Chapter 2
Nonlinear Oscillators

In this book the pure nonlinear oscillator is considered. The pure nonlinear oscillator has a pure nonlinearity. The nonlinear function $f(x)$, which depends on the variable $x \in (-\infty, +\infty)$ and is a continual one, is defined as a pure nonlinear if it satisfies the condition that it has no linear approximation in any neighborhood of $x = 0$ (Mickens 2010). In general the pure nonlinearity is expressed as

$$f(x) = c_\alpha^2 x \left|x\right|^{\alpha-1}, \tag{2.1}$$

where c_α^2 is a positive constant which need not to be small, and the order of nonlinearity $\alpha \in \mathbb{R}_+$ is the positive rational number written as a termination decimal or as an exact fraction, $\alpha \in \mathbb{Q}_+ = \left\{\frac{m}{n} > 0 : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\right\}$ and $\mathbb{Z}$ is integer. The absolute value of the variable x is used in order to ensure the function (2.1) to be an odd one: for $x > 0$ the nonlinearity $f(x)$ is positive, and for $x < 0$ the nonlinearity is negative. Namely, $|\pm x| > 0$, independently on the sign of x. Analyzing the first derivative $C \equiv df/dx = \alpha c_\alpha^2 \left|x\right|^{\alpha-1}$, two types of pure nonlinearities are evident: hard, for $\alpha > 1$ when C continually increases with x, and soft, for $\alpha < 1$ when C continually decreases. For $\alpha = 1$, we have $C = c_\alpha^2 = const.$ Thus, for various types of nonlinearity (2.1), the pure nonlinear oscillators with hard, soft and linear properties are obtained.

In this chapter the examples for various nonlinearities (2.1) and corresponding pure nonlinear oscillators are given. The mathematical modeling of the pure nonlinear oscillator is expressed. Solution procedures for solving pure nonlinear differential equation for some special cases of nonlinearity are presented.

L. Cveticanin, *Strong Nonlinear Oscillators*, Mathematical Engineering,
DOI 10.1007/978-3-319-58826-1_2

2.1 Physical Models

Experimental investigation on a significant number of materials, for example: aluminum, titanium and other aircraft materials (Prathap and Varadan 1976), copper and copper alloys (Lo and Gupta 1978), aluminum alloys and annealed copper (Lewis and Monasa 1982), wood (Haslach 1985), ceramic materials (Colm and Clark 1988), hydrophilic polymers (Haslach 1992), composites (Chen and Gibson 1998), polyurethane foam (Patten et al. 1998), felt (Russell and Rossing 1998), etc., show that force decreases or increases more rapidly than the deflection, i.e. the stress–strain properties of the material are strong nonlinear. Then, the nonlinear dependence of the restoring force on the deflection is usually modelled as a polynomial whose exponent is any positive integer. However, there are examples of the systems, for which this exponent can be of non-integer order. Very often the experimentally obtained stress–strain diagrams are mathematically approximated with an one term of a polynome whose coefficient and order correspond to experimentally obtained data. The convenient function has the form (2.1). The order of nonlinearity is an integer or a non-integer, i.e., any rational number. In praxis, due to simplicity, it is assumed that the elastic properties of the materials are linear, or weak nonlinear. It has been emphasized that the assumption of the linear elasticity is correct only for small deformation and is not correct for large deformation force.

The nonlinearity (2.1) must not be the result of physical properties of the material (as it is previously mentioned), but also of the geometry of the system: shape and dimensions of the body, type of laying, loading etc. The examples with geometrical nonlinearity are the helicoidal and the conical springs made of the material with linear property, but due to geometry, the function (2.1) is nonlinear (Lou et al. 2009). The nonlinearities may be caused by physical effects such as the contacting of coils in a compressed coil spring, or by excessively straining the spring material (Beards 1995). The concept of the passive vibration isolator is based on the geometric nonlinear property of the system. Thus, three linear elastic springs fixed at one end and connected to each other at the other end represent a passive vibration isolator with quasi-zero stiffness characteristic whose property is nonlinear (Alabudzev et al. 1989; Carela et al. 2007; Kovacic et al. 2008a; Gatti et al. 2010). Such a nonlinear effect can be realized by application of the permanent magnets, too (Nijse 2001, Xing et al. 2005). The order of nonlinearity is usually assumed to be quadratic. Hence, Ravindra and Mallik (1994) applied the pure cubic nonlinearity for vibroisolating system. Dymnikov (1972) supposed the deformation characteristic of a radially loaded rubber cylinder to be cubic, too. Rivin (2003) presented the vibration isolators made of wire-mesh and felt materials, while Ibrahim (2008) introduces cable isolators for vibrations. The passive vibration isolators with their nonlinear properties are important for protecting of the buildings during the earthquake (Araky et al. 2010). Besides, these isolators can protect the ships from the sea waves excited vibrations (Xiong et al. 2005). Finally, it is important to mention that in contrary to the physical nonlinearity of the material, the geometric nonlinearity can be eliminated by proper design (Jutte 2008).

The practical application of the integer or non-integer order nonlinearity is evident in engineering (in micro-electro-mechanical systems (MEMS), nano-electro-mechanical devices, nanometer switches, vibration-, acoustic- and impact isolators (Bondar 1978; Pilipchuk 2007, 2010; Afsharfard and Farshidianfar 2012), snap-through mechanisms, etc.), but also for explaining phenomena in structural mechanics, nanotechnology, chemistry and physics.

Nowadays, the relevance of nanotechnology is well recognized, so new developments and applications based on nonlinear dynamics are reached in an interdisciplinary framework. The most common structure which is applied in nanotechnology is the system of nano-oscillators which represents the micro-electro-mechanical system (MEMS). The term MEMS refers to mechanical microstructures (on the order to 10–1000 μm), such as sensors, valves, gears, gyroscopes microbridges, electric microactuators etc. The MEMS are suitable to be modeled by one or more nano-masses connected by one or more nonlinear springs (see for example, Polo et al. 2009, 2010; Jones and Nenadic 2013). Usually, the nonlinearity in these systems is assumed to be cubic (Mojahedi et al. 2001) or quadratic, for example for micromirrors (Burns and Bright 1997). In microactuators the order of nonlinearity is in the interval [2,7] (Cortopassi and Englander 2010). The simulation values obtained for the system are compared with results measured on experimental devices (see de Sudipto and Aluru 2006a, b). The difference between the results is obvious due to the fact that the nonlinear property is not modelled in the correct manner, i.e., the order of nonlinearity is neither quadratic nor cubic, as is usually considered. Such an assumption does not correspond to the order of nonlinearity of the real system. The proposal is to design an adequate model of the system and to obtain the position and velocity time distribution with nanometer accuracy, which would be the starting parameters for control of actual MEMS devices. Only the correct input parameters with excellent precision would give the correct control laws and accurate motion of the MEMS. The mechanical model has to be improved and the accurate order of the nonlinearity (which may be a positive integer and/or non-integer) to be considered. Such modification in the mechanical model of MEMS would give not only the correct qualitative behavior of the system but also the most accurate quantitative results.

The nonlinearity described with (2.1) is registered in vehicle hanging, seats, and vehicle tires. In vehicle hanging the nonlinearity is of order 3/2 (Zhu and Ishitoby 2004), and for tires it is in the interval [2.5,3] (Dixon 1996). Supports for machines, cutting machines with periodical motion of the cutting tools, presses, etc., have also nonlinear properties. Nonlinearity is detected in music instruments (hammers in piano, for example). The human voice producing folds (voice cords) exhibit nonlinear property, too.

The vibrating system with nonlinearity (2.1) represents the pure nonlinear oscillator whose investigation is of prior interest in science and engineering.

2.2 Mathematical Models

Since 1918, when Duffing published his results in oscillator with cubic nonlinearity, a significant number of investigation in the free vibrations of the one-degree-of-freedom strong nonlinear undamped and damped oscillators is done. The most often investigated oscillator is with pure cubic nonlinearity ($\alpha = 3$), whose mathematical model is

$$\ddot{x} + c_3^2 x^3 = \varepsilon f(x, \dot{x}), \tag{2.2}$$

with the initial conditions

$$x(0) = A, \qquad \dot{x}(0) = 0. \tag{2.3}$$

where c_3^2 is a positive constant and εf is a small additional function in comparison to the strong nonlinearity. Usually, it is impossible to find the exact analytical solution of (2.2) with initial conditions (2.3), and various approximate analytical solving methods are developed. Let us mention some of them: elliptic perturbation method (Mickens and Oyedeji 1985), averaging procedure (Coppola and Rand 1990), variable amplitude and phase method (Yuste and Bejarano 1986, 1990; Cveticanin 2009a), Galerkin's method (Chen and Gibson 1998), harmonic balance method (Chen 2003; Leung et al. 2012), modified Lindstedt–Poincar é method (Cheng et al. (1991), series expansion method (Kovacic and Brennan 2008), etc. The pure Duffing oscillator with additional damping term is also widely investigated. Trueba et al. (2000) and Sharma et al. (2012) considered the Duffing oscillator with positive linear and cubic term and small linear and cubic damping. The influence of the linear damping is investigated by Waluya and van Horssen (2003) and also by Cveticanin (2011). Cveticanin (2004, 2008, 2009b) and Akinpelu (2011) considered the influence of the quadratic damping on the vibrations of the Duffing oscillator. Siewe et al. (2009) and also Kanai and Yabuno (2012) extended the investigation to the so called Rayleigh–Duffing oscillator with cubic and linear damping terms. The role of nonlinear damping in soft Duffing oscillator with a simultaneous presence of viscous damping has been discussed in Ravindra and Mallik (1994) and Sanjuan (1999). Baltanas et al. (2001) have studied the effect of a nonlinear damping term, proportional to the power of velocity, on the dynamics of the double-well Duffing oscillator.

Finally, it has to be mentioned that a wide range of approximate solutions for (2.2) is given in Chap. 4 "Analysis Technique for the Various Forms of the Duffing Equation" in the book entitled: "The Duffing Equation: Nonlinear Oscillators and their Behaviour" edited by Brennan and Kovacic.

Cveticanin in her papers (1998, 2001, 2005a, b) considered the Duffing oscillator with complex-valued function. She adopted the previously mentioned methods for the differential equation with complex variable.

The oscillator with quadratic nonlinearity is also intensively studied. In Chen et al. (1998) and Chen and Cheung (1996) the elliptic perturbation method is applied for solving of the second order differential equation with quadratic nonlinearity

$$\ddot{x} + c_1^2 x + c_2^2 x^2 = \varepsilon f(x, \dot{x}), \tag{2.4}$$

where c_1^2 and c_2^2 are the constants of the linear and quadratic term. Cveticanin (2003) considered the oscillator with quadratic term and an additional constant F

$$\ddot{x} + c_1^2 x + c_2^2 x\,|x| = F.$$

For the small nonlinearity the approximate solution is assumed in the form of a trigonometric function. A similar model was considered by Mickens (1981)

$$\ddot{x} + c_1^2 x - c_2^2 x^2 = F, \tag{2.5}$$

who gives the solution by applying of the power series solution procedure. The expression (2.5) describes some phenomena in general reliability and also in solid-state physics. The solution for the pure nonlinear oscillator with quadratic term

$$\ddot{x} + c_1^2 x + c_2^2 x\,|x| = \varepsilon f(x, \dot{x}), \tag{2.6}$$

is determined assuming the same methods as for the Duffing equation (Cveticanin 2004).

Combining the both nonlinearities, the quadratic and cubic one, the mixed parity oscillator is formed (Hu 2007). The harmonic balance method is seen to be the most simple solution procedure.

Recently, the more general type of pure nonlinear oscillators is investigated: the nonlinearity is an integer order and $\alpha > 1$. Thus, Andrianov (2002), Andrianov and Awrejcewicz (2003a), Cveticanin (2011), Kovacic (2011) and Cveticanin et al. (2012) consider the oscillators with pure nonlinearity of any integer, specially of higher order i.e., $\alpha \gg 1$. Opposite, Awrejcewicz and Andrianov (2002) considered the oscillations of the nonlinear system where the order of nonlinearity is extremely small, i.e., $\alpha \approx 0$. Applying the harmonic balance method the approximate analytic solution is obtained. Andrianov and Awrejcewicz (2003b) analyzed the asymptotic behavior of the oscillator with damping and high power form nonlinearity.

Mickens (2001) was the first to investigate the pure nonlinear oscillators with non-integer order. Cooper and Mickens (2002) considered the oscillator with $x^{4/3}$ potential. The generalized harmonic balance/numerical method for determining analytical approximations is applied. Ozis and Yildirm (2007) applied the modified Lindsedt–Poincaré method for solving the differential equation with nonlinearity of order $1/3$. van Horssen (2003) generalized the problem and investigated the oscillator with the order of nonlinearity $\alpha < 1$. The mathematical model of the oscillator is

$$\ddot{x} + x^{1/(2n+1)} = 0, \tag{2.7}$$

where $n = 1, 2, 3, \ldots$ is a positive integer. van Hoorsen assumed the approximate analytical solution in the form of a trigonometric function. Applying the harmonic balance method, he obtained the approximate value of the frequency of vibration as

$$\Omega(x_0) = \left[\frac{2^{2n}}{\binom{2n+1}{n} x_0^{2n}} \right]^{1/(4n+2)}. \tag{2.8}$$

Belendez et al. (2007) obtained higher-order approximations applying a modified He's homotopy perturbation method. Hu and Xiong (2003) extended the consideration to the system

$$\ddot{x} + x^{(2m+2)/(2n+1)} = 0, \tag{2.9}$$

where $n = 1, 2, 3, \ldots$, and $m = 1, 2, 3, \ldots$ are positive integers. Using the same solution procedure the following approximate frequency of vibration is obtained

$$\Omega(x_0) = \left[\frac{2^{2(2n-m)} \binom{2(m-n)+1}{m-n}}{\binom{2n+1}{n} x_0^{2(2n-m)}} \right]^{1/(4n+2)}. \tag{2.10}$$

Gottlieb (2003) compared this value with the exact one. Andrianov and Awrejcewicz (2003a) and also Andrianov and van Horssen (2006) considered the nonlinear oscillator (2.9) extended with a negative damping of Van der Pol type.

Nowadays, the oscillator model (2.9) is generalized and the nonlinearity of any rational number α is studied. Various methods for obtaining the frequency of oscillation are developed: the improved Lindstedt–Poincaré method (Amore and Aranda 2005), the series expansion method (Kovacic and Brennan 2008), the adopted Lindstedt–Poincaré method (Belendez et al. 2007; Cveticanin et al. 2010), the non-simultaneous variational approach (Kovacic et al. 2010), Hamiltonian approach (Cveticanin et al. 2010), the modified Lindstedt–Poincaré method (Cheng et al. 1991; He 2002a, b; Ozis and Yildirm 2007), the decomposition method (Kermani and Dehestani 2013), etc. Very often the solution procedures applied for pure nonlinear oscillators require addition of a linear term into the differential equation. The equation is transformed to the model with strong linearity and a weak nonlinearity, which is already widely investigated and a numerous solution procedures are developed. The most often applied solution methods are: the Krylov and Bogolubov (1943), Bogolubov and Mitropolski (1963) methods, the multiple scale method (Nayfeh and Mook 1979), perturbation method for certain nonlinear oscillators (Burton 1984), the method of straightforward expansion, the Lindstedt–Poinceré method, the homotopy perturbation technique (He 1998a, b; Cveticanin 2006, 2009a), the homotopy analysis method (Liao and Tan 2007), combined equivalent linearization and averaging perturbation method (Mickens and Oyedeji 1985; Mickens 2003), the iteration procedure for calculating approximations to periodic solutions (Mickens 2005, 2006), the method of slowly varying amplitude and phase (Cveticanin 2009b; Mickens 2010), etc. The mentioned methods are appropriate for solving strong linear and additional weak nonlinear complex-valued differential equations of vibration (Cveticanin 1992, 1993), too.

Unfortunately, there are numerous oscillators where the nonlinearity is much stronger than the linearity and even the oscillator is purely nonlinear. For such systems the application of the aforementioned methods is not possible. Namely, the differential equation is without a linear term and also the linearization of the equation is not possible due to the property of the system. These oscillators are not the perturbed versions of the linear ones and their behavior is far of those obtained for linear ones. To exceed this problem, in this book the solution procedures for pure strong nonlinear differential equation, which describe the oscillatory motion of pure nonlinear oscillator, are presented.

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Strong Nonlinear Oscillators
Analytical Solutions
Cveticanin, L.
2018, XII, 317 p. 93 illus., 21 illus. in color., Hardcover
ISBN: 978-3-319-58825-4