p_1 previously was zero. After we have constructed the initial profile, evaluation of profiles at later t and waveforms at any x follows the procedure described in Figs. 2.11 and 2.12.

The operations are depicted on Fig. 2.13. The initial pressure $P(x)$ and particle velocity $V(x)$ are described in Fig. 2.13a, where the velocity has been multiplied by the characteristic impedance in order to plot both functions on the same scale. In Fig. 2.13b the initial value functions have been added and subtracted according to Eq. (2.5.17) in order to generate the $f(x)$ and $g(x)$ functions. Figure 2.13c constructs the spatial profiles of p_1 and p_2 at $t = 0$. The portion of p_1 in $x < 0$ is the mirror image of $g(x)$ multiplied by the reflection coefficient, which is set at $R = 0.5$. This image describes the portion of p_1 that is generated by reflection of p_2 . In contrast, the portion of p_1 in $x > 0$ is the direct result of the initial conditions being nonzero. It would be observed even if there were no reflection. Figure 2.13d shows the construction of the p_1 and p_2 waveforms at $x = 0$.

From Fig. 2.13c, the spatial profiles at any t may be obtained by shifting the p_1 profile along the x axis by ct to the right, and the p_2 profile to the left by ct . Waveforms at any x may be obtained from Fig. 2.13d by shifting the p_1 wavefront to the right along the ct axis by x , and the p_2 axis to the left by x . The total pressure is obtained by adding the contributions of both simple waves, but only in the intervals $x \geq 0$ and $t \geq 0$.

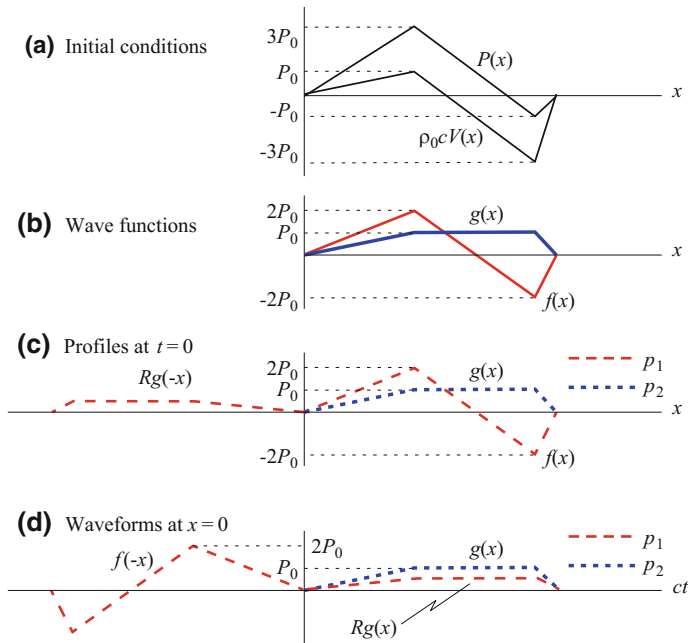


Fig. 2.13 Application of the method of wave images to generate the initial spatial profile and waveform in a semi-infinite waveguide. The initial pressure distribution is $P(x)$ and the initial particle velocity distribution is $V(x)$, and $R = 0.5$

EXAMPLE 2.5 The diagram shows a very long cylindrical tube within which a cylindrical container of length L may move. The container is filled with air at the same density and temperature as the ambient conditions within the tube. Prior to $t = 0$ the cylinder was translating to the left at constant speed v_c , such that the distance to the container's left end from some reference is $-v_c t$. At $t = 0$, a magnet is activated, thereby stopping the container almost instantaneously. Concurrently with halting the container, a high current circuit rapidly melts the plastic wall at the trailing (right) end, thereby merging the air inside and outside the container. The left wall remains intact. Construct spatial profiles of p and v in the region $x > 0$ at $t = 0.75L/c$ and $t = 1.5L/c$. Also construct waveforms of p and v for the interval $t > 0$ at $x = 0.75L$ and $x = 1.5L$.

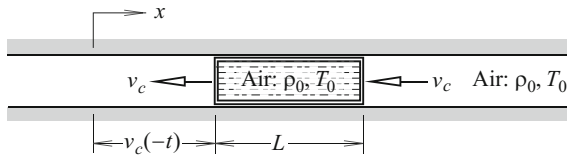


Figure 1.

Significance

The initial state will be seen to be somewhat unusual. This will give rise to waves that are readily described, yet emphasize the fundamental aspects of the combined effects of general initial conditions and reflections. The results will be interesting, and to some extent counterintuitive.

Solution

At the outset, we must assume that $v_c \ll c$. This allows us to avoid treating a moving boundary condition.⁴ Subject to this assumption, it is a straightforward matter to determine the field for $t < 0$, which is the interval prior to disintegration of the container's wall. The right side of the container is at $v_c(-t) + L$. It is the near boundary for the semi-infinite wave guide to the right. The container was translating to the left for $t < 0$. Hence the boundary condition is $v = -v_c$ at $x = -v_c t + L$. This motion generates a wave that propagates in the direction of increasing x . There is no wave coming from infinity. Because $p = \rho_0 c v$ for a forward wave, it must be that $p = -\rho_0 c v_c$ and $v = -v_c$ for $x > L - ct$ when $t < 0$.

At $t = 0$, the container's right wall disintegrates, so the fluid within the container merges with the fluid to the right. The pressure within the container for $t < 0$ was the ambient value, and all particles were moving to the left at v_c . Neither of these variables can change instantaneously, so the field within the container at $t = 0$ is

⁴The case where movement of the boundary is significant requires consideration of nonlinear effects. This issue is discussed in Chap. 13.

$p = 0$ and $v = -v_c$. Combining this field and the field to the right at $t = 0$ shows that the initial state is

$$p(x, 0) \equiv P(x) = -\rho_0 c_0 v_c h(x - L) \text{ and } v(x, 0) \equiv V(x) = -v_c h(x), \quad x > 0 \quad (1)$$

These initial conditions are the profiles in Fig. 2.

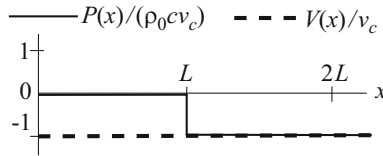


Figure 2.

In addition, it is given that the container comes to rest simultaneously with disintegration of the right end. The left end remains intact, so it becomes a stationary boundary. Therefore, we must satisfy $v = 0$ at $x = 0$ for $t \geq 0$.

This is our first encounter with discontinuous pressure and velocity fields. According to Euler's equation, a discontinuous pressure profile would require infinite acceleration. Also, a discontinuous velocity profile would correspond to an infinite shear rate, which would mean that viscosity cannot be ignored. A proper description of such behavior requires the study of shocks, which is a topic in Chap. 13 on nonlinear effects. For our studies of linear acoustics a discontinuity serves as a simplified model of a large change of state that occurs over an extremely short distance and a very brief time interval.

The profile functions $f(x)$ and $g(x)$ for the forward and backward waves are found by substituting Eq. (1) into (2.5.17). Doing so yields

$$f(x) = -\frac{1}{2}\rho_0 c v_c [h(x) + h(x - L)], \quad g(x) = +\frac{1}{2}\rho_0 c v_c [h(x) - h(x - L)], \quad x > 0 \quad (2)$$

The reflection coefficient at $x = 0$ is $R = 1$ because the particle velocity at that location is zero. These considerations complete the preliminaries required to implement the method of wave images.

The first step is to create initial profiles that replace the boundary with wave images. In the physical region $x > 0$ there is a forward propagating wave p_1 whose profile at $t = 0$ is $f(x)$. There also is a backward propagating wave p_2 whose profile at $t = 0$ is $g(x)$. Reflection at $x = 0$ of the backward propagating wave is accounted for with an image in the region $x < 0$. This image belongs to p_1 . It is formed by left/right flipping of $g(x)$ relative to the origin, then multiplying the result by R . This forms $Rg(-x)$. Thus, the initial profile of p_1 consists of $f(x)$ in $x > 0$ and $Rg(-x)$ in $x < 0$. The initial profile of p_2 consists of $g(x)$ for $x > 0$ and nothing in $x < 0$. Both are depicted in the upper graph in Fig. 3. The initial velocity profiles in the lower graph are $v_1 = p_1/(\rho_0 c)$ and $v_2 = -p_2/(\rho_0 c)$, so the former replicates p_1 and the latter inverts p_2 .

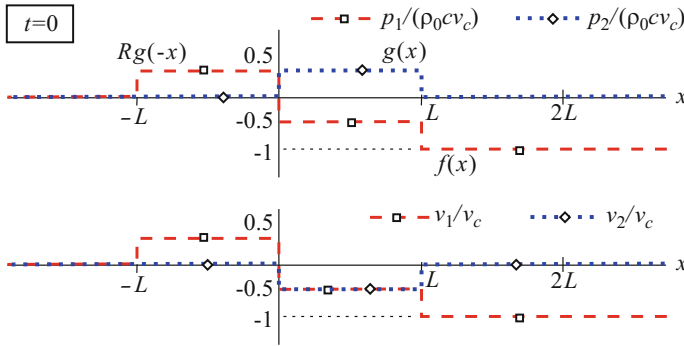


Figure 3.

Spatial profiles at any instant t are obtained by translating the profiles in Fig. 2 by distance ct . The p_1 and v_1 curves are shifted in the direction of increasing x , while the p_2 and v_2 curves are shifted in the direction of decreasing x . The translated curves for $t = 0.75L/c$ are described by the first two graphs in Fig. 4. In the third graph the values of the forward and backward curves are added. Only the portion in the physical domain $x > 0$ is retained. The fact that $p/(\rho_0cv_c) = v/v_c$ for $x > 0.25L$ comes about because only the forward propagating wave exists in that region.

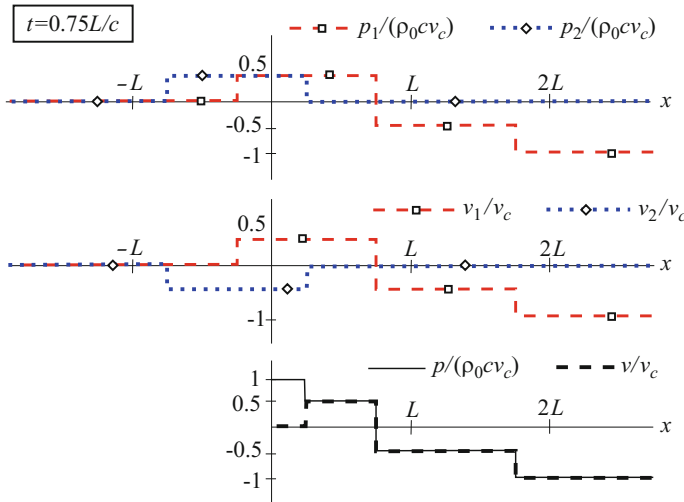


Figure 4.

The same steps as those in Fig. 4 lead to the spatial profiles for $t = 1.5L/c$ in Fig. 5. The shift of the initial profiles in this case is $1.5L$. For any instant after L/c , the p_2 wave has been fully reflected. Consequently, all that remains is the wave that propagates to the right, with $p = \rho_0cv$.

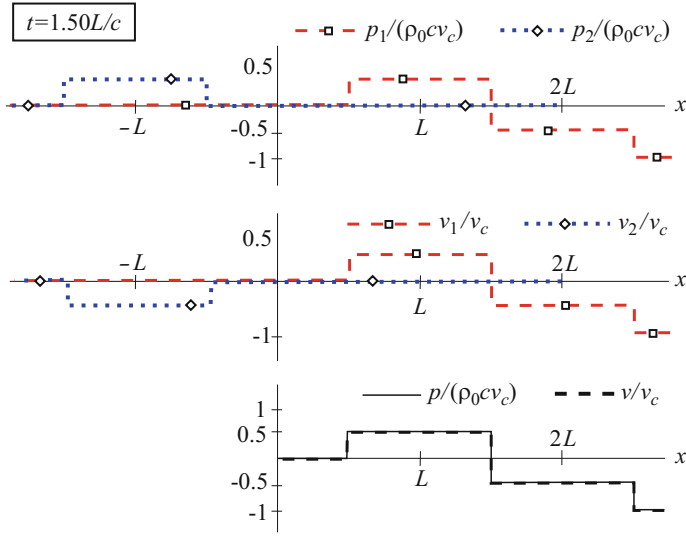


Figure 5.

Construction of waveforms for $x > 0$ follows the standard procedure for an initial value problem in an infinite domain. It begins by converting the initial spatial profiles of p_1 and p_2 in Fig. 3 to waveforms at $x = 0$. The waveforms of p_1 and v_1 are obtained by left/right swapping of the respective initial profiles relative to the origin. The p_2 and v_2 waveforms are obtained by replicating the initial profiles. These operations are shown in Fig. 6.

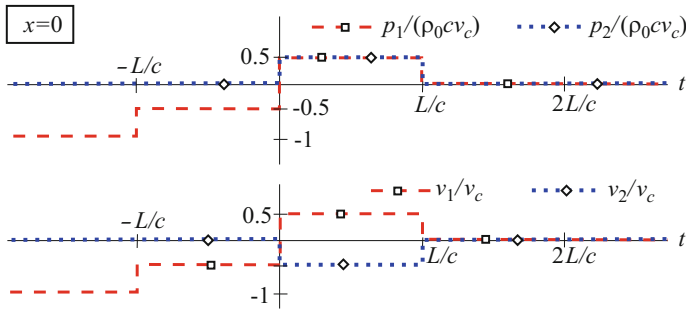


Figure 6.

A waveform at any $x > 0$ is obtained by shifting the p_1 and v_1 waveforms in Fig. 6 in the sense of increasing t by x/c , corresponding to a delayed arrival. The p_2 and v_2 waveforms are shifted by the same amount in the sense of decreasing t , corresponding to an advanced arrival time. The waveforms for the total signal consist of the sum of the individual contributions. The result of these operations for $x = 0.75L/c$ appears in Fig. 7. It is helpful to refer to the initial profiles in Fig. 3 to explain these results. The initial state within the container results in forward and backward waves. The time required for the trailing edge of the backward wave, which initially was at $x = L$, to travel to $x = 0.75L$ is $0.25L/c$. After that time, only the

forward wave contributes. At $t = 0.75L/c$, the trailing edge of the p_1 wave, which began at $x = 0$, passes $x = 0.75L$. That also is the arrival time of the leading edge of the reflected wave. Then, at $t = 1.75L$, the trailing edge of the reflected wave passes $x = 0.75L$. After that instant there is no disturbance.

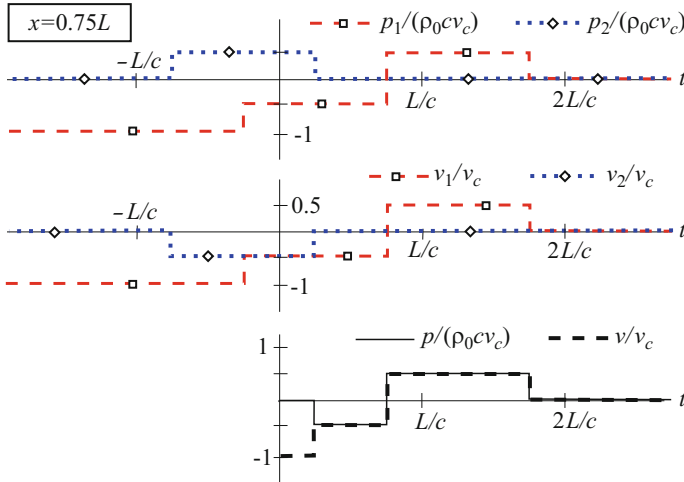


Figure 7.

The same construction steps as those described by Fig. 7, with a time delay of $1.5L/c$ leads to the waveform at $x = 1.5L$. The result is shown in Fig. 8. This location is outside the region that experiences the initial backward wave. Prior to $t = 0.5L/c$ the pressure is unchanged from its initial value. At this instant the phase of the directly generated part of p_1 that initially was at $x = L$ has traveled to $x = 1.5L$. The reflected wave simultaneously arrives. Then at $t = 2.5L$, the last nonzero part of the reflected wave passes $x = 1.5L$. After that instant no disturbance is present.

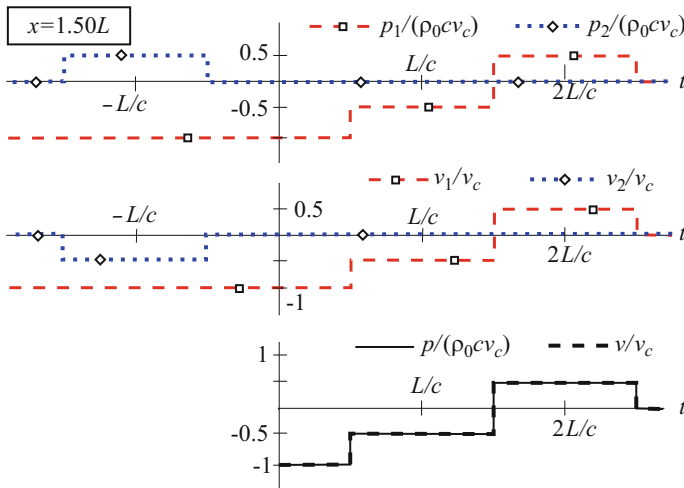


Figure 8.

2.5.3 Plane Waves in a Finite Waveguide

A semi-infinite domain is a highly idealized model. It does not account for the fact that even if the far end of a waveguide is very distant, any forward propagating wave will eventually reach that end, and return. Not only does this observation identify the limitation of the model of a semi-infinite domain, it tells us under what condition the model may be used. Specifically, at a specified location a semi-infinite model is adequate until the wave that is reflected from the far end returns to that location.

Allowance for reflection at the far end leads to a physical process of multiple reflections. A forward propagating wave is reflected at the far end, $x = L$, as a backward propagating wave that is reflected at $x = 0$ as another forward propagating wave. The process continues *ad infinitum*. The description of this multiple reflection process, and its implications for a variety of system configurations, provides closure for our study of time domain solutions for plane waves.

Initial Value Problem

The system consists of a closed waveguide, such as a tube, that is bounded by walls at $x = 0$ and $x = L$. Reflections at $x = 0$ are described by Eq. (2.5.15). We require a similar description of the reflection process at $x = L$. The incident wave in this case is p_1 , and the location of the boundary has been shifted to $x = L$. Furthermore, the fluid is situated in $x < L$. Nevertheless, the physical nature of the reflected wave should be unaltered by placing the reflection boundary at a different location. This allows us to introduce a coordinate transformation that makes this end look like $x = 0$, specifically, $x' = L - x$. Thus, we replace x with x' in Eq. (2.5.14) and swap F and G because F represents the incident wave. Doing so gives

$$G(x' - ct) = RF(ct - x') \quad (2.5.18)$$

Introduction of the coordinate transformation to eliminate x' yields the form we seek

$$G(L - x - ct) = R_L F(x + ct - L), \quad 0 \leq x \leq L \quad (2.5.19)$$

A form that explicitly gives $G(x)$ will be more useful to the construction of an overall solution. We obtain such a representation by subtracting L from the argument in both sides, and then multiplying each argument by -1 . Setting $t = 0$ yields the initial profile of p_2 ,

$$\boxed{G(x) = R_L F(2L - x), \quad 0 \leq x \leq L} \quad (2.5.20)$$

In other words, if we know $F(x)$ to be the incident wave, the initial profile of the reflected wave may be obtained by replacing x with $2L - x$ in the argument and multiplying the function by the reflection factor. Although the mathematical appearance of this rule is different from the one at $x = 0$, the physical and visual aspects when viewed as position in front of, and behind, the boundary, are unchanged.

To see that this is so consider the phase of $F(x)$ that is at a specified distance x from the left boundary. This point is at distance $L - x$ from the right boundary. A point at the same distance behind the right boundary is at $x = L + (L - x)$. Therefore, Eq. (2.5.20) tells us the $G(x)$ is the image of $F(x)$, flipped left to right relative to $x = L$ and multiplied by R .

These observations lead to the recognition that the initial profile of the reflected wave may be visually constructed in two steps, regardless of where the boundary is situated. First a copy of the incident wave's initial profile is mirror-imaged by swapping it left and right relative to the boundary. The second step multiplies the reflected wave's image by the reflection factor.

Now that we can describe how waves are reflected from both ends, we may proceed to satisfy the initial conditions. In the absence of boundaries the d'Alembert solution corresponding to an initial pressure field $P(x)$ and velocity field $V(x)$ is described by Eq. (2.5.2). The p_1 and p_2 waves that are described there undergo the sequence of reflections. Each reflection is described by adding a wave image to the profiles directly generated by the initial conditions. To distinguish waves that are generated from images, let us denote the initial functions as $f(x)$ for p_1 and $p_2(x)$ for p_2 . Both are defined only within the confines of the waveguide, $0 < x < L$. The method of wave images extends all profiles infinitely in both directions. We can do so by using a combination of step functions $h(x) - h(x - L)$ as a factor. This factor is one in the physical domain and zero for $x < 0$ and $x > L$. Thus the initial pressure profiles induced by the initial conditions are written as

$$\begin{aligned} f(x) &\equiv \frac{1}{2} [P(x) + \rho_0 c V(x)] [h(x) - h(x - L)] \\ g(x) &\equiv \frac{1}{2} [P(x) - \rho_0 c V(x)] [h(x) - h(x - L)] \end{aligned} \quad (2.5.21)$$

The $f(x)$ portion propagates forward, so it is reflected at $x = L$, whereas the $g(x)$ portion propagates backward and is reflected at $x = 0$. It is easier to track the multiple reflections that ensue if we consider each initial wave individually. Linearity of the governing equations allows us to superpose the individual contributions. The notation we have adopted sets $F(x)$ as the overall initial profile formed from all contributions to the forward propagating wave. Therefore it is logical to designate the initial wave by itself as $F_0(x) = f(x)$.

The concept now is to identify images based on the reflection equations, and trace each image back to $f(x)$. According to Eq. (2.5.20), reflection of $F_0(x)$ at $x = L$ is described by adding image #1 to the profile of p_2 . It is described by $G_1(x) = R_L F_0(2L - x) \equiv R_L f(2L - x)$. This reflected wave propagates backward until it is reflected at $x = 0$. This reflection is described by adding image #2 to the initial profile of p_1 . According to Eq. (2.5.15), this image is $F_2(x) = R_0 G_1(-x) = R_0 [R_L f(2L - (-x))]$. Reflection at $x = L$ of this portion of the forward propagating wave generates image #3 that adds $G_3(x)$ to p_2 . The reflection law at $x = L$ gives $G_3(x) = R_L F_2(2L - x) = R_L [R_0 R_L f(2L + (2L - x))]$. This contribution to p_2 is reflected at $x = 0$, which leads to image #4. The reflection law at that end gives $F_4(x) = R_0 G_3(-x) \equiv R_0 [R_0 R_L^2 f(4L - (-x))]$.

We could go on generating many more images, but a pattern is now apparent. We construct $F(x)$ by adding to the initial part, $f(x)$, images #2, #4, and their successors. The additional images are obtained by multiplying the previous one by $R_0 R_L$, corresponding to reflection at both ends, and adding $2L$ to the argument of the predecessor, corresponding to the distance the image must travel down the waveguide and back to $x = 0$. Similarly, $G(x)$ consists of images #1, #3, and their successors. The first image is obtained by satisfying the law for reflection of the initial wave $f(x)$ at $x = L$. Successive images also are obtained by multiplying the predecessor by $R_0 R_L$ and adding $2L$ to the argument. A compact representation of the combination of all images is

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} (R_0 R_L)^n f(x + 2nL) \\ G(x) &= \sum_{n=0}^{\infty} (R_0 R_L)^n [R_L f(-x + 2(n+1)L)] \end{aligned} \quad (2.5.22)$$

Figure 2.14 provides a graphic description of this process. It will be observed that because $f(x)$ is defined over an interval whose length is L and each image is shifted by $2L$, in the negative x direction for $F(x)$ and in the positive x direction for $G(x)$, the images are separated by gaps of length L . Also, the front of a reflected wave's image is at the same distance behind the reflecting boundary as the distance from the front of the corresponding incident wave's image to the boundary, in accord with the general discussion of reflection. For example, $F_2(x)$ is reflected at $x = L$, which results in G_3 . The front of $G_3(x)$ is at $x = 3L$, and the front of $F_2(x)$ is at $x = -L$, so the distance from each front to $x = L$ is $2L$.

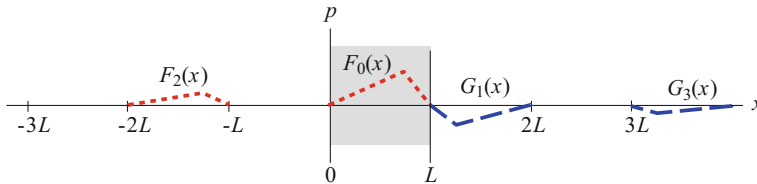


Fig. 2.14 Initial pressure profile in a finite waveguide generated by the portion of the initial wave that propagates in the sense of increasing x

The same reasoning leads to a description of the contribution of $g(x)$ to the initial profiles. This function describes the portion of the p_2 wave generated by the initial conditions, so we denote it as $G_0(x) = g(x)$. This wave is incident at $x = 0$, so its reflection adds image #1 to p_1 , with $F_1(x) = R_0 G_0(-x) \equiv R_0 g(-x)$. In turn, $F_1(x)$ is reflected at $x = L$, which leads to image #2 for the p_2 wave that is reflected. This image is $G_2(x) = R_L F_1(2L - x) \equiv R_L [R_0 g(-(2L - x))]$. The p_2 wave described by $G_2(x)$ propagates backward to the origin, where it is reflected, thereby generating image #3 for p_1 . This image is $F_3(x) = R_0 G_2(-x) \equiv R_0 [R_L R_0 g(-2L + (-x))]$. The p_1 wave associated with image #3 propagates for-

ward until it reflects at $x = L$, with that reflection being described as $G_4(x) = R_L F_3(2L - x) \equiv R_L [R_L R_0^2 g(-2L - (2L - x))]$, and so on. A pattern similar to that associated with $f(x)$ has emerged. The first contribution to $G(x)$ is $g(x)$, which is generated by the initial condition. Successive images for $G(x)$ multiply the predecessor by $R_0 R_L$ and subtract $2L$ from the function's argument. The first contribution to $F(x)$ consists of the reflection of the initial wave at $x = 0$, which is $R_0 g(-x)$. Successive images are obtained by multiplying the predecessor by $R_0 R_L$ and subtracting $2L$ from the function's argument. The mathematical representation of the combined set of waves is

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} (R_0 R_L)^n [R_0 g(-x - 2nL)] \\ G(x) &= \sum_{n=0}^{\infty} (R_0 R_L)^n g(x - 2nL) \end{aligned} \quad (2.5.23)$$

This construction is depicted in Fig. 2.15. Once again, the images are separated by intervals of length L . These gaps match the intervals in which $f(x)$ generates images.

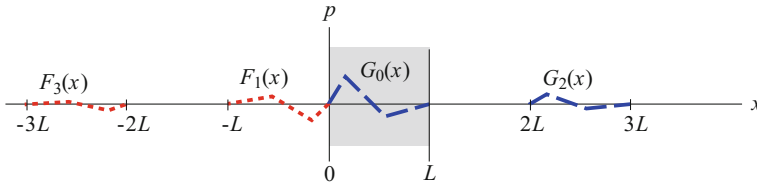


Fig. 2.15 Initial pressure profile in a finite waveguide generated by the portion of the initial wave that propagates in the sense of increasing x

The initial profile that is consistent with the reflection laws and satisfies the initial conditions is the sum of the profiles associated with $f(x)$ and $g(x)$, as depicted in Fig. 2.16. All images behind the boundary at $x = 0$ are part of the forward propagating wave, and all images behind the boundary at $x = L$ are part of the backward propagating wave. The oppositely traveling simple waves initially only overlap in $0 \leq x \leq L$.

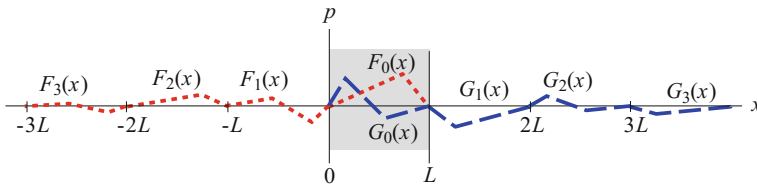


Fig. 2.16 Initial pressure profile in a finite waveguide generated by initial conditions

The general solution is obtained by summing Eqs. (2.5.22) and (2.5.23). Then because $\xi = x$ at $t = 0$, we replace x with $x - ct$ in $F(x)$. Similarly, $\eta = x$ at $t = 0$, so we replace x with $x + ct$ in $G(x)$. For the sake of completeness the definition of the $f(x)$ and $g(x)$ functions is repeated in the description of the full signal.

$$\begin{aligned}
 p &= p_1 + p_2 \\
 p_1(x, t) &= F(x - ct) = \sum_{n=0}^{\infty} (R_0 R_L)^n [f(x - ct + 2nL) \\
 &\quad + R_0 g(-x + ct - 2nL)] \\
 p_2(x, t) &= G(x + ct) = \sum_{n=0}^{\infty} (R_0 R_L)^n [g(x + ct - 2nL) \\
 &\quad + R_L f(-x - ct + 2(n+1)L)] \\
 f(x) &\equiv \frac{1}{2} [P(x) + \rho_0 c V(x)] [h(x) - h(x - L)] \\
 g(x) &\equiv \frac{1}{2} [P(x) - \rho_0 c V(x)] [h(x) - h(x - L)]
 \end{aligned} \tag{2.5.24}$$

Image constructions like Fig. 2.16 are a great aid when we wish to interpret why various phenomena occur, but the sketches become difficult if $f(x)$ or $g(x)$ are complicated functions. When we need to obtain quantitative results, the preferable route is to use mathematical software to evaluate Eq. (2.5.24). Both approaches require selection of the number N at which the summation of images contributions may be terminated. To determine this number we observe that halting the summations at N places images in $-2NL \leq x \leq (2N+1)L$ at $t = 0$. This is an adequate representation if images outside this range, which have not been constructed, arrive at the location of interest after the maximum time of interest, $T_{\max}$. The distance from the tail of the last image for F or G to the nearest wall is $2NL$, so the time required to propagate the farthest tail to the closest boundary is $2NL/c$, which must be greater than $T_{\max}$. Thus, the index at which the summations in Eq. (2.5.24) are halted must be such that

$$N \geq \text{ceil} \left(\frac{cT_{\max}}{2L} \right) \tag{2.5.25}$$

where “ceil” is a function that gives the smallest integer that is greater than or equal to the argument.

Some interesting phenomena occur in situations where $|R| = 1$ at both ends. The reflection process in such cases does not attenuate the amplitude of a reflected wave relative to the incident, so reflected waves are either the positive or negative mirror image of the incident. We could use results from the study of acoustic energy in Chap. 4 to prove that no energy is dissipated in the reflection, but that property is apparent from the sameness of the waves.

There are three combinations of boundary conditions for which $|R| = 1$ at both ends, because the system is physically the same if $R_0 = 1$ and $R_L = -1$ or vice versa. Typical initial profiles for the case where both walls of the waveguide are rigid, $R_0 = R_L = 1$, are depicted in Fig. 2.17.

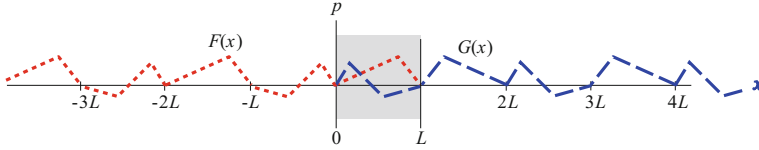


Fig. 2.17 Initial pressure profiles in a finite waveguide whose ends are rigid, $R_0 = R_L = 1$

Because the amplitudes are unchanged, the profiles of $F(x)$ and $G(x)$ can be seen to be periodic, with the pattern repeating in an interval of $2L$. This periodicity property only applies to $x < L$ and $x > 0$, respectively. Furthermore, $F(x)$ for $x < 0$ is the mirror image of $G(x)$ for $x > 0$, and $G(x)$ for $x > L$ is the mirror image of $F(x)$ for $x < L$. These properties are stated mathematically as

$$\begin{aligned} F(x - 2L) &= F(x), \quad x < L \\ G(x + 2L) &= G(x), \quad x > 0 \\ F(x) &= G(-x), \quad x < 0 \\ G(x) &= F(2L - x); \quad x > L \end{aligned} \quad (2.5.26)$$

Note that the first and third properties taken together state that $G(-x) = F(x - 2L)$. Changing the sign of the arguments in this relation yields the fourth property, so it is not independent.

A function that is periodic in x may be expanded in a Fourier series in x . Aside from the change of the variables name, the procedures previously used for periodic functions of time are equally applicable here. Furthermore, it follows from the third of Eq. (2.5.26) that changing x to $-x$ in the series for $F(x)$ will give $G(x)$, so that

$$\begin{aligned} F(x) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} F_n \exp\left(\frac{in\pi x}{L}\right), \quad F_{(-n)} = F_n^*; \quad x < L \\ G(x) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} F_n \exp\left(-\frac{in\pi x}{L}\right); \quad x > 0 \end{aligned}$$

The characteristic variables for F and G are ξ and η , respectively, and both equal x at $t = 0$. Thus replacing x with the appropriate characteristic yields descriptions of the p_1 and p_2 waves,

$$\begin{aligned} p_1(x, t) &= F(x - ct) = \frac{1}{2} \sum_{n=-\infty}^{\infty} F_n \exp\left(\frac{in\pi}{L}(x - ct)\right) \\ p_2(x, t) &= G(x + ct) = \frac{1}{2} \sum_{n=-\infty}^{\infty} F_n \exp\left(-\frac{in\pi}{L}(x + ct)\right) \end{aligned} \quad (2.5.27)$$

No specification of the range of x and t is indicated in this representation because the series for F and G are always applicable in the physical domain, $0 < x < L$.

The total pressure is the sum of the pressures in the forward and backward waves, so we have

$$p(x, t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} F_n \left[\exp\left(\frac{in\pi x}{L}\right) + \exp\left(-\frac{in\pi x}{L}\right) \right] \exp\left(-\frac{in\pi ct}{L}\right) \quad (2.5.28)$$

Our convention for description of a harmonic oscillation uses a complex exponential with a positive frequency. We can convert the preceding to such a form by setting $n = -n'$. The prime symbol may be deleted after the substitution, and we know that $F_{(-n)} = F_n^*$. The representation of the pressure field becomes

$$p(x, t) = \frac{1}{2} \sum_{n=-\infty}^{\infty} F_n^* \left[2 \cos\left(\frac{n\pi x}{L}\right) \right] \exp\left(\frac{in\pi ct}{L}\right) \quad (2.5.29)$$

Each term for negative n is the complex conjugate of the term for positive n , so the pressure reduces to

$$p(x, t) \equiv F_0 + \sum_{n=1}^{\infty} \operatorname{Re} \left[2F_n^* \exp\left(\frac{in\pi ct}{L}\right) \right] \cos\left(\frac{n\pi x}{L}\right) \quad (2.5.30)$$

This is an extremely important representation of the signal. Consider a specific n in the summation. Because $\cos(n\pi x/L)$ is real, we conclude that the pressure contribution of the n th term consists of an oscillation at frequency $n\pi c/L$, and that the pressure at all locations is in-phase, with an amplitude that is proportional to $\cos(n\pi x/L)$. Each term represents a *standing wave*, which is an oscillation in which the profile maintains a fixed spatial pattern while its magnitude fluctuates. The initial conditions only influence the value of the complex amplitudes F_n , from which it follows that the harmonic time dependence and the cosinusoidal spatial variation of each term are fundamental properties of the waveguide. In fact, we say that the $n\pi c/L$ values are the system's *natural frequencies* ω_n , and the $\cos(n\pi x/L)$ functions constitute its *natural modes* $\phi_n(x)$. Thus, the method of images has led us to the recognition that the signal generated by initial conditions within a closed, rigid-rigid, waveguide is composed of an infinite series of modes whose spatial distribution is an integer multiple of half-cosine segments, with natural frequencies that are integer multiples of the fundamental frequency $\pi c/L$, that is,

If $R_0 = R_L = 1$, then

$$\omega_n = n \frac{\pi c}{L} \text{ and } \phi_n = \cos\left(\frac{n\pi x}{L}\right), \quad n = 0, 1, 2, \dots$$

(2.5.31)

Each natural mode function has the property that $d\phi_n/dx = 0$ at $x = 0$ and $x = L$. According to Euler's equation, this means that each mode's contribution to the particle velocity is such that $dv/dt = 0$ at both ends. Integration of a zero acceleration gives $v = 0$ at $x = 0$ and $x = L$, which is the condition imposed by rigidity. Clearly, the $n = 0$ mode, whose distribution is constant and whose natural frequency is zero, fits this description.

Typical initial profiles for the case where both ends are pressure-release are depicted in Fig. 2.18.

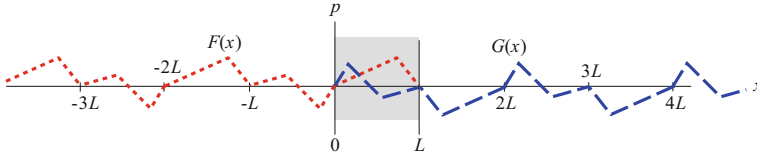


Fig. 2.18 Initial pressure profiles in a finite waveguide whose ends are pressure-release, $R_0 = R_L = -1$

The periodicity and mirror image properties exhibited by this figure are

$$\begin{aligned} F(x - 2L) &= F(x); & x < L \\ G(x + 2L) &= G(x); & x > 0 \\ F(x) &= -G(-x); & x < 0 \\ G(x) &= -F(2L - x), & x > L \end{aligned} \quad (2.5.32)$$

Here too, the fourth property is redundant, because it can be obtained from the first and third properties.

These periodicity conditions lead to a Fourier series that is different from Eq. (2.5.31) for a waveguide with rigid ends. The task in Homework Exercise 2.17 is to demonstrate that the initial condition response in a waveguide whose end are pressure-release is

$$p(x, t) \equiv \sum_{n=1}^{\infty} \operatorname{Re} \left[\frac{2}{i} F_n^* \exp \left(\frac{in\pi ct}{L} \right) \right] \sin \left(\frac{n\pi x}{L} \right) \quad (2.5.33)$$

It follows that the natural frequencies are the same as those in the rigid case, except that there is no zero frequency mode. The natural modes now consist of an integer number of lobes of a sine function, specifically

If $R_0 = R_L = -1$, then

$$\omega_n = n \frac{\pi c}{L} \text{ and } \phi_n = \sin \left(\frac{n\pi x}{L} \right), \quad n = 1, 2, \dots$$

(2.5.34)

The fact that the natural modes are such that $\phi_n = 0$ at $x = 0$ and $x = L$ shows that the solution is consistent with the pressure-release condition at each end.

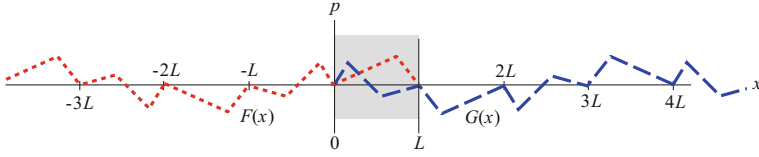


Fig. 2.19 Initial pressure profiles in a finite waveguide in which one end is rigid and the other is pressure-release, $R_0 = 1$, $R_L = -1$

A typical initial profile of $F(x)$ and $G(x)$ for the case where $x = 0$ is rigid and $x = L$ is pressure release appears in Fig. 2.19. Both functions are periodic, but the period now is $4L$. This change relative to the previous configurations stems from the fact that reflection of the wave at the pressure-release end inverts it. Two trips back and forth are required to recover the original sign. The periodicity and mirror image properties now are

$$\begin{aligned}
 F(x - 4L) &= -F(x - 2L) = F(x), \quad x < L \\
 G(x + 4L) &= -G(x + 2L) = G(x), \quad x > 0 \\
 F(x) &= G(-x), \quad x < 0 \\
 G(x) &= -F(2L - x), \quad x > L
 \end{aligned} \tag{2.5.35}$$

Once again, the fourth property is redundant.

The Fourier series that results from these properties are like Eq. (2.5.30), except that the period is doubled and only odd harmonics are nonzero,

$$p(x, t) \equiv \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \operatorname{Re} \left[2F_n^* \exp \left(\frac{in\pi ct}{2L} \right) \right] \cos \left(\frac{n\pi x}{2L} \right) \tag{2.5.36}$$

Here, the image construction leads to the recognition that the modes consist of an odd multiple of quarter-cosine segments, with the natural frequencies being that multiple of the fundamental $\pi c/2L$, that is,

If $R_0 = 1$ and $R_L = -1$, then

$$\omega_n = (2n - 1) \frac{\pi c}{2L} \text{ and } \phi_n = \cos \left(\frac{(2n - 1) \pi x}{2L} \right), \quad n = 1, 2, \dots$$

(2.5.37)

An overview of the three cases where the boundary conditions do not dissipate energy shows that the repetitive nature of the reflection process, which causes the reflected wave to be either a positive or negative mirror image of the incident wave, leads to spatial periodicity of the initial profiles of the forward and backward waves. Such behavior is manifested by representation of the response to any set of initial conditions as a set of natural modes, which represent standing waves patterns, each of

which features a pressure oscillation at a natural frequency. Modal decomposition is an essential feature of the acoustical analysis of many systems, especially enclosures. Such decompositions are usually identified by solving partial differential equations.

When either end is dissipative, there is no spatial periodicity in the initial profile functions $F(x)$ and $G(x)$. This situation was depicted in Fig. 2.16. When a wave leaves one end, then undergoes reflections at the far end and then returns to the end from which it departed, it is attenuated by the product of the reflection coefficients at both ends. This factor applies to the profile in each interval of $2L$. This observation leads to the conclusion that

$$\begin{aligned} F(x) &= R_0 R_L F(x + 2L); \quad x < L \\ G(x) &= R_0 R_L G(x - 2L); \quad x > 0 \end{aligned} \quad (2.5.38)$$

Because of their similarity to the periodicity conditions for nondissipative ends, it would be appropriate to refer the preceding as periodic attenuation conditions. They describe a decay given by

$$\begin{aligned} F(x) &= (R_0 R_L)^{n_F} F_0(x + 2n_F L); \quad x < L \\ G(x) &= (R_0 R_L)^{n_G} G_0(x - 2n_G L); \quad x > 0 \end{aligned} \quad (2.5.39)$$

where $F_0(x)$ and $G_0(x)$ are the functions on the first $2L$ intervals.

General descriptions of the forward and backward simple waves are obtained by replacing x in $F(x)$ and $G(x)$ with the characteristic associated with each function. Furthermore, the physical domain is restricted to $0 < x < L$, so the argument of each function passes to the next interval as t increases. We indicate this behavior by associating n_F and n_G with the time dependence, with the result that Eq. (2.5.39) becomes

$$\begin{aligned} p_1(x, t) &= (R_0 R_L)^{n_F} F_0\left(x - c\left(t - n_F \frac{2L}{c}\right)\right) \\ p_2(x, t) &= (R_0 R_L)^{n_G} G_0\left(x + c\left(t - n_G \frac{2L}{c}\right)\right) \end{aligned} \quad (2.5.40)$$

Note that it is no longer necessary to indicate the range of x and t for which these expressions apply, because our interest now is restricted to $0 < x < L$ and $t > 0$.

Evaluation of p_1 and p_2 at a specified x, t requires knowing what values of n_F and n_G should be used. In Eq. (2.5.39), the integer n_F is selected to maintain the argument of the F_0 function in the range $-L < x < L$, while n_G is selected to maintain the argument of the G_0 function in the range $0 < x < 2L$. Because the range of x in Eq. (2.5.40) is restricted, the arguments of F_0 and G_0 in this equation go out of range if t increases too much. For example in the case of F_0 , we would find that if $ct > L + x$, then $x - ct < -L$. The number of time intervals to subtract from $ct - x - L$ increase as t increases. The argument of G_0 behaves in a similar manner. Both numbers can be determined from the integer function $\text{floor}(u)$, which

is the largest integer that is less than u . The specific expressions are

$$n_F = \text{floor} \left(\frac{ct - x - L}{2L} \right), \quad n_G = \text{floor} \left(\frac{x + ct}{2L} \right) \quad (2.5.41)$$

The combination of Eqs. (2.5.40) and (2.5.41) state that the waveforms at a specific x are obtained by multiplying the pressure in an initial time interval by an attenuation factor. Let us use t' to denote the elapsed time in this interval, so that

$$t' = t - n_F \frac{2L}{c} \quad (2.5.42)$$

Also, if we are willing to ignore the step-wise manner in which the pressure changes, the floor functions for both integers may be approximated as the continuous function $ct/(2L)$. This allows Eq. (2.5.40) to be approximated as

$$\begin{aligned} p_1(x, t) &= \text{sign}(R_0 R_L) (|R_0 R_L|)^{ct/(2L)} F_0(x - ct') \\ p_2(x, t) &= \text{sign}(R_0 R_L) (|R_0 R_L|)^{ct/(2L)} G_0(x + ct') \end{aligned} \quad (2.5.43)$$

where the sign function gives plus one if its argument is positive, and minus one if its argument is negative. The behavior exhibited above is an exponential decay, as can be seen by setting

$$(|R_0 R_L|)^{ct/(2L)} = e^{-\beta t} \quad (2.5.44)$$

This replacement is correct if the decay constant is

$$\beta = -\frac{c}{2L} \ln(|R_0 R_L|) \quad (2.5.45)$$

The corresponding representation of the waveforms of each simple wave at any location is

$$\begin{aligned} p_1(x, t) &= \text{sign}(R_0 R_L) e^{-\beta t} F_0(x - ct') \\ p_2(x, t) &= \text{sign}(R_0 R_L) e^{-\beta t} G_0(x + ct') \end{aligned} \quad (2.5.46)$$

Thus, the waveforms in intervals of duration $2L/c$ may be evaluated by multiplying the waveform in the first interval by the exponential attenuation factor.

In addition to providing a general picture for the decay associated with dissipation, the preceding leads to the concept of *reverberation time* T_{rev} , which is defined as the time interval required for the mean-squared pressure to be attenuated by a factor of 10^{-6} . This corresponds to a decay of 0.001 for the instantaneous pressure amplitude. Equating the exponential factor in Eq. (2.5.44) to 0.001 leads to

$$T_{\text{rev}} = -\frac{6 \ln(10)}{\ln(|R_0 R_L|)} \frac{L}{c} \quad (2.5.47)$$

This form is reminiscent of the Norris–Eyring reverberation time.⁵ Reverberation time is one of the primary factors affecting the acoustical quality of rooms and auditoria. A room with a very large reverberation time hinders speech comprehension, because the sound of a previously spoken word would still be audible when the next word is spoken. In contrast, a very small reverberation time for a symphony hall would make the music seem stifled and dull.

EXAMPLE 2.6 A tube of length L is closed at one end by a resistive liner whose impedance is $\zeta = 2.5$, and the other end is covered by a mylar membrane that is only slightly tensioned, so it effectively is a pressure-release termination. Differential heating of the walls of the tube has created an initial pressure distribution that is parabolic, $p(x, 0) = 4\varepsilon(x/L - x^2/L^2)$, with the fluid at rest. Use the method of wave images as the basis for a graphical construction of spatial profiles of the pressure in the forward and backward propagating waves at $t = 5.25L/c$. Then write a computer program to evaluate spatial profiles of the directly generated waves and their reflected images, as well as temporal waveforms at arbitrarily selected locations within the waveguide. Use this program to verify the spatial profile at $t = 5.25L/c$, and also evaluate the waveform at $x = L/2$.

Significance

This example provides additional practice in the graphical construction of pressure profiles and waveforms and their interpretation. However, quantitative evaluations for complicated initial conditions are best done with computer software. Minor modifications of the program that is developed would provide a general tool to explore other configurations.

Solution

The graphical construction follows the procedure employed in Figs. 2.14, 2.15 and 2.16. We are free to take the pressure-release condition to be either end; we will let it be $x = L$. We set $R_0 = 0.427$ corresponding to $\zeta = 2.5$, and $R_L = -1$. The initial velocity field is zero, so $f(x) = g(x) = 2\varepsilon(x/L - x^2/L^2)[h(x) - h(x - L)]$. Before we construct the initial profiles, we must determine how many images are required. To answer this, we observe that the profile of p_1 at an arbitrary instant t will be shifted in the direction of increasing x by distance ct . For a complete picture, the point in the initial profile at the most negative x must still be in the region $x < 0$ when $t = 5.25L/c$. (Otherwise, it would be as though the p_1 wave was cutoff in $0 < x < L$, even though its images are endless.) If N is the total number of images for p_1 (from both the $f(x)$ and $g(x)$ functions) then the smallest x for our picture of p_1 will be $-NL$. A similar consideration applies to p_2 except distances now are measured beyond $x = L$. Thus, we require that $NL > ct$, so we set $N = 6$.

⁵C.F. Eyring, "Reverberation time in 'dead' rooms," J. Acoust. Soc. Am. **1**, 217–241 (1930). See also A.D. Pierce, *Acoustics* (McGraw-Hill, New York, reprinted by the Acoustical Society of America, Melville, NY), 20–22, 263–265 (1981).

Figure 1 was generated with the aid of drawing software that scaled and joined the various images. Like Fig. 2.14, the first graph describes the images associated with $f(x)$. Images for reflections at $x = L$ are obtained by inverting and flipping the incident image about the vertical axis. (If one does this construction using graphics software, the shape of the new image can be obtained by applying scale factors of -1 for the x direction and $R_L = -1$ for the y direction.) The new image is placed such that its leading edge is at the same distance behind $x = L$ as the distance from the leading edge of the incident wave to that boundary. Each image beyond $x = L$ begets another image behind $x = 0$. These images are obtained by multiplying the incident wave's height by R_0 and flipping the incident image about the vertical axis. (For graphics software the scale factors would be -1 in the x direction and R_0 in the y direction.) The distances for placing these images is based on distances to leading edges measured from $x = 0$. The second graph describes the images associated with the reflections of the waves generated by $g(x)$. In all respects the construction procedure is the same as in the first graph. The third graph assembles both sets of images, thereby forming $p_1(x, 0)$ and $p_2(x, 0)$. The fourth graph obtains the profiles at the desired instant by shifting $p_1(x, 0)$ in the positive x direction and $p_2(x, 0)$ in the negative x direction. The distance for both shifts is ct .

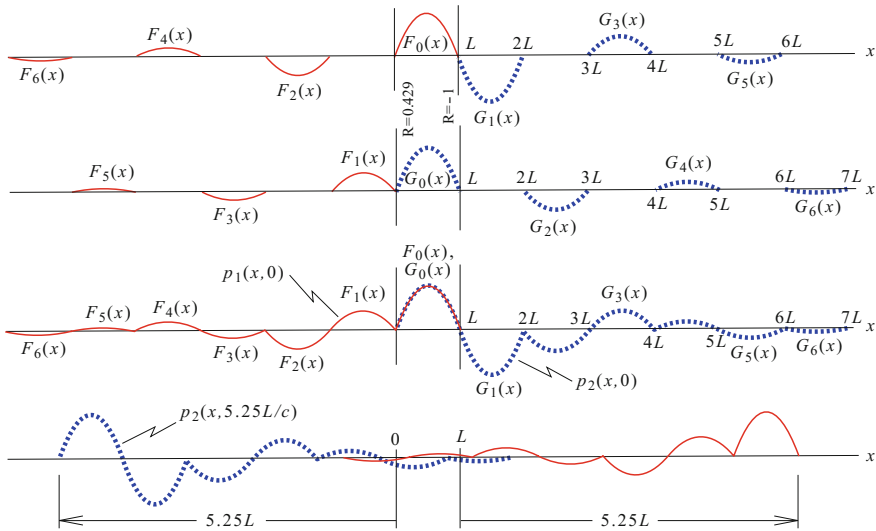


Figure 1. Manual construction of pressure profiles at $ct/L = 0$ and $ct/L = 5.25$.

The computer program we shall create uses nondimensional variables, that is, x represents x/L , t represents ct/L , and p represents p/ε . Thus the only parameters for a computation are R_0 , R_L and the number of images N . Equation (2.5.24) contains four summations. Profiles call for computing these sums for a range of x values at fixed t , whereas waveforms entail their evaluation for a range of t at fixed x . In view of the repetitive nature of this computation it is useful to perform the main computations within a subprogram. In Matlab this can be done with a function M-file, which we will call `p_fg_ic.m`. It returns the four sum values corresponding to input vectors of ξ and

η , the reflection coefficients, and the number of images to be computed. The following listing begins by defining the f and g functions corresponding to the specified initial conditions. These definitions are in the form of anonymous functions in order to evaluate them for various arguments. This is the only part of this subprogram that requires change to handle a different set of initial conditions. After that, the sums are initialized and then accumulated by successive passes through a loop. The specific program is

```
function [p_1_F p_1_G p_2_F p_2_G] = p_f_g_ic(xi,eta,R_0,R_L,N_image)
f_image = @(xi) 2 * (xi - xi.^2).*((xi > 0) - (xi > 1));
g_image = @(eta) 2 * (eta - eta.^2).*((eta > 0) - (eta > 1));
p_1_F=zeros(length(xi)); p_1_G=p_1_F; p_2_F=p_1_F; p_2_G=p_1_F;
for n=0:N_image;
    p_1_F=p_1_F + (R_0*R_L)^n*f_image(xi+2*n);
    p_2_G=p_2_G + (R_0*R_L)^n*g_image(eta-2*n);
    p_1_G=p_1_G + (R_0*R_L)^n*(R_0*g_image(-xi-2*n));
    p_2_F=p_2_F + (R_0*R_L)^n*(R_L*f_image(-eta+2*(n+1)));
end
```

Preceding operators with a period in Matlab implements the operation on an element by element basis.

To evaluate profiles the value of t and a set of x locations are defined in the main program. The waveguide is situated between zero and one in terms of the nondimensional x , and that interval is augmented by the extent of images at either end. Thus we set $x = [-N_image:0.01:N_image+1]$; . The nondimensional characteristics are $xi=x-t$; $eta=x+t$; . This data is the input to the `p_f_g_ic` function. The output `p_1_F`, etc., may be plotted individually, then combined to evaluate the forward and backward waves according to $p_1=p_1_F+p_1_G$; $p_2=p_2_F+p_2_G$; . Addition of these waves yields the total pressure between the boundaries, $p = (p_1+p_2) * ((x > 0) - (x > 1))$; . There is no need to display the individual p_1 and p_2 because they closely match Fig. 1 after compensation for different scaling factors. Figure 2 is the profile that results from evaluating $p = p_1 + p_2$, then eliminating the part that is not in $0 < x < L$.

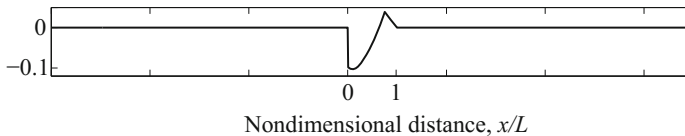


Figure 2. Computer generated profile of total pressure for $0 < x/L < 1$ when $ct/L = 5.25$

Waveforms are obtained by defining a vector of nondimensional time values. The time interval in which the construction is valid begins at $t = 0$ and ends when the last image has transited the waveguide. The computation ceases to be valid when the tail of the most distant image enters the domain of the waveguide. The distance to the tail of the last image on both sides NL , so the longest permissible window for a waveform is $0 \leq t < NL/c$. The corresponding vector of nondimensional

time instants is $t = [0:0.01:N_image];$. The value of x in the nondimensional range $0 \leq x/L \leq 1$ is set in the main program and used to evaluate vectors of characteristic values $x_1 = x - t;$ $x_2 = x + t;$. These vectors are inputs to the `p_fg_ic` function.

The program results are displayed in Fig. 3. The first graph shows p_1 and p_2 versus t at $x = 0$, which are the starting point in the graphical construction of waveforms, but are not needed for computations. Aside from differences in scale factors, this and the other plots are consistent with the waveforms that would be obtained if the graphical method were applied to the initial profiles in Fig. 1. The second graph depicts the pressure in the p_1 and p_2 waves at $x/L = 0.5$. It is evident that relative to the first graph, p_1 is retarded by $ct/L = x/L = 0.5$, and p_2 is advance by that amount. The total pressure is plotted in the third graph. This plot indicates that the reverberation time is very small.

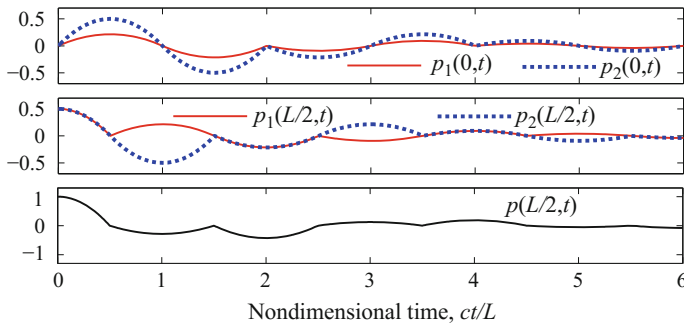


Figure 3. Computer generated waveforms at $x/L = 0$ and $x/L = 0.5$.

Excitation at a Boundary

The response within a waveguide resulting from nonzero initial conditions is not typically a concern. Rather, the important question usually is what pressure waves are induced in a finite length waveguide when the velocity at one end is a specified function of time $\dot{u}(t)$? Before we extend the wave image formulation to address this question, it is useful to contemplate what happens from a physical viewpoint. The boundary at which v is specified is taken to be $x = 0$, which is the same as the setup described by Eq. (2.5.6). Thus, movement of the boundary induces a forward propagating wave. The boundary condition at $x = L$ is an impedance $\rho_0 c \zeta_L$, so Eq. (2.5.19) governs the reflected wave. This is a backward propagating wave, so it returns to $x = 0$ where it is reflected again. The question is: What boundary condition governs this reflection? To answer this we observe that the first part of the p_1 wave, which is the one induced by movement of the boundary, already has a particle velocity at $x = 0$ that equals $\dot{u}$. Hence, there can be no additional contributions to the fluid's particle velocity from the presence of additional waves. In other words, all waves that are incident at $x = 0$ as a result of their prior reflection at $x = L$ see $x = 0$ as being rigid. This means that their reflection is governed by the reflection law in Eq. (2.5.15) with $R_0 = 1$.

These considerations allow us to extend the wave image formulation. The forward propagating wave generated by the boundary motion is described by Eq. (2.5.8), because it is a direct response to the excitation, and it is unaffected by the presence of a far boundary. Hence, the first contribution to the initial spatial profile is

$$F_0(x) = \rho_0 c \dot{u}(-x/c) h(-x) \quad (2.5.48)$$

Successive waves are obtained by constructing images whose profiles alternately satisfy the reflection laws at $x = L$ and $x = 0$. This construction is like that in Fig. 2.16, with two differences: Here, the F_0 function is nonzero only for $x < 0$. The other difference is that all images previously occupied a distance L in the solution to the initial value problem, whereas here F_0 , and therefore all of the images, have lengths of ct_F , where t_F is the largest time at which $\dot{u}$ is nonzero. If $ct_F > 2L$, images will overlap. Such a situation is depicted in Fig. 2.20, where the leading edge of each image is marked by a dot.

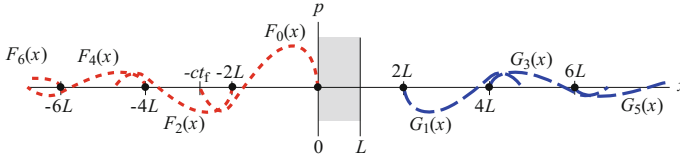


Fig. 2.20 Construction of the initial pressure profiles resulting from movement of the boundary at $x = 0$ in a finite waveguide, $R_L = -0.6$

A mathematical representation of the general response is obtained by following steps like those for the analysis of the incident wave case. We begin by constructing the initial spatial profile. The directly generated wave $F_0(x)$ is reflected at $x = L$, thereby generating a backward propagating wave described by $G_1(x) = R_L F_0(2L - x)$. This wave is reflected at $x = 0$, which generates a reflected forward propagating wave. The reflection factor for waves that are incident at $x = 0$ was shown previously to be $R_0 = 1$ because the directly generated wave already matches the wall motion of the boundary. Thus the first image for the forward propagating wave is $F_2(x) = G_1(-x) = R_L F_0(2L - (-x))$. Reflection of this wave at $x = L$ generates another contribution to the backward propagating wave, which is described by image $G_3(x) = R_L F_2(2L - x) = (R_L)^2 F_0(2L + (2L - x))$, *ad infinitum*. The initial profiles of p_1 and p_2 are the sum of the respective images, so that

$$\begin{aligned} p_1(x, 0) &= \sum_{n=0}^{\infty} (R_L)^n F_0(x + 2nL) \\ p_2(x, 0) &= \sum_{n=1}^{\infty} (R_L)^n F_0(-x + 2nL) \end{aligned}, \quad (2.5.49)$$

To obtain general descriptions for arbitrary x, t we recall that p_1 depends solely on the forward characteristic ξ , and that p_2 depends solely on the backward characteristic η . The preceding expressions apply for $t = 0$, at which time $\xi = x$

and $\eta = x$. Hence, expressions for arbitrary t result from replacing x by ξ in p_1 , while x in p_2 is replaced by η . The next step is to use Eq. (2.5.48) to describe the F_0 function. The occurrence of this function in the expression for p_1 is $F_0(\xi + 2nL) = \rho_0 c \dot{u}(-(\xi + 2nL)/c)$, whereas the term in the summation for p_2 is $F_0(-\eta + 2nL) = \rho_0 c \dot{u}(-(-\eta + 2nL)/c)$. Replacing the characteristics by their definitions in terms of x and t leads to

$$\left. \begin{aligned} p &= p_1 + p_2 \\ p_1 &= \rho_0 c \sum_{n=0}^{\infty} (R_L)^n \dot{u}(t - x/c - 2nL/c) \\ p_2 &= \rho_0 c \sum_{n=1}^{\infty} (R_L)^n \dot{u}(t + x/c - 2nL/c) \end{aligned} \right\}, \quad 0 < x < L, \quad t > 0 \quad (2.5.50)$$

This representation tells us that successive images that contribute to p_1 and p_2 depend on a retarded time that is the true time less the time required to propagate the leading edge of each image to position x .

This representation is useful for computer evaluations, but we can learn much from the pictorial representation of the initial spatial profile. One general feature already is apparent. Figure 2.20 depicts the situation for a pulse excitation. We know that each reflection at $x = L$ attenuates the amplitude of the reflected wave by the factor R_L . Even though individual images overlap, the result is that the both p_1 and p_2 decrease with increasing t . The manner in which this decrease occurs is the same as for the case where initial conditions are specified. Consequently, unless $x = L$ is rigid or pressure-release, the overall p_1 and p_2 waves essentially decay exponentially at large t , with the decay constant being β in Eq. (2.5.45). A corollary is that the notion of a reverberation time and the associated Norris–Eyring formula, Eq. (2.5.47), are equally applicable here. This is physically what should happen, because it would not make sense that sound produced by movement of the boundary behaves differently from sound that was generated prior to $t = 0$.

These assertions are only true if the excitation $\dot{u}$ at the boundary ceases at $t > t_F$. Such an excitation is said to be *transient*. A continuous excitation goes on forever, or at least, later than the time at which we cease our observations. A special case is a periodic excitation, which can produce some interesting phenomena. In Chap. 1 we found that Fourier series and transfer functions may be used to determine the response to such excitations. Here we will use the method of wave images. We consider a wall motion $\dot{u}$ that is periodic in time for $t > 0$, such that $\dot{u}(t + T) = \dot{u}(t)$. Because the image $F_0(x)$ directly generated by this type of motion extends to $x \rightarrow -\infty$, all subsequent images associated with reflection of the forward and backward propagating waves also extend to $|x| \rightarrow \infty$, so the number of images that overlap will increase as we look at smaller $x < 0$ for the forward propagating waves, or at larger $x > L$ for the backward waves. This will make it difficult to pick out distinctive features in a single image. An aid to doing so is to use a very simple form for $\dot{u}$, specifically, a scalene triangle lasting for an interval τ less than a period, followed by an interval of zero. The one to be used here is

$$\dot{u} = \dot{u}_{\max} \left[3 \frac{t}{\tau} h(t) - \frac{9}{2} \left(\frac{t}{\tau} - \frac{1}{3} \right) h \left(t - \frac{\tau}{3} \right) + \frac{3}{2} \left(\frac{t}{\tau} - 1 \right) h(t - \tau) \right], \quad 0 \leq t < T \quad (2.5.51)$$

where $\dot{u}_{\max}$ is the amplitude of this motion. The value of τ for all evaluations will be $0.75T$.

The directly generated initial profile is

$$F_0(x) = \rho_0 c \dot{u}(-x/c) h(-x) \quad (2.5.52)$$

This profile is spatially periodic in $x < 0$, with its pattern repeating over distance cT . The spacing between images for successive reflections matches the arrival times for the fronts of the incident and reflected waves, so it is $2L$ behind each boundary. This suggests that a crucial issue is whether cT equals $2L$. It does not for Fig. 2.21a. Successive images grow smaller because $|R_L| < 1$. The interference between images is not constructive because their spatial spacing is $2L$ but their individual wavelength is $cT = 1.42L$. These observations suggest that the amplitude of the total pressure in the steady state condition will be comparable to that of the directly generated wave, which is $\rho_0 c \dot{u}_{\max}$.

The situation in Fig. 2.21b, for which $cT = 2L$, is quite different. Behind their front, all forward propagating images are in-phase with $F_0(x)$, and all backward propagating images behind their front are in-phase with $G_0(x)$. Even though successive images are attenuated, Fig. 2.21b suggests that the combined amplitude will be substantially larger than the amplitude $\rho_0 c \dot{u}_{\max}$ of the directly generated wave. We can predict the amplitudes of p_1 and p_2 without constructing profiles. A $2L$ increase in $|x|$ adds another image to each profile. For the forward propagating wave, in the interval $-2(n+1)L < x < -2nL$ there is $F_0(x)$ plus $n-1$ images associated with reflections at $x = 0$. Because $x = 0$ is effectively rigid for reflections, each of these images is attenuated by a factor R_L relative the previous image, so that the j th image is $(R_L)^j F_0(x) = (R_L)^j \rho_0 c \dot{u}(-x/c)$ in this interval. It follows that the initial profile of p_1 is such that

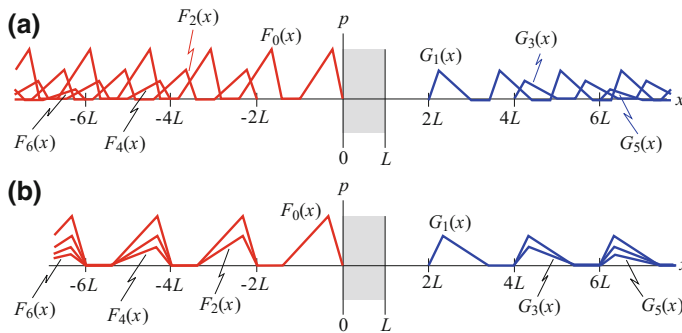


Fig. 2.21 Initial spatial profiles in a finite waveguide resulting from a periodic vibration of the boundary at $x = 0$, $R_L = 0.6$. **a** $T = 1.42L/c$, **b** $T = 2L/c$

$$p_1(x, 0) = \left[\sum_{j=0}^n (R_L)^j \right] \rho_0 c \dot{u}(-x/c), \quad -2(n+1)L < x < -2nL \quad (2.5.53)$$

A general expression for p_1 results when x is replaced in the above with the characteristic associated with forward propagation, $\xi = x - ct$, which gives

$$p_1(x, t) = \left[\sum_{j=0}^n (R_L)^j \right] \rho_0 c \dot{u}\left(t - \frac{x}{c}\right), \quad -2(n+1)L < x - ct < -2nL \quad (2.5.54)$$

The steady-state response may be obtained by letting $n \rightarrow \infty$ in the above expression. In this limit the summation is the binomial series expansion of $1/(1 - R_L)$, so that

$$\lim_{t \rightarrow \infty} p_1(x, t) = \frac{1}{1 - R_L} \rho_0 c \dot{u}\left(t - \frac{x}{c}\right), \quad 0 < x < L, \quad t > 0 \quad (2.5.55)$$

The images forming the initial profile of p_2 behave in a similar manner. As exhibited by Fig. 2.21b, there is no direct wave, and the first image of $F_0(x)$ is $G_0(x) = R_L F_0(2L - x) = R_L \rho_0 c \dot{u}(-(2L - x)/c)$. Each subsequent image multiplies the previous image by R_L and shifts it by distance $2L$ farther behind $x = L$. Thus, we find that the backward propagating waves initial profile is given by

$$p_2(x, 0) = \left[\sum_{j=1}^n (R_L)^j \right] \rho_0 c \dot{u}\left(\frac{x - 2L}{c}\right), \quad 2nL < x < 2(n+1)L, \quad n > 0 \quad (2.5.56)$$

Replacement of x in the initial profile of p_2 with $\eta = x + ct$ leads to yields the general description,

$$p_2(x, 0) = \left[\sum_{j=1}^n (R_L)^j \right] \rho_0 c \dot{u}\left(t + \frac{x - 2L}{c}\right), \quad 2nL < x + ct < 2(n+1)L, \quad n > 0 \quad (2.5.57)$$

Aside from the absence of the $n = 0$ term, this summation is the same as that for the p_1 profile. Thus, as $n \rightarrow \infty$, the summation approaches $1/(1 - R_L) - 1 \equiv R_L/(1 - R_L)$.

$$p_2(x, t) \rightarrow \frac{R_L}{1 - R_L} \rho_0 c \dot{u}\left(t + \frac{x - 2L}{c}\right), \quad 0 < x < L, \quad t > 0 \quad (2.5.58)$$

This analysis is confirmed by computations of waveforms. Figure 2.22 depicts the waveforms of p_1 and p_2 at $x = 0$, and the waveform of the total pressure at $x = L/2$. The period for this case is $cT = 1.42L$, which is not expected to lead to

significant enhancement of the pressure. The steady-state amplitudes of p_1 and p_2 are substantially less than $\rho_0 c \dot{u}_{\max}$. The pressure amplitude obtained from their sum at $x = L/2$ is approximately $\rho_0 c \dot{u}_{\max}/2$. The oscillation is relative to a mean value that results from $\dot{u}$ having a nonzero mean.

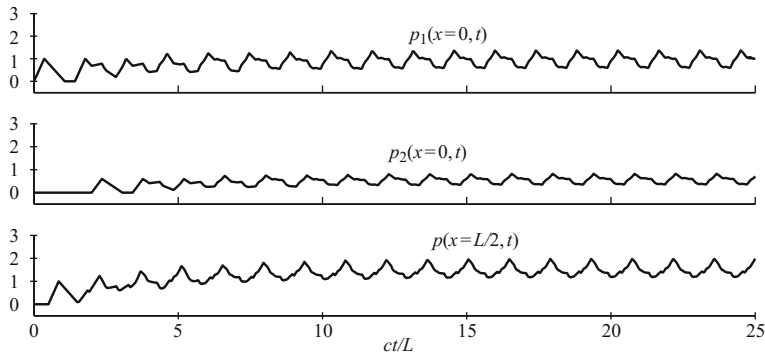


Fig. 2.22 Waveforms in a finite waveguide generated by vibration of the boundary at $x = 0$ when $T = 1.42L/c$, $R_L = 0.6$

The situation for $cT = 2L$ is described by Fig. 2.23. The waveforms of p_1 and p_2 attain their steady-state value in the vicinity of $ct = 12L$, and the steady-state amplitudes of each agree with Eqs. (2.5.55) and (2.5.58). The result is a much larger amplitude for the pressure at $x = L/2$.

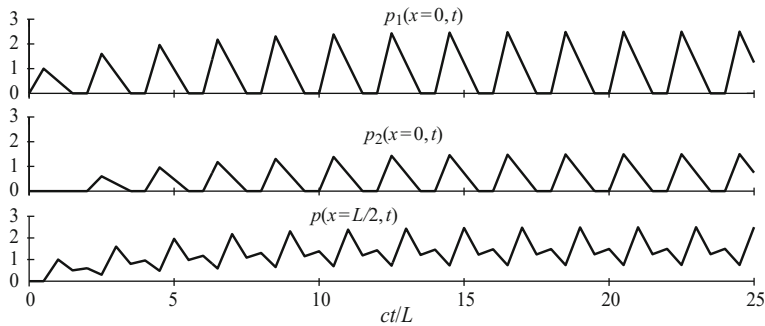


Fig. 2.23 Waveforms in a finite waveguide generated by vibration of the boundary at $x = 0$ when $T = 2L/c$, $R_L = 0.6$

An interesting question now arises. Suppose $R_L = 1$ and $cT = 2L$. According to Eqs. (2.5.55) and (2.5.58), this means that the late time versions of p_1 and p_2 will be infinite. Can this be correct? Before we view computed waveforms, let us consider how letting $R_L = 1$ would change Fig. 2.21b. The image for every reflected wave would be situated as it is in this figure, except but all images would be the

same as that of $F_0(x)$ for the directly generated wave. Thus, the spatial profiles of p_1 and p_2 would grow with increasing $|x|$ as more and more images are added. This means that the waveforms of these waves would grow in magnitude essentially linearly with increasing t . This line of reasoning leads us to the conclusion that the waves will approach infinite amplitude, but only as $t \rightarrow \infty$. Figure 2.24 confirms this expectation.

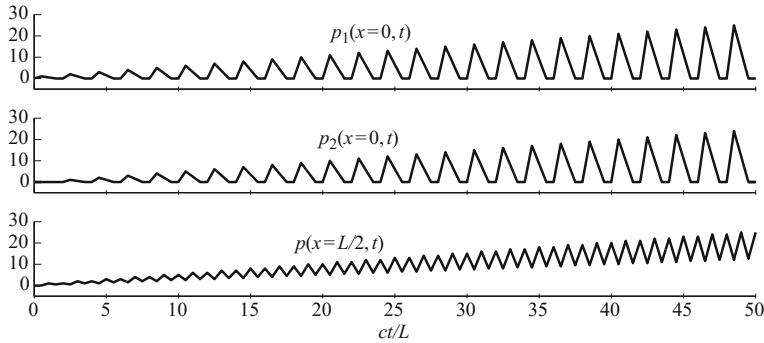


Fig. 2.24 Waveforms in a finite waveguide generated by vibration of the boundary at $x = 0$ when $T = 2L/c$, $R_L = 1$

Of course, this is a highly idealized situation. If the pressure amplitudes were to grow to large levels, nonlinear effects would become significant, thereby invalidating the entire analysis. Furthermore, no energy is dissipated in the model of a rigid wall, which obviously is unrealistic. If R_L is extremely close to one, the denominator in Eqs. (2.5.55) and (2.5.58) will be finite, but very small. In that case the late time versions of p_1 and p_2 will have a very large, but finite, steady-state amplitude. The situation where R_L is close to or equal to one and $cT = 2L$ is said to be a *resonance*. We previously encountered this phenomenon in Example 1.1 of Chap. 1. There the system had a single natural frequency, and the excitation was periodic. Resonance occurred if the frequency of any of the Fourier harmonics matched the system's natural frequency. In the case of a finite length waveguide with forced motion at $x = 0$, waves that are incident at that end perceive the reflection coefficient to be $R_0 = 1$. The specific system we have considered has $R_L = 1$, so its analog for an initial value problem is a waveguide with rigid ends. Hence, the natural frequencies of this system are given by Eq. (2.5.31). The fundamental (that is, lowest) natural frequency, is $\pi c/L$. The fundamental frequency of $\dot{u}$ is $2\pi/T$. Thus, the condition $cT = 2L$ corresponds to an unabated system resonance. If we reduce R_L slightly, some energy is removed in each reflection at $x = L$, thereby inhibiting unlimited growth.

A corollary of this discussion is that resonance can be expected to occur in systems for which R_L is close to one whenever the frequency of any harmonic in the Fourier series for $\dot{u}$ matches any natural frequency. That is, a resonance will occur if there are two integers j and n for which $j(2\pi/T) = n(\pi c/L)$. These observations lead to the question of whether resonance is unique to the case where $x = L$ is rigid? The answer is no! Its occurrence requires that there be little dissipation, which is satisfied if $|R_L|$

is close to one. Thus, the pressure-release condition also can lead to resonance. The effective reflection coefficient at $x = 0$ is $R_0 = 1$ because the wall motion is specified, so setting $R_L = -1$ yields the rigid/pressure-release configuration described by Eq. (2.5.37). Like the behavior of the system in the previous case, resonance of this system occurs if any harmonic of periodic $\dot{u}$ has a period that matches one of the system's natural frequencies, that is, if $j(2\pi/T) = (2n - 1)\pi c/2L$. The lowest resonance corresponds to $j = n = 1$, which gives $cT = 4L$. Another configuration that can exhibit resonance is $R_0 = R_L = -1$. Motion at $x = L$ cannot be imposed in this case, but a resonance could be attained if the excitation at $x = 0$ were a specified pressure as a function of time.

In most situations resonance is a phenomenon to be avoided. For example, in a symphony hall instruments that play notes in the vicinity of resonant frequencies will be too loud relative to other instruments. Furthermore, these instruments will not sound the way they should, because one or more of the harmonics they generate will be too large relative to the others. However, the resonance phenomenon can be useful. It is essential to the response characteristics of every musical instrument, and it is exploited to make some highly efficient transducers.

EXAMPLE 2.7 A waveguide, whose length is L , is filled with water. The specific impedance at one end is ζ , while the other end executes a periodic square wave vibration in which the velocity is $v(t) = \varepsilon c$ if $0 < t < T/2$, $v(t) = -\varepsilon c$ if $T/2 < t < T$, $v(t) = v(t - T)$ if $t > T$. Parameters for the system are $L = 2$ m, $\rho_0 = 1000$ kg/m³, $c = 1480$ m/s, and $\varepsilon = 10^{-5}$. (a) Determine the values of T for which the pressure within the waveguide will exhibit a resonance. (b) Write a computer program to evaluate waveforms at any x . (c) Use the computer program to evaluate the waveform at $x = 0$. Cases of interest are $T = 4L/(3c)$ and $\zeta = 0$; $T = 4L/(3c)$ and $\zeta = 0.02$; $T = (0.99)4L/(3c)$ and $\zeta = 0$. For each T and ζ combination examine the waveform to determine the time at which the pressure excursion, positive or negative, will first exceed 400 Pa.

Significance

The computer program that we will develop may be used as an independent tool for the exploration of a variety of phenomena. Its application here will provide a more detailed picture of how the excitation frequency and dissipation affect the severity of resonances.

Solution

Part (a) In each case the impedance at $x = L$ is very small, so the end is effectively pressure-release. Because the velocity at $x = 0$ is specified, that end is rigid for considerations pertaining to natural frequencies and modes. Hence, we can anticipate that resonance will occur when the any multiple of the excitation frequency equals any natural frequency of a rigid/pressure-release waveguide whose length is L . Equation

(2.5.37) indicates that the natural frequencies of a waveguide with $R_0 = 1$ and $R_L = -1$ are $\omega_n = (2n - 1) \pi c / (2L)$. The square wave is periodic in intervals of T , so its fundamental frequency is $2\pi/T$. The frequencies of the harmonics in its Fourier series are $j (2\pi/T)$. Equating one of each type of frequency gives

$$\frac{2\pi j}{T} = \frac{(2n - 1) \pi c}{2L} \implies \frac{cT}{L} = \frac{4j}{(2n - 1)}$$

The first harmonic of the excitation ($j = 1$) matches a natural frequency of the waveguide if $T = 4L/c, 4L/(3c), \dots = 5.405, 1.802, \dots$ ms. Furthermore, the j th harmonic matches a natural frequency if T is j times any of these values. The period that is specified is either equal or close to $4L/(3c)$. Thus, the specified period corresponds to the first harmonic of the excitation resonantly exciting the second mode of the waveguide.

The preceding resonance criterion applies for an arbitrary periodic $\dot{u}$, but another condition must be met. The preceding resonance condition assumes that the j th Fourier series coefficient of $\dot{u}$ is nonzero. In the case of the specified square wave, the Fourier coefficients for even j are identically zero. Thus, resonances occur only if the first, third, fifth, *etc.* harmonic of $\dot{u}$ matches the frequency of any mode.

Part (b) The issue here is computerized evaluation of waveforms, so there is no need to determine the initial profile from which the graphical procedure begins. It will be necessary to compute $\dot{u}$ for many different sets of arguments, and the computation is best done in a two step process. A separate function is most suitable to this purpose. The square wave is periodic, with the first half of each period being $\dot{u} = \varepsilon c$ and the second half being $\dot{u} = -\varepsilon c$. The argument at which $\dot{u}$ is to be evaluated might fall into any period. It is placed in an interval $0 \leq t' < T$ by employing the modulus function, whose definition is $\text{mod}(a, b) \equiv a - b \text{ floor}(a/b)$. Also, we must set $\dot{u} = 0$ if $t < 0$. In Matlab, these operations may be implemented as a function M-file, `u_dot.m`, whose code is

```
function v_boundary=u_dot(t,T_period)
t_prime=mod(t, T_period);
v_boundary=(t >= 0).*((t_prime > 0)-2*(t_prime > T_period/2));
```

Let us temporarily forego consideration of the number of images that must be incorporated to the analysis. The basic algorithm for evaluating p_1 and p_2 is like the one for an initial value problem in the previous Example. Here the task is to evaluate a single waveform corresponding to each set of parameters, so all operations can be done in the main program. The two sums in Eq. (2.5.50) are computed in a loop. In the following program fragment the variable `tau_1` is $t - x/c$ and `tau_2` is $t + x/c$. The summation for p_1 is initialized with the $n = 0$ term, and the loop increments the sum for each $n = 1, \dots, N$. Similar steps evaluate p_2 . There is no $n = 0$ term in the summation for p_2 , so it is initialized by a set of zeros. Matlab code that performs these operations is:

```

tau_1=t-x/c; tau_2=t+x/c;
p_1=eps*rho*c*u_dot(tau_1,T); p_2=0*p_1;
for n=1:N
    p_1=p_1+eps*rho*c*R_L^n*u_dot(tau_1-2*n*L,T_period);
    p_2=p_2+eps*rho*c*R_L^n*u_dot(tau_2-2*n*L,T_period);
end

```

Prior to arriving at this program fragment, a vector of values of t for the waveform, as well as the values of x , R_L and T , must be defined.

The maximum time for computation of waveforms is specified indirectly by the requirement that we should determine the time at which the pressure rises to 400 Pa. Let this time be $t_{\max}$, and let the time increment be Δt . The latter should be sufficiently small to represent the discontinuities of the square wave. The results that follow were obtained with $\Delta t = T/1000$. The instants at which the waveform will be evaluated are $t_j = j\Delta t$. The number of images N must be sufficiently large that $t_{\max}$ for evaluation of a waveform is less than the time required for the leading edge of image $N + 1$, which is the first to be omitted, to arrive at the waveguide. The leading edge of image n for p_1 is at $2nL$ behind $x = 0$, whereas image n for p_2 is $(2n - 1)L$ beyond $x = L$. Thus, we require that $2(N + 1)L/c > t_{\max}$, which means that $N > ct_{\max}/(2L) - 1$. Trial and error was used to identify the value of $t_{\max}$ for which the pressure reaches the 400 Pa. Eventually, it was extended in order to exhibit some interesting effects.

Part (c) The reflection coefficient is $R_L = (\zeta - 1)/(\zeta + 1)$, so the three parameter combinations each correspond to R_L being close to minus one. The first combination, $T = 4L/(3c)$ and $\zeta = 0$, corresponds to an uncontrolled resonance because R_L is exactly minus one. Consequently, the images are not attenuated. The result is Fig. 1. It can be seen that the pressure at $x = 0$ grows in amplitude linearly with elapsed time. The instant when $|p|$ at $x = 0$ first exceeds 400 Pa may be found by scanning the output, which is more accurate than identifying that value in the graph. The result is 37.85 ms.

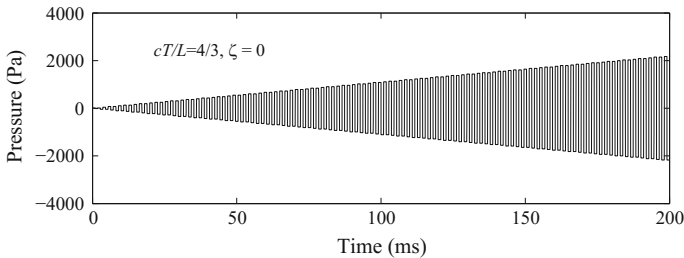


Figure 1.

Another case where the fundamental frequency of $\dot{u}$ exactly equals the second natural frequency is described by Fig. 2, for which $\zeta = 0.02$. The reflection coefficient is $R_L = -0.961$, so successive images are attenuated. Initially, the pressure amplitude increases, but at a slower rate than it does in Fig. 1. The pressure first

surpasses 400 Pa at $t = 52.26$ ms. Eventually, the amplitude approaches a steady value.

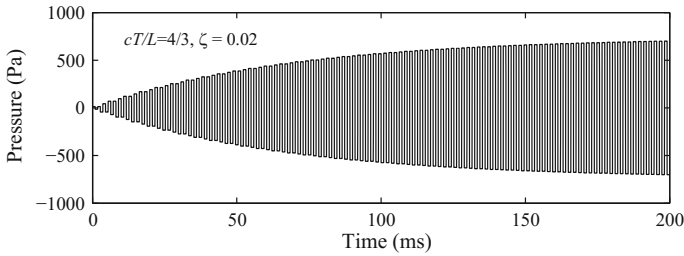


Figure 2.

The behavior when $R_L = -1$ and a harmonic of $\dot{u}$ is very close to, but not equal to, a natural frequency is unlike either of the previous cases. Indeed, even experienced individuals find it surprising. Figure 3 shows a waveform that is reminiscent of a beating signal. Indeed, this is the process, but proving it requires a different line of analysis. The beating occurs between many harmonics of $\dot{u}$ and many modes. Note in this regard that because $cT/L = 4/3$, the resonance criterion identified above states that resonances occur if $3j = 2n - 1$. Recall that j must be odd because the even harmonics of $\dot{u}$ are zero. Thus, the resonances are mode 2 by harmonic 1, mode 5 by harmonic 3, mode 8 by harmonic 5, and so on. The fact that the envelope of the beat is diamond shaped is a consequence of the simultaneous occurrence of these multiple resonances. If $\dot{u}$ were a sine function at the same pressure, the envelope would also be a sine. A 400 Pa pressure is first reached 38.21 ms, which is only slightly later than the value for Fig. 1.

An overview of these three cases suggests methods to limit acoustic resonances. One possibility is to cut off the excitation before an unacceptable state is attained. However, the pressure builds up quickly, so one must be prepared to act before the pressure level is unacceptable. If doing so is not feasible, the only recourse is to alter the parameters. The easiest alteration is to shift the excitation's period sufficiently to reduce the steady-state pressure amplitude to an acceptable value. As evidenced by Fig. 3, the detuning of the excitation frequency relative to a natural frequency must be reasonably substantial. However, in some situations the excitation's frequency is

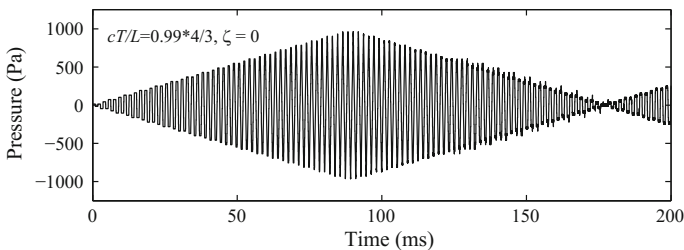


Figure 3.

mandated by other considerations. In that case raising the amount of dissipation is the only recourse. This is done in general by using dissipative liner materials, like acoustical ceiling tiles, within the waveguide.

2.6 Analogous Vibratory Systems

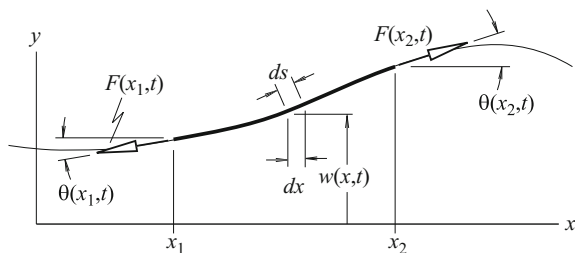
It was mentioned previously that the wave equation is not unique to acoustics. Two occurrences as the governing equation of vibratory systems are of particular interest because in some situations acoustic response in a fluid is intertwined with their vibration. A cable that is prestretched by a large tensile force, and an elastic bar provide accessible prototypes for investigating the interaction of acoustic waves in fluids and the vibration of solid bodies. It therefore is appropriate that we understand the commonality of these systems, which superficially appear to be quite dissimilar. Identification of the analogies between these systems and acoustic signals in a waveguide will provide a global perspective for the wave equation.

2.6.1 Stretched Cable

Field Equations

Linear vibration theory for solid media is like linear acoustics, in that both rely on some core assumptions and approximations. One difference in the approach to the derivation is it is possible to track the particles of a solid object, for example, by marking points with ink. Figure 2.25 shows a segment of a cable bounded by two interior locations x_1 and x_2 that are arbitrarily selected. The cable is assumed to displace solely in the transverse y direction, which is the first primary assumption on which the derivation lies. We seek an equation of motion governing the displacement field $w(x, t)$ measured from the x axis where the cable is situated when it is in static equilibrium. The situation addressed here is one in which the cable carries a very large tensile force, even if it is at static equilibrium. The angle θ locates the tangent to the curve formed by plotting w as a function of x at an arbitrary t .

Fig. 2.25 Isolated segment of a stretched cable showing the internal tensile force



We begin with the time derivative of the momentum-impulse equation for the system of particles of which the isolated cable segment is composed. Isolating the cable segment in Fig. 2.25 exposes the tensile force F applied by the other portions of the cable. Allowance is made for the tensile force to be variable, but we will soon see that the actual situation is much simpler. Gravity is excluded from the formulation, which can be justified either on the basis that the motion occurs in the horizontal plane, or more generally by recognizing that gravity has a static effect that would merely add a time-invariant displacement that is known as the sag. In regard to inertial effects, the mass per unit length of the cable is the constant value μ . The infinitesimal arclength ds is measured along the cable, so dx is the projection of ds onto the x axis. Thus, $ds = dx / \cos(\theta)$ and the differential element of mass is $\mu dx / \cos(\theta)$.

The acceleration of a differential element is $\partial^2 w / \partial t^2$ in the y direction. The acceleration in the x direction is zero, so there must be a force balance in that direction,

$$\sum F_x = [F \cos(\theta)]|_{x_2} - [F \cos(\theta)]|_{x_1} = 0 \quad (2.6.1)$$

The second assumption is introduced here. Specifically, the maximum of the displacement is taken to be much smaller than the distance between the cable's ends. In that condition a profile of w as a function of x would show that the cable everywhere is close to the x axis, so θ is much less than one in magnitude. Consequently, we may use

$$\cos(\theta) \approx 1, \quad \sin(\theta) \approx \theta \quad (2.6.2)$$

The approximation for $\sin \theta$ will soon be used for the y momentum equation. The approximation for $\cos(\theta)$ converts the force balance in the x direction to a statement that $F(x_2, t) = F(x_1, t)$. Because x_1 and x_2 were selected arbitrarily, it must be that the same tensile force occurs at all locations at any instant. The third assumption simplifies the situation further. It was postulated at the outset that the cable carries a large tensile force F_0 when it is situated in its undisplaced configuration along the x axis. Such a large force would stretch the cable considerably before any displacement occurs. A displacement field $w(x, t)$ would stretch the cable further, but we have restricted our considerations to situations where $w(x, t)$ is very small. In that case the additional stretch is also small, so we can take the tensile force at any instant to be unchanged from the force within the cable at static equilibrium,

$$F(x, t) \approx F_0 \quad (2.6.3)$$

We use this approximation to formulate the y momentum equation, which is

$$[F_0 \sin(\theta)]|_{x_2} - [F_0 \sin(\theta)]|_{x_1} = \int_{x_1}^{x_2} \left(\mu \frac{dx}{\cos(\theta)} \right) \frac{\partial^2 w}{\partial t^2} \quad (2.6.4)$$

Equation (2.6.2) yields a linearized version of this equation of motion. We also invoke the one-dimensional divergence theorem to convert the terms at x_1 and x_2 to an integral, specifically,

$$[F_0 \sin(\theta)]|_{x_2} - [F_0 \sin(\theta)]|_{x_1} \approx [F_0 \theta]|_{x_2} - [F_0 \theta]|_{x_1} = \int_{x_1}^{x_2} F_0 \frac{\partial \theta}{\partial x} dx \quad (2.6.5)$$

The result of these operations is

$$\int_{x_1}^{x_2} \left[F_0 \frac{\partial \theta}{\partial x} - \mu \frac{\partial^2 w}{\partial t^2} \right] dx = 0 \quad (2.6.6)$$

The integral will vanish for any combination of x_1 and x_2 only if the integrand vanishes. The result is

$$\boxed{F_0 \frac{\partial \theta}{\partial x} - \mu \frac{\partial^2 w}{\partial t^2} = 0} \quad (2.6.7)$$

Another equation is needed to eliminate θ . For an acoustic signal, Euler's equation is supplemented by the continuity condition. The latter is a kinematical requirement that prevents the occurrence of gaps. Similarly, θ is eliminated by a kinematical condition. If θ were discontinuous, there would be a kink, that is a sharp bend, in the shape of the $w(x, t)$ curve. This possibility is eliminated by recognizing that θ is the angle of the tangent to the $w(x, t)$ curve from the x axis, but that angle can be found from the slope $\partial w / \partial x$. Thus,

$$\tan(\theta) \approx \theta = \frac{\partial w}{\partial x} \quad (2.6.8)$$

Substitution of this relation into the momentum-impulse equation leads to the field equation we seek,

$$\boxed{F_0 \frac{\partial^2 w}{\partial x^2} - \mu \frac{\partial^2 w}{\partial t^2} = 0} \quad (2.6.9)$$

This is the one-dimensional wave equation for the vibration of a cable.

Analogies

The process of identifying how the equations governing motion of a stretched cable are analogous to acoustic relations entails comparison of similar types of equations. Our first step compares the momentum equation, Eq. (2.6.7), and Euler's equation, which may be written as

$$\frac{\partial p}{\partial x} + \rho_0 \frac{\partial v}{\partial t} = 0 \quad (2.6.10)$$

Because $\partial w/\partial t$ is a velocity, the two have analogous forms, with $F_0\theta$ being like $-p$ if we consider μ to be analogous to ρ_0 . (The difference in sign may be traced to the definition of F_0 as a tension, whereas positive p is compressional.)

The analogies for the continuity equation are recognizable by differentiating Eq. (2.6.8) with respect to t , then multiplying the equation by F_0 . The relation for acoustics is given by Eq. (2.2.11). The comparable forms are

$$\frac{\partial}{\partial t} (F_0\theta) - F_0 \frac{\partial}{\partial x} \left(\frac{\partial w}{\partial t} \right) = 0 \iff \frac{\partial p}{\partial t} + \rho_0 c^2 \frac{\partial v}{\partial x} = 0 \quad (2.6.11)$$

The analogy of $-F_0\theta$ to p holds if we identify F_0 as the bulk modulus $K = \rho_0 c^2$. This also reinforces identification of the transverse velocity of the cable, $\partial w/\partial t$, as the analog to the fluid particle velocity v . The analogies we have established are summarized in the follow table.

Acoustic fluid	p	v	K	ρ_0
Flexible cable	$-F_0 (\partial w/\partial x)$	$\partial w/\partial t$	F_0	μ

A basic difference between the one-dimensional wave equation for a cable and for a fluid is that the former governs displacement, whereas the latter governs a force-like quantity. Either equation is readily transformed to the type of variable for the other equation. For example, if we differentiate Eq. (2.6.9) with respect to x , then multiply it by F_0 , we obtain a wave equation governing the transverse force component, $F_0 (\partial w/\partial x)$. Comparing that equation to the equation for a fluid leads to

$$\frac{\partial^2}{\partial x^2} \left(F_0 \frac{\partial w}{\partial x} \right) - \frac{\mu}{F_0} \frac{\partial^2}{\partial t^2} \left(F_0 \frac{\partial w}{\partial x} \right) = 0 \iff \frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \quad (2.6.12)$$

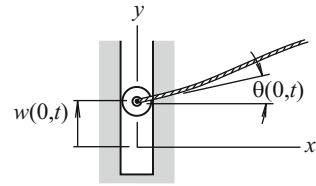
These forms are similar. The phase speed of transverse waves is found by matching the equations, which leads to

$$c = \left(\frac{F_0}{\mu} \right)^{1/2} \quad (2.6.13)$$

Boundary conditions describe the manner in which the cable is supported at its end. The physical arrangement must be such that the axial tension is sustained. This requirement is inherently met for a fixed end, at which the cable is anchored to an immovable object, like a wall. An anchor prevents displacement. Because $w = 0$ for all t at that end, it must be that $\partial w/\partial t = 0$, which is analogous to a rigid termination in an acoustic waveguide. It also is possible that the end is shaken in the transverse direction while the tension is provided by pulling the end in the axial direction. This would correspond an acoustic waveguide in which the particle velocity is specified at an end.

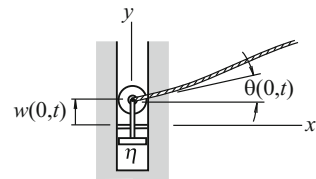
Other types of boundary conditions are more difficult to obtain. This is so because they must allow for transverse movement of the end concurrently with resisting the axial force. The analogy of a pressure-release end in a waveguide is $F_0\theta = 0$. A realization of this condition can be obtained with a roller that moves within a groove that is aligned transversely. This is the configuration depicted in Fig. 2.26. The string is tied to the axle of the roller. The displacement of the roller from the reference position is $w(\xi = 0, t)$ and the slope is $\theta(0, t)$. Based on the assumption that the roller's inertia is negligible, the vertical component of the force exerted by the cable on the roller must be zero. In other words, $F_0\theta = 0$ at this end.

Fig. 2.26 A roller support at the end of a cable that produces the analog to a pressure-release boundary condition



A small modification of the arrangement in Fig. 2.26 leads to an impedance boundary condition. In an acoustical waveguide this condition is attained if the pressure applied to a wall produces a proportional velocity. The analog for a cable is a transverse velocity that is proportional to the vertical component of the tensile force. The nonzero force component may be obtained by attaching a viscous dashpot to the roller. Such a configuration is depicted in Fig. 2.27 for the end at $x = 0$, wherein the end of the cable is tied to the center of a roller.

Fig. 2.27 A dashpot whose viscous resistance is η leads to an impedance boundary condition for a stretched cable



The movement of the roller is opposed by the dashpot, whose resistance (force/velocity) is denoted as χ . Upward movement of the roller is $w(x = 0, t)$. The force exerted by the dashpot on the roller is $\chi \partial w(0, t) / \partial t$, downward if the velocity is upward. If the inertia of the roller is negligible, the cable must exert an upward force that balances the force applied by the dashpot. The resulting boundary condition is $\chi(\partial w / \partial t) = F_0\theta$ at $x = 0$. In an acoustical waveguide, a specific impedance ζ at $x = 0$ leads to $p = -\rho_0 c \zeta v$ at that end. That this is analogous to the condition in the cable is evident by the analogy of p to $-F_0\theta$. The dashpot resistance χ is analogous to $\rho_0 c \zeta$. If χ is infinite, it must be that $\partial w / \partial t = 0$ at that end for all t , which leads to $w = 0$ at that end. Thus, any boundary condition governed by a velocity impedance may be written as

$$\chi_0 \frac{\partial w}{\partial t} = F_0 \frac{\partial w}{\partial x} \text{ at } x = 0, \quad \chi_L \frac{\partial w}{\partial t} = -F_0 \frac{\partial w}{\partial x} \text{ at } x = L, \quad \chi_0, \chi_L \geq 0 \quad (2.6.14)$$

Application of the Method of Wave Images

It is awkward to implement the analogies when we wish to use the method of wave images. This is so because the typical initial conditions are specification of the cable's displacement and transverse velocity. To use the analogies we would need to work with the cable's slope, rather than its displacement. For this reason we shall rederive the basic formulation. We decompose the displacement of the cable into forward and backward waves,

$$w = f(x - ct) + g(x + ct) \quad (2.6.15)$$

The corresponding description of the transverse velocity is

$$\frac{\partial w}{\partial t} = -cf'(x - ct) + cg'(x + ct) \quad (2.6.16)$$

Let $W(x)$ and $V(x)$ respectively denote the initial displacement and velocity profiles. Matching these functions to the preceding general solution evaluated at $t = 0$ gives

$$W(x) = f(x) + g(x), \quad V(x) = -cf'(x) + cg'(x) \quad (2.6.17)$$

If we are to implement the d'Alembert solution in the same manner as we did for a pressure field, we must determine the f and g functions, rather than their derivatives. To that end we integrate the velocity initial condition, which we may do because its sole independent variable is x . Doing so yields

$$U(x) = \int V(x)dx + C = -cf(x) + cg(x) + C \quad (2.6.18)$$

where C is a constant of integration. The constant of integration C is not arbitrary. Rather it must be selected such that the displacement is continuous at the leading edges, which are the junction of the disturbed and undisturbed regions. This consideration is addressed in the next Example. The last step prior to constructing images is to solve the equations relating W and U to f and g . The result is

$$f(x) = \frac{1}{2} \left[W(x) - \frac{1}{c} U(x) \right], \quad g(x) = \frac{1}{2} \left[W(x) + \frac{1}{c} U(x) \right] \quad (2.6.19)$$

If the cable's extent were infinite, these relations would be sufficient to evaluate the displacement at arbitrary x and t . Other configurations require that we consider reflections. To do so we require that Eq. (2.6.15) satisfy the general boundary

condition at $x = 0$ in Eq. (2.6.14). Thus,

$$\chi_0 \frac{\partial w}{\partial t} \Big|_{x=0} = F_0 \frac{\partial w}{\partial x} \Big|_{x=0} \implies \chi_0 c [-f'(-ct) + g'(ct)] = F_0 [f'(-ct) + g'(ct)] \quad (2.6.20)$$

The incident wave is $g(x + ct)$, so we solve for the reflected wave,

$$f'(-ct) = R_0 g'(ct) \quad (2.6.21)$$

where the reflection coefficient is

$$R_0 = \frac{c\chi_0 - F_0}{c\chi_0 + F_0} \quad (2.6.22)$$

To generalize the reflection rule to describe the forward wave at arbitrary x , we observe that in general f is solely a function of $\xi = x - ct$. At $x = 0$, $\xi = -ct$, so we replace $-ct$ with $x - ct$ in the reflection equation, which leads to

$$f'(x - ct) = R_0 g'(ct - x) \quad (2.6.23)$$

This is the same relation as that which governs the forward and backward pressures in a waveguide. However, the analysis is not complete because we wish to describe the displacement. This requires that we integrate the preceding. Because $(\partial/\partial x) f(x - ct) = f'(x - ct)$, whereas $(\partial/\partial x) g(ct - x) = -g'(ct - x)$, the integration leads to

$$f(x - ct) = -R_0 g(ct - x) \quad (2.6.24)$$

It might seem as though the law is reversed, in that a fixed end corresponds to $R_0 = 1$, which means that the reflected wave is inverted. In contrast, a free end, for which $R_0 = -1$, results in upright reflection. This apparent paradox is a consequence of the analogy of p to the slope $\partial w/\partial x$, whereas we are interested in w .

EXAMPLE 2.8 A cable is fixed at one end. Its other end is extremely distant, so that reflections from that end are irrelevant in the time interval of interest. Thus, the cable may be considered to be semi-infinite. At $t = 0$, an initial velocity field is imparted to it, such that $\partial w/\partial t = v_0 \sin(\pi x/L)$ if $0 \leq x \leq L$, and $\partial w/\partial t = 0$ if $x > L$. The initial displacement is zero. (a) Determine the profiles of the forward and backward displacement waves at $t = 0$. (b) Determine the profile of the displacement at $ct = L/3$ and $ct = 4L/3$. (c) Determine the displacement waveform at $x = L/3$ and $x = 4L/3$.

Significance

The similarities and differences of the method of wave images for stretched cables and acoustic waveguides will be evident as the analysis proceeds.

Solution

The given initial velocity field is $V(x)$. Integrating it leads to

$$U(x) = -\frac{Lv_0}{\pi} \cos\left(\frac{\pi x}{L}\right) [h(x) - h(x-L)] + C$$

The initial displacement is zero, so $f(x) = -U(x)/c$ and $g(x) = U(x)/c$. If C were zero, both functions would be discontinuous at $x = 0$ and $x = L$, so we must select it to yield a displacement that is continuous. The forward wave, which corresponds to $f(x)$, is the sole contributor to the displacement field in $x > L$. The displacement immediately ahead of this wave is zero according to the stated initial conditions. Continuity requires $f(x)$ be zero at its leading edge. This leads to the condition that $U(L) = 0$, which is met by selecting C . Doing so leads to

$$U(x) = -\frac{Lv_0}{\pi} \left[1 + \cos\left(\frac{\pi x}{L}\right)\right] [h(x) - h(x-L)] = -cf(x) = cg(x)$$

It is evident that $f(x)$ and $g(x)$ are continuous at $x = L$. Their discontinuity at $x = 0$ is addressed in the reflection process.

In the method of wave images the function representing the initial profiles for all values of x , negative and positive, are denoted as $F(x)$ and $G(x)$. The preceding descriptions of $f(x)$ and $g(x)$ respectively represent these function for $x > 0$, that is,

$$F(x) = f(x) \text{ and } G(x) = g(x) \text{ if } x > 0$$

Because $f(x)$ represents a wave that propagates in the direction of increasing x , it is not reflected at $x = 0$. Therefore, it does not lead to an image behind the boundary. This is contrasted by $g(x)$, which represents a wave that is reflected at $x = 0$. This reflection is described by adding an image to $F(x)$. Equation (2.6.24) describes this image. The reflection coefficient for a rigid termination is $R_0 = 1$. Hence, the image in $x < 0$ is

$$F(x) = -R_0 g(-x) = -\frac{1}{c} U(-x), \quad x < 0$$

Thus, we have established that wave functions for the initial profiles of the forward and backward waves are

$$F(x) = \frac{1}{c} [-U(x)h(x) - U(-x)h(-x)]$$

$$G(x) = \frac{1}{c} U(x)h(x)$$

Figure 1 graphs both functions.

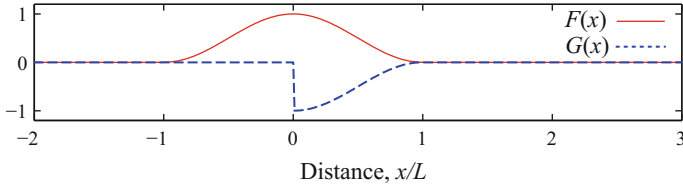


Figure 1.

The remainder of the analysis proceeds as it would for an acoustic wave. A profile at a specified t is obtained by translating $F(x)$ by ct in the positive x direction, which produces $F(x - ct)$. Similarly, $G(x)$ is translated by ct in the direction of decreasing x , thereby producing $G(x + ct)$. The displacement is obtained by adding these functions, then deleting the region $x < 0$, which is not part of the actual cable. The result of this process for the two requested instants is depicted in Fig. 2.

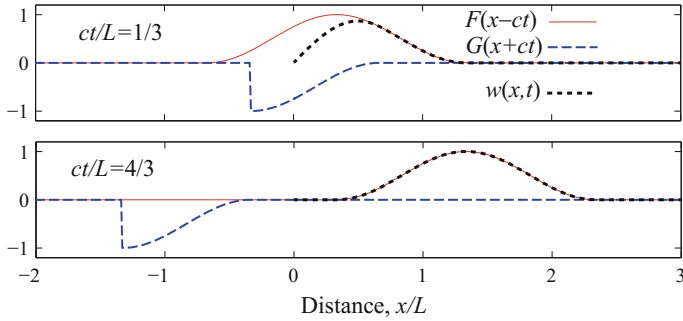


Figure 2.

Construction of waveforms begins by graphing $G(ct)$ and $F(-ct)$ along the ct axis. These are the contributors to the displacement waveform at $x = 0$. According to the given initial conditions, this waveform should be zero. This attribute is readily verified. Waveforms at a specified $x > 0$ are obtained by translating $F(-ct)$ by $+x$, and $G(ct)$ by $-x$, then adding the shifted functions. The result for $x = L/3$ and $x = 4L/3$ appear in Fig. 3.

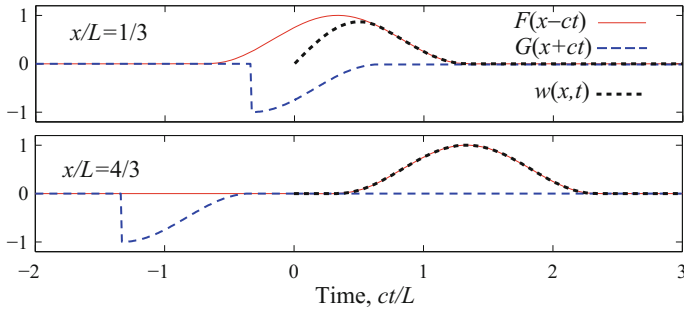


Figure 3.

The preceding analysis illustrates the main alterations required to employ the method of wave images to describe wave propagation along a stretched cable. The simplest is reversal of sign for the rule governing reflections for the comparable acoustic waveguide. The sign change in the formulas for the initial wave functions $f(x)$ and $g(x)$ relative to the acoustic relations also is easy to introduce. The task requiring careful consideration is integration of the initial velocity field consistent with continuity conditions. After the initial profiles have been constructed by combining the wave images, evaluation of displacement of a cable proceeds in the same way as it does for pressure in a waveguide.

2.6.2 Extensional Waves in an Elastic Bar

Field Equations

Our concern here is with the axial displacement $u(x, t)$ that occurs within a straight elastic bar when the internal axial stress σ is unsteady. It might seem that this system is directly analogous to a one-dimensional acoustic waveguide, because axial stress is like pressure. However, the dependent variable of primary interest for a bar is the displacement of cross-sections, so the equations will be closer to those for a stretched cable.

We begin with the momentum-impulse law for a segment of the bar between two arbitrary locations. In Fig. 2.28 a tensile force $F = \sigma \mathcal{A}$ acts at each end. At an arbitrary instant a cross section that was at axial position x has displaced by $u(x, t)$ to its current location. The velocity of this cross section is $\partial u / \partial t$.

The mass per unit length of the bar is the product of its density ρ_0 in the unstretched state and the cross-sectional area $\mathcal{A}$. The element of mass is contained between two cross-sections that were separated by distance dx prior to displacement. Equating the rate of change of the momentum of all mass elements to the force acting on this system yields

$$\sigma \mathcal{A}|_{x_2} - \sigma \mathcal{A}|_{x_1} = \int_{x_1}^{x_2} (\rho_0 \mathcal{A} dx) \frac{\partial^2 u}{\partial t^2} \quad (2.6.25)$$

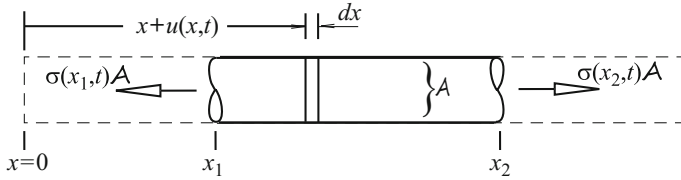


Fig. 2.28 Free body diagram describing the longitudinal forces acting on an isolated segment of an elastic bar

Application of the one-dimensional divergence theorem converts this relation to

$$\int_{x_1}^{x_2} \left(A \frac{\partial \sigma}{\partial x} - \rho_0 A \frac{\partial^2 u}{\partial t^2} \right) dx = 0 \quad (2.6.26)$$

As was asserted previously, the integral relation will hold for any x_1 and x_2 only if the integrand vanishes, so the momentum equation becomes

$$\boxed{\frac{\partial \sigma}{\partial x} - \rho_0 \frac{\partial^2 u}{\partial t^2} = 0} \quad (2.6.27)$$

We seek a field equation for u , which means that σ must be eliminated. If the strain is not excessive, many materials behave elastically according to a linear stress-strain law. In the one-dimensional case of a bar the relation is Hooke's law, $\sigma = E \varepsilon$, where E is Young's modulus and ε is the axial strain. A small stress implies that the strain also is small, so we may use the linearized definition of strain $\varepsilon = \partial u / \partial x$. Thus the role of the continuity equation is fulfilled here by the combination of the stress-strain and strain-displacement equations, which give

$$\sigma = E \frac{\partial u}{\partial x} \quad (2.6.28)$$

Substitution of this relation into the momentum equation produces the wave equation for extensional waves in a bar,

$$\boxed{E \frac{\partial^2 u}{\partial x^2} - \rho_0 \frac{\partial^2 u}{\partial t^2} = 0} \quad (2.6.29)$$

The phase speed of extensional waves is

$$\boxed{c = \left(\frac{E}{\rho_0} \right)^{1/2}} \quad (2.6.30)$$

A consequence of the similarity of the equation for elastic bars and stretched cables is the different way the reflection coefficient is applied. As was true for a cable, an incident displacement wave is reflected with the negative of the coefficient for a pressure wave. Thus, reflection at a fixed end leads to an inverted displacement wave, and reflection at a free end leads to an upright reflected wave. An interesting aspect is that it is easier to implement physically a free or impedance boundary condition for a bar. This is so because the bar can maintain its shape without external action. In contrast, the axial force must be applied at the end of a cable, regardless of other aspects of the end condition. An impedance boundary condition is quite realizable for an elastic bar. Suppose a dashpot aligned axially resists movement of the end $x = 0$ of an elastic bar. A positive strain corresponds to elongation. Thus if the end is moving in the positive x direction, so that $\partial u / \partial t$ is positive, the dashpot will exert a tensile force $\chi (\partial u / \partial t)$ on the bar. The factor χ is the resistance; its units are force/velocity. The stress resultant $\sigma \mathcal{A}$ at this end must match the externally applied force, so the boundary condition

$$\sigma \mathcal{A} = \chi \frac{\partial u}{\partial t} \quad (2.6.31)$$

Very small χ is analogous to a pressure-release termination of a waveguide, and extremely large χ is like a rigid end.

Analogies

The equations governing extensional waves and those for a stretched cable have the same mathematical structure. Consider the momentum equation. The relation for a bar is Eq. (2.6.27). It can be obtained by changing the labels for the dependent variables and the parameters in Eq. (2.6.7). Specifically, $F_0 \theta$ becomes σ , w becomes u , and μ becomes ρ_0 . Note that this is not merely a mathematical similarity. Rather, the quantities are physically alike. Both u and w are displacements. The stress σ is the horizontal force per unit area acting on a cross section, whereas $F_0 \theta$ is the vertical force at a cross section. Correspondingly, ρ_0 is a mass per unit volume, while μ is the mass per unit length.

Multiplication of the continuity equation for a stretched cable, Eq. (2.6.8), by F_0 gives $F_0 \theta = F_0 \partial w / \partial x$. A comparison of this relation and Eq. (2.6.28) for a bar reinforces the identification that σ is like $F_0 \theta$, and u is like w . Then it must be that E is like F_0 . This is true because F_0 was identified as being analogous to the bulk modulus $K = \rho_0 c^2$ of a fluid. Note that F_0 is a force, whereas E has units of force per unit area, which is consistent with the different units of $F_0 \theta$ and σ .

The final element in identification of the analogy of stress waves in a bar and transverse vibration of a stretched cable comes from consideration of the respective equations for the phase speed. Recognition that they are analogous follows from the identification that E is like F_0 and ρ_0 is like μ . It is interesting to note that the phase speeds of extensional waves in a bar and transverse waves in a stretched cable usually have vastly different magnitudes, and that both speeds differ significantly from the speed of sound in fluids. For example, E for steel is approximately 200

GPa and $\rho_0 = 7800 \text{ kg/m}^3$, which gives a phase speed in excess of 5 km/s. This is much greater than what is observed in liquids. To compare these properties to those for a cable, consider a 20 mm diameter steel cable whose internal stress is half the yield strength for some types of steel, $\sigma = 140 \text{ MPa}$. The corresponding values are $F_0 = \sigma A = 44000 \text{ N}$ and $\mu = \rho_0 A = 25 \text{ kg/m}$, which gives a phase speed of approximately 40 m/s. This is far slower than the speed of stress waves in a bar or the speed of sound in a fluid.

In view of these observations, extensional waves may be incorporated into the prior tabulation of analogies. We now have

Acoustic fluid	p	v	K	ρ_0
Flexible cable	$-F_0 (\partial w / \partial x)$	$\partial w / \partial t$	F_0	μ
Elastic bar	$-\sigma$	$\partial u / \partial t$	E	ρ_0

This tabulation has the paradoxical effect of clarifying how the systems are related, while also confusing implementation of the method of wave images. For a cable the displacement usually is the state variable of interest. The displacement is the time integral of velocity, and it is the spatial integral of the slope. These differences lead to a reversal of sign of the reflection coefficients relative those for an acoustics problem. For stress waves in an elastic bar, there are two alternatives. If we seek to determine the displacement of cross sections, then the formulation for a cable is suitable. In that case we begin with initial conditions for the transverse displacement w and velocity $\partial w / \partial t$.

Young's modulus for solids is quite large, so bar displacements are quite small. Instrumentation to measure stress facilitates investigating this variable. For this reason, stress typically is the primary variable of interest. (Of course, the other variable can be found if either the displacement or stress field is known.) It is possible to determine the stress field directly from the method of images for acoustic waves. In such a formulation, the appropriate initial conditions are σ and $\partial u / \partial t$ as functions of x . The development must account for the sign difference in the analogy of pressure and stress. Let us decompose the stress into forward and backward waves. The stress-strain law states that $\sigma = E (\partial u / \partial x)$ and the particle velocity is $\partial u / \partial t$. The consequence is that the particle velocities have reversed signs when they are related to the stress in the respective waves, that is,

$$\begin{aligned} \sigma_1 &= F(x - ct), \quad \sigma_2 = G(x + ct), \quad \sigma = \sigma_1 + \sigma_2 \\ v_1 &= -\frac{c}{E} F(x - ct), \quad v_2 = \frac{c}{E} G(x + ct), \quad v = v_1 + v_2 \end{aligned} \quad (2.6.32)$$

Note that $c^2 = E / \rho_0$, so that $E / c = \rho_0 c$, which is analogous to the fluid impedance. The change of sign is manifested by an alteration in the relations for the wave functions in terms of the initial conditions. In particular, Eq. (2.5.2) become

$$F(x) = \frac{1}{2} \left[\sigma - \rho_0 c \frac{\partial u}{\partial t} \right]_{t=0}, \quad G(x) = \frac{1}{2} \left[\sigma + \rho_0 c \frac{\partial u}{\partial t} \right]_{t=0} \quad (2.6.33)$$

2.7 Closure

We began this chapter by deriving basic equations governing one-dimensional acoustic response in the time domain. The field equation describing the pressure as a function of position and time is the wave equation, and the Euler equation provides the means by which particle velocity and pressure may be related. The only solution of the wave equation that we derived was the general one attributed to d'Alembert. It tells us that the signal within the fluid consists of two waves that propagate in opposite directions without changing the shape of their spatial profile and temporal waveforms. Such waves are said to be simple, but they also are known as nondispersive waves because they do not spread out or contract as they propagate. In the case where the domain is unbounded, these waves are set by the initial conditions. We extended the d'Alembert solution to describe the response of semi-infinite and bounded waveguides to initial condition and boundary excitation. These extensions were achieved by implementing the method of wave images. We encountered a wealth of phenomena, including reflections, attenuation, reverberation time, natural frequencies and modes, and multiple resonances. These phenomena also are encountered in other situations, as we saw in analyses of the transverse displacement of a stretched cable and extensional deformation of an elastic bar. Similar phenomena also occur in far more complicated acoustical systems. Unfortunately the relative simplicity of the method of wave images stems from restricting our attention to a highly idealized model of a reflecting boundary, in which the velocity at a termination is proportional to the pressure that is applied to it. Generalization of the boundary condition requires that we pursue frequency domain solutions. Nevertheless, the present developments provide a unique perspective that will enhance our understanding of the response features of more complicated systems.

2.8 Homework Exercises

Exercise 2.1 Consider a fluid whose ambient state consists of a homogeneous uniform flow at constant velocity U in the x direction, with corresponding pressure p_0 , density ρ_0 , and temperature T_0 . Let the acoustic perturbation of this flow be represented by pressure p' , density ρ' , and particle velocity v' , with $p' \ll \rho_0 c^2$, $\rho' \ll \rho_0$, and $v' \ll U$. Derive the modified linearized wave equation governing the dependence of pressure on position x and time t .

Exercise 2.2 Consider an isentropic process in an ideal gas. Derive an expression for the temperature change T' from the ambient value T_0 when the pressure increases from p_0 to $p_0 + p'$. Then linearize that relation.

Exercise 2.3 Some gases when compressed greatly show a relationship between pressure, density, and pressure that deviates significantly from the ideal gas law. Laws that account for these deviations are known as virial equations. Derive an

expression for the speed of sound in the case where the virial equation describes T explicitly as a function of p and ρ , that is, $T = F(p, \rho)$.

Exercise 2.4 An important dissipation mechanism for many fluids is relaxation. The linearized equation of state for such a fluid is

$$(p - p_0) - c_0^2 (\rho - \rho_0) + \tau \left(\frac{\partial p}{\partial t} - c_\infty^2 \frac{\partial \rho}{\partial t} \right) = 0$$

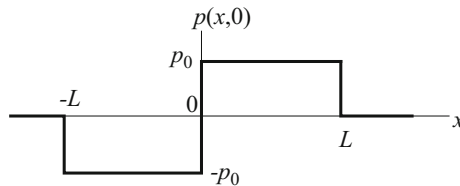
where τ is a constant called the relaxation time, and c_0 and c_∞ are two different sound speeds, usually close together. The continuity and momentum equations may be taken to be the same as those for an ideal fluid. Show that the corresponding wave equation is

$$\frac{\partial^2 p'}{\partial t^2} - c_0^2 \frac{\partial^2 p'}{\partial x^2} + \tau \left[\frac{\partial^3 p'}{\partial t^3} - c_\infty^2 \frac{\partial}{\partial t} \left(\frac{\partial^2 p'}{\partial x^2} \right) \right] = 0, \quad p' = p - p_0$$

Exercise 2.5 Consider a situation in which the walls of a circular tube are porous. The radius of the tube is a . The ambient conditions in the exterior are the same as those inside. The consequence is that there is a mass flow per unit length through the tube's wall that is proportional to the acoustic pressure. The result is that the mass flowing out of the wall between x to $x + dx$ in an interval dt is $\dot{m}_{\text{wall}} dx dt$, where $\dot{m}_{\text{wall}} = \sigma p(x, t)$. If this flow is sufficiently small, it is permissible to assume that the acoustical pressure and particle velocity within the tube depend only on x and t , as in the case where the tube is impermeable. (a) Determine how the porosity affects the continuity and momentum equations. (b) Derive the linearized field equation governing pressure.

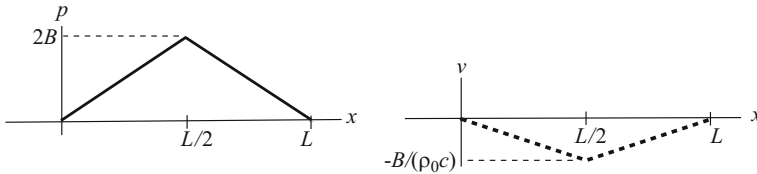
Exercise 2.6 The signal within a duct consists of a pair of oppositely propagating plane waves, as described by the d'Alembert solution. Suppose the acoustic pressure and particle velocity waveforms are measured at $x = 0$ as a function of time, so that $p(0, t) = p_A(t)$ and $v(0, t) = v_A(t)$ are known. Derive an expression for $p(x, t)$ in terms of the functions p_A and v_A .

Exercise 2.7 When $t = 0$ the pressure within a tube is observed to depend on the distance along the tube in the manner shown in the graph. The fluid within the tube is not moving at $t = 0$. Sketch spatial profiles of the pressure and particle velocity at $t = 0.5L/c$ and $1.5L/c$.



Exercise 2.7

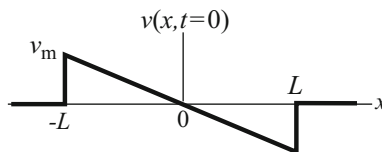
Exercise 2.8 The initial pressure and velocity distributions for a plane wave are depicted below. (a) For the instant when $t = 0$, draw a graph showing the x dependence of the initial acoustic pressure in the individual left and right propagating plane waves generated by this set of initial conditions. (b) Determine the spatial profile of the pressure when $t = 0.5L/c$ and $t = L/c$. (c) Determine the pressure waveforms at $x = L/2$ and $x = L$.



Exercise 2.8

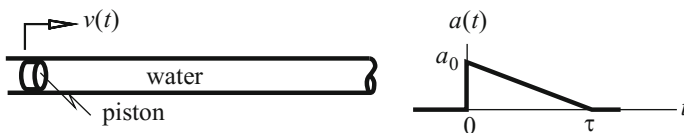
Exercise 2.9 The initial velocity particle velocity in an infinite waveguide is $v(x, 0) = V \sin(\pi x/L)$ for $-L < x < L$. (a) Determine the pressure and particle velocity profiles at $t = L/c$. (b) Determine the waveforms at $x = 0, \pm L$.

Exercise 2.10 When $t = 0$ the initial particle velocity field within an infinitely long tube is the N-wave shown below. The initial pressure is zero. Sketch p and v as functions of x when $t = L/2c$. Then sketch the waveforms of p and v as functions of t at $x = L/2$.



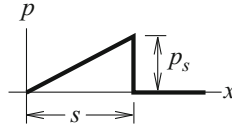
Exercise 2.10

Exercise 2.11 A piston at the left end of a tube filled with water is given the acceleration $\ddot{v} = a(t)$ graphed below. (a) Determine the pressure, particle velocity, and particle displacement temporal waveforms at the location $x = c\tau$. (b) Determine the spatial profiles of pressure, particle velocity, and particle displacement at $t = 2\tau$. (c) The cross-sectional area of the tube is A and the mass of the piston is m . Determine the force $F(t)$ that must be applied to the piston to achieve this motion.



Exercise 2.11

Exercise 2.12 At $t = 0$ the initial pressure perturbation in a semi-infinite waveguide, $0 \leq x < \infty$, depends on the distance from the closed end in the manner described by the graph. The end $x = 0$ is terminated by a material whose impedance is $Z = 0.20\rho_0 c$, independent of frequency. The initial particle velocity is zero. (a) Sketch graphs of the pressure perturbation and particle velocity as functions of x when $t = s/2c$. (b) Sketch a graph of the pressure perturbation as a function of time t at $x = s/2$. Be sure to mark critical values in the graphs.

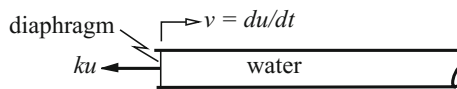


Exercise 2.12

Exercise 2.13 If the termination of a waveguide is actively controlled, it is possible to attain a negative impedance. Consider a waveguide for which the specific impedance at $x = 0$ is $\zeta = -0.5$. The initial pressure distribution is $p(x, 0) = A \exp(-\beta x)$ for $x \geq 0$. Determine the pressure distribution when $t = 1/(\beta c)$ and the pressure waveform observed at $x = 1/\beta$.

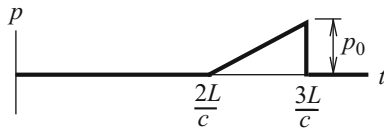
Exercise 2.14 The pressure and particle within a semi-infinite waveguide at $t = 0$ are $p = -\rho_0 c v = A[h(x - L) - h(x - L/2)]$. It also is known that the pressure distribution at an unspecified instant after the wave has reflected is $p = 0.2[h(x - L/2) - h(x - L)]$. (a) Determine the time at which the reflected signal was measured. (b) Determine the reflection coefficient. (c) Determine the waveforms at $x = L/2$ and $x = L$.

Exercise 2.15 A wave $p = g(t + x/c)$ propagates to the left inside the tube. The left end, $x = 0$, is closed by a diaphragm that exerts a restoring force ku to the left when it displaces u to the right. This force must balance the pressure force exerted by the incident and reflected waves. Derive a differential equation whose solution would yield the time dependence of the pressure wave that is reflected from the membrane.



Exercise 2.15

Exercise 2.16 The termination of a semi-infinite tube has a specific impedance $\zeta = 3$. The reflected waveform observed at $x = 1.5L$ is the function in the graph. The incident wave is not known. (a) Determine the spatial profile of the incident wave at $t = 0$. (b) Determine the waveform of the total pressure (incident plus reflected) at $x = 1.5L$.



Exercise 2.16

Exercise 2.17 Both ends of a waveguide whose length is L are pressure-release. Prove that Eq. (2.5.33) is the Fourier series describing the response of this system to arbitrary initial conditions.

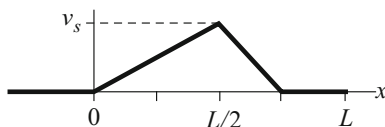
Exercise 2.18 The end $x = 0$ of a waveguide is pressure-release and $x = L$ is rigid. Prove that Eq. (2.5.36) is the Fourier series describing the response of this system to arbitrary initial conditions. Then use that equation to identify the natural frequencies and modes for the case where the ends are swapped, so that $p = 0$ at $x = 0$ and $v = 0$ at $x = L$.

Exercise 2.19 The initial pressure in a finite waveguide is $p = B[h(x - L/2)]$. The particles are initially at rest. The end at $x = 0$ is pressure-release and the end at $x = L$ is rigid. Graphically construct the pressure profile at $t = L/c$ and $t = 2L/c$.

Exercise 2.20 Perform the analysis requested in Exercise 2.19 for the case where both ends are pressure-release.

Exercise 2.21 The end at $x = 0$ of a tube is rigid, while the reflection coefficient at $x = L$ is $R_L = 0.5$. Membranes stretched across the tube at $x = L/4$ and $x = 3L/4$ hold the pressure perturbation in the region between them at value B , while the pressure in $0 < x < L/4$ and $3L/4 < x < L$ is zero. The fluid is at rest in this condition. At $t = 0$ the membranes are destroyed. (a) Determine the spatial pressure profile over $0 < x < L$ at $t = L/c, 2L/c, 3L/c$. (b) Determine the waveform at $x = L/2$ for $0 < t < 4L/c$.

Exercise 2.22 It hypothetically is possible to induce an arbitrary initial velocity field in a magnetic fluid. Such a velocity field in a waveguide at $t = 0$ appears in the graph, where $v > 0$ indicates that a particle is moving in the positive x direction. The initial pressure perturbation relative to ambient conditions is zero. The end $x = 0$ is rigid and the impedance at $x = L$ is $Z_L = 0.5\rho_0 c$. (a) Determine the pressure profile at $t = 1.5L/c$. (b) Determine the pressure waveform at $x = L$ for $0 \leq t \leq 3L/c$. Use graphical constructions to address both tasks.



Exercise 2.22

Exercise 2.23 Use mathematical software to solve Exercise 2.22.

Exercise 2.24 The particle velocity at $x = 0$ in a waveguide is a pulse that is a ramp function whose duration is L/c , that is, $v(0, t) = at [h(t) - h(t - L/c)]$. The end at $x = L$ is rigid. Use graphical construction methods to determine the pressure profile at $t = 2L/c$ and the pressure waveforms at $x = 0$ and $x = L$.

Exercise 2.25 Use mathematical software to solve Exercise 2.24.

Exercise 2.26 The end at $x = L$ of a waveguide is rigid. Beginning at $t = 0$, the pressure at $x = 0$ decays exponentially according to $p = A \exp(-\beta t)$. (a) Derive an expression for the pressure within the waveguide. (b) Create a program that will evaluate pressure profiles and waveforms. (c) Evaluate pressure profiles when $t = 2L/c$ and $t = 2.5L/c$. (d) Evaluate waveforms at $x = 0$, $L/2$, and L . The decay constant is $\beta = 2c/L$.

Exercise 2.27 Perform the analysis requested in Exercise 2.26 for the case where the impedance at $x = L$ is $\zeta_L = 0.4$.

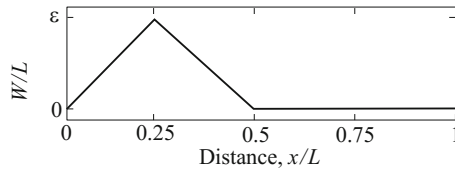
Exercise 2.28 The particle velocity at $x = 0$ in a waveguide is zero until $t = 0$. Then it varies harmonically at $V_0 \sin(\omega t)$. The impedance at $x = L$ is ζ . Develop a computer program to evaluate the pressure waveform at an arbitrary location. Use it to determine the waveform at $x = L/2$ for the following combination of parameters: (a) $\omega = \pi c/(3L)$ and $\zeta = 0.95$, (b) $\omega = \pi c/(2.95L)$ and $\zeta = 1$, (c) $\omega = \pi c/(3L)$ and $\zeta = 1$.

Exercise 2.29 Consider the situation where a waveguide is excited at $x = L$ and the reflection coefficient at $x = 0$ is R_0 . Modify Eq. (2.5.50) to describe the pressure field at arbitrary x, t (a) if the velocity at $x = L$ is specified to be an arbitrary function $v_L(t)$, (b) if the pressure at $x = L$ is specified to be an arbitrary function $p_L(t)$.

Exercise 2.30 The end $x = 0$ of a waveguide is rigid. The pressure at $x = L$ is zero until $t = 0$, at which time it is made to vary as a sequence of positive lobes of a sine function, so that $p(x = L, t) = 0$ if $t < 0$, $p(x = L, t) = B \sin(2\pi t/T)$ if $0 < t < T/2$, $p(x = L, t) = 0$ if $T/2 < t < T$, $p(x = L, t) = p(x = L, t - T)$ if $t \geq T$. (a) Develop a software program to evaluate waveforms at an arbitrary x . (b) Cases of interest are $T = 2L/c$ and $T = 4L/c$. Validate the program by verifying that, for both values of T , the result for the pressure waveform at $x = L$ is the specified function. (c) Evaluate the waveform at $x = L/2$ for both values of T .

Exercise 2.31 The A string of a cello is tensioned such that its fundamental natural frequency is 220 Hz. The length of the string is 695 mm and its mass per unit length is 2.2 gm/m. What is the tensile force?

Exercise 2.32 A cable is tensioned, then tied between two fixed posts. The string is pulled upward at the quarter-point and prevented from moving at the midpoint, so the initial displacement is the triangular distribution graphed below. The string is held stationary in this state, then released. (a) Determine the displacement of the cable at $t = 3.5L/c$. (b) Determine the displacement waveform at the midpoint of the cable for $t \leq 4L/c$.



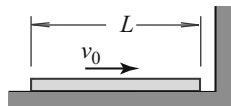
Exercise 2.32

Exercise 2.33 Solve Exercise 2.32 in the case where the cable at $x = L$ is free to move vertically while the tension force is applied. The end $x = 0$ is fixed.

Exercise 2.34 A 600 mm long steel rod has a cross-sectional diameter of 20 mm. The end at $x = 0$ is welded to a rigid wall, and the right end is attached to a dashpot that is aligned with the longitudinal axis of the rod. Determine the viscous resistance coefficient of the dashpot for which the displacement reflection coefficient at $x = L$ is -0.5 .

Exercise 2.35 A pulse excitation applied to one end of an aluminum rod consists of a 1 kN axial stress whose duration is 2 ms. The reflection coefficient for displacement at the far end is 0.9. The rod has a square cross section 40 mm $\times$ 40 mm, and the length of the rod is 4 m. Determine the stress waveform observed at the midpoint and far end.

Exercise 2.36 The elastic bar in the sketch is sliding at speed v_0 along the floor with negligible frictional resistance. At $t = 0$, it strikes the wall, which may be considered to be rigid. Contact with the wall is maintained as long as the stress at the end is compressive. Thus, the interval of contact extends from $t = 0$ until $t = t_f$ when the compressive stress at the end is zero. After t_f , the end condition becomes zero stress, corresponding to the bar rebounding from the impact. (a) Determine t_f . (b) Determine the stress and particle velocity profile at an instant that slightly precedes t_f . From that, describe how the bar rebounds from the collision.



Exercise 2.36

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