Chapter 2
Kinematics

This chapter first introduces multi-body systems in conceptual terms. It then describes the concept of a Euclidean frame in the material world, following the concept of a *Euclidean structure* introduced in [5]. The Euclidean frame is used to define the set of all possible configurations. This set of configurations is assumed to have the mathematical structure of an embedded finite-dimensional manifold. Differential equations or vector fields on a configuration manifold describe the kinematics or velocity relationships; especially in the case of rotational kinematics, these kinematics relationships are sometimes referred to as *Poisson's* equations. Kinematics equations are obtained for several interesting connections of mass particles and rigid bodies. This chapter provides important background for the subsequent results on the dynamics of Lagrangian and Hamiltonian systems. A classical treatment of kinematics of particles and rigid bodies is given in [99].

2.1 Multi-Body Systems

The concept of a multi-body system is a familiar one in physics and engineering. It consists of a collection of rigid and deformable bodies; these bodies may be physically connected and/or they may interact through forces that arise from a potential. Multi-body systems, as interpreted here and throughout the physics and engineering literature, represent idealizations of real mechanical systems.

Throughout this book, several different categories of multi-body systems are studied. In some cases, an individual body is idealized as consisting of a rigid straight line, referred to as a link, with mass concentrated at one or more points on the link. Such an idealization is a convenient approximation, especially for cases where the physical body is slender and rotational motion

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about its slender axis and the associated kinetic energy can be ignored. Such approximations are sometimes referred to as lumped mass bodies or bodies defined by mass particles. On the other hand, the concept of a rigid body assumes a full three-dimensional body with spatially distributed mass. Rigidity implies that distances between material points in the body remain constant. Possible interconnection constraints between two bodies include: rotational joints that allow constrained relative rotation between the bodies and prismatic joints that allow constrained relative translation between the bodies. This also allows elastic or gravitational connections that arise from a mutual potential field between the bodies. We do not attempt to provide a theoretical framework for multi-body systems; rather the subsequent development provides numerous instances of multi-body systems that arise as approximations in physics and engineering.

Our interest in multi-body systems is to understand their possible motions, not the structural or design features of multi-body systems. It is convenient to distinguish between multi-body kinematics and multi-body dynamics. Kinematics describe relationships between configuration variables and velocity variables; dynamics describe acceleration relationships. In this chapter, we consider kinematics issues; the remainder of this book presents results on multi-body dynamics, from a Lagrangian and Hamiltonian perspective that make use of the methods of variational calculus.

2.2 Euclidean Frames

In order to describe, in mathematical terms, the mechanics of objects such as ideal mass particles, rigid bodies, and deformable bodies, it is convenient to introduce the concepts of spatial vectors and Euclidean frames that are used to define the motion of objects in the material world or forces that act on those objects.

A spatial vector in mechanics has a direction and magnitude in the material world. A Euclidean frame can be viewed as a construction in the three-dimensional material world consisting of three mutually orthogonal direction vectors that we associate with the standard basis vectors of $\mathbb{R}^3$. We think of the three orthogonal directions as defining (positively) directed axes or spatial vectors in the Euclidean frame, thereby inducing Euclidean coordinates for spatial vectors in the material world. Thus, any spatial vector can be expressed as a linear combination of the three basis vectors of the Euclidean frame, so that a spatial vector is uniquely associated with a vector in $\mathbb{R}^3$. Any direction in the material world is defined by a nonzero vector, typically scaled to be of unit length, in $\mathbb{R}^3$. If we also specify the location of the origin of the Euclidean frame in the material world then the location of any point in the material world is represented by a spatial vector and by a corresponding vector in $\mathbb{R}^3$, expressed with respect to the selected Euclidean frame. It is convenient to order the directed axes of the Euclidean frame according to the usual *right-hand rule*: if the first directed axis is rotated in the direction of the second directed axis according to the fingers of the right hand, then the thumb points in the direction of the third directed axis.

In summary, we have the following. For any given Euclidean frame, we associate a spatial vector in the material world with a vector in $\mathbb{R}^3$; if the Euclidean frame has a specified origin, we can associate the location of a point in the material world by a vector in $\mathbb{R}^3$. These associations imply that the geometry of the material world can be described mathematically in terms of $\mathbb{R}^3$, viewed as a linear vector space with an inner product. Consequently, the following developments do not emphasize the spatial vectors of mechanics but rather the developments are built upon their representations with respect to one or more Euclidean frames.

In the case that the motion of points of a physical object can be characterized to lie within a two-dimensional plane fixed in a three-dimensional Euclidean frame, we often select the three-dimensional frame so that the objects can be easily described with respect to a two-dimensional Euclidean frame consisting of two orthogonal direction vectors; these ideas are natural and are used in examples that are subsequently introduced.

We make use of several categories of Euclidean frames. A Euclidean frame may be fixed or stationary with respect to a background in the material world; such frames are said to be inertial. We refer to such frames as fixed frames or inertial frames. In some cases, we introduce Euclidean frames that are attached to a rigid body, that is the frame translates and rotates with the body; such frames are said to be body-fixed frames. In some cases, it is convenient to introduce a reference Euclidean frame that is neither stationary nor body-fixed but is physically meaningful as a reference. In situations where several Euclidean frames are introduced, it is important to maintain their distinction. We do not introduce any special notation that identifies a specific frame or frames, but rather we hope that this is always clear from the context.

2.3 Kinematics of Ideal Mass Particles

The motion of an ideal mass element or mass particle, viewed as an abstract point in the material world at which mass is concentrated, is naturally characterized with respect to an inertial Euclidean frame. The position of the mass particle in the Euclidean frame, at one instant, is represented by a vector in $\mathbb{R}^3$ where the components in the vector are defined with respect to the standard basis vectors for the Euclidean frame.

If the particle is in motion, then its position vector changes with time t and is represented by a vector-valued function of time $t \to x(t) \in \mathbb{R}^3$. We refer to the position vector $x \in \mathbb{R}^3$ as the configuration of the particle; the space of configurations is represented by the vector space $\mathbb{R}^3$. We can differentiate the position vector once to obtain the velocity vector $v(t) = \frac{dx(t)}{dt}$ of the moving particle, and we observe that the velocity is an element of the tangent space

$\mathsf{T}_x\mathbb{R}^3$, which we can identify in this case with $\mathbb{R}^3$ itself. The tangent bundle $\mathsf{T}\mathbb{R}^3$ consists of all possible pairs of position vectors and velocity vectors, which in this case is identified with $\mathbb{R}^3 \times \mathbb{R}^3$. This characterization defines the kinematics of an ideal mass particle. In fact, this characterization can be used to describe the motion of any fixed point on a body, whether or not this point corresponds to a concentrated mass.

These concepts can easily be extended to a finite number n of interacting particles that are in motion in the material world. Suppose that no constraints are imposed on the possible motions of the n particles; for example, we do not prohibit two or more particles from occupying the same location in the material world. We introduce a Euclidean frame in the material world to describe the motion of these interacting particles. The motion of n interacting particles can be described by an n-tuple, consisting of the position vectors of the n particles, that is functions of time $t \to x_i \in \mathbb{R}^3$, $i = 1, \ldots, n$. Thus, the configuration of the n interacting particles is described by the vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^{3n}$. We can differentiate the configuration vector once to obtain the velocity vector $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n) \in \mathsf{T}_x\mathbb{R}^{3n}$ of the n particles. The time derivative of the configuration vector, or the velocity vector, is an element of the tangent space $\mathsf{T}_x\mathbb{R}^{3n}$, which we identify in this case with $\mathbb{R}^{3n}$ itself. The tangent bundle consists of all possible pairs of position vectors and velocity vectors, which in this case is identified with $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$.

On the other hand, suppose n interacting particles in motion are subject to algebraic constraints, which can be represented by an embedded manifold M in $\mathbb{R}^{3n}$. The configuration of the n interacting particles is the vector $x = (x_1, \ldots, x_n) \in M$. We can differentiate the configuration vector once to obtain the velocity vector $\dot{x} = (\dot{x}_1, \ldots, \dot{x}_n) \in \mathsf{T}_xM$ of the n particles. That is, the time derivative of the configuration vector, also referred to as the velocity vector, is an element of the tangent space T_xM. The tangent bundle consists of all possible pairs of position vectors and velocity vectors, which is identified with the tangent bundle $\mathsf{T}M$.

2.4 Rigid Body Kinematics

The concept of a rigid body is an idealization of real bodies, but it is a useful approximation that we adopt in the subsequent developments. A rigid body is defined as a collection of material particles, located in the three-dimensional material world. The material particles of a rigid body have the property that the relative distance between any two particles in the body does not change. That is, the body, no matter what forces act on the rigid body or what motion it undergoes, does not deform.

We sometimes consider rigid bodies consisting of an interconnection of a finite number of ideal particles, where the particles are connected by rigid links. Such rigid bodies may be a good approximation to real rigid bodies in the material world, and they have simplified inertial properties. More commonly, we think of a rigid body in the material world as consisting of a spatially distributed mass continuum.

The key in defining rigid body kinematics is the definition of the configuration of the rigid body. The choice of the configuration of a rigid body, and its associated configuration manifold, depends on the perspective and the assumptions or constraints imposed on the rigid body motion. Kinematic relationships describe the rate of change of the configuration as it depends on translational and rotational velocity variables and the configuration. The role of the geometry of the configuration manifold is emphasized in the subsequent development.

As shown subsequently, several configuration manifolds are commonly employed. If the rigid body is constrained to rotate so that each of its material points moves on a circle within a fixed plane, then the attitude configuration of the rigid body can be represented by a point on the configuration manifold S^1. If the rigid body is constrained to rotate so that each of its material points moves on the surface of a sphere in $\mathbb{R}^3$, then the attitude configuration of the rigid body can be represented by a point on the configuration manifold S^2. If the rigid body can rotate arbitrarily in $\mathbb{R}^3$, then the attitude configuration of the rigid body can be represented by a point on the configuration manifold $\mathsf{SO(3)}$.

In addition to rotation of rigid bodies, we also consider translation of a rigid body, often characterized by the motion of a selected point in the body, such as its center of mass. If the rigid body is constrained to translate so that each of its material points moves within a fixed plane, then the translational configuration of the rigid body can be selected to lie in the configuration manifold $\mathbb{R}^2$. If the rigid body can translate arbitrarily in $\mathbb{R}^3$, then the translational configuration of the rigid body can be selected to lie in the configuration manifold $\mathbb{R}^3$.

Finally, general rigid body motion, or *Euclidean motion*, can be described by a combination of rotation and translation. For example, a rigid body may be constrained to translate and rotate so that each of its material points lies in a fixed plane or a rigid body may rotate and translate arbitrarily in $\mathbb{R}^3$. We consider each of these situations subsequently, in each case identifying the configuration manifold and the corresponding rigid body kinematics.

2.5 Kinematics of Deformable Bodies

It is challenging to study the kinematics of deformable bodies. The configuration variables and the configuration manifold of a deformable body in motion must be carefully selected to characterize the translational, rotational, and spatial deformation features. This choice is strongly influenced by the assumptions about how the body is spatially distributed and how it may deform in three dimensions. A full treatment of the kinematics and dynamics of deformable bodies requires the use of infinite-dimensional configuration manifolds and is beyond the scope of this book. However, we do treat several challenging examples of finite-dimensional deformable bodies in Chapter 10 that can be viewed as finite element approximations of infinite-dimensional deformable bodies.

2.6 Kinematics on a Manifold

As seen above, the motion of particles, rigid bodies, and deformable bodies is naturally described in terms of the time evolution of configuration variables within a configuration manifold. Kinematics relationships are the mathematical representations that describe this evolution. The time derivative of the configuration variables is necessarily an element of the tangent space at each instant. Thus, the kinematics are the differential equations, and possibly associated algebraic equations, that describe the time derivative of the configuration as it depends on the configuration.

The kinematics relations are typically viewed as describing the time evolution of the configuration or the flow in the configuration manifold. In this chapter, variables that generate the flow within the configuration manifold are often left unspecified. In the subsequent chapters where issues of dynamics are included, the variables that describe the flow within the configuration manifold are specified as part of the dynamics. Alternatively, these flow variables can be specified as having constant values or even as functions of the configuration; mathematical models of this latter type are sometimes referred to as *closed loop kinematics* or kinematics with *feedback*.

Additional structure arises when the configuration manifold is (a) a Lie group manifold, (b) a homogeneous manifold, or (c) a product of Lie groups and homogeneous manifolds. In these cases, the kinematics can be expressed in terms of the Lie algebra of the Lie group manifold or of the Lie group associated with the homogeneous manifold. All of the kinematics examples studied subsequently arise from configuration manifolds of this or a closely related form.

The motion of a particular physical system may be described in different ways, for example by selecting one of several possible configuration manifolds. We demonstrate this possibility in some of the examples to follow. Typically, we would like to select configuration variables and configuration manifolds that are parsimonious in describing the physical features, while providing physical insight without excessive analytical or computational baggage. One of the important features of this book is demonstration of the important role of the geometry of the configuration manifold.

We emphasize an important point made in the Preface: configuration manifolds are fundamental in describing kinematics (and dynamics); although we make use of a Euclidean frame and associated coordinates for the material world, we do *not* use local coordinates to describe the configuration manifolds. In this sense, the subsequent development is said to be *coordinate-free*. This approach is important in obtaining globally valid descriptions of kinematics (and dynamics).

2.7 Kinematics as Descriptions of Velocity Relationships

Several examples of mechanical systems are introduced. In each case, the physical description and assumptions are made clear. The configurations are selected and a configuration manifold is identified. Kinematics relationships are obtained by describing the time derivative of the configuration variables using differential equations, and possibly algebraic equations, on the configuration manifold.

It is important to emphasize that the formulation in each of the following examples leads to representations for the global kinematics; that is, there are no singularities or ambiguities in the expressions for the kinematics. Consequently, this formulation is suitable to describe extreme motions, without requiring an ad-hoc fix or adjustment as is necessary when using local coordinates.

2.7.1 Translational Kinematics of a Particle on an Inclined Plane

A particle is constrained to move in an inclined plane in $\mathbb{R}^3$ with respect to an inertial Euclidean frame. The plane is given by the linear manifold $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 - 1 = 0\}$. A schematic of the particle on an inclined plane is shown in Figure 2.1.

The manifold M is viewed as the configuration manifold. Since the dimension of the configuration manifold is two, the translational motion of a particle in the plane is said to have two degrees of freedom. We first construct a basis for the tangent space T_xM; note that M is the zero level set of the constraint function $f(x) = x_1 + x_2 + x_3 - 1$ and the gradient of this constraint function is $[1, 1, 1]^T \in \mathbb{R}^3$, which is normal to M. Thus, the tangent space $T_xM = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 + y_2 + y_2 = 0\}$ is a subspace of $\mathbb{R}^3$ that does not depend on $x \in \mathbb{R}^3$. A basis for the tangent space is easily selected, for example as $\{[1, -1, 0]^T, [0, 1, -1]^T\}$. Suppose that a function of time $t \to x \in M$ represents a translational motion of the particle in the inclined plane. Since $x \in M$, it follows that the time derivative $\dot{x} \in \mathsf{T}_xM = \text{span}\{[1, -1, 0]^T, [0, 1, -1]^T\}$. This implies there is a vector-valued function of time $t \to (v_1, v_2) \in \mathbb{R}^2$ such that

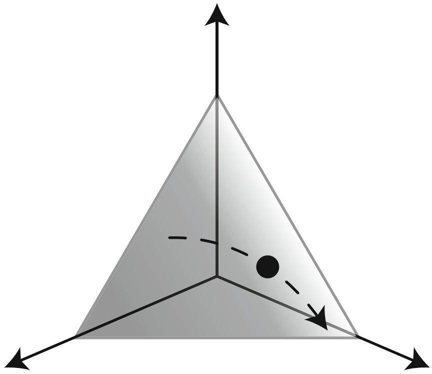

Fig. 2.1 Translational kinematics of a particle on an inclined plane

$$\dot{x} = v_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \tag{2.1}$$

This vector differential equation (2.1) is referred to as the translational kinematics for a particle on the inclined plane. This describes the rate of change of the configuration, namely $\dot{x} \in \mathsf{T}_x M$, in terms of $v = [v_1, v_2]^T \in \mathbb{R}^2$. The vector $v \in \mathbb{R}^2$ is referred to as the translational velocity vector of the particle in the linear manifold. Thus, the translational kinematics of a particle on an inclined plane can be viewed through the evolution of $(x, \dot{x}) \in \mathsf{T}M$ in the tangent bundle or through the evolution of $(x, v) \in M \times \mathbb{R}^2$.

Now, suppose that the translational velocity vector $v \in \mathbb{R}^2$ is a smooth function of the configuration. The translational kinematics equations (2.1) can be viewed as defining a smooth vector field on the linear manifold M. Initial-value problems can be associated with the translational kinematics. The following result can be shown to hold: for any initial-value $x(t_0) = x_0 \in M$, there exists a unique solution of (2.1) satisfying the specified initial-value and this unique solution satisfies $x(t) \in M$ for all t.

2.7.2 Translational Kinematics of a Particle on a Hyperbolic Paraboloid

A particle is constrained to move in a smooth surface in $\mathbb{R}^3$, namely a hyperbolic paraboloid. Its motion is described with respect to an inertial Euclidean frame. The surface is described by the manifold

$$M = \left\{ x \in \mathbb{R}^3 : x_3 = -(x_1)^2 + (x_2)^2 \right\},$$

which we select as the configuration manifold. A schematic of the particle on the surface is shown in Figure 2.2.

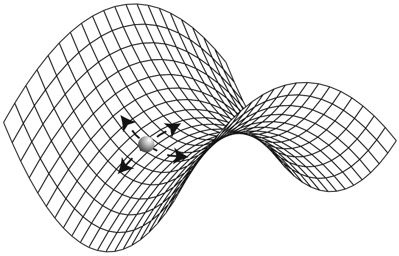

Fig. 2.2 Translational kinematics of a particle on a hyperbolic paraboloid

We determine the tangent space and the tangent bundle of the configuration manifold M. This is done indirectly by first focusing on the kinematics of $[x_1, x_2]^T \in \mathbb{R}^2$ which is the projected position vector of the particle, with respect to the inertial Euclidean frame. Since the dimension of the configuration manifold is two, the translational motion of a particle on the surface is said to have two degrees of freedom.

Suppose that a function of time $t \to [x_1, x_2, x_3]^T \in M$ represents a translational motion of the particle on the surface. It follows that $t \to [x_1, x_2]^T \in \mathbb{R}^2$ and the time derivative $[\dot{x}_1, \dot{x}_2]^T \in \mathsf{T}_{[x_1,x_2]^T}\mathbb{R}^2 = \text{span}\left\{[1,0]^T, [0,1]^T\right\}$ using the standard basis for $\mathbb{R}^2$. This implies there is a vector-valued function of time $t \to [v_1, v_2]^T \in \mathbb{R}^2$ such that

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = v_1 \begin{bmatrix}1\\0\end{bmatrix} + v_2 \begin{bmatrix}0\\1\end{bmatrix}.$$

Since $\dot{x}_3 = -2x_1\dot{x}_1 + 2x_2\dot{x}_2$, it follows that

$$\dot{x} = v_1 \begin{bmatrix}1\\0\\-2x_1\end{bmatrix} + v_2 \begin{bmatrix}0\\1\\2x_2\end{bmatrix}. \tag{2.2}$$

This vector differential equation (2.2) describes the translational kinematics for a particle on a hyperbolic paraboloid in $\mathbb{R}^3$. This describes the rate of change of the configuration $\dot{x} \in \mathsf{T}_x M$ in terms of $v = [v_1, v_2]^T \in \mathbb{R}^2$. The vector $v \in \mathbb{R}^2$ is the translational velocity vector of the particle projected onto the x_1x_2-plane. Thus, the translational kinematics of a particle on the surface can be viewed through the evolution of $(x, \dot{x}) \in \mathsf{T}M$ in the tangent bundle or through the evolution of $(x, v) \in M \times \mathbb{R}^2$.

Now, suppose that the translational velocity vector is a smooth function of the configuration. The translational kinematics equations (2.2) can be viewed as defining a smooth vector field on the manifold M. The following result can be shown to hold for an initial-value problem associated with (2.2): for any initial-value $x(t_0) = x_0 \in M$, there exists a unique solution of (2.2) satisfying the specified initial-value and this unique solution satisfies $x(t) \in M$ for all t.

2.7.3 Rotational Kinematics of a Planar Pendulum

A planar pendulum is an inextensible, rigid link that rotates about a fixed axis of rotation, referred to as the pivot. The pivot is fixed in an inertial Euclidean frame. Each material point in the pendulum is constrained to move along a circular path within an inertial two-dimensional plane; thus, it is referred to as a planar pendulum.

The configuration of the planar pendulum is the direction vector of the pendulum link $q \in \mathsf{S}^1$ defined in a two-dimensional Euclidean frame, where the two axes define the plane of rotation and the origin of the frame is located at the pivot. For simplicity, the plane of rotation is viewed as defined by the first two axes of a three-dimensional Euclidean frame. The configuration manifold is S^1. Since the dimension of the configuration manifold is one, planar rotations are said to have one degree of freedom. A schematic of the planar pendulum is shown in Figure 2.3.

Suppose that a function of time $t \to q \in \mathsf{S}^1$ represents a rotational motion of the planar pendulum. Since $q \in \mathsf{S}^1$, it follows that the time derivative $\dot{q} \in \mathsf{T}_q\mathsf{S}^1$ is necessarily a tangent vector of S^1 at q. Thus, $\dot{q}$ is orthogonal to q, that is $(\dot{q} \cdot q) = 0$. This implies that there is a scalar-valued function of time $t \to \omega \in \mathbb{R}^1$ such that

$$\dot{q} = \omega \mathrm{S} q, \tag{2.3}$$

where S is the 2×2 constant skew-symmetric matrix

$$\mathrm{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

introduced in (1.6). Note that when S acts on a vector by multiplication, it rotates the vector by $\frac{\pi}{2}$ counterclockwise. It is easy to verify that equation (2.3) is consistent with the familiar expression in three dimensions for velocity of a point in terms of the cross product of the angular velocity vector and the position vector if we embed $\mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. This embedding is given in terms of the matrix

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then equation (2.3) is equivalent to

$$Q\dot{q} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times (Qq).$$

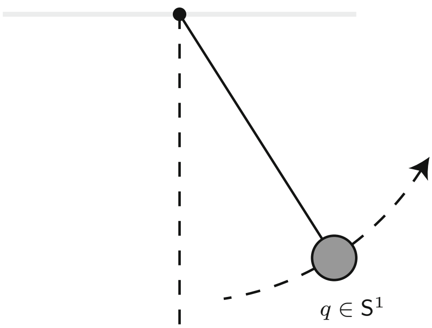

Fig. 2.3 Planar pendulum

This vector differential equation (2.3) is referred to as the rotational kinematics for a planar pendulum. It describes the rate of change of the configuration $\dot{q} \in \mathsf{T}_q\mathsf{S}^1$ in terms of $\omega \in \mathbb{R}^1$. The scalar ω is referred to as the angular velocity of the planar pendulum. Thus, the rotational kinematics of a planar pendulum can be viewed through the evolution of $(q, \dot{q}) \in \mathsf{TS}^1$ in the tangent bundle or through the evolution of $(q, \omega) \in \mathsf{S}^1 \times \mathbb{R}^1$.

Suppose the link length of the planar pendulum is L, so that the end of the link is located at $LQq \in \mathbb{R}^3$. Then the velocity vector of the mass particle is $LQ\dot{q} \in \mathbb{R}^3$. It is easy to see that the velocity vector of any material point on the link is proportional to $Q\dot{q} \in \mathbb{R}^3$. Thus, the rotational kinematics of the planar pendulum can be used to characterize the velocity of any point on the planar pendulum.

Now, suppose that the angular velocity is a smooth function of the configuration. The rotational kinematics equation (2.3) can be viewed as defining a smooth vector field on the manifold S^1. We are interested in initial-value problems associated with the rotational kinematics equation (2.3). The following result can be shown to hold: for any initial-value $q(t_0) = q_0 \in \mathsf{S}^1$, there exists a unique solution of (2.3) satisfying the specified initial-value and this unique solution satisfies $q(t) \in \mathsf{S}^1$ for all t.

2.7.4 Rotational Kinematics of a Spherical Pendulum

A spherical pendulum is an inextensible, rigid link that can rotate about an inertially fixed point on the link, referred to as the pivot. The origin of the inertial Euclidean frame is chosen to be the location of the fixed pivot point. Each material point on the pendulum link is therefore constrained to move on a sphere centered about the pivot, which is why it is referred to as a spherical pendulum. The direction vector $q \in \mathsf{S}^2$ of the link specifies the configuration of the spherical pendulum. The configuration manifold is S^2. Since the dimension of the configuration manifold is two, rotations of the spherical pendulum are said to have two degrees of freedom. This is shown in a schematic of the spherical pendulum in Figure 2.4.

Suppose that the function of time $t \to q \in \mathsf{S}^2$ represents a rotational motion of the spherical pendulum. Since $q \in \mathsf{S}^2$, it follows that the time derivative $\dot{q} \in \mathsf{T}_q\mathsf{S}^2$ is a tangent vector of S^2 at $q \in \mathsf{S}^2$. Thus, $\dot{q}$ is orthogonal to q, that is $(\dot{q} \cdot q) = 0$. This implies that there is a vector-valued function of time $t \to \omega \in \mathbb{R}^3$ such that

$$\dot{q} = \omega \times q,$$

or equivalently

$$\dot{q} = S(\omega)q, \tag{2.4}$$

where

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

is a 3×3 skew-symmetric matrix function introduced in (1.8). There is no loss of generality in requiring that $(\omega \cdot q) = 0$. This observation is true since $\omega \in \mathbb{R}^3$ can be decomposed into the sum of a part in the direction of q and a part that is orthogonal to q and the former part does not contribute to the value of $\dot{q}$ in equation (2.4).

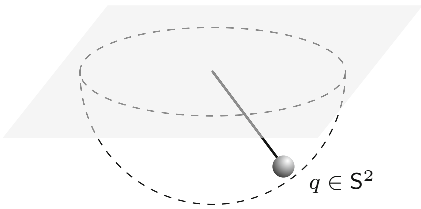

Fig. 2.4 Spherical pendulum

This vector differential equation (2.4) is referred to as the rotational kinematics for a spherical pendulum. It describes the rate of change of the configuration $\dot{q} \in \mathsf{T}_q\mathsf{S}^2$ in terms of the vector-valued function $\omega \in \mathbb{R}^3$. The vector ω is referred to as the angular velocity vector of the spherical pendulum. Thus, the rotational kinematics of a spherical pendulum can be viewed through the evolution of $(q, \dot{q}) \in \mathsf{TS}^2$ in the tangent bundle or through the evolution of $(q, \omega) \in \mathsf{S}^2 \times \mathbb{R}^3$. Notice that the dimensions of TS^2 and $\mathsf{S}^2 \times \mathbb{R}^3$ are different, which is because of the ambiguity in ω mentioned above. As before, this is resolved by requiring that $(\omega \cdot q) = 0$ for each $(q, \omega) \in \mathsf{S}^2 \times \mathbb{R}^3$.

Suppose the link length of the spherical pendulum is L, so that the end of the link is located at $Lq \in \mathbb{R}^3$ and the velocity vector of the point at the end of the link is $L\dot{q} \in \mathbb{R}^3$. It is easy to see that the velocity vector of any point along the massless link of the spherical pendulum is proportional to $\dot{q} \in \mathbb{R}^3$. Thus, the rotational kinematics of the spherical pendulum can be used to characterize the velocity of any material point on the spherical pendulum.

Suppose that the angular velocity vector is a smooth function of the configuration. The rotational kinematics equation (2.4) can be viewed as defining a smooth vector field on the manifold S^2. We are interested in initial-value problems associated with the rotational kinematics equation (2.4). The following result can be shown to hold: for any initial-value $q(t_0) = q_0 \in \mathsf{S}^2$, there exists a unique solution of (2.4) satisfying the specified initial-value and this unique solution satisfies $q(t) \in \mathsf{S}^2$ for all t.

2.7.5 Rotational Kinematics of a Double Planar Pendulum

The double planar pendulum is an interconnection of two planar pendulum links, with each link constrained to rotate in a common fixed vertical plane. The attitude of each link is defined by a direction vector in the fixed vertical plane. The first link rotates about a fixed one degree of freedom rotational joint or pivot while the second link is connected by a one degree of freedom rotational joint to the end of the first link. A schematic of the double planar pendulum is shown in Figure 2.5.

Here we introduce the attitude of a double planar pendulum as a pair of direction vectors of the two links, with each direction vector in the unit circle S^1. Thus, the configuration of the double planar pendulum is $q = (q_1, q_2) \in (\mathsf{S}^1)^2$. These two direction vectors are defined with respect to a fixed two-dimensional Euclidean frame, where the two axes define the plane of rotation of the two pendulums. The origin of the frame is located at the pivot. For simplicity, the plane of rotation is viewed as defined by the first two axes of a three-dimensional Euclidean frame. It follows that $(\mathsf{S}^1)^2$ is the two-dimensional configuration manifold of the double planar pendulum, which has two degrees of freedom.

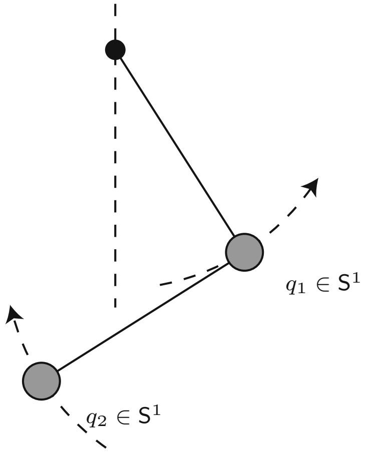

Fig. 2.5 Double planar pendulum

Suppose that a function of time $t \to q = (q_1, q_2) \in (\mathsf{S}^1)^2$ represents a rotational motion of the double planar pendulum. Since $q_i \in \mathsf{S}^1$, $i = 1, 2$, it follows that the time derivative $\dot{q}_i \in \mathsf{T}_{q_i}\mathsf{S}^1$ is a tangent vector of S^1 at q_i. Thus, $\dot{q}_i$ is orthogonal to q_i, that is $(\dot{q}_i \cdot q_i) = 0$, $i = 1, 2$. This implies that there is a function of time $t \to \omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ such that

$$\dot{q}_1 = \omega_1 \mathrm{S} q_1, \tag{2.5}$$

$$\dot{q}_2 = \omega_2 \mathrm{S} q_2. \tag{2.6}$$

As previously, S denotes the 2×2 skew-symmetric matrix given by (1.6).

These vector differential equations (2.5) and (2.6) are referred to as the rotational kinematics for a double planar pendulum. It describes the rate of change of the configuration $\dot{q} \in \mathsf{T}_q(\mathsf{S}^1)^2$ in terms of $\omega \in \mathbb{R}^2$. The vector $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ is referred to as the angular velocity vector of the double planar pendulum. Thus, the rotational kinematics of a double planar pendulum can be viewed through the evolution of $(q, \dot{q}) \in \mathsf{T}(\mathsf{S}^1)^2$ in the tangent bundle or through the evolution of $(q, \omega) \in (\mathsf{S}^1)^2 \times \mathbb{R}^2$.

Suppose the link lengths are L_1 and L_2 for the two planar pendulums. The end of the second planar pendulum, in the fixed Euclidean frame, is located at $L_1 Q q_1 + L_2 Q q_2 \in \mathbb{R}^3$, where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

so that the velocity vector of the end of the second spherical pendulum is $L_1 Q \dot{q}_1 + L_2 Q \dot{q}_2 \in \mathbb{R}^3$. It is easy to see that the velocity of any selected body-fixed point on either link of the double planar pendulum is a linear combination of $Q\dot{q}_1 \in \mathbb{R}^3$ and $Q\dot{q}_2 \in \mathbb{R}^3$. Thus, the rotational kinematics of the double planar pendulum can be used to characterize the velocity vector of any material point on either link.

Now, suppose that the angular velocity vector is a smooth function of the configuration. The rotational kinematics equations (2.5) and (2.6) can be viewed as defining a smooth vector field on the manifold $(\mathsf{S}^1)^2$. We are interested in initial-value problems associated with the rotational kinematics equation (2.5) and (2.6). The following result can be shown to hold: for any initial-value $q(t_0) = q_0 \in (\mathsf{S}^1)^2$, there exists a unique solution of (2.5) and (2.6) satisfying the specified initial-value and this unique solution satisfies $q(t) \in (\mathsf{S}^1)^2$ for all t.

2.7.6 Rotational Kinematics of a Double Spherical Pendulum

The double spherical pendulum is an interconnection of two spherical pendulum links, with each link able to rotate in three dimensions. The first link rotates about a fixed spherical pivot while the second link is connected by a spherical pivot located at some point on the first link. A schematic of the double spherical pendulum is shown in Figure 2.6.

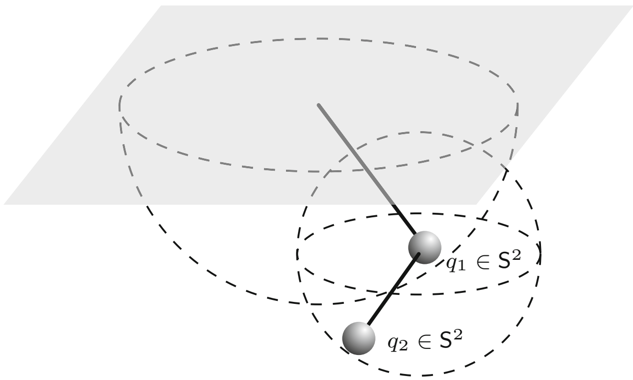

Fig. 2.6 Double spherical pendulum

Here we introduce the attitude of a double spherical pendulum as a pair of direction vectors for the two links, with each direction vector in the unit sphere S^2. Thus, the configuration of the double spherical pendulum is $q = (q_1, q_2) \in (\mathsf{S}^2)^2$. These two direction vectors are defined with respect to an inertially fixed three-dimensional Euclidean frame. The origin of the frame is located at the inertially fixed pivot. It follows that $(\mathsf{S}^2)^2$ is the four-dimensional configuration manifold of the double spherical pendulum, which has four degrees of freedom.

Suppose that a function of time $t \to q = (q_1, q_2) \in (\mathsf{S}^2)^2$ represents a rotational motion of the double spherical pendulum. Since $q_i \in \mathsf{S}^2$, $i = 1, 2$, it follows that the time derivative $\dot{q}_i \in \mathsf{T}_{q_i}\mathsf{S}^2$ is a tangent vector of S^2 at q_i. Thus, $\dot{q}_i$ is orthogonal to q_i, that is $(\dot{q}_i \cdot q_i) = 0$, $i = 1, 2$. This implies that there is a function of time $t \to \omega = (\omega_1, \omega_2) \in (\mathbb{R}^3)^2$ satisfying $(\omega_i \cdot q_i) = 0$, $i = 1, 2$, such that

$$\dot{q}_1 = S(\omega_1)q_1, \tag{2.7}$$

$$\dot{q}_2 = S(\omega_2)q_2, \tag{2.8}$$

where $S(\omega)$ is the 3×3 skew-symmetric matrix function introduced in (1.8).

These vector differential equations (2.7) and (2.8) are referred to as the rotational kinematics for a double spherical pendulum. It describes the rate of change of the configuration $\dot{q} \in \mathsf{T}_q(\mathsf{S}^2)^2$ in terms of $\omega \in (\mathbb{R}^3)^2$. The vector $\omega = (\omega_1, \omega_2) \in (\mathbb{R}^3)^2$ is referred to as the angular velocity vector of the double spherical pendulum. Thus, the rotational kinematics of a double spherical pendulum can be viewed through the evolution of $(q, \dot{q}) \in \mathsf{T}(\mathsf{S}^2)^2$ in the tangent bundle or through the evolution of $(q, \omega) \in (\mathsf{S}^2)^2 \times (\mathbb{R}^3)^2$.

Suppose the link lengths are L_1 and L_2 for the two spherical pendulums. The end of the second spherical pendulum, in the fixed Euclidean frame, is located at $L_1 q_1 + L_2 q_2 \in \mathbb{R}^3$, so that the velocity vector of the end of the second spherical pendulum is $L_1 \dot{q}_1 + L_2 \dot{q}_2 \in \mathbb{R}^3$. It is easy to see that the velocity of any selected body-fixed point on either link of the double spherical pendulum is a linear combination of $\dot{q}_1 \in \mathbb{R}^3$ and $\dot{q}_2 \in \mathbb{R}^3$. Thus, the rotational kinematics of the double spherical pendulum can be used to characterize the velocity vector of any material point on either link.

Now, suppose that the angular velocity vector is a smooth function of the configuration. The rotational kinematics equations (2.7) and (2.8) can be viewed as defining a smooth vector field on the manifold $(\mathsf{S}^2)^2$. We are interested in initial-value problems associated with the rotational kinematics equation (2.7) and (2.8). The following result can be shown to hold: for any initial-value $q(t_0) = q_0 \in (\mathsf{S}^2)^2$, there exists a unique solution of (2.7) and (2.8) satisfying the specified initial-value and this unique solution satisfies $q(t) \in (\mathsf{S}^2)^2$ for all t.

2.7.7 Rotational Kinematics of a Planar Pendulum Connected to a Spherical Pendulum

We consider a planar pendulum that can rotate in a fixed plane about a one degree of freedom rotational pivot whose axis is perpendicular to the plane of rotation. One end of a spherical pendulum is connected to the end of the planar pendulum by a two degree of freedom rotational pivot; the spherical pendulum can rotate in three dimensions. An inertially fixed Euclidean frame in three dimensions is constructed so that its first two axes define the plane of rotation of the planar pendulum and its third axis is orthogonal. A schematic of the connection of a planar pendulum and a spherical pendulum is shown in Figure 2.7.

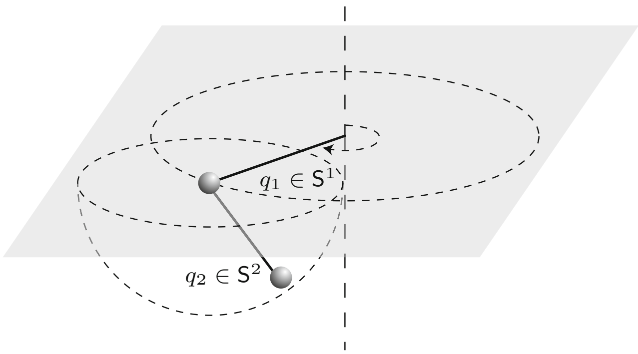

Fig. 2.7 Connection of a planar pendulum and a spherical pendulum

The configuration of the planar pendulum $q_1 \in \mathsf{S}^1$ is defined by the direction vector of the planar link, with respect to the first two axes of the Euclidean frame that define the plane of rotation of the link. Thus, S^1 is the configuration manifold of the planar pendulum. The configuration of the spherical pendulum $q_2 \in \mathsf{S}^2$ is defined as the direction vector of the spherical link, with respect to the Euclidean frame. Thus, S^2 is the configuration manifold of the spherical pendulum. It follows that $\mathsf{S}^1 \times \mathsf{S}^2$ is the three-dimensional configuration manifold of the connection of the planar pendulum and the spherical pendulum. The connection has three degrees of freedom.

Suppose the function of time $t \to (q_1, q_2) \in \mathsf{S}^1 \times \mathsf{S}^2$ represents a motion of the connection of a planar pendulum and a spherical pendulum. As seen previously for a planar pendulum, the time derivative $\dot{q}_1 \in \mathsf{T}_{q_1}\mathsf{S}^1$ is a tangent vector of S^1 at q_1. Thus, $\dot{q}_1$ is orthogonal to q_1. This implies that there is a scalar-valued function of time $t \to \omega_1 \in \mathbb{R}^1$, referred to as the scalar angular velocity of the planar pendulum about its joint axis, such that

$$\dot{q}_1 = \omega_1 \mathrm{S} q_1. \tag{2.9}$$

Here, S denotes the 2×2 skew-symmetric matrix given by (1.6) that rotates a vector by $\frac{\pi}{2}$ counterclockwise.

As seen previously for the spherical pendulum, the time derivative $\dot{q}_2 \in \mathsf{T}_{q_2}\mathsf{S}^2$ is a tangent vector of S^2 at q_2. Thus, $\dot{q}_2$ is orthogonal to q_2. This implies that there is a vector-valued function of time $t \to \omega_2 \in \mathbb{R}^3$, referred to as the angular velocity vector of the spherical pendulum, such that

$$\dot{q}_2 = S(\omega_2)q_2. \tag{2.10}$$

The skew-symmetric matrix function is defined by (1.8).

These vector differential equations (2.9) and (2.10) are referred to as the rotational kinematics for a connection of a planar pendulum and a spherical pendulum. They describe the rate of change of the configuration $\dot{q} = (\dot{q}_1, \dot{q}_2) \in \mathsf{T}_{(q_1,q_2)}(\mathsf{S}^1 \times \mathsf{S}^2)$ in terms of the angular velocities $\omega = (\omega_1, \omega_2) \in \mathbb{R}^1 \times \mathbb{R}^3$. The rotational kinematics can be viewed through the evolution of $(q, \dot{q}) \in \mathsf{T}(\mathsf{S}^1 \times \mathsf{S}^2)$ in the tangent bundle or through the evolution of $(q, \omega) \in \mathsf{S}^1 \times \mathsf{S}^2 \times \mathbb{R}^1 \times \mathbb{R}^3$.

Suppose the link length from the inertially fixed pivot to the pivot connecting the two links is L_1 for the planar pendulum and the link length from the spherical pivot to the end of the spherical pendulum is L_2. The position vector of the end of the spherical pendulum, in the fixed Euclidean frame, is $L_1 Q q_1 + L_2 q_2 \in \mathbb{R}^3$, where

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which defines an embedding of $\mathbb{R}^2$ into $\mathbb{R}^3$. Then the velocity vector of the end of the spherical pendulum is $L_1 Q \dot{q}_1 + L_2 \dot{q}_2 \in \mathbb{R}^3$. It is easy to see that the velocity of any body-fixed point on either link of the connection of a planar pendulum and a spherical pendulum is a linear combination of $Q\dot{q}_1 \in \mathbb{R}^3$ and $\dot{q}_2 \in \mathbb{R}^3$. Thus, the rotational kinematics of the connection of a planar pendulum and a spherical pendulum can be used to characterize the velocity vector of any material point in either link.

Suppose that the angular velocities are smooth functions of the configuration. The rotational kinematics equations can be viewed as defining a smooth vector field on the manifold $\mathsf{S}^1 \times \mathsf{S}^2$. The following result for the initial-value problem holds: for any initial-value $q(t_0) = q_0 \in \mathsf{S}^1 \times \mathsf{S}^2$, there exists a unique solution of (2.9) and (2.10) satisfying the specified initial-value and this unique solution satisfies $q(t) \in \mathsf{S}^1 \times \mathsf{S}^2$ for all t.

2.7.8 Kinematics of a Particle on a Torus

Consider an ideal particle that is constrained to move on the surface of a torus or *doughnut* in $\mathbb{R}^3$, where the torus is the surface of revolution generated by revolving a circle about an axis, coplanar with the circle, that does not touch the circle. Without loss of generality, it is assumed that the torus has major radius $R > 0$ which is the distance from the axis of the torus to the center of the circle and minor radius $0 < r < R$ which is the radius of the revolved circle. A schematic of the particle on a torus is shown in Figure 2.8.

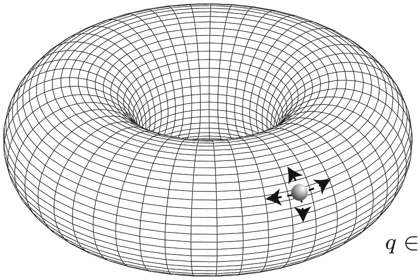

Fig. 2.8 Particle constrained to a torus

An inertially fixed Euclidean frame is constructed so that the center of the revolved circle is located at $(R, 0, 0) \in \mathbb{R}^3$, the circle lies in the plane defined by the first and third axes and the axis of the torus is the third axis of the Euclidean frame.

The position vector of the particle on the torus is denoted by $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The configuration of the particle on the torus can be selected as $q = (q_1, q_2) \in (\mathsf{S}^1)^2$, and we show that this uniquely determines the position vector of the particle on the torus. Thus, the configuration manifold is $(\mathsf{S}^1)^2$ and the particle has two degrees of freedom. We describe the kinematics of the particle on the torus by expressing these kinematics in terms of the evolution on the configuration manifold.

This example should be contrasted with the double planar pendulum example which has the same configuration manifold. It is common to visualize the flow of the double planar pendulum on a torus in $\mathbb{R}^3$, but the values of the major radius and the minor radius of the torus are irrelevant to this visualization. In contrast, the kinematics of the particle on a torus depend on the major radius and the minor radius of the torus, as is subsequently shown. The formulation of the kinematics of a particle on a torus in $\mathbb{R}^3$, as developed here, seems not to have appeared previously in the published literature.

The torus in $\mathbb{R}^3$ can be defined parametrically in terms of two angles

$$\begin{aligned} x_1 &= (R + r\cos\phi)\cos\theta, \\ x_2 &= (R + r\cos\phi)\sin\theta, \\ x_3 &= r\sin\phi, \end{aligned}$$

but this description leads to an ambiguity in the description of the kinematics.

Alternatively, the torus in $\mathbb{R}^3$ can be defined implicitly by the constraint equation

$$\left(R - \sqrt{x_1^2 + x_2^2}\right)^2 + x_3^2 - r^2 = 0.$$

This formulation can be the basis for describing the kinematics of a particle moving on a torus in terms of a constraint manifold embedded in $\mathbb{R}^3$. This leads to kinematics equations described in terms of differential-algebraic equations on this constraint manifold. We do not develop this formulation any further.

Now, we describe a geometric approach in terms of the configuration manifold $(\mathsf{S}^1)^2$. We first express the position of the particle on the torus $x \in \mathbb{R}^3$ in terms of the configuration $q = (q_1, q_2) \in (\mathsf{S}^1)^2$. The geometry of the torus implies that an arbitrary vector $x \in \mathbb{R}^3$ on the torus can be uniquely decomposed, in the Euclidean frame, into the sum of a vector from the origin to the center of the embedded circle on which x lies and a vector from the center of this embedded circle to x. This decomposition can be expressed as

$$x = \begin{bmatrix} (R + r(e_1^T q_2))(e_1^T q_1) \\ (R + r(e_1^T q_2))(e_2^T q_1) \\ r(e_2^T q_2) \end{bmatrix}, \tag{2.11}$$

where e_1, e_2 denote the standard basis vectors in $\mathbb{R}^2$. It is easy to see that this is consistent with the parametric representation in terms of two angles given above, but avoids the ambiguity of the angular representation associated with angles that differ by multiples of 2π. This decomposition demonstrates that the position vector of the particle on the torus depends on the configuration of the particle and on the values of the major radius and minor radius of the torus.

The kinematics for the motion of a particle on a torus in $\mathbb{R}^3$ are easily obtained. The velocity vector of the particle on the torus is described by

$$\dot{x} = \begin{bmatrix} (R + r(e_1^T q_2))e_1^T \\ (R + r(e_1^T q_2))e_2^T \\ 0 \end{bmatrix} \dot{q}_1 + \begin{bmatrix} r(e_1^T q_1)e_1^T \\ r(e_2^T q_1)e_1^T \\ re_2^T \end{bmatrix} \dot{q}_2. \tag{2.12}$$

As we have seen previously, there exists an angular velocity vector that is a function of time $t \to \omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ such that the configuration kinematics for $q = (q_1, q_2) \in (\mathsf{S}^1)^2$ are given by

$$\dot{q}_1 = \omega_1 \mathrm{S} q_1, \tag{2.13}$$

$$\dot{q}_2 = \omega_2 \mathrm{S} q_2, \tag{2.14}$$

where S is the 2×2 skew-symmetric matrix that rotates a vector by $\frac{\pi}{2}$ counterclockwise. Thus, the velocity vector of the particle on the torus can also be described by

$$\dot{x} = \begin{bmatrix} -(R + r(e_1^T q_2))e_2^T q_1 \\ (R + r(e_1^T q_2))e_1^T q_1 \\ 0 \end{bmatrix} \omega_1 + \begin{bmatrix} -r(e_1^T q_2)e_2^T q_2 \\ -r(e_1^T q_2)e_2^T q_2 \\ r(e_1^T q_2) \end{bmatrix} \omega_2. \tag{2.15}$$

The vector differential equation (2.12) or equivalently the vector differential equation (2.15), together with (2.13) and (2.14), are referred to as the kinematics of a particle on a torus. They describe the rates of change of the configuration $\dot{q} \in \mathsf{T}_q(\mathsf{S}^1)^2$ and the particle velocity vector $\dot{x} \in \mathbb{R}^3$ on the torus. The scalars ω_1 and ω_2 are referred to as the angular velocities of the particle on the torus. Thus, the translational kinematics of the particle on a torus can be viewed through the evolution of $(q, \dot{q}) \in \mathsf{T}(\mathsf{S}^1)^2$ in the tangent bundle or through the evolution of $(q, \omega) \in (\mathsf{S}^1)^2 \times (\mathbb{R}^1)^2$.

Suppose that the angular velocities are a smooth function of the configuration. The rotational kinematics (2.13) and (2.14) can be viewed as defining a smooth vector field on the manifold $(\mathsf{S}^1)^2$. We are interested in initial-value problems associated with these kinematics equations. The following results can be shown to hold: for any initial-value $q(t_0) = q_0 \in (\mathsf{S}^1)^2$, there exists a unique solution of (2.13) and (2.14) satisfying the specified initial-value and this unique solution satisfies $q(t) \in (\mathsf{S}^1)^2$ for all t. Each such solution results in a unique solution of (2.12) or (2.15) which has the property: if $x(t_0) \in \mathbb{R}^3$ is on the torus, then $x(t) \in \mathbb{R}^3$ remains on the torus for all t.

2.7.9 Rotational Kinematics of a Free Rigid Body

A rigid body is free to rotate in three dimensions without constraint. In addition to an inertially fixed Euclidean frame, it is convenient to define another Euclidean frame that is fixed to the rigid body. That is, the body-fixed Euclidean frame rotates with the body; in fact, we describe the rotation of the body through the rotation of the body-fixed Euclidean frame. In formal terms, the configuration of a rotating rigid body in three dimensions is the linear transformation that relates the representation of a vector in the body-fixed Euclidean frame to the representation of that vector in the inertially fixed Euclidean frame.

The matrix representing this linear transformation can be constructed by computing the direction cosines between the unit vectors that define the axes of the inertially fixed Euclidean frame and the unit vectors that define the axes of the body-fixed Euclidean frame. This attitude configuration is referred to as a rotation matrix, attitude matrix, or direction cosine matrix of the rigid body. It can be shown that such matrices are orthogonal with determinant $+1$, so that each rotation or attitude matrix $R \in \mathsf{SO}(3)$. Hence, the configuration manifold is $\mathsf{SO}(3)$, which, as seen previously, is a Lie group. Since the dimension of the configuration manifold is three, rigid body rotations are said to have three degrees of freedom. A schematic of a rotating free rigid body is shown in Figure 2.9.

Suppose that the function of time $t \to R \in \mathsf{SO}(3)$ represents a rotational motion of a free rigid body. Since R is an orthogonal matrix, it follows that $R^T R = I_{3\times 3}$. Differentiating, we obtain $\dot{R}^T R = -R^T \dot{R}$ which implies that

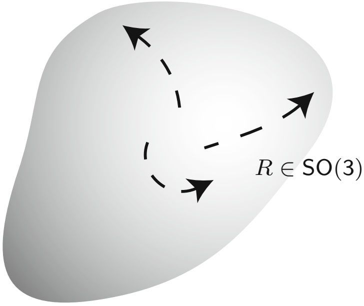

Fig. 2.9 Rotating free rigid body

$R^T\dot{R}$ is always skew-symmetric. This implies that there is a vector-valued function of time $t \to \omega \in \mathbb{R}^3$ such that $R^T\dot{R} = S(\omega)$ where we have used (1.8). This can be written as

$$\dot{R} = RS(\omega), \tag{2.16}$$

which shows that $\dot{R} \in \mathsf{T}_R\mathsf{SO}(3)$, that is $\dot{R}$ is in the tangent space of $\mathsf{SO}(3)$ at $R \in \mathsf{SO}(3)$. This should come as no surprise, since the set of tangent vectors to a curve that takes values on a manifold are the very definition of the tangent space to a manifold. Recall that $S(\omega) \in \mathfrak{so}(3)$, where $\mathfrak{so}(3)$ is the space of skew-symmetric matrices, which is the Lie algebra associated with the Lie group $\mathsf{SO}(3)$. Thus, the time derivative $\dot{R}$ can be expressed as shown in (2.16) so that $S(\omega) = R^T\dot{R} \in \mathfrak{so}(3)$.

Alternatively, we can use the matrix identity $S(R\omega) = RS(\omega)R^T$ to obtain the rotational kinematics for a rotating rigid body given by

$$\dot{R} = S(R\omega)R. \tag{2.17}$$

This alternate form of the rotational kinematics describes the rate of change of the configuration in terms of $R\omega$ which is the angular velocity vector of the rigid body represented in the inertially fixed Euclidean frame, sometimes referred to as the spatial angular velocity of the rigid body. Although (2.16) and (2.17) are equivalent, the form (2.16) using the body-fixed angular velocity is most convenient; we will most often use the rotational kinematics expressed in terms of the body-fixed angular velocity.

Yet another perspective is to view the attitude configuration $R \in \mathsf{SO}(3)$ by partitioning into three rows or equivalently by partitioning $R^T \in \mathsf{SO}(3)$ into three columns, that is

$$R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}, \quad R^T = \begin{bmatrix} r_1 \ r_2 \ r_3 \end{bmatrix}.$$

The attitude or rotational kinematics of a rotating rigid body can also be described by the three vector differential equations

$$\dot{r}_i = S(r_i)\omega, \quad i = 1, 2, 3. \tag{2.18}$$

The matrix differential equation (2.16) or (2.17), or equivalently the vector differential equations (2.18), are referred to as the rotational kinematics of a free rigid body. They describe the rates of change of the configuration $\dot{R} \in \mathsf{T}_R\mathsf{SO}(3)$ in terms of the angular velocity $\omega \in \mathbb{R}^3$ represented in the body-fixed frame. Thus, the rotational kinematics of a free rigid body can be viewed through the evolution of $(R, \dot{R}) \in \mathsf{TSO}(3)$ in the tangent bundle of $\mathsf{SO}(3)$ or through the evolution of $(R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$.

Suppose that the angular velocity is a smooth function of the configuration. The rotational kinematics can be viewed as defining a smooth vector field on the Lie group manifold $\mathsf{SO}(3)$. We are interested in initial-value problems associated with these rotational kinematics equations. The following results can be shown to hold: for any initial-value $R(t_0) = R_0 \in \mathsf{SO}(3)$, there exists a unique solution of the kinematics differential equations satisfying the specified initial-value and this unique solution satisfies $R(t) \in \mathsf{SO}(3)$ for all t.

2.7.10 Rotational and Translational Kinematics of a Rigid Body Constrained to a Fixed Plane

Planar Euclidean motion of a rigid body occurs if each point in the body is constrained to move in a fixed two-dimensional plane. Select a point fixed in the rigid body and define a two-dimensional inertial Euclidean frame for this fixed plane within which the selected point moves. Additionally, define a body-fixed Euclidean frame centered at the selected point and spanned by unit vectors e_1 and e_2.

Here, the configuration is taken as $(q, x) \in \mathsf{S}^1 \times \mathbb{R}^2$, where $q \in \mathsf{S}^1$ denotes the unit vector e_1 with respect to the inertially fixed Euclidean frame and $x \in \mathbb{R}^2$ denotes the position vector of the selected point in the Euclidean frame. In this case, the configuration manifold has a product form. A schematic of a rotating and translating rigid body constrained to a plane is shown in Figure 2.10.

Suppose the function of time $t \to (q, x) \in \mathsf{S}^1 \times \mathbb{R}^2$ represents a rotational and translational motion of the constrained rigid body. Since $q \in \mathsf{S}^1$ there is a scalar function $t \to \omega \in \mathbb{R}^1$, referred to as the scalar angular velocity of the body and there is a translational velocity vector for the body-fixed point that is a function of time $t \to v \in \mathbb{R}^2$ such that

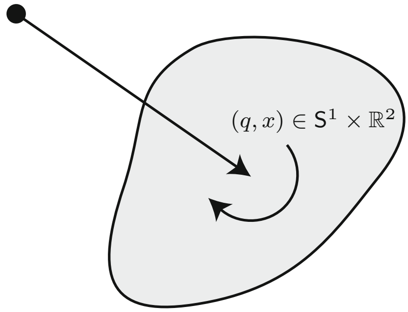

Fig. 2.10 Rotating and translating rigid body in planar motion

$$\dot{q} = \omega \mathrm{S} q, \tag{2.19}$$

$$\dot{x} = \left[q, \mathrm{S}q\right] v, \tag{2.20}$$

where S denotes the 2×2 skew-symmetric matrix (1.6) that rotates a vector by $\frac{\pi}{2}$ counterclockwise and the 2×2 partitioned matrix in (2.20) consists of columns $q \in \mathsf{S}^1$ and $\mathrm{S}q \in \mathsf{S}^1$. Note that $v \in \mathbb{R}^2$ represents the translational velocity of the selected body-fixed point. The rotational and translational kinematics (2.19) and (2.20) can be viewed through the evolution of $(q, x, \dot{q}, \dot{x}) \in \mathsf{T}(\mathsf{S}^1 \times \mathbb{R}^2)$ in the tangent bundle or through the evolution of $(q, x, \omega, v) \in \mathsf{S}^1 \times \mathbb{R}^2 \times \mathbb{R}^1 \times \mathbb{R}^2$.

Now suppose that the angular velocity scalar ω and the translational velocity vector v are smooth functions of the configurations. The kinematics equations can be viewed as defining a smooth vector field on the configuration manifold $\mathsf{S}^1 \times \mathbb{R}^2$. We are interested in initial-value problems associated with the kinematics equations (2.19) and (2.20). The following result can be shown to hold: for any initial-values $(q(t_0), x(t_0)) = (q_0, x_0) \in \mathsf{S}^1 \times \mathbb{R}^2$, there exists a unique solution of (2.19) and (2.20) satisfying the specified initial-values and this unique solution satisfies $(q(t), x(t)) \in \mathsf{S}^1 \times \mathbb{R}^2$ for all t.

2.7.11 Rotational and Translational Kinematics of a Free Rigid Body

A rigid body is free to translate and rotate in three dimensions. As previously, the configuration is defined in terms of an inertially fixed Euclidean frame and a body-fixed Euclidean frame. The configuration of a translating and rotating rigid body in three dimensions consists of an ordered pair $(R, x) \in \mathsf{SE}(3)$, where $R \in \mathsf{SO}(3)$ is a rotation matrix describing the attitude of the rigid body and $x \in \mathbb{R}^3$ is the vector describing the position of a point in the body, typically the origin of the body-fixed frame, in the inertial frame. The pair $(R, x) \in \mathsf{SE}(3)$ can be viewed as a homogenous matrix in $\mathsf{GL}(4)$. Hence, the configuration manifold is $\mathsf{SE}(3)$, which is a Lie group. Since the dimension of the configuration manifold is six, rigid body rotations and translations are said to have six degrees of freedom. A schematic of a rotating and translating free rigid body is shown in Figure 2.11.

Suppose that the function of time $t \to R(t) \in \mathsf{SO}(3)$ is a rotational motion and the position vector of the origin of the body-fixed Euclidean frame with respect to the inertial Euclidean frame is described by the function of time $t \to x(t) \in \mathbb{R}^3$, which defines a translational motion of the rigid body. This implies that there are vector-valued functions of time $t \to \omega \in \mathbb{R}^3$ and $t \to v \in \mathbb{R}^3$ such that the kinematics for Euclidean motion of a rigid body are given by

$$\dot{R} = RS(\omega), \tag{2.21}$$

$$\dot{x} = Rv, \tag{2.22}$$

which shows that $(\dot{R}, \dot{x}) \in \mathsf{T}_{(R,x)}\mathsf{SE}(3)$. We have used the 3×3 skew-symmetric matrix function given by (1.8).

As in the prior section, it can be shown that if $r_i = R^T e_i$, $i = 1, 2, 3$ then $r_i \in \mathsf{S}^2$, $i = 1, 2, 3$ and the kinematics can be expressed as

$$\dot{r}_i = S(r_i)\omega, \quad i = 1, 2, 3, \tag{2.23}$$

$$\dot{x} = \begin{bmatrix} r_1^T v \\ r_2^T v \\ r_3^T v \end{bmatrix}, \tag{2.24}$$

which also implies that $(\dot{R}, \dot{x}) \in \mathsf{T}_{(R,x)}\mathsf{SE}(3)$.

Equations (2.21) and (2.22), or equivalently equations (2.23) and (2.24), reflect the fact that the Lie algebra $\mathfrak{se}(3)$ associated with the Lie group $\mathsf{SE}(3)$ can be identified with $\mathfrak{so}(3) \times \mathbb{R}^3$.

These differential equations are referred to as the Euclidean kinematics or the rotational and translational kinematics for a rigid body. The matrix differential equation (2.21) describes the rate of change of the rotational configuration in terms of the angular velocity vector $\omega \in \mathbb{R}^3$, represented in the body-fixed frame. The vector differential equation (2.22) is referred to as the translational kinematics for a rigid body. It relates the translational velocity vector $\dot{x} \in \mathbb{R}^3$ represented in the inertial Euclidean frame to the translational velocity vector v represented in the body-fixed Euclidean frame. Thus, the rotational and translational kinematics of a free rigid body can be viewed through the evolution of $(R, x, \dot{R}, \dot{x}) \in \mathsf{TSE}(3)$ in the tangent bundle or through the evolution of $(R, x, \omega, v) \in \mathsf{SE}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$.

Suppose that the rotational and translational velocity vectors $\omega \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ are smooth functions of the configuration. The translational and rotational kinematics equations (2.21) and (2.22) can be viewed as defining a smooth vector field on the Lie group $\mathsf{SE}(3)$. We are interested in initial-value problems. The following result can be shown to hold: for any initial-value

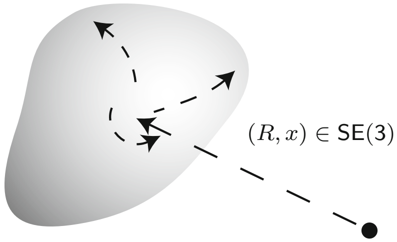

Fig. 2.11 Rotating and translating free rigid body

$(R(t_0), x(t_0)) = (R_0, x_0) \in \mathsf{SE}(3)$, there exists a unique solution of (2.21) and (2.22) satisfying the specified initial-values and this unique solution satisfies $(R(t), x(t)) \in \mathsf{SE}(3)$ for all t.

2.7.12 Translational Kinematics of a Rigid Link with Ends Constrained to Slide Along a Straight Line and a Circle in a Fixed Plane

One end of a rigid link is constrained to slide along a straight line; the other end of the link is constrained to slide along a circle of radius r. For simplicity, assume that the straight line and the circle lie in a common plane, with the straight line passing through the center of the circle. The length of the rigid link is L and assume $r < L$.

Since the ends of the link are constrained, this physical system is referred to as a slider-crank mechanism that can be used to transform circular motion of one end of the link to translational motion of the other end of the link, or vice versa. We now study the kinematics of the slider-crank mechanism. A schematic of this mechanism is shown in Figure 2.12.

A two-dimensional inertially fixed Euclidean frame is constructed with the first axis along the straight line and the second axis orthogonal to the first axis; the origin of the frame is located at the center of the circle.

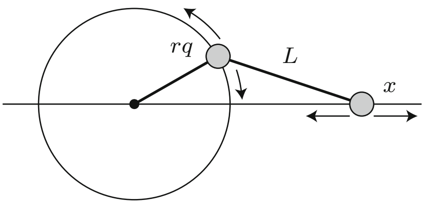

Fig. 2.12 Rigid link with ends constrained to slide along a straight line and a circle

Let $q \in \mathsf{S}^1$ denote the direction vector of the end of the link that moves on the circle in the fixed frame; thus, rq is the position vector of the other end of the link in the fixed frame. Let $x \in \mathbb{R}^1$ denote the position of the end of the link that moves on the straight line in the fixed frame. Thus, $(q, x) \in \mathsf{S}^1 \times \mathbb{R}^1$, but this is constrained by the fixed length of the rigid link. This constraint is given by

$$\|rq - xe_1\|^2 = L^2.$$

It can be shown that

$$M = \left\{(q, x) \in \mathsf{S}^1 \times \mathbb{R}^1 : \|rq - xe_1\|^2 - L^2 = 0\right\}$$

is a manifold that characterizes all possible configurations of the physical mechanism. This configuration manifold has dimension one; thus, the mechanism has one degree of freedom.

Suppose that the function of time $t \to (q, x) \in M$ defines a motion for the mechanism. Since one end of the link is constrained to the circle, that is $q \in \mathsf{S}^1$, it follows that there is a scalar angular velocity $t \to \omega \in \mathbb{R}^1$ such that

$$\dot{q} = \omega \mathrm{S} q,$$

where S is the 2×2 skew-symmetric matrix given by (1.6) that rotates a vector by $\frac{\pi}{2}$ counterclockwise.

Further, since $(\dot{q}, \dot{x}) \in \mathsf{T}_{(q,x)} M$, the time derivative of the configuration must satisfy

$$(rq - xe_1)^T (r\omega \mathrm{S} q - e_1 \dot{x}) = 0.$$

Since $\dot{x}$ is a scalar, some algebra shows that the time derivative of the configuration can be expressed as

$$\begin{bmatrix} \dot{q} \\ \dot{x} \end{bmatrix} = \omega \begin{bmatrix} I_{2\times 2} \\ \frac{-rxe_1^T}{(re_1^T q - x)} \end{bmatrix} \mathrm{S} q. \tag{2.25}$$

This defines the kinematics for the mechanism by describing the time derivative of the configuration as a tangent vector of the configuration manifold M. Note that the assumption that $r < L$ guarantees that $re_1^T q - x \neq 0$ on M.

Thus, (2.25) guarantees that $(\dot{q}, \dot{x}) \in \mathsf{T}_{(q,x)} M$. The kinematics can be viewed through the evolution of $(q, x, \dot{q}, \dot{x}) \in \mathsf{T} M$ in the tangent bundle of M or through the evolution of $(q, x, \omega, \dot{x}) \in M \times \mathbb{R}^2$.

If the angular velocity $\omega \in \mathbb{R}^1$ is a smooth function of the configuration, the differential equations (2.25) define a smooth vector field on the configuration manifold M. It follows that for each initial condition $(q(t_0), x(t_0)) = (q_0, x_0) \in M$ there exists a unique solution $(q(t), x(t)) \in M$ for all t.

2.7.13 Rotational and Translational Kinematics of a Constrained Rigid Rod

A thin rigid rod is viewed as a rigid body in three dimensions; the end points of the rigid rod are constrained to move on a fixed, rigid sphere. An inertially fixed Euclidean frame is constructed so that its origin is located at the center of the fixed sphere. A body-fixed Euclidean frame is constructed so that its origin is located at the centroid of the rod and the third body-fixed axis is along the minor principal axis of the rod. The length of the rod along its minor principal axis is L. The radius of the fixed sphere is r and we assume that $r > \frac{L}{2}$. The two ends of the rigid rod are constrained to move in contact with the fixed sphere. This gives rise to two scalar constraints that the ends of the rigid rod maintain contact with the fixed sphere. A schematic of the constrained rigid rod is given in Figure 2.13.

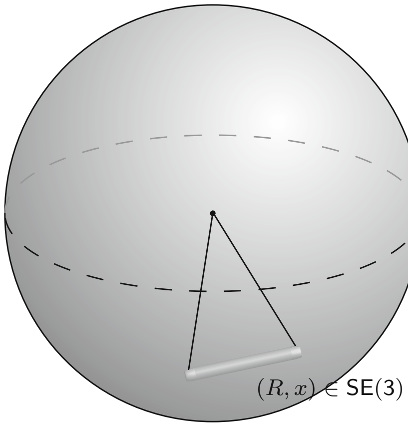

Fig. 2.13 A constrained rigid rod

Let $R \in \mathsf{SO}(3)$ denote the attitude of the rigid rod and let $x \in \mathbb{R}^3$ denote the position vector of the center point of the rod in the fixed Euclidean frame. The two constraint equations that encode the contact between the ends of the rod and the sphere are

$$\|x\|^2 - r^2 + \left(\frac{L}{2}\right)^2 = 0, \tag{2.26}$$

$$x^T R e_3 = 0. \tag{2.27}$$

The position vector to one end of the rod is $x_1 = x + \frac{L}{2} R^T e_3 \in \mathbb{R}^3$ while the position vector to the other end of the rod is $x_2 = x - \frac{L}{2} R^T e_3 \in \mathbb{R}^3$. Simple computations show that $\|x_1\|^2 = r^2$ and $\|x_2\|^2 = r^2$ so that each end of the rod is in contact with the sphere.

Thus, the configuration manifold is

$$M = \left\{ (R, x) \in \mathsf{SE}(3) : \|x\|^2 - r^2 + \left(\frac{L}{2}\right)^2 = 0,\ x^T R e_3 = 0 \right\}. \tag{2.28}$$

It can be shown that the dimension of the configuration manifold is four so that the rigid rod, constrained to move on the sphere, has four degrees of freedom.

Consider a motion of the constrained rigid rod given by $t \to (R, x) \in M$. This motion implies that there is an angular velocity vector $\omega \in \mathbb{R}^3$ and a translational velocity vector $v \in \mathbb{R}^3$ such that

$$\dot{R} = RS(\omega), \tag{2.29}$$

$$\dot{x} = Rv, \tag{2.30}$$

using the 3×3 skew-symmetric matrix function given by (1.8). Here, $\omega \in \mathbb{R}^3$ is the angular velocity vector of the rigid rod in the body-fixed frame and $v \in \mathbb{R}^3$ is the translational velocity vector of the rigid rod in the body-fixed frame.

Differentiating the constraints (2.26) and (2.27) and using the kinematics results in

$$x^T R v = 0, \tag{2.31}$$

$$e_3^T v + e_3^T R^T S(x) R \omega = 0. \tag{2.32}$$

Thus, if $(\dot{R}, \dot{x})$ satisfy (2.29) and (2.30) and the velocities $(\omega, v) \in \mathbb{R}^6$ satisfy (2.31) and (2.32), then $(\dot{R}, \dot{x}) \in \mathsf{T}_{(R,x)}M$. The rotational and translational kinematics of the constrained rigid rod can be viewed through the evolution of $(R, x, \dot{R}, \dot{x}) \in \mathsf{T}M$ in the tangent bundle of M or through the evolution of $(R, x, \omega, v) \in M \times \mathbb{R}^6$ where (ω, v) are subject to the constraints (2.31) and (2.32). These differential equations define the constrained kinematics on the configuration manifold M.

Suppose that the angular velocity vector and the translational velocity vector $\omega \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ are smooth functions of the configuration $(R, x) \in M$ that satisfy (2.31) and (2.32). It can be shown that if the initial conditions $R(t_0) = R_0 \in \mathsf{SO}(3)$, $x(t_0) = x_0 \in \mathbb{R}^3$ satisfy $(R_0, x_0) \in M$, then there exists a unique solution $(R(t), x(t)) \in M$ for all t.

2.8 Problems

2.1. A particle moves on a curve embedded in $\mathbb{R}^2$ described by the manifold $M = \left\{x \in \mathbb{R}^2 : x_1 - (x_2)^2 = 0\right\}$.

(a) Describe the geometry of the manifold.
(b) Describe the translational kinematics of the particle on the curve in terms of a single kinematics parameter.
(c) Interpret the geometric significance of the kinematics parameter you selected.

2.2. A particle moves on a plane embedded in $\mathbb{R}^3$ described by the manifold $M = \left\{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 - 1 = 0\right\}$.

(a) Describe the geometry of the manifold.
(b) Describe the translational kinematics of the particle on the plane in terms of two kinematics parameters.
(c) Interpret the geometric significance of the two kinematics parameters you selected.

2.3. A particle moves on a plane embedded in $\mathbb{R}^3$ described by the manifold $M = \left\{x \in \mathbb{R}^3 : x_1 - x_2 + x_3 - 1 = 0\right\}$.

(a) Describe the geometry of the manifold.
(b) Describe the translational kinematics of the particle on the plane in terms of two kinematics parameters.
(c) Interpret the geometric significance of the two kinematics parameters you selected.

2.4. A particle moves on a surface embedded in $\mathbb{R}^3$ described by the manifold $M = \left\{x \in \mathbb{R}^3 : (x_1)^2 - x_2 + x_3 = 0\right\}$.

(a) Describe the geometry of the manifold.
(b) Describe the translational kinematics of the particle on the surface in terms of two kinematics parameters.
(c) Interpret the geometric significance of the two kinematics parameters you selected.

2.5. A particle is constrained to move on a surface embedded in $\mathbb{R}^3$ described by the manifold $M = \left\{x \in \mathbb{R}^3 : (x_1)^2 + (x_2)^2 - x_3 = 0\right\}$.

(a) Describe the geometry of the manifold.
(b) Describe the translational kinematics of the particle on the surface in terms of two kinematics parameters.
(c) Interpret the geometric significance of the two kinematics parameters you selected.

2.6. A particle is constrained to move on a line embedded in $\mathbb{R}^3$ described by the manifold $M = \left\{x \in \mathbb{R}^3 : x_1 - x_2 = 0,\, x_1 + x_2 - x_3 = 0\right\}$.

(a) Describe the geometry of the manifold.
(b) Describe the translational kinematics of the particle on the line in terms of one kinematics parameter.
(c) Interpret the geometric significance of the one kinematics parameter you selected.

2.7. A particle is constrained to move on a curve embedded in $\mathbb{R}^3$ described by the manifold $M = \left\{x \in \mathbb{R}^3 : x_1 - x_2 = 0,\ (x_1)^2 + (x_2)^2 - x_3 = 0\right\}$.

(a) Describe the geometry of the manifold.
(b) Describe the translational kinematics of the particle on the curve in terms of one kinematics parameter.
(c) Interpret the geometric significance of the one kinematics parameter you selected.

2.8. Let $R > 0$ and $L > 0$. A particle moves on a curve embedded in $\mathbb{R}^3$ given by the manifold $M = \left\{x \in \mathbb{R}^3 : x_1 = R\cos\left(\frac{2\pi x_3}{L}\right),\ x_2 = R\sin\left(\frac{2\pi x_3}{L}\right)\right\}$.

(a) Describe the geometry of the manifold. What is the geometric interpretation of the parameters R and L?
(b) Describe the translational kinematics of the particle on the curve in terms of one kinematics parameter.
(c) Interpret the geometric significance of the one kinematics parameter you selected.

2.9. Let $a > 0$, $b > 0$. A particle is constrained to move on an elliptical curve embedded in $\mathbb{R}^2$ that is given by $M = \{q \in \mathbb{R}^2 : \{\frac{q_1}{a}\}^2 + \{\frac{q_2}{b}\}^2 - 1 = 0\}$. Show that the kinematics of the particle can be expressed in terms of the kinematics on S^1 using the global diffeomorphism $\phi : \mathsf{S}^1 \to M$ given by $\phi(q) = (\frac{q_1}{a}, \frac{q_2}{b})$.

2.10. Let $a > 0$, $b > 0$, $c > 0$. A particle is constrained to move on an ellipsoidal surface given by $M = \{q \in \mathbb{R}^3 : \{\frac{q_1}{a}\}^2 + \{\frac{q_2}{b}\}^2 + \{\frac{q_3}{c}\}^2 - 1 = 0\}$. Show that the kinematics of the particle can be expressed in terms of the kinematics on S^2 using the global diffeomorphism $\phi : \mathsf{S}^2 \to M$ given by $\phi(q) = (\frac{q_1}{a}, \frac{q_2}{b}, \frac{q_3}{c})$.

2.11. Show the following results hold.

(a) The angular velocity $\omega \in \mathbb{R}^1$ of the planar pendulum satisfies $\omega = -q^T \mathrm{S}\dot{q}$.
(b) The angular velocity $\omega \in \mathbb{R}^3$ of the spherical pendulum satisfies $\omega = S(q)\dot{q}$.
(c) The angular velocities $\omega_1 \in \mathbb{R}^1$ and $\omega_2 \in \mathbb{R}^3$ for the connection of a planar pendulum and a spherical pendulum satisfy $\omega_1 = -q_1^T \mathrm{S}\dot{q}_1$ and $\omega_2 = S(q_2)\dot{q}_2$.
(d) The angular velocity $\omega \in \mathbb{R}^3$ of the rotating free rigid body satisfies $\omega = (R^T \dot{R})^\vee$, where $(\cdot)^\vee$ denotes the inverse of the hat map defined by (1.8).

2.12. Show that the following results are valid for the planar pendulum, the spherical pendulum, the double planar pendulum, and the double spherical pendulum.

(a) The velocity vector of any material point on the link of a planar pendulum is proportional to $\dot{q} \in \mathsf{T}_q\mathsf{S}^1$.
(b) The velocity vector of any material point on the link of a spherical pendulum is proportional to $\dot{q} \in \mathsf{T}_q\mathsf{S}^2$.
(c) The velocity vector of any material point on either link of a double planar pendulum is a linear combination of $\dot{q}_1 \in \mathsf{T}_{q_1}\mathsf{S}^1$ and $\dot{q}_2 \in \mathsf{T}_{q_2}\mathsf{S}^1$.
(d) The velocity vector of any material point on either link of a double spherical pendulum is a linear combination of $\dot{q}_1 \in \mathsf{T}_{q_1}\mathsf{S}^2$ and $\dot{q}_2 \in \mathsf{T}_{q_2}\mathsf{S}^2$.

2.13. Consider a rotating rigid body with a body-fixed point located at the origin of the body-fixed frame; this body-fixed point is also assumed to be fixed in the inertial frame at the origin of the inertial frame. Assuming the rotational motion of the rigid body is given by $t \to R(t) \in \mathsf{SO}(3)$, define the vector $q(t) = R(t)a \in \mathbb{R}^3$, where $a \in \mathbb{R}^3$ denotes an identified point on the rigid body in the body-fixed frame.

(a) Confirm that $q \in \mathbb{R}^3$ is the position vector of the identified point on the rigid body in the inertial frame.
(b) Determine an expression for the angular velocity vector of this identified point in the inertial frame.
(c) Show that this identified point moves on the surface of a sphere in the inertial frame.

2.14. Use the rotational kinematics of a rigid body to show that the angular velocity vector $\omega \in \mathbb{R}^3$, in the body-fixed frame, is constant in time if and only if the angular velocity vector $R\omega \in \mathbb{R}^3$, in the inertial frame, is constant in time.

2.15. Verify that the rotational kinematics of a rotating rigid body, expressed in terms of the transpose of the rows of the attitude matrix $r_i = R^T e_i$, $i = 1, 2, 3$, satisfy $r_i \in \mathsf{S}^2$, $i = 1, 2, 3$, and

$$\dot{r}_i = S(r_i)\omega, \quad i = 1, 2, 3.$$

2.16. In contrast with the prior development in this chapter, show that the rotational and translational kinematics of a rigid body constrained to undergo planar Euclidean motion can be described as follows. The configuration manifold is $\mathsf{SE}(2) = \mathsf{SO}(2) \times \mathbb{R}^2$; the configuration is $(R, x) \in \mathsf{SE}(2)$ where $R \in \mathsf{SE}(2)$ denotes the planar rotation matrix of the body and $x \in \mathbb{R}^2$ denotes the position vector of a selected point in the body expressed in a fixed two-dimensional Euclidean frame. Show that the rotational and translational kinematics are given by

$$\dot{R} = R\mathrm{S}\omega,$$
$$\dot{x} = Rv,$$

where $\omega \in \mathbb{R}^1$ is the scalar angular velocity of the rigid body and $v \in \mathbb{R}^2$ is the velocity vector of the selected point on the body expressed in the body-fixed Euclidean frame.

2.17. Consider the three-dimensional translational kinematics of a rigid link with ends constrained to slide along a straight line and a sphere of radius r; assume the straight line passes through the center of the sphere and the length of the link is $L > r$.

(a) Show that the configuration manifold

$$M = \left\{(q, x) \in \mathsf{S}^2 \times \mathbb{R}^1 : \|rq - xe_1\|^2 - L^2 = 0\right\}$$

is a differentiable manifold under the given assumptions.

(b) What are the resulting kinematics of the rigid link on M expressed in terms of the angular velocity vector?

2.18. Consider the planar pendulum with rotational kinematics given by

$$\dot{q} = \omega \mathrm{S} q,$$

where $q \in \mathsf{S}^1$, $\omega \in \mathbb{R}^1$ and

$$\mathrm{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Suppose that the angular velocity is given in terms of the configuration by $\omega = \sin q_2$. This defines a *closed loop* kinematics system.

(a) What are the closed loop kinematics? Show that they define a continuous vector field on S^1.

(b) Show that there are two equilibrium solutions.

(c) Determine linearized equations at each equilibrium solution. What are the stability properties of each equilibrium solution?

(d) Describe the physical motions of the planar pendulum as governed by the closed loop kinematics.

2.19. Consider the spherical pendulum with rotational kinematics given by

$$\dot{q} = S(\omega) q,$$

where $q \in \mathsf{S}^2$, $\omega \in \mathbb{R}^3$, and

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Suppose that the angular velocity vector is given in terms of the configuration by

$$\omega = \begin{bmatrix} 0 \\ \sin q_3 \\ -\sin q_2 \end{bmatrix}.$$

This defines a *closed loop* kinematics system.

(a) What are the closed loop kinematics? Show that they define a continuous vector field on S^2.
(b) Show that there are two equilibrium solutions.
(c) Determine linearized equations at each equilibrium solution. What are the stability properties of each equilibrium solution?
(d) Describe the physical motions of the spherical pendulum as governed by the closed loop kinematics.

2.20. Consider the rotational kinematics of a rigid body given by

$$\dot{R} = RS(\omega),$$

where $R \in \mathsf{SO}(3)$, $\omega \in \mathbb{R}^3$ and

$$S(\omega) = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Suppose that the angular velocity vector is given in terms of the configuration by

$$\omega = \sum_{i=1}^{3} e_i \times R^T e_i,$$

where e_1, e_2, e_3 denote the standard unit basis vectors in $\mathbb{R}^3$. This defines a *closed loop* kinematics system.

(a) What are the closed loop kinematics? Show that they define a continuous vector field on $\mathsf{SO}(3)$.
(b) Show that there are four equilibrium solutions.
(c) Determine linearized equations at each equilibrium solution. What are the stability properties of each equilibrium solution?
(d) Describe the rotational motions of the rigid body as governed by the closed loop kinematics.

2.21. Consider the translational kinematics of a rigid link with ends constrained to slide along a straight line and a circle in a fixed plane. The radius of the circle is r and the length of the rigid link is $L > r$.

(a) Assume the angular velocity of the end of the link constrained to the circle is identically zero. Describe the possible configurations of the rigid link.

(b) Assume the angular velocity of the end of the link constrained to the circle is a nonzero constant. Describe the motion of the link. What is the translational motion of the end of the rigid link that is constrained to move along a straight line?
(c) Assume the translational velocity of the end of the link constrained to the straight line is sinusoidal. Describe the motion of the link. What is the rotational motion of the end of the link that is constrained to move along the circle?

2.22. Consider the kinematics of a rigid link that is constrained to move within a fixed plane described by a two-dimensional Euclidean frame. Let A, B, and C denote three fixed points on the rigid link: point A of the link is constrained to translate along the x-axis of the Euclidean frame while point B of the link is constrained to translate along the y-axis of the Euclidean frame. The distance between points A and B is denoted by L, while the distance between point B and C is denoted by D. This mechanism is referred to as the *Trammel of Archimedes* [4] .

(a) Let $(x, y) \in \mathbb{R}^2$ denote the position vector of point C on the link in the Euclidean frame. What is the algebraic constraint that this position vector must satisfy? What is the configuration manifold embedded in $\mathbb{R}^2$? Describe the geometry of the configuration manifold.
(b) Describe the kinematics relationship of point C on the link, by expressing the time derivative of the configuration in terms of the scalar angular velocity $\omega \in \mathbb{R}^1$ of the rigid link and the configuration.
(c) Suppose the angular velocity of the link is constant; describe the resulting motion of point C on the link.

2.23. A knife-edge can slide on a horizontal plane without friction; the knife-edge is assumed to have a single point of contact with the plane. The motion of the point of contact of the knife-edge is constrained so that its velocity vector is always in the direction of the axis of the knife-edge. This constraint on the direction of the velocity vector is an example of a nonholonomic or non-integrable constraint . The motion of the knife-edge is controlled by the axial speed of the knife-edge and the rotation rate of the knife-edge about its point of contact. A two-dimensional Euclidean frame is introduced for the horizontal plane, so that $x \in \mathbb{R}^2$ denotes the position vector of the contact point of the knife-edge; let $q \in \mathsf{S}^1$ denote the direction vector of the knife-edge in the horizontal plane. Let $V \in \mathbb{R}^1$ be the scalar speed of the knife-edge and let $\omega \in \mathbb{R}^1$ be the scalar rotation rate of the knife-edge.

(a) Show that the nonholonomic constraint can be expressed as

$$\dot{x} = Vq,$$

and the rotational kinematics of the knife-edge are

$$\dot{q} = \omega S q.$$

(b) Show that these equations of motion can be written in the standard nonlinear control form

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = g_1(x, q)V + g_2(x, q)\omega.$$

This is an example of a drift-free nonlinear control system with control vector fields $g_1(x, q)$ and $g_2(x, q)$ defined on the manifold $\{x, q) : x \in \mathbb{R}^2,\ q \in \mathsf{S}^1\}$; what are the control vector fields?

(c) Suppose that the speed of the knife-edge is a positive constant and the rotation rate of the knife-edge is zero. Describe the motion of the point of contact of the knife-edge in the horizontal plane.

(d) Suppose that the speed of the knife-edge is a positive constant and the rotation rate of the knife-edge is a positive constant. Describe the motion of the point of contact of the knife-edge in the horizontal plane.

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